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Some inverse scattering problems on star-shaped graphs

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ABSTRACT

Having in mind applications to the fault-detection/diagnosis of lossless electrical networks, here we consider some inverse scattering problems for Schrödinger operators over star-shaped graphs. We restrict ourselves to the case of minimal experimental setup consisting in measuring, at most, two reflection coefficients when an infinite homogeneous (potential-less) branch is added to the central node. First, by studying the asymptotic behavior of only one reflection coefficient in the high-frequency limit, we prove the identifiability of the geometry of this star-shaped graph: the number of edges and their lengths. Next, we study the potential identification problem by inverse scattering, noting that the potentials represent the inhomogeneities due to the soft faults in the network wirings (potentials with bounded H^1-norms). The main result states that, under some assumptions on the geometry of the graph, the measurement of two reflection coefficients, associated to two different sets of boundary conditions at the external vertices of the tree, determines uniquely the potentials; it can be seen as a generalization of the theorem of the two boundary spectra on an interval.

1. Introduction

The rather extensive literature concerning the "inverse scattering problem" and the "inverse Sturm–Liouville problem" on graphs have mostly followed separate pathways except for a very few results [15,14,1]. In the following paragraphs, we briefly recall the previous results on these subjects and at the end we will situate the result of this paper with respect to the others. Indeed, as it will be seen later, the inverse Sturm–Liouville problem considered in this paper raises from the necessity of finding a minimal setup for solving the inverse scattering problem.

A first set of results deals with inverse scattering problems over graphs. The article [17] considers a star-shaped graph consisting of N infinite branches and solves the inverse scattering problem assuming the measurement of $N-1$ reflection coefficients. Next, in [18], Harmer provides an extension of the previous result with general self-adjoint boundary conditions at the central node. This however necessitates the knowledge of N reflection coefficients. The paper [24] studies the relation between the scattering data and the topology of the graph. The authors show that the knowledge of the scattering matrix is not enough to determine uniquely the topological structure of a generic graph.

In [9], Brown and Weikard prove that the knowledge of the whole Dirichlet-to-Neumann map for a tree determines uniquely the potential on that tree.

In [1], Avdonin and Kurasov consider a star-shaped graph with N finite branches. They prove that the knowledge of a diagonal element of the response operator allows one to reconstruct the graph, i.e. the total number of edges and their lengths. This result is very similar to Theorem 1 of the present paper and can be seen as a time-domain version of Theorem 1 (see the remarks after Theorem 1 for further details). Furthermore, they prove, through the same paper [1], that

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the knowledge of the diagonal elements of the response operator over all but one external nodes is enough to identify the potentials on the branches. At last they prove an extension of the result to the more generic tree case where they need the whole response operator.

The two more recent papers [3,2] consider other types of inverse scattering problems on trees. The first paper considers the case of potential-free Schrödinger operators over the branches of a co-planar tree where the matching conditions at the internal nodes of the graph depends explicitly on the angles between the branches. The authors prove, for the case of star graph, that the knowledge of the diagonal elements of the Titchmarsh–Weyl matrix at the external nodes is enough to reconstruct the lengths and the angles between the branches. This result is then extended to the more generic tree case, where further elements of Titchmarsh–Weyl matrix are needed. The second paper [2], based on a previous one [23], considers the inverse problem of characterizing the matching condition for the internal node of a star graph through the knowledge of a part of the scattering matrix.

As mentioned above, in parallel to the research on inverse scattering problems, another class of results consider the inverse spectral problem for Sturm–Liouville operators on compact graphs. These results can be seen as extensions of the classical result provided by Borg [7], on the recovering of the Sturm–Liouville operator from two spectra on a finite interval.

A first set of results has been obtained by Yurko [32–34]. The article [32] deals with the inverse spectral problem on a tree. It provides a generalization of the Borg's result in the following sense: for a tree with n boundary vertices, it is sufficient to know n spectra, corresponding to n different settings for boundary conditions at the external nodes, to retrieve the potentials on the tree. In a recent work [34], the same kind of result is proposed for a star-shaped graph including a loop joined to the central node. Finally, [33] provides a generalization of [32] to higher order differential operators on a star-shaped graph.

Pivovarchik and co-workers provide a next set of results in this regard [25–27,8]. In particular, in [27], the author proves that under some restrictive assumptions on the spectrum of a Sturm–Liouville operator on a star-shaped graph with some fixed boundary conditions, the knowledge of this spectra can determine uniquely the Sturm–Liouville operator.

A third set of results deals with the problem of identifying the geometry of the graph [16,30]. In particular, [16] provides a well-posedness result for the identification of the lengths of the branches through the knowledge of the spectrum. This result is to be compared with Theorem 1 of this paper. While [16] considers a more general setting of generic graphs, it assumes the $\mathbb{Q}$-independence of lengths, an assumption that has been removed in Theorem 1 for the simpler case of a star-shaped graph.

Belishev considers the potential-free case over a tree and proves that the knowledge of the eigenvalues and the normal derivatives of the Dirichlet eigenfunctions at the external node is enough to identify the geometry of the tree up to a spatial isometry [4]. Together with his co-workers, he further provides an identification algorithm and numerical simulations [5]. Carlson considers the potential-free case over a directed graph and provides information on the boundary conditions at the external nodes as well as the lengths through the spectrum of the operator [11]. Finally Kursov and Nowaczyk consider the potential-free case over a finite graph and similarly to [16] treat the problem of identifying the geometry through the spectral data, provided that the branch lengths are rationally independent [22].

In this paper, we consider a class of inverse scattering problems on star-shaped graphs, having in mind certain applications such as the fault-detection/diagnosis of electrical networks through reflectometry-type experiments. Even though a part of the obtained results (Theorems 1 and 2) can be directly applied to such applications, some of them (see Theorem 3 and assumption **A2**) remain preliminary results and need significant improvement. However, from a theoretical insight all the results are original and provide some new uniqueness results for the solutions of inverse scattering problems on networks. Note that, similarly to the case of a simple line [12], the existence of a solution to the inverse scattering problem (i.e. classifying the scattering data for which there exists a solution to the inverse scattering problem) remains for itself a complete subject apart and we do not consider here such existence problems. In other words, we assume that the scattering data (and notably the reflection coefficient) are precisely obtained from a real physical system and therefore the existence of the solution to the inverse scattering problem is ensured by the existence of the physical system. Before announcing the main results of this paper, let us briefly explain how the reflectometry of an electrical network can be related to inverse scattering problems.

The electric signal transmission through a lossless wired network is, generally, modeled with the "Telegrapher's equation" and characterized by the parameters L and C (functions of the space position z along the transmission lines) representing, respectively, the inductance and the capacitance. In the harmonic regime, this Telegrapher's equation may be written as

$$\begin{cases} \dfrac{d}{dz}V(k,z) - \imath k L(z) I(k,z) = 0, \\ \dfrac{d}{dz}I(k,z) - \imath k C(z) V(k,z) = 0, \end{cases} \tag{1}$$

assuming the parameters $L(z)$ and $C(z)$ to be strictly positive and twice continuously differentiable with respect to z. Following [20], we apply the Liouville transformation $x(z) = \int_0^z \sqrt{L(z)C(z)}\,ds$ and we also use the convention $C(x) \equiv C(z(x))$, $L(x) \equiv L(z(x))$, $V(k,x) \equiv V(k,z(x))$ and $I(k,x) \equiv I(k,z(x))$. Setting

$$y(x,k) = \left(\frac{C(x)}{L(x)}\right)^{\frac{1}{4}} V(k,x)$$

the Telegrapher equation (1) becomes

$$-\frac{d^2}{dx^2}y(x,k)+q(x)y(x,k)=k^2y(x,k),$$

where $q(x)=(\frac{C(x)}{L(x)})^{-\frac{1}{4}}\frac{d^2}{dx^2}(\frac{C(x)}{L(x)})^{\frac{1}{4}}$. Note, in particular that, in these new terms, the electrical current is given by

$$I(k,x)=\frac{1}{ik}\left(\frac{1}{2}\frac{Z_c'(x)}{Z_c^{3/2}(x)}y(x,k)+\frac{1}{Z_c^{1/2}(x)}\frac{d}{dx}y(x,k)\right),$$

where $Z_c(x)=\sqrt{L(x)/C(x)}$ denotes the characteristic impedance over the transmission line.

To cope with the network case, we translate the Kirchhoff rules at the nodes of the network within this new modeling framework. Considering v a vertex of the network and $\mathcal{E}(v)$ the set of edges joining at v, the Kirchhoff's matching condition can be written as

$$\sum_{e\in\mathcal{E}(v)}I_e(v,k)=0\quad\text{and}\quad V_e(v,k)=V_{e'}(v,k),\quad\forall e,e'\in\mathcal{E}(v),$$

where I_e and V_e denote the current and the tension over the branch e and where the sum is an algebraic sum (direction of current is needed to be taken into account). Assuming the continuity of the characteristic impedance $Z_c(x)=\sqrt{L(x)/C(x)}$ at the nodes of the graph (an assumption that we will make everywhere through this paper), the above Kirchhoff rules give rise to the following matching conditions after the Liouville transformation:

$$\begin{aligned}&y_e(v,k)=y_{e'}(v,k)=:\bar{y}(v,k),\quad\forall e,e'\in\mathcal{E}(v),\\&\sum_{e\in\mathcal{E}_{out}(v)}y_e'(v,k)-\sum_{e\in\mathcal{E}_{in}(v)}y_e'(v,k)=-\frac{1}{2}\frac{\sum_{e\in\mathcal{E}(v)}(Z_c^e)'(v)}{Z_c(v)}\bar{y}(v,k)\end{aligned}$$

where $\mathcal{E}_{out}(v)$ (resp. $\mathcal{E}_{in}(v)$) denotes the set of edges in $\mathcal{E}(v)$ such that the current's direction is outward (resp. inward) with respect to v. Furthermore, Z_c^e denotes the characteristic impedance over the edges $e\in\mathcal{E}(v)$, admitting the same value $Z_c(v)$ at the vertex v.

