J. Math. Anal. Appl. 385 (2012) 414–422

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of periodic solutions of ordinary differential equations

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ARTICLE INFO

Article history:
Received 10 March 2011
Available online 24 June 2011
Submitted by W. Sarlet

Keywords:
Ordinary differential equations
Periodic solutions
Euler iterates
Brouwer fixed point theorem
Forced oscillations
Solow equation
Forced nonlinear dissipative pendulum

ABSTRACT

We prove the existence of a periodic solution, $y \in C^1(\mathbb{R}, \mathbb{R}^\ell)$, of a first-order differential equation $\dot{y} = f(t, y)$, where f is periodic with respect to t and admits a star-shaped compact set that is invariant under the Euler iterates of the equation with sufficiently small time-step. As in Peano's Theorem for the Cauchy problem, the only required regularity condition on f is continuity. We present two nontrivial examples that illustrate the usefulness of this theorem in applications related to forced oscillations.

1. Introduction and problem setting

We consider the problem of existence of classical periodic solutions of the ordinary differential equation

$$\frac{dy}{dt} = f(y, t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $f : D \times \mathbb{R} \to \mathbb{R}^\ell$ for some open subset D of $\mathbb{R}^\ell$. Throughout this article, we consider ℓ a fixed positive natural number. Our purpose is to study forced oscillations, and so we require that f is T-periodic with respect to the variable t.

The study of forced oscillations has always been relevant, particularly in applications. Nevertheless, the question of finding general conditions that guarantee its existence has not been comprehensively solved. It is usual to adopt a case by case approach, carried out with the aid of tools from functional analysis that may demand unnecessarily strong conditions. The main purpose of this work is to derive a general result that is both sharper and of easier application.

The classical study of the Cauchy problem for Eq. (1.1) showed that the weakest regularity condition on f that can possibly be demanded is continuity. Following the method of Peano introduced in [11] (see [10], or a more recent account in [5]), our convergence tool is the Ascoli–Arzelà Theorem ([1] and [2]). As such, we do not require the usual local Lipshitz continuity estimate of f with respect to y. This condition has been of vital importance in the classical theory, since the Lipshitz bounds and uniqueness allow the use of Schauder fixed point theory as a convergence tool in periodic solution problems.

However, it is known since Banach that the space of Lipshitzian functions on $[0, 1]$ is not topologically generic in the space of continuous functions on $[0, 1]$ (see [3,7]), nor generic in a measure-theoretic sense (introduced in [8]). Hence, the problem of existence of strong periodic solutions of (1.1) when f may only be continuous seems relevant. The method we

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[1] This research has been supported by Fundação para a Ciência e Tecnologia, FCT, program FEDER/POCI 2010.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.048

use is inspired on Peano's proof of existence of solution for the Cauchy problem. The problem at hand makes its Euler schemes implicit, and so we use Brouwer fixed-point theory (introduced in [4], but see [12] for a comprehensive account on this subject) to show existence of solutions of the approximate problems.

It is trivial to see that T-periodicity and continuity of f are not sufficient to establish existence of periodic solutions. What fails, on counterexamples, is the existence of a compact set, $K \subset \mathbb{R}^\ell$ that is *invariant*, in some sense, with respect to the solutions of (1.1) starting in K. Our use of the Euler scheme as an approximation tool makes the following a good choice of invariance condition.

Definition 1.1. Let $K \subset \mathbb{R}^\ell$ be compact and $f : K \times \mathbb{R} \to \mathbb{R}^\ell$. We say that K is invariant under the Euler iterates of f (in short, K is E-invariant under f) if there exists $\epsilon > 0$ such that, for all $t \in \mathbb{R}$, $y \in K$ and $\Delta t \in \mathbb{R}^+$:

$$0 < \Delta t < \epsilon \quad \Rightarrow \quad y + f(y,t)\Delta t \in K.$$

An obvious difficulty with this E-invariance notion is that it excludes cases where f might be tangent to ∂K. For this reason, we weaken it in the following way:

Definition 1.2. Let $K \subset \mathbb{R}^\ell$ be compact and $f : K \times \mathbb{R} \to \mathbb{R}^\ell$. We say that K is invariant under f if there exists a sequence of continuous functions $f_n : K \times \mathbb{R} \to \mathbb{R}^\ell$, that converge uniformly to f on compact subsets of $K \times \mathbb{R}$ and such that, for all $n \in \mathbb{N}$, K is E-invariant under f_n.