The faults, in which we are interested here, are represented by the lengths of the branches (hard faults) and by the heterogeneities of $q(x)$ along the branches (soft faults). Indeed, in the perfect situation, the parameters $L(z)$ and $C(z)$ are constant on the network and therefore the potential $q(x)$ is uniformly zero on the whole network. While, we consider the particular case of a star-shaped network, the reflectometry experiment is based on a far-field method consisting in adding a uniform (constants L and C) infinite wire joined to the network at its central node. In practice, connecting a matched charge to the external node of a finite line is sufficient to emulate the electrical propagation through an infinite line (we refer to the preprint version of this paper [29] for further details in this regard).

A preliminary version of this paper can be found in [29] where some more details on the above mentioned applications are provided. More recently, applying the same kind of approaches as in [29], Yang has considered an inverse spectral problem on a star-shaped graph [31]. In particular, the author shows how to reconstruct the potential on a fixed edge from the knowledge of some spectra once the potentials on all other edges is known.

2. Main results

Throughout this paper, Γ represents a compact star-shaped network consisting of segments $(e_j)_{j=1}^N$ of lengths l_j joining at a central node. It will be convenient to take the same positive orientation on all branches, from the central node at $x=0$ toward the increasing x. Γ^+ is the extended graph where a uniform (potential-less) semi-infinite branch e_0 is also added to the graph Γ with the reverse orientation $(-\infty,0]$.

Consider the Schrödinger operator on Γ^+

$$\mathcal{L}_{\mathcal{N},\mathcal{D}}^{+}=\bigotimes_{j=0}^{N}\left(-\frac{d^2}{dx^2}+q_j(x)\right)\quad\big(q_0(x)\equiv 0\big)\tag{2}$$

acting on the domain

$$D\big(\mathcal{L}_{\mathcal{N},\mathcal{D}}^{+}\big)=\text{closure of }C_{\mathcal{N},\mathcal{D}}^{\infty}\quad\text{in }H^2\big(\Gamma^+\big),\tag{3}$$

where $C_{\mathcal{N}}^{\infty}(\Gamma^+)$ and $C_{\mathcal{D}}^{\infty}(\Gamma^+)$ denote the spaces of infinitely differentiable functions $f=\bigotimes_{j=0}^N f_j$ defined on Γ^+ both satisfying the boundary conditions at the central node

$$\begin{aligned}&f_j(0)=f_0(0),\quad j=1,\dots,N,\\&\sum_{j=1}^{N}f_j'(0)-f_0'(0)=Hf_0(0).\end{aligned}\tag{4}$$

Here

$$H = -\frac{1}{2}\frac{\sum_{j=1}^{N}(Z_c^j)'(0)}{Z_c(0)}, \tag{5}$$

where Z_c^j denotes the characteristic impedance over the branch number j and $Z_c(0)$ is the common value of the impedances at the central node.

Moreover for $C_{\mathcal{N}}^{\infty}(\Gamma^+)$ (resp. for $C_{\mathcal{D}}^{\infty}(\Gamma^+)$), we assume Neumann condition (resp. Dirichlet condition) at all boundary vertices:

$$f_j'(l_j) = 0 \quad \text{for } f \in C_{\mathcal{N}}^{\infty}\bigl(\Gamma^+\bigr) \quad \text{and} \quad f_j(l_j) = 0 \quad \text{for } f \in C_{\mathcal{D}}^{\infty}\bigl(\Gamma^+\bigr), \tag{6}$$

for $j = 1, \dots, N$.

Remark 1. The operators $(\mathcal{L}_{\mathcal{N},\mathcal{D}}^+, D(\mathcal{L}_{\mathcal{N},\mathcal{D}}^+))$ are essentially self-adjoint. To prove this fact we observe first that these operators are a compact perturbation of the operators $\bigotimes_{j=0}^{n}(-\frac{d^2}{dx^2})$ with the same boundary conditions. Now, we apply a general result by Carlson [10] on the self-adjointness of differential operators on graphs. Indeed, following Theorem 3.4 of [10], we only need to show that at a node connecting m edges, we have m linearly independent linear boundary conditions. At the terminal nodes of $\{e_j\}_{j=1}^{N}$ this is trivially the case as there is one branch and one boundary condition (Dirichlet or Neumann). At the central node it is not hard to verify that (4) define $N+1$ linearly independent boundary conditions as well. This implies that the operators $(\mathcal{L}_{\mathcal{N},\mathcal{D}}^+, D(\mathcal{L}_{\mathcal{N},\mathcal{D}}^+))$ are essentially self-adjoint and therefore that they admit a unique self-adjoint extension on $L^2(\Gamma^+)$.

The reflection coefficients $R_{\mathcal{N},\mathcal{D}}(k)$ for $\mathcal{L}_{\mathcal{N},\mathcal{D}}^+$ are defined by the following proposition:

Proposition 1. *Assume $q = \bigotimes_{j=1}^{N} q_j \in C^0(\Gamma)$ and $q_j(0) = 0$ for $j = 1, \dots, N$ (this ensures the continuity of $q(x)$ at the central node when the uniform branch e_0 is added). Then for almost every $k \in \mathbb{R}$ there exists a unique solution*

$$\Psi_{\mathcal{N},\mathcal{D}}(x,k) = \bigotimes_{j=0}^{N} y_{\mathcal{N},\mathcal{D}}^j(x,k),$$

of the scattering problem and associated to it, a unique reflection coefficient $R_{\mathcal{N},\mathcal{D}}(k)$. This means that for almost every $k \in \mathbb{R}$, there exist a unique function $\bigotimes_{j=0}^{N} y_{\mathcal{N},\mathcal{D}}^j(x,k)$ and a unique constant $R_{\mathcal{N},\mathcal{D}}(k)$ satisfying

- $-\frac{d^2}{dx^2} y_{\mathcal{N},\mathcal{D}}^j(x,k) + q_j(x) y_{\mathcal{N},\mathcal{D}}^j(x,k) = k^2 y_{\mathcal{N},\mathcal{D}}^j(x,k)$ *for* $j = 0, \dots, N$;
- $(y_{\mathcal{N},\mathcal{D}}^j(x,k))_{j=0}^{N}$ *satisfy the boundary conditions* (4) *and* (6);
- $y_{\mathcal{N},\mathcal{D}}^0(x,k) = e^{+ikx} + R_{\mathcal{N},\mathcal{D}}(k)e^{-ikx}$.

Finally, the reflection coefficient $R_{\mathcal{N},\mathcal{D}}(k)$ can be extended by continuity to all $k \in \mathbb{R}$.

A proof of this proposition will be given in Section 3.1.

As a first inverse problem, we consider the inversion of the geometry of the graph. In fact, we will prove the well-posedness of the inverse problem of finding the number of branches N and the lengths $(l_j)_{j=1}^{N}$ of a star-shaped graph through only one reflection coefficient $R_{\mathcal{N}}(k)$ (the case of Dirichlet reflection coefficient can be treated similarly).

Theorem 1. *Consider a star-shaped network Γ composed of n_j branches of length l_j ($j = 1, \dots, m$) all joining at a central node so that the whole number of branches N is given by $\sum_{j=1}^{m} n_j$. Assume for the potential q on the network to be $C^0(\Gamma)$ and that it takes the value zero at the central node. Then the knowledge of the Neumann reflection coefficient $R_{\mathcal{N}}(k)$ determines uniquely the parameters $(n_j)_{j=1}^{m}$ and $(l_j)_{j=1}^{m}$.*

The problem of identifying the geometry of a graph through the knowledge of the reflection coefficient has been previously considered in [16,22,1]. Through the two first papers, the authors consider a more general context of any graph and not only a star-shaped one. However, in order to ensure a well-posedness result, they need to assume a strong assumption on the lengths consisting in their $\mathbb{Q}$-independence. The third result [1] states a very similar result to that of Theorem 1 for time-domain reflectometry (see Lemma 2 of [1]). The authors also provide a frequency-domain version of their result (see Lemma 3 of [1]); however their proof is strongly based on the proof of the time-domain result. We believe that the proof provided in the present paper, exploring the high-frequency regime of the reflection coefficient and providing a frequency-based constructive method, can be useful from an engineering point of view, where we are interested in detecting the faults without stopping the normal activity of the transmission network (we therefore need to apply test frequencies that are much higher than the activity frequencies of the transmission network). This proof will be given in Section 3.2.

A second inverse problem can be formulated as the identification of the potentials on the branches. The following theorem provides a global uniqueness result concerning the quantities $\bar{q}_j := \int_0^{l_j} q_j(s)\,ds$.

Theorem 2. *Assume for the star-shaped graph Γ that*

B1 $l_j \neq l_{j'}$ *for any $j, j' \in \{1, \dots, N\}$ such that $j \neq j'$.*

If there exist two potentials $q = \bigotimes_{j=1}^N q_j$ and $\tilde{q} = \bigotimes_{j=1}^N \tilde{q}_j$ in $H^1(\Gamma)$, satisfying $q_j(0) = \tilde{q}_j(0) = 0$, and giving rise to the same reflection coefficient, $R_{\mathcal{N}}(k) \equiv \tilde{R}_{\mathcal{N}}(k)$, one necessarily has

$$\int_0^{l_j} q_j(s)\,ds = \int_0^{l_j} \tilde{q}_j(s)\,ds, \quad j = 1, \dots, N.$$

Remark 2. Getting back to the transmission line parameters, the moment $\int_0^{l_j} q_j(s)\,ds$ can be written as

$$\int_0^{l_j} q_j(s)\,ds = \frac{1}{4}\int_0^{l_j} \frac{|(Z_c^j)'(s)|^2}{|Z_c^j(s)|^2}\,ds - \frac{1}{2}\left(\frac{(Z_c^j)'(l_j)}{Z_c^j(l_j)} - \frac{(Z_c^j)'(0)}{Z_c^j(0)}\right), \tag{7}$$

where Z_c^j denotes the characteristic impedance over the branch j.

This theorem allows us to identify the situations where the soft faults in the network cause a change of the quantities $\bar{q}_j$. In particular, it allows us to identify the branches on which these faults have happened. A next test, by analyzing these branches separately, will then allow the engineer to identify more precisely the faults. A proof of this theorem will be provided in Section 3.4.

Next, we will consider the situations where the faults in the network, do not affect the quantities $\bar{q}_j$. Keeping in mind the application to the transmission line network, this means that:

A1 $\bar{q}_j = \int_0^{l_j} q_j(s)\,ds = 0$ for $j = 1, \dots, N$;

as for the perfect setting, we had assumed uniform transmission lines: L and C constant.

In order to provide a well-posedness result for such situations, we need more restrictive assumptions on the geometry of the graph:

A2 For any $j, j' \in \{1, \dots, N\}$ such that $j \neq j'$, $l_j/l_{j'}$ is an algebraic irrational number.

Under this assumption, the value

$$M(\Gamma) := \max\left\{m \in \mathbb{N} \;\Big|\; \left|\frac{l_i}{l_j} - \frac{1}{m}\right| < \frac{1}{m^3}, \text{ for some } i \neq j\right\} \tag{8}$$

is well defined and is finite. In fact, by Thue–Siegel–Roth Theorem [28], for any irrational algebraic number α, and for any $\delta > 0$, the inequality

$$|\alpha - p/q| < 1/|q|^{2+\delta}, \tag{9}$$

has only a finite number of integer solutions p, q $(q \neq 0)$.

Before stating the final theorem, we give a lemma on the asymptotic behavior of the eigenvalues of the Sturm–Liouville operator $-\frac{\partial^2}{\partial x^2} + q(x)$ on the segment $[0, l]$, with Dirichlet boundary condition at 0 and Neumann boundary condition at l (the case of Dirichlet–Dirichlet boundary condition can be treated similarly). This lemma allows us to define a constant $C_0(l)$ which will be used in the statement of the final theorem.

Lemma 1. *Assume for the potential $q(x) \in H^1(0, l)$ that $q(0) = 0$, that $\|q\|_{L^\infty(0,l)} < \frac{\pi^2}{4l^2}$ and that $\int_0^l q(s)\,ds = 0$. Then, there exists a constant $C_0(l)$ such that λ_n, the n-th eigenvalue of the operator $-\frac{\partial^2}{\partial x^2} + q(x)$ on the segment $[0, l]$, with Dirichlet boundary condition at 0 and Neumann boundary condition at l, satisfies*

$$\left|\lambda_n - \frac{(2n-1)^2\pi^2}{4l^2}\right| \leqslant C_0(l)\frac{\|q\|_{H^1(0,l)}}{2n-1}.$$

A proof of this lemma, based on the perturbation theory of linear operators [21], will be given in Appendix A.

In order to state the final theorem, we define the following constants only depending on the geometry of the graph Γ (lengths of branches):

$$C_1(\Gamma) := \min\left\{ \frac{\pi^2}{4l_j^{5/2}} \;\Big|\; j = 1, \dots, N \right\}, \tag{10}$$

$$C_2(\Gamma) := \min\left\{ \frac{\pi^2}{4l_i l_j (C_0(l_i) + C_0(l_j))} \;\Big|\; i \neq j,\ i, j = 1, \dots, N \right\}, \tag{11}$$

$$C_3(\Gamma) := \min\left\{ \frac{\pi^2}{C_0(l_i) + C_0(l_j)} \cdot \left| \frac{(2n-1)^2}{l_j^2} - \frac{(2n'-1)^2}{l_i^2} \right| \;\Bigg|\; \begin{array}{l} n = 1, 2 \dots, M(\Gamma) \\ n' = 1, 2, \dots \\ i \neq j,\ i, j = 1, \dots, N \end{array} \right\}, \tag{12}$$

$$C(\Gamma) := \min\bigl(C_1(\Gamma), C_2(\Gamma), C_3(\Gamma)\bigr). \tag{13}$$

Note, in particular that $C_3(\Gamma)$ is strictly positive as the lengths l_i and l_j are two-by-two $\mathbb{Q}$-independent. We have the following theorem:

Theorem 3. *Consider a star-shaped graph Γ satisfying the geometrical assumption* **A2**. *Take the strictly positive constant $C(\Gamma)$ as defined by* (13) *and consider two potentials q and $\tilde{q}$ belonging to $H^1(\Gamma)$, satisfying $q_j(0) = \tilde{q}_j(0) = 0$, the assumption* **A1**, *and*

$$\|q\|_{H^1(\Gamma)} < C(\Gamma) \quad \textit{and} \quad \|\tilde{q}\|_{H^1(\Gamma)} < C(\Gamma).$$

If they give rise to the same Neumann and Dirichlet reflection coefficients:

$$R_{\mathcal{N}}(k) \equiv \tilde{R}_{\mathcal{N}}(k) \quad \textit{and} \quad R_{\mathcal{D}}(k) \equiv \tilde{R}_{\mathcal{D}}(k),$$

then $q \equiv \tilde{q}$.

A proof of this theorem will be given in Section 3.5. We end this section by a remark on the assumption **A2**:

Remark 3. The assumption **A2** seems rather restrictive and limits the applicability of Theorem 3 in real settings. In fact, such kind of assumptions have been previously considered in the literature for the exact controllability of the wave equations on networks [35]. In general, removing this kind of assumptions, one can ensure approximate controllability results rather than the exact controllability ones. Theorem 3 can be seen in the same vein as providing a first exact identifiability result. However, in order to make it applicable to real settings one needs to consider improvements by relaxing the assumption **A2** and looking instead for approximate identifiability results. This will be considered in future work.

Finally, we note that the only place, where we need the assumption **A2**, is to ensure that there exists at most a finite number of co-prime factors $(p, q) \in \mathbb{N} \times \mathbb{N}$, such that the Diophantine approximation (9) holds true. However, this is a classical result of the Borel–Cantelli Lemma that for almost all (with respect to Lebesgue measure) positive real α's this Diophantine approximation has finite number of solutions. Therefore the assumption **A2** can be replaced by the weaker assumption of $l_j/l_{j'}$ belonging to this set of full measure.

3. Proofs of the statements

3.1. Direct scattering problem

This subsection has for goal to give a proof of Proposition 1 ensuring the well-posedness of the direct scattering problem and allowing us to define the reflection coefficients $R_{\mathcal{N},\mathcal{D}}(k)$.

Proof of Proposition 1. This proof gives us a concrete method for obtaining scattering solutions. Indeed, we will propose a solution and we will show that it is the unique one.

In this aim, we need to use Dirichlet/Neumann fundamental solutions of a Sturm–Liouville boundary problem.

Definition 1. Consider the potentials q_j as before and extend them by 0 on $(-\infty, 0)$ so that they are defined on the intervals $(-\infty, l_j]$. The Dirichlet (resp. Neumann) fundamental solution $\varphi_{\mathcal{D}}^j(x, k)$ (resp. $\varphi_{\mathcal{N}}^j(x, k)$), is a solution of the equation,

$$\begin{aligned}
&-\frac{d^2}{dx^2}\varphi_{\mathcal{D},\mathcal{N}}^j(x,k) + q_j(x)\varphi_{\mathcal{D},\mathcal{N}}^j(x,k) = k^2\varphi_{\mathcal{D},\mathcal{N}}^j(x,k), \quad x \in (-\infty, l_j), \\
&\varphi_{\mathcal{D}}^j(l_j, k) = 0, \qquad \bigl(\varphi^j\bigr)'_{\mathcal{D}}(l_j, k) = 1, \\
&\varphi_{\mathcal{N}}^j(l_j, k) = 1, \qquad \bigl(\varphi^j\bigr)'_{\mathcal{N}}(l_j, k) = 0.
\end{aligned}$$

Consider, now, the function

$$\Psi_{\mathcal{D},\mathcal{N}}(x,k)=\bigotimes_{j=0}^{N}\Psi^{j}_{\mathcal{D},\mathcal{N}}(x,k),$$

where

$$\Psi^{0}_{\mathcal{D},\mathcal{N}}(x,k)=e^{+\imath kx}+R_{\mathcal{D},\mathcal{N}}(k)e^{-\imath kx},\quad x\in(-\infty,0],$$
$$\Psi^{j}_{\mathcal{D},\mathcal{N}}(x,k)=\alpha^{j}_{\mathcal{D},\mathcal{N}}(k)\varphi^{j}_{\mathcal{D},\mathcal{N}}(x,k),\quad x\in[0,l_j],\ j=1,\dots,N.$$

Here the coefficients $R_{\mathcal{D},\mathcal{N}}$ and $\alpha^{j}_{\mathcal{D},\mathcal{N}}$ are given by the boundary conditions (4) at the central node:

$$R_{\mathcal{D},\mathcal{N}}(k)+1=\alpha^{j}_{\mathcal{D},\mathcal{N}}(k)\varphi^{j}_{\mathcal{D},\mathcal{N}}(0,k),\quad j=1,\dots,N, \tag{14}$$

$$\sum_{j=1}^{N}\alpha^{j}_{\mathcal{D},\mathcal{N}}(k)\big(\varphi^{j}_{\mathcal{D},\mathcal{N}}\big)'(0,k)+\imath k\big(1-R_{\mathcal{D},\mathcal{N}}(k)\big)=H\big(R_{\mathcal{D},\mathcal{N}}(k)+1\big). \tag{15}$$

One sees that this $\Psi_{\mathcal{D},\mathcal{N}}$ is in $D(\mathcal{L}^{+}_{\mathcal{N},\mathcal{D}})$, the domain of the operator, and satisfies the conditions of the proposition. This, trivially, provides the existence of a scattering solution. Here, we show that $\Psi_{\mathcal{D},\mathcal{N}}$ is actually the unique one.