If K is invariant but not E-invariant under f then ϵ (from Definition 1.1), depends on n and, moreover, $\epsilon \downarrow 0$ as $n \to \infty$. Suppose $\ell \geqslant 2$ and ∂K is a differentiable $(\ell - 1)$-manifold on a neighborhood of some point $y \in \partial K$. For any sufficiently small $\Delta t > 0$, let $n_{\Delta t}$ be the largest n such that the corresponding ϵ of Definition 1.1 satisfies $\Delta t < \epsilon$. Note that $n_{\Delta t} \to \infty$ as $\Delta t \downarrow 0$. Definition 1.2 implies that the distance of the Euler iterate of y, $y + f(y,t)\Delta t$, to K (hence to ∂K) is bound by:

$$\left| f(y,t) - f_{n_{\Delta t}}(y,t) \right| \Delta t = o(\Delta t)\Delta t, \quad \text{as } \Delta t \downarrow 0.$$

This estimate allows the direction field, f, of Eq. (1.1) to be tangent to ∂K. Note that, in a setting using Schauder theory as a convergence tool, the obvious invariance condition — solutions of the Cauchy problem starting on K remain on K up to some time — requires uniqueness of solution, at least in situations where f may be tangent to ∂K.

We may now present our problem setting. Let D be an open subset of $\mathbb{R}^\ell$ and consider the first-order differential equation (1.1), where $f : D \times \mathbb{R} \to \mathbb{R}$ verifies the following conditions:

($f1$) There exists $T \in \mathbb{R}^+$ such that f is T-periodic with respect to t, that is, $f(y,\cdot)$ is T-periodic for all $y \in D$.
($f2$) f is continuous in $D \times \mathbb{R}$.
($f3$) There exists some star-shaped compact set K in $\mathbb{R}^\ell$ such that K is invariant under f.

Remark 1.1. In this paper, we consider a function of real variable T-periodic if it has a period $T > 0$, where T may not be its smallest period. In particular, a constant function is T-periodic for all $T > 0$. We call a function strictly T-periodic if T is its smallest period.

The main result of this paper can be stated as follows: *if f verifies conditions ($f1$), ($f2$) and ($f3$), then the differential equation $y' = f(y,t)$ admits a T-periodic solution.*

In Section 2, we describe the Euler discretization scheme used as an approximation tool for the differential equation. In Section 3 we use the Brouwer fixed point theorem to show that the approximate solutions exist and use Peano's method to prove an existence lemma for an equivalent integral equation in the E-invariant case. In Section 4 we show the main result and in Sections 6 and 7 we apply it to the Solow equation and the forced nonlinear dissipative pendulum.

The invariance condition requires the existence of a uniformly convergent sequence of approximations of f; hence, it may seem technically hard to use in practice. In Section 5 we address this concern, providing simple criteria for invariance of compact subsets of $\mathbb{R}^\ell$, of the prescribed form, under f.

2. A discretization for the differential equation in the circle

Let $I = [0,T]$, where $t = 0$ is identified with $t = T$; this makes I topologically equivalent to a circle. The derivative of a function $y : I \to \mathbb{R}$ is defined from the appropriate side derivatives of y at $t = 0$ and $t = T$. The periodic extension of a solution of

$$\frac{dy}{dt} = f(y,t), \quad t \in I, \tag{2.1}$$

satisfies the original problem.

We now construct a simple discretized version of problem (2.1). Choose $N \in \mathbb{N}$, let $\Delta t = \frac{T}{N}$, and:

$$I_N = \{0, \Delta t, 2\Delta t, \dots, (N-1)\Delta t, T\} = \Delta t\, (\mathbb{Z} \bmod N).$$

Consistently with our interpretation of I_N as a discrete version of I, we define addition in I_N as it must: given any $t = m\Delta t$ and $s = n\Delta t$ in I_N, let:

$$t + s = \big((m+n) \bmod N\big)\Delta t.$$

Notice that addition is well-defined in I_N; in particular, both $t + \Delta t$ and $t - \Delta t$ are in I_N for all $t \in I_N$.