Assume that there exists another $Y_{\mathcal{D},\mathcal{N}}=\bigotimes_{j=0}^{N}Y^{j}_{\mathcal{D},\mathcal{N}}(x,k)$ solution of the scattering problem. Since $Y^{j}_{\mathcal{D},\mathcal{N}}(\cdot,k)$ and $\Psi^{j}_{\mathcal{D},\mathcal{N}}(\cdot,k)$ are solutions of the same Sturm–Liouville equation over each branch and their derivatives vanish at l_j, $Y^{j}_{\mathcal{D},\mathcal{N}}(\cdot,k)$ and $\Psi^{j}_{\mathcal{D},\mathcal{N}}(\cdot,k)$ are co-linear:

$$Y^{j}_{\mathcal{D},\mathcal{N}}(x,k)=\beta^{j}_{\mathcal{D},\mathcal{N}}(k)\varphi^{j}_{\mathcal{D},\mathcal{N}}(x,k),\quad x\in[0,l_j],\ j=1,\dots,N.$$

Over the branch e_0, as $Y^{0}_{\mathcal{D},\mathcal{N}}(\cdot,k)$ satisfies a homogeneous Sturm–Liouville equation ($q_0=0$), it necessarily admits the following form

$$Y^{0}_{\mathcal{D},\mathcal{N}}(x,k)=e^{+\imath kx}+\tilde{R}_{\mathcal{D},\mathcal{N}}(k)e^{-\imath kx}.$$

We need to show that one necessarily has $\tilde{R}_{\mathcal{D},\mathcal{N}}(k)\equiv R_{\mathcal{D},\mathcal{N}}(k)$ and similarly $\beta^{j}_{\mathcal{D},\mathcal{N}}(k)\equiv\alpha^{j}_{\mathcal{D},\mathcal{N}}(k)$. Indeed, for almost all $k\in\mathbb{R}$, Eqs. (14) and (15) provide $N+1$ linear relations for the $N+1$ unknown coefficients $R_{\mathcal{D},\mathcal{N}}$ and $(\alpha^{j}_{\mathcal{D},\mathcal{N}})_{j=1}^{N}$. Trivially, as soon as, the coefficients $(\varphi^{j}_{\mathcal{D},\mathcal{N}}(0,k))_{j=1}^{N}$ are non-zero, these linear relations are independent and there exists a unique solution for the unknowns $R_{\mathcal{D},\mathcal{N}}$ and $(\alpha^{j}_{\mathcal{D},\mathcal{N}})_{j=1}^{N}$. However, the zeros of each one of the coefficients $(\varphi^{j}_{\mathcal{D},\mathcal{N}}(0,k))_{j=1}^{N}$ correspond to isolated values of k (square-root of the eigenvalues of the operator $-\frac{d^2}{dx^2}+q_j(x)$ with Dirichlet boundary condition at $x=0$ and Dirichlet or Neumann boundary condition at $x=l_j$).

We can compute explicitly these coefficients for all k except for a set $\mathcal{K}$ of isolated values: dividing (15) by $(1+R_{\mathcal{D},\mathcal{N}}(k))$ and inserting (14), we find

$$\frac{1-R_{\mathcal{D},\mathcal{N}}(k)}{R_{\mathcal{D},\mathcal{N}}(k)+1}=\frac{H}{\imath k}-\frac{1}{\imath k}\sum_{j=1}^{N}\frac{(\varphi^{j}_{\mathcal{D},\mathcal{N}})'(0,k)}{\varphi^{j}_{\mathcal{D},\mathcal{N}}(0,k)},\quad\forall k\in\mathbb{R}\backslash\mathcal{K}. \tag{16}$$

Finally, inserting the value of $R_{\mathcal{D},\mathcal{N}}(k)$ into (14), we find

$$\alpha^{j}_{\mathcal{D},\mathcal{N}}(k)=\frac{R_{\mathcal{D},\mathcal{N}}(k)+1}{\varphi^{j}_{\mathcal{D},\mathcal{N}}(0,k)},\quad\forall k\in\mathbb{R}\backslash\mathcal{K}.$$

What remains to be shown is the extendibility of reflection coefficient $R_{\mathcal{D},\mathcal{N}}(k)$ to whole real axis. Let $\bar{k}\in\mathcal{K}$ be one of the isolated values where $R_{\mathcal{D},\mathcal{N}}$ is not defined: $\varphi^{j}_{\mathcal{N},\mathcal{D}}(0,\bar{k})=0$ for some j. Then we have to show the continuity of $R_{\mathcal{D},\mathcal{N}}(k)$ at $\bar{k}$, i.e.

$$\lim_{k\to\bar{k}^{+}}R_{\mathcal{D},\mathcal{N}}(k)=\lim_{k\to\bar{k}^{-}}R_{\mathcal{D},\mathcal{N}}(k)=:R_{\mathcal{D},\mathcal{N}}(\bar{k}),$$

with $|R_{\mathcal{D},\mathcal{N}}(\bar{k})|<\infty$ (even more $|R_{\mathcal{D},\mathcal{N}}(\bar{k})|=1$ here).

Indeed, through (16), and by the fact that fundamental solutions are analytic with respect to k, the reflection coefficient $R_{\mathcal{D},\mathcal{N}}(k)$ can be written as a fraction of two analytic functions, at least for k's where it is well defined. Furthermore, for these k's we have $|R_{\mathcal{D},\mathcal{N}}(k)|=1$. These two facts, together, ensure the existence of the limit when $k\to\bar{k}$ and that $|R_{\mathcal{D},\mathcal{N}}(\bar{k})|=1$. $\square$

3.2. Identifiability of geometry

This subsection has for goal to give a proof of Theorem 1 ensuring the uniqueness of the geometry of the graph giving rise to a measured reflection coefficient $R_{\mathcal{N}}(k)$. The method is rather constructive and one can think of an algorithm to identify the lengths, at least approximately. The proof is based on an asymptotic analysis in high-frequency regime of the reflection coefficient and some classical results from the theory of almost periodic functions (in Bohr sense).

Before, proving Theorem 1, we need the following lemma:

Lemma 2. *Consider a star-shaped network Γ composed of n_j branches of length l_j $(j = 1, \dots, m)$ all joining at a central node so that the whole number of branches N is given by $\sum_{j=1}^m n_j$. Assume the potential q on the network to be 0 $(q \equiv 0)$. Then the knowledge of the Neumann reflection coefficient $R_{\mathcal{N}}(k)$ determines uniquely the parameters $(n_j)_{j=1}^m$ and $(l_j)_{j=1}^m$.*

Proof. We need to apply the explicit computation of the reflection coefficient provided by (16). The fundamental solutions are given, simply, by $\varphi_{\mathcal{N}}^j(x,k) = \cos(k(l_j - x))$. Therefore:

$$\frac{1 - R_{\mathcal{D},\mathcal{N}}(k)}{R_{\mathcal{N}}(k) + 1} = \frac{1}{\imath k} H - \frac{1}{\imath k} \sum_{j=1}^m n_j \frac{k \sin(l_j k)}{\cos(l_j k)}.$$

The knowledge of $R_{\mathcal{N}}(k)$ determines uniquely the signal:

$$f(k) := \sum_{j=1}^m n_j \tan(k l_j).$$

Assuming, without loss of generality, that the lengths l_j are ordered increasingly $l_1 < \dots < l_m$, the first pole of the function $f(k)$ coincides with $\pi/2l_m$ and therefore determines l_m. Furthermore,

$$n_m = \lim_{k \to \pi/2l_m} \cos(k l_m) f(k),$$

and therefore one can also determine n_m. Now, considering the new signal $g(k) = f(k) - n_m \tan(k l_m)$, one removes the branches of length l_m and exactly in the same manner, one can determine l_{m-1} and n_{m-1}. The proof of the lemma follows then by a simple induction. □

Now we are able to prove the main theorem.

Proof of Theorem 1. Assume that, there exist two graph settings $(l_j, q_j)_{j=1}^N$ and $(\tilde{l}_j, \tilde{q}_j)_{j=1}^{\tilde{N}}$ (the lengths l_j are not necessarily different) giving rise to the same Neumann reflection coefficients: $R_{\mathcal{N}}(k) \equiv \tilde{R}_{\mathcal{N}}(k)$. By the explicit formula (16), we have

$$\frac{1}{k} \sum_{j=1}^N \frac{(\varphi_{\mathcal{N}}^j)'(0,k)}{\varphi_{\mathcal{N}}^j(0,k)} \equiv \frac{1}{k} \sum_{i=1}^{\tilde{N}} \frac{(\tilde{\varphi}_{\mathcal{N}}^i)'(0,k)}{\tilde{\varphi}_{\mathcal{N}}^i(0,k)}.$$

This is equivalent to:

$$\prod_{j=1}^{\tilde{N}} \tilde{\varphi}_{\mathcal{N}}^j(0,k) \left(\sum_{i=1}^N (\varphi_{\mathcal{N}}^i)'(0,k) \prod_{l \neq i} \varphi_{\mathcal{N}}^l(0,k) \right) - \prod_{j=1}^N \varphi_{\mathcal{N}}^j(0,k) \left(\sum_{i=1}^{\tilde{N}} (\tilde{\varphi}_{\mathcal{N}}^i)'(0,k) \prod_{l \neq i} \tilde{\varphi}_{\mathcal{N}}^l(0,k) \right) = 0. \tag{17}$$

Now, we use the fact that the high-frequency behavior of the Neumann fundamental solutions $(\varphi_{\mathcal{N}}^j)_{j=1}^N$ is given as follows (see [13, p. 4]):

$$\begin{aligned} &\varphi_{\mathcal{N}}^j(0,k) = \cos(k l_j) + \mathcal{O}\left(\frac{1}{k}\right) \quad \text{as } k \to \infty, \\ &(\varphi_{\mathcal{N}}^j)'(0,k) = k \sin(k l_j) + \mathcal{O}(1) \quad \text{as } k \to \infty. \end{aligned} \tag{18}$$

Defining the function:

$$F(k) := \prod_{j=1}^{\tilde{N}} \cos(k\tilde{l}_k) \left(\sum_{i=1}^N \sin(k l_i) \prod_{l \neq i} \cos(k l_l) \right) - \prod_{j=1}^N \cos(k l_j) \left(\sum_{i=1}^{\tilde{N}} \sin(k\tilde{l}_i) \prod_{l \neq i} \cos(k\tilde{l}_l) \right).$$

The asymptotic formulas (18) together with (17) imply

$$F(k) = \mathcal{O}(1/k) \quad \text{as } k \to \infty.$$

However, the function $F(k)$ is a trigonometric polynomial and almost periodic in the Bohr's sense [6]. The function $F^2(k)$ is, also, almost periodic and furthermore, we have

$$M(F^2) := \lim_{k\to\infty} \frac{1}{k}\int_0^k F^2(k)\,dk = \lim_{k\to\infty}\frac{1}{k}\left(\int_0^1 F^2(k)\,dk + \int_1^k F^2(k)\,dk\right)$$
$$\leqslant \lim_{k\to\infty}\frac{1}{k}\left(C_1 + C_2\int_1^k \frac{1}{k^2}\,dk\right) = 0.$$

This, trivially, implies that $F = 0$ (one only needs to apply the Parseval's Theorem to the generalized Fourier series of the function F). However, the relation $F(k) \equiv 0$ is equivalent to

$$\sum_{j=1}^{N} \tan(kl_j) = \sum_{j=1}^{\tilde N} \tan(k\tilde l_j),$$

and therefore, by Lemma 2, the two settings are equivalent and the theorem follows. □

3.3. From inverse scattering to inverse spectral problem

Here we present some auxiliary propositions that we will need for the proof of Theorems 2 and 3. The main objective of this subsection is to show the equivalence between the inverse scattering problem on Γ^+ and some inverse spectral problem on Γ.

So, as before, we consider a general star-shaped graph Γ (of N finite branches) and a potential $q = \bigotimes_{j=1}^N q_j$ belonging to $H^1(\Gamma)$. We will see that the knowledge of the reflection coefficient $R_{\mathcal N}(k)$ for $\mathcal{L}^+_{\mathcal N}$ (resp. $R_{\mathcal D}(k)$ for $\mathcal{L}^+_{\mathcal D}$) is equivalent to the knowledge of different positive spectra of Sturm–Liouville operators defined on Γ with Neumann (resp. Dirichlet) boundary conditions at terminal nodes and for various boundary conditions at the central node. In fact, defining the function

$$h_{\mathcal N,\mathcal D}(k) = H + \frac{\imath k(1 - R_{\mathcal D,\mathcal N}(k))}{(R_{\mathcal N,\mathcal D}(k)+1)},$$

where H is given by (5), we have the following result.

Proposition 2. *Fix $k \in \mathbb{R}$ and define the Schrödinger operators $\mathcal{L}_{\mathcal N,\mathcal D}(k)$ on the compact graph Γ as follows*:

$$\mathcal{L}_{\mathcal N,\mathcal D}(k) = \bigotimes_{j=1}^{N}\left(-\frac{d^2}{dx^2} + q_j(x)\right),$$
$$D(\mathcal{L}_{\mathcal N,\mathcal D}(k)) = \text{closure of } C^\infty_{k;\mathcal N,\mathcal D}(\Gamma) \quad \text{in } H^2(\Gamma),$$

where $C^\infty_{k;\mathcal N}(\Gamma)$ (resp. $C^\infty_{k;\mathcal D}(\Gamma)$) denotes the space of infinitely differentiable functions $f = \bigotimes_{j=1}^N f_j$ defined on Γ satisfying the boundary conditions

$$f_j(0) = f_{j'}(0) =: \bar f, \quad j, j' = 1,\ldots,N,$$
$$\sum_{j=1}^{N} f_j'(0) = h_{\mathcal N,\mathcal D}(k)\bar f,$$
$$f_j'(l_j) = 0 \quad \left(f_j(l_j) = 0 \text{ for } C^\infty_{k;\mathcal D}(\Gamma)\right), \quad j = 1,\ldots,N.$$

Then we are able to characterize the positive spectrum of $\mathcal{L}_{\mathcal N,\mathcal D}(k)$ as a level set of the function $h_{\mathcal N,\mathcal D}(k)$:

$$\sigma^+(\mathcal{L}_{\mathcal N,\mathcal D}(k)) = \{\xi^2 \mid \xi \in \mathbb{R},\ h_{\mathcal N,\mathcal D}(\xi) = h_{\mathcal N,\mathcal D}(k)\}.$$

Remark 4. As it can be seen through the proof, the above proposition holds for the generic case of any compact graph, where a test branch of infinite length is added to an arbitrary node of the graph.

Proof. We prove the proposition for the case of Neumann boundary conditions. The Dirichlet case can be treated exactly in the same manner. We start by proving the inclusion

$$\sigma^+(\mathcal{L}_{\mathcal N}(k)) \subseteq \{\xi^2 \mid \xi \in \mathbb{R},\ h_{\mathcal N}(\xi) = h_{\mathcal N}(k)\}.$$

Let $\xi^2 \in \sigma^+(\mathcal{L}_\mathcal{N}(k))$, then there exists Ψ eigenfunction of the operator $\mathcal{L}_\mathcal{N}(k)$ associated to ξ^2. In particular, it satisfies

$$\sum_{j=1}^{N} \Psi_j'(0) = h_\mathcal{N}(k)\bar{\Psi},$$

where $\bar{\Psi}$ is the common value of Ψ at the central node.

Now we extend Ψ to the extended graph Γ^+, such that Ψ^+ is a scattering solution for $\mathcal{L}_\mathcal{N}^+$ (see Proposition 1). In particular, the function Ψ^+ must satisfy, at the central node,

$$\Psi_j^+(0) = \Psi_0^+(0), \quad j = 1, \dots, N,$$
$$\sum_{j=1}^{N} \bigl(\Psi_j^+\bigr)'(0) - \bigl(\Psi_0^+\bigr)'(0) = H\Psi_0^+(0).$$

Noting that Ψ is an eigenfunction of $(\mathcal{L}_\mathcal{N}(k), D(\mathcal{L}_\mathcal{N}(k)))$, we have

$$h_\mathcal{N}(k)\Psi_0^+(0) - \bigl(\Psi_0^+\bigr)'(0) = \sum_{j=1}^{N} \bigl(\Psi_j^+\bigr)'(0) - \bigl(\Psi_0^+\bigr)'(0) = H\Psi_0^+(0). \tag{19}$$

Now, noting that Ψ^+ over the infinite branch admits the following form

$$\Psi_0^+(x) = R_\mathcal{N}(\xi)e^{-\imath\xi x} + e^{+\imath\xi x}, \quad x \in (-\infty, 0],$$

the relation (19) yields to

$$h_\mathcal{N}(k)\bigl(R_\mathcal{N}(\xi) + 1\bigr) - \imath\xi\bigl(1 - R_\mathcal{N}(\xi)\bigr) = H\bigl(R_\mathcal{N}(\xi) + 1\bigr),$$

or equivalently

$$h_\mathcal{N}(k) = H + \frac{\imath\xi(1 - R_\mathcal{N}(\xi))}{(R_\mathcal{N}(\xi) + 1)} = h_\mathcal{N}(\xi).$$

This proves the first inclusion. Now, we prove that

$$\sigma^+\bigl(\mathcal{L}_\mathcal{N}(k)\bigr) \supseteq \bigl\{\xi^2 \bigm| \xi \in \mathbb{R},\ h_\mathcal{N}(\xi) = h_\mathcal{N}(k)\bigr\}.$$

Let $\xi \in \mathbb{R}$ be such that $h_\mathcal{N}(\xi) = h_\mathcal{N}(k)$. We consider a scattering solution Ψ^+ of the extended operator $\mathcal{L}_\mathcal{N}^+$ (defined by (3)) associated to the frequency ξ^2. We, then, prove that the restriction of Ψ^+ to the compact graph Γ is an eigenfunction of $\mathcal{L}_\mathcal{N}(k)$ associated to the eigenvalue ξ^2. This trivially implies that $\xi^2 \in \sigma^+(\mathcal{L}_\mathcal{N}(k))$.

In this aim, we only need to show that this restriction of Ψ^+ to Γ is in the domain $D(\mathcal{L}_\mathcal{N}(k))$. Indeed, this is equivalent to proving that the boundary condition

$$\sum_{j=1}^{N} \bigl(\Psi_j^+\bigr)'(0) = h(k)\Psi_0^+(0) \tag{20}$$

is satisfied. As Ψ^+ is a scattering solution of $\mathcal{L}_\mathcal{N}^+$, it satisfies

$$\sum_{j=1}^{N} \bigl(\Psi_j^+\bigr)'(0) = H\Psi_0^+(0) + \bigl(\Psi_0^+\bigr)'(0) = \left(H + \frac{(\Psi_0^+)'(0)}{\Psi_0^+(0)}\right)\Psi_0^+(0).$$

Furthermore,

$$\Psi_0^+(0) = R_\mathcal{N}(\xi) + 1 \quad \text{and} \quad \bigl(\Psi_0^+\bigr)'(0) = \imath\xi\bigl(1 - R_\mathcal{N}(\xi)\bigr),$$

and so

$$\sum_{j=1}^{N} \bigl(\Psi_j^+\bigr)'(0) = \left(H + \frac{\imath\xi(1 - R_\mathcal{N}(\xi))}{R_\mathcal{N}(\xi) + 1}\right)\Psi_0^+(0) = h(\xi)\Psi_0^+(0) = h(k)\Psi_0^+(0).$$

This proves (20) and finishes the proof of the proposition. $\square$

The following proposition provides the characteristic equation permitting to identify the eigenvalues of the operator $\mathcal{L}_{\mathcal{N},\mathcal{D}}(k)$:

Proposition 3. *The real $\lambda^2 > 0$ is an eigenvalue of the operator $\mathcal{L}_{\mathcal{N},\mathcal{D}}(k)$ if and only if*

$$\Psi_{\mathcal{N},\mathcal{D}}(\lambda) = h_{\mathcal{N},\mathcal{D}}(k)\Phi_{\mathcal{N},\mathcal{D}}(\lambda),$$

where

$$\Phi_{\mathcal{N},\mathcal{D}}(\lambda) := \prod_{j=1}^{N} \varphi^j_{\mathcal{N},\mathcal{D}}(0,\lambda) \quad \text{and} \quad \Psi_{\mathcal{N},\mathcal{D}}(\lambda) := \frac{d}{dx}\left(\prod_{j=1}^{N} \varphi^j_{\mathcal{N},\mathcal{D}}(x,\lambda)\right)\Bigg|_{x=0}, \tag{21}$$

$\varphi^j_{\mathcal{N},\mathcal{D}}(x,\lambda)$ being the fundamental solutions on different branches.

Proof. We give the proof for the Neumann boundary conditions, noting that the Dirichlet case can be treated, exactly, in the same manner. Assume λ^2 to be a positive eigenvalue of $\mathcal{L}_{\mathcal{N}}(k)$. The associated eigenfunction, $y_\lambda(x) = \bigotimes_{j=1}^N y^j_\lambda(x)$, has necessarily the following form:

$$y^j_\lambda(x) = \alpha_j \varphi^j_{\mathcal{N}}(x,\lambda),$$

where α_j's are real constants and the vector $(\alpha_1,\dots,\alpha_N)$ is different from zero. The function y_λ, being in the domain $D(\mathcal{L}_{\mathcal{N}}(k))$, it should satisfy the associated boundary condition at the central node. This implies that the vector $(\alpha_1,\dots,\alpha_N)$ is in the kernel of the matrix:

$$M := \begin{pmatrix} \varphi^1_{\mathcal{N}}(0,\lambda) & -\varphi^2_{\mathcal{N}}(0,\lambda) & 0 & \cdots & 0 \\ 0 & \varphi^2_{\mathcal{N}}(0,\lambda) & -\varphi^3_{\mathcal{N}}(0,\lambda) & \cdots & 0 \\ 0 & 0 & \varphi^3_{\mathcal{N}}(0,\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -h_{\mathcal{N}}(k)\varphi^1_{\mathcal{N}}(0,\lambda) + \psi^1_{\mathcal{N}}(0,\lambda) & \psi^2_{\mathcal{N}}(0,\lambda) & \psi^3_{\mathcal{N}}(0,\lambda) & \cdots & \psi^N_{\mathcal{N}}(0,\lambda) \end{pmatrix}$$

where $\psi^j_{\mathcal{N}}(0,\lambda)$ denotes $\frac{d}{dx}\varphi^j_{\mathcal{N}}(x,\lambda)|_{x=0}$. This means that the determinant $\det(M)$ is necessarily 0. Developing this determinant we find

$$\Psi_{\mathcal{N}}(\lambda) = h_{\mathcal{N}}(k)\Phi_{\mathcal{N}}(\lambda). \qquad \square$$

Corollary 1. *Consider two potentials $q = \bigotimes_{j=1}^N q_j$ and $\tilde{q} = \bigotimes_{j=1}^N \tilde{q}_j$ and denote by $\mathcal{L}^+_{\mathcal{N}}$ and $\tilde{\mathcal{L}}^+_{\mathcal{N}}$, the associated Neumann Schrödinger operators defined on the extended graph Γ^+. Assuming that the reflection coefficients $R_{\mathcal{N}}(k)$ and $\tilde{R}_{\mathcal{N}}(k)$ are equivalent $R_{\mathcal{N}}(k) \equiv \tilde{R}_{\mathcal{N}}(k)$, we have*

$$\Phi_{\mathcal{N}}(k)\tilde{\Psi}_{\mathcal{N}}(k) = \tilde{\Phi}_{\mathcal{N}}(k)\Psi_{\mathcal{N}}(k), \quad \forall k \in \mathbb{R}, \tag{22}$$

where $\Phi_{\mathcal{N}}$, $\Psi_{\mathcal{N}}$, $\tilde{\Phi}_{\mathcal{N}}$ and $\tilde{\Psi}_{\mathcal{N}}$ are defined through (21) *for the potentials q and $\tilde{q}$.*

Proof. By Proposition 2, k^2 is an eigenvalue of the operators $\mathcal{L}_{\mathcal{N}}(k)$ and $\tilde{\mathcal{L}}_{\mathcal{N}}(k)$. Applying Proposition 3, this means that

$$\Psi_{\mathcal{N}}(k) = h_{\mathcal{N}}(k)\Phi_{\mathcal{N}}(k) \quad \text{and} \quad \tilde{\Psi}_{\mathcal{N}}(k) = \tilde{h}_{\mathcal{N}}(k)\tilde{\Phi}_{\mathcal{N}}(k).$$

As $R_{\mathcal{N}}(k) \equiv \tilde{R}_{\mathcal{N}}(k)$, we have $h_{\mathcal{N}}(k) \equiv \tilde{h}_{\mathcal{N}}(k)$ and thus the above equation yields to (22). $\square$

The above corollary is also valid when we replace the Neumann by Dirichlet boundary conditions. Finally, this corollary yields to the following proposition on the difference between the two potentials q and $\tilde{q}$.

Proposition 4. *Consider two potentials $q = \bigotimes_{j=1}^N q_j$ and $\tilde{q} = \bigotimes_{j=1}^N \tilde{q}_j$ and denote by $\mathcal{L}^+_{\mathcal{N}}$ and $\tilde{\mathcal{L}}^+_{\mathcal{N}}$, the associated Neumann Schrödinger operators defined on the extended graph Γ^+. Assuming that the reflection coefficients $R_{\mathcal{N}}(k)$ and $\tilde{R}_{\mathcal{N}}(k)$ are equivalent $R_{\mathcal{N}}(k) \equiv \tilde{R}_{\mathcal{N}}(k)$, we have*

$$\sum_{j=1}^{N} \prod_{i \neq j} \varphi^i_{\mathcal{N}}(0,k)\tilde{\varphi}^i_{\mathcal{N}}(0,k) \int_0^{l_j} \hat{q}_j(x)\varphi^j_{\mathcal{N}}(x,k)\tilde{\varphi}^j_{\mathcal{N}}(x,k)\,dx = 0, \quad \forall k \in \mathbb{R}, \tag{23}$$

where $\hat{q}_j = \tilde{q}_j - q_j$.

Proof. For $j = 1,\dots,N$, we have

$$\int_0^{l_j} \tilde{q}_j(x)\tilde{\varphi}^j_{\mathcal{N}}(x,k)\varphi^j_{\mathcal{N}}(x,k)\,dx - \int_0^{l_j} q_j(x)\varphi^j_{\mathcal{N}}(x,k)\tilde{\varphi}^j_{\mathcal{N}}(x,k)\,dx$$

$$= \varphi^j_{\mathcal{N}}(x,k)\frac{d}{dx}\tilde{\varphi}^j_{\mathcal{N}}(x,k)\bigg|_{x=0}^{x=l_j} - \frac{d}{dx}\varphi^j_{\mathcal{N}}(x,k)\tilde{\varphi}^j_{\mathcal{N}}(x,k)\bigg|_{x=0}^{x=l_j}$$
$$= \psi^j_{\mathcal{N}}(0,k)\tilde{\varphi}^j_{\mathcal{N}}(0,k) - \varphi^j_{\mathcal{N}}(0,k)\tilde{\psi}^j_{\mathcal{N}}(0,k). \tag{24}$$

Here the second line has been obtained from the first one, replacing $q_j(x)\varphi^j_{\mathcal{N}}(x,k)$ by $\frac{d^2}{dx^2}\varphi^j_{\mathcal{N}}(x,k) + k^2\varphi^j_{\mathcal{N}}(x,k)$ and integrating by parts. Using (22) and the above equation, we have

$$\sum_{j=1}^N \prod_{i\neq j} \varphi^i_{\mathcal{N}}(0,k)\tilde{\varphi}^i_{\mathcal{N}}(0,k)\int_0^{l_j} \hat{q}_j(x)\varphi^j_{\mathcal{N}}(x,k)\tilde{\varphi}^j_{\mathcal{N}}(x,k)\,dx = \Psi_{\mathcal{N}}(k)\tilde{\Phi}_{\mathcal{N}}(k) - \Phi_{\mathcal{N}}(k)\tilde{\Psi}_{\mathcal{N}}(k) = 0. \qquad \square$$

Before finishing this subsection, note that, once more, the above proposition is also valid for the case of Dirichlet boundary conditions and $R_{\mathcal{D}}(k) \equiv \tilde{R}_{\mathcal{D}}(k)$ implies

$$\sum_{j=1}^N \prod_{i\neq j} \varphi^i_{\mathcal{D}}(0,k)\tilde{\varphi}^i_{\mathcal{D}}(0,k)\int_0^{l_j} \hat{q}_j(x)\varphi^j_{\mathcal{D}}(x,k)\tilde{\varphi}^j_{\mathcal{D}}(x,k)\,dx = 0, \quad \forall k \in \mathbb{R}. \tag{25}$$

We are now ready to prove Theorems 2 and 3.

3.4. Proof of Theorem 2

We prove Theorem 2 applying the characteristic equation (23) and high-frequency behavior of $\varphi^j_{\mathcal{N},\mathcal{D}}(x,k)$. Again, for simplicity sakes, we give the proof only for the case of Neumann boundary conditions, noting that the Dirichlet case can be done similarly.

We know the asymptotic behavior of fundamental solutions $\varphi^j_{\mathcal{N}}(x,k)$

$$\varphi^j_{\mathcal{N}}(x,k) = \cos\big(k(l_j - x)\big) + \mathcal{O}\left(\frac{1}{k}\right).$$

In particular the product writes

$$\varphi^j_{\mathcal{N}}(x,k)\tilde{\varphi}^j_{\mathcal{N}}(x,k) = \cos^2\big(k(l_j - x)\big) + \mathcal{O}\left(\frac{1}{k}\right).$$

Applying the characteristic equation (23) and developing the products $\varphi^j_{\mathcal{N}}(x,k)$, $\tilde{\varphi}^j_{\mathcal{N}}(x,k)$, we have

$$\sum_{j=1}^N \left(\prod_{i\neq j}\cos^2(kl_i)\right)\int_0^{l_j} \hat{q}_j(s)\cos^2\big(k(l_j - s)\big)\,ds = \mathcal{O}\left(\frac{1}{k}\right),$$
$$\sum_{j=1}^N \left(\prod_{i\neq j}\cos^2(kl_i)\right)\int_0^{l_j} \hat{q}_j(s)\left(\frac{1+\cos 2(k(l_j - s))}{2}\right)ds = \mathcal{O}\left(\frac{1}{k}\right),$$
$$\sum_{j=1}^N \left(\prod_{i\neq j}\cos^2(kl_i)\right)\frac{1}{2}\int_0^{l_j} \hat{q}_j(s)\,ds = \mathcal{O}\left(\frac{1}{k}\right). \tag{26}$$

In the last passage, we applied the fact that $\int_0^{l_j} \hat{q}_j(s)\cos 2(k(l_j - s))\,ds = \mathcal{O}(1/k)$, since $\hat{q}$ is in $H^1(\Gamma)$.

The left side of (26) is an almost periodic function with respect to k, in the Bohr's sense. Following the same arguments as those of Theorem 1 we obtain

$$\sum_{j=1}^N \left(\prod_{i\neq j}\cos^2(kl_i)\right)\frac{1}{2}\int_0^{l_j} \hat{q}_j(s)\,ds = 0, \quad \forall k \in \mathbb{R}.$$

Assume, without loss of generality, that the lengths are ordered increasingly $l_1, \dots, l_N$ and choose $k_N = \pi/2l_N$:

$$\cos(k_N l_j) \neq 0 \quad \text{for } j \neq N.$$

Indeed, we have

$$\prod_{i\neq N}\cos^2(k_N l_i)\int_0^{l_N}\hat{q}_N(s)\,ds=0 \quad\Rightarrow\quad \int_0^{l_N}\hat{q}_N(s)\,ds=0.$$

Then, the characteristic equation can be rewritten

$$\cos^2(kl_N)\sum_{j=1}^{N-1}\left(\prod_{i\neq j}\cos^2(kl_i)\right)\frac{1}{2}\int_0^{l_j}\hat{q}_j(s)\,ds=0,$$

and, since it is a product of two analytic functions w.r.t. k, we have

$$\sum_{j=1}^{N-1}\left(\prod_{i\neq j}\cos^2(kl_i)\right)\frac{1}{2}\int_0^{l_j}\hat{q}_j(s)\,ds=0,\quad \forall k\in\mathbb{R},$$

and we finish the proof of Theorem 2, repeating the same argument $N-1$ times.

3.5. Proof of Theorem 3

In this subsection, we consider two potentials $q=\bigotimes_{j=1}^N q_j$ and $\tilde{q}=\bigotimes_{j=1}^N \tilde{q}_j$, satisfying the assumptions of Theorem 3. Assuming that they give rise to the same Neumann and Dirichlet reflection coefficients, $R_{\mathcal{N}}(k)\equiv\tilde{R}_{\mathcal{N}}(k)$ and $R_{\mathcal{D}}(k)\equiv\tilde{R}_{\mathcal{D}}(k)$, we have the characteristic equations (23) and (25).

Let us define the operators $\mathcal{L}^j_{\mathcal{N},\mathcal{D}}$ to be the operator $-\frac{d^2}{dx^2}+q_j(x)$ over $[0,l_j]$ with the domain

$$D\big(\mathcal{L}^j_{\mathcal{N},\mathcal{D}}\big)=\text{closure of } C^\infty_{\mathcal{N},\mathcal{D}}(0,l_j)\quad\text{in } H^2(0,l_j),$$

where $C^\infty_{\mathcal{N}}(0,l_j)$ (resp. $C^\infty_{\mathcal{D}}(0,l_j)$) denotes the space of infinitely differentiable functions f defined on $[0,l_j]$ satisfying Dirichlet boundary condition at 0 and Neumann (resp. Dirichlet) boundary condition at l_j.

Noting that, we have assumed for the potential $q_j(x)$ to satisfy $\|q_j\|_{H^1(0,l_j)}<C(\Gamma)\leqslant C_1(\Gamma):=\min_{j=1,\dots,N}\frac{\pi^2}{4l_j^{5/2}}$ and that $q_j(0)=0$, we have $\|q_j\|_{L^\infty(0,l_j)}<\frac{\pi^2}{4l_j^2}$ (one has the Sobolev injection $\|q_j\|_{L^\infty(0,l_j)}\leqslant\sqrt{l_j}\|q_j\|_{H^1(0,l_j)}$). This implies that the eigenvalues of $\mathcal{L}^j_{\mathcal{N}}$ remain positive. In fact, $\frac{\pi^2}{4l_j^2}$ is the minimum eigenvalue of the potential-less Schrödinger operator (with Neumann boundary conditions) and therefore by adding a potential whose L^∞-norm is smaller than this eigenvalue, the eigenvalues of $\mathcal{L}^j_{\mathcal{N}}$ remain positive.

Considering $((\lambda^j_n)^2)_{n=1}^\infty$ ($\lambda^j_n>0$) the sequence of eigenvalues of $\mathcal{L}^j_{\mathcal{N}}$, (23) implies for each $j=1,\dots,N$,

$$\prod_{i\neq j}\varphi^i_{\mathcal{N}}\big(0,\lambda^j_n\big)\tilde{\varphi}^i_{\mathcal{N}}\big(0,\lambda^j_n\big)\int_0^{l_j}\hat{q}_j(x)\varphi^j_{\mathcal{N}}\big(x,\lambda^j_n\big)\tilde{\varphi}^j_{\mathcal{N}}\big(x,\lambda^j_n\big)\,dx=0,\quad\forall n=1,2,\dots \tag{27}$$

where we have applied the fact that $\varphi^j_{\mathcal{N}}(0,\lambda^j_n)=0$.

At this point, we will use the assumption **A2** on the lengths l_j to obtain a lemma on the non-overlapping of the eigenvalues for different branches:

Lemma 3. *Under the assumptions of Theorem* 3*, for all* $j=1,\dots,N$*,*

$$\prod_{i\neq j}\varphi^i_{\mathcal{N}}\big(0,\lambda^j_n\big)\tilde{\varphi}^i_{\mathcal{N}}\big(0,\lambda^j_n\big)\neq 0,\quad\forall n\in\mathbb{N}.$$

Proof. In order to prove this lemma, we only need to show that $(\lambda^j_n)^2$ is not an eigenvalue of $\mathcal{L}^i_{\mathcal{N}}$ nor $\tilde{\mathcal{L}}^i_{\mathcal{N}}$ for $i\neq j$.

In this aim, we first show that, if $\|q\|_{H^1},\|\tilde{q}\|_{H^1}<C_2(\Gamma)$ and assumption **A2** holds, then there are at most a finite number of overlapping eigenvalues for different branches. Indeed, for $M(\Gamma)$ defined by (8), we show that taking $n_1,n_2>M(\Gamma)$, $\lambda^i_{n_1}$ is different from $\lambda^j_{n_2}$ and $\tilde{\lambda}^j_{n_2}$ the eigenvalues of $\mathcal{L}^j_{\mathcal{N}}$ and $\tilde{\mathcal{L}}^j_{\mathcal{N}}$ ($j\neq i$). Assume, contrarily, that there exist $n_1,n_2>M(\Gamma)$ and $i\neq j$, such that

$$\lambda^i_{n_1}=\lambda^j_{n_2}\quad\text{or}\quad\lambda^i_{n_1}=\tilde{\lambda}^j_{n_2}. \tag{28}$$

Without loss of generality, we consider the first case. Applying Lemma 1, we have

$$\begin{aligned}&\left|\lambda^i_{n_1}-\frac{(2n_1-1)^2\pi^2}{4l_i^2}\right|<\frac{C_0(l_i)C_2(\Gamma)}{2n_1-1},\\&\left|\lambda^j_{n_2}-\frac{(2n_2-1)^2\pi^2}{4l_j^2}\right|<\frac{C_0(l_j)C_2(\Gamma)}{2n_2-1}.\end{aligned}\tag{29}$$

Therefore, the relation (28) implies that,

$$\left|\frac{(2n_1-1)^2\pi^2}{4l_i^2}-\frac{(2n_2-1)^2\pi^2}{4l_j^2}\right|<C_2(\Gamma)\left(\frac{C_0(l_i)}{2n_1-1}+\frac{C_0(l_j)}{2n_2-1}\right).$$

Taking (without loss of generality) $n_1\leqslant n_2$, and dividing the above inequality by $(2n_1-1)^2\pi^2/4l_j^2$, we have

$$\left|\frac{l_j^2}{l_i^2}-\frac{(2n_2-1)^2}{(2n_1-1)^2}\right|<C_2(\Gamma)\frac{4l_j^2(C_0(l_i)+C_0(l_j))}{\pi^2}\frac{1}{(2n_1+1)^3}.$$

Applying the trivial inequality

$$\left|\frac{l_j^2}{l_i^2}-\frac{(2n_2-1)^2}{(2n_1-1)^2}\right|=\left|\frac{l_j}{l_i}+\frac{2n_2-1}{2n_1-1}\right|\left|\frac{l_j}{l_i}-\frac{2n_2-1}{2n_1-1}\right|>\frac{l_j}{l_i}\left|\frac{l_j}{l_i}-\frac{2n_2-1}{2n_1-1}\right|,$$

we have

$$\left|\frac{l_j}{l_i}-\frac{2n_2-1}{2n_1-1}\right|<C_2(\Gamma)\frac{4l_il_j(C_0(l_i)+C_0(l_j))}{\pi^2}\frac{1}{(2n_1+1)^3}\leqslant\frac{1}{(2n_1+1)^3},$$

where we have applied the definition of $C_2(\Gamma)$. This leads to a contradiction with the definition of $M(\Gamma)$.

Now assume that, there exist $n_1\in\{1,\dots,M(\Gamma)\}$ and $n_2\in\mathbb{N}$ such that for some $i\neq j$, $\lambda^i_{n_1}=\lambda^j_{n_2}$ and we will find a contradiction with the fact that $\|q\|_{H^1}<C_3(\Gamma)$ (the case of $\lambda^i_{n_1}=\tilde\lambda^j_{n_2}$ can be treated exactly in the same manner). In this aim, we apply once again Lemma 1. If $\lambda^i_{n_1}=\lambda^j_{n_2}$, we have

$$\begin{aligned}\left|\frac{(2n_1-1)^2\pi^2}{4l_i^2}-\frac{(2n_2-1)^2\pi^2}{4l_j^2}\right|&<C_3(\Gamma)\left(\frac{C_0(l_i)}{2n_1-1}+\frac{C_0(l_j)}{2n_2-1}\right)\\&\leqslant C_3(\Gamma)\big(C_0(l_i)+C_0(l_j)\big).\end{aligned}$$

This, trivially, is in contradiction with the definition of $C_3(\Gamma)$. □

Applying Lemma 3 to (27), we have

$$\int_0^{l_j}\hat q_j(x)\varphi^j_{\mathcal N}\big(x,\lambda^j_n\big)\tilde\varphi^j_{\mathcal N}\big(x,\lambda^j_n\big)\,dx=0,\quad\forall j=1,\dots,N,\ \forall n\in\mathbb{N}.$$

From Eq. (24) we have

$$\int_0^{l_j}\hat q_j(x)\tilde\varphi^j_{\mathcal N}\big(x,\lambda^j_n\big)\varphi^j_{\mathcal N}\big(x,\lambda^j_n\big)\,dx=\psi^j_{\mathcal N}\big(0,\lambda^j_n\big)\tilde\varphi^j_{\mathcal N}\big(0,\lambda^j_n\big)-\varphi^j_{\mathcal N}\big(0,\lambda^j_n\big)\tilde\psi^j_{\mathcal N}\big(0,\lambda^j_n\big)=0.$$

This leads to $\psi^j_{\mathcal N}(0,\lambda^j_n)\tilde\varphi^j_{\mathcal N}(0,\lambda^j_n)=0$, since $(\lambda^j_n)^2$ is an eigenvalue of $\mathcal{L}^j_{\mathcal N}$. Furthermore, the value $\psi^j_{\mathcal N}(0,\lambda^j_n)$ is different from 0, because otherwise we would have a non-zero fundamental solution $\varphi^j_{\mathcal N}(x,\lambda^j_n)$ with three zero boundary conditions. Thus $\tilde\varphi^j_{\mathcal N}(0,\lambda^j_n)=0$ which implies that $(\lambda^j_n)^2$ is also an eigenvalue of $\tilde{\mathcal{L}}^j_{\mathcal N}$. Therefore, the eigenvalues of $\mathcal{L}^j_{\mathcal N}$ and $\tilde{\mathcal{L}}^j_{\mathcal N}$ coincide. In the same manner, we can show that the eigenvalues of $\mathcal{L}^j_{\mathcal D}$ and $\tilde{\mathcal{L}}^j_{\mathcal D}$ coincide as well.

It is well-known result [7,19] that the specification of two spectra of Sturm–Liouville boundary value problem uniquely determines the potential on the segment e_j, i.e.

$$\hat q_j(x)\equiv 0,\quad\forall x\in[0,l_j],\ j=1,\dots,N.$$

This completes the proof of Theorem 3.

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Appendix A. Proof of Lemma 1

We, basically, use a classical result from the perturbation theory of linear operators (see [21, Chapter VII, Example 2.17]). The assumption $\|q\|_{L^\infty(0,l)} < \frac{\pi^2}{4l^2}$ allows us to apply the Taylor expansion of the eigenvalues of the above operator as a perturbation of the Laplacian operator with the same boundary conditions. Therefore, following [21], we have

$$\left|\lambda_n - \frac{(2n-1)^2\pi^2}{4l^2} + 2\int_0^l q(s)\sin^2\left(\frac{(2n-1)\pi}{2l}s\right)ds\right| \leqslant c_0(l)\frac{\|q\|_{L^\infty}^2}{2n-1},$$

for some constant $c_0(l)$, only depending on the length l. This leads to

$$\begin{aligned}
\left|\lambda_n - \frac{(2n-1)^2\pi^2}{4l^2}\right| &\leqslant c_0(l)\frac{\|q\|_{L^\infty}^2}{2n-1} + 2\left|\int_0^l q(s)\sin^2\left(\frac{(2n-1)\pi}{2l}s\right)ds\right| \\
&= c_0(l)\frac{\|q\|_{L^\infty}^2}{2n-1} + 2\left|\int_0^l q(s)\frac{1-\cos(\frac{(2n-1)\pi}{l}s)}{2}\,ds\right| \\
&< c_0(l)\frac{\pi^2}{4l^{3/2}}\frac{\|q\|_{H^1(0,l)}}{2n-1} + \left|\int_0^l q(s)\cos\left(\frac{(2n-1)\pi s}{l}\right)ds\right| \\
&= c_0(l)\frac{\pi^2}{4l^{3/2}}\frac{\|q\|_{H^1(0,l)}}{2n-1} + \frac{l}{(2n-1)\pi}\left|\int_0^l q'(s)\sin\left(\frac{(2n-1)\pi s}{l}\right)ds\right| \\
&\leqslant \left(c_0(l)\frac{\pi^2}{4l^{3/2}} + \frac{l^{3/2}}{\pi\sqrt{2}}\right)\frac{\|q\|_{H^1(0,l)}}{2n-1}.
\end{aligned}$$

In the above computations, for passing from the second to the third line, we have applied the facts that $\|q\|_{L^\infty} < \frac{\pi^2}{4l^2}$, that $\|q\|_{L^\infty} \leqslant \sqrt{l}\|q\|_{H^1}$ (as $q(0)=0$) and that $\int_0^l q(s)\,ds = 0$. For passing from the third to the fourth line, we have integrated by parts and finally for passing from the fourth line to last one, we have applied a Cauchy–Schwartz inequality. Therefore, the constant $C_0(l)$ of the lemma is given as follows:

$$C_0(l) = c_0(l)\frac{\pi^2}{4l^{3/2}} + \frac{l^{3/2}}{\pi\sqrt{2}}.$$

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