The Euler scheme for problem (2.1) is the family of problems (indexed in $N \in \mathbb{N}$) given by:

$$Y_N(t + \Delta t) = Y_N(t) + f\big(Y_N(t), t\big)\Delta t, \quad t \in I_N. \tag{2.2}$$

The definition of I_N makes this scheme implicit; for each N, (2.2) forms a nonlinear system of N equations in the N unknowns $Y_N(t)$, $t \in I_N$. We now derive a fixed point problem that, under our hypothesis, is equivalent to the Euler scheme equations.

For any $N \in \mathbb{N}$, define a map

$$\big(\mathbb{R}^\ell\big)^{I_N} \ni Y \overset{\Phi_N}{\longmapsto} \Phi_N(Y) \in \big(\mathbb{R}^\ell\big)^{I_N}$$

by:

$$\Phi_N(Y)(t) = Y(t) + f\big(Y(t), t\big)\Delta t \tag{2.3}$$

for each $Y \in (\mathbb{R}^\ell)^{I_N}$ and $t \in I_N$. Consider also the usual shift map, given by:

$$\sigma(Y)(t) = Y(t + \Delta t).$$

From this point on, whenever we are working with the Euler scheme problem (2.2) for some fixed N, we will omit the subscript N from Y_N and Φ_N.

If $N \in \mathbb{N}$ is fixed, then problem (2.2) can be written as:

$$\sigma(Y)(t) = Y(t + \Delta t) = Y(t) + f\big(Y(t), t\big)\Delta t = \Phi(Y)(t),$$

for all $t \in I_N$. This is equivalent to the fixed point problem

$$Y = \sigma^{-1}\Phi(Y) \tag{2.4}$$

in $(\mathbb{R}^\ell)^{I_N}$. Thus, the following is a consequence of the previous definitions.

Lemma 2.1. *Let $N \in \mathbb{N}$. If $Y \in (\mathbb{R}^\ell)^{I_N}$ is a fixed point of $\sigma^{-1}\Phi$ then Y satisfies the implicit Euler scheme* (2.2) *in I_N.*

3. Existence of a T-periodic weak solution

We start by proving existence of solutions for the implicit Euler scheme.

Lemma 3.1. *Let K be E-invariant under $f : K \times [0, T] \to K$ where, for all $t \in [0, T]$, $f(\cdot, t)$ is continuous on K. If N be large enough so that $\Delta t = \frac{T}{N} < \epsilon$, where $\epsilon > 0$ is given by the condition of E-invariance of K under f, $(f3)$, then $\sigma^{-1}\Phi$ has a fixed point in K^{I_N}.*

Proof. If $B_N = K^{I_N}$, we claim that:

a) *B_N is star-shaped.*

Since K is star-shaped, there exists $p \in K$ such that the line segment connecting any $y \in K$ with p is in K. Let $P \in B_N$ be the constant function $P(t) \equiv p$ and $Y \in B_N$ be arbitrary. If $0 \leqslant \theta \leqslant 1$ and $t \in I_N$ then $\theta Y(t) + (1-\theta)P(t) \in K$.

b) *If $Y \in B_N$ then $\Phi(Y) \in B_N$.*

If $Y \in B_N$ and $t \in I_N$ then, by E-invariance of K under f, $\Phi(Y)(t) = Y(t) + f(Y(t), t) \in K$.

c) *If $Y \in B_N$ then $\sigma^{-1}\Phi(Y) \in B_N$.*

It is obvious that for any $Z \in B_N$, $\sigma^{-1}(Z) \in B_N$. The result now follows from b).

The uniform convergence of $f(\cdot,t)$, for every $t \in I_N$, implies that Φ (and thus $\sigma^{-1}\Phi$) is continuous in K^{I_N}. Applying the Brouwer fixed point theorem to the map $\sigma^{-1}\Phi$ on the star-shaped set B_N, we can conclude that there exists $Y \in B_N$ such that $Y = \sigma^{-1}\Phi(Y)$. $\square$

Next, we present an existence proof for the integral equation problem associated to Eq. (1.1).

Lemma 3.2. *Let K be a star-shaped compact subset of $\mathbb{R}^\ell$, and $f : K \times [0,T] \to \mathbb{R}^\ell$ be continuous, and such that K is E-invariant under f. Then the equation*

$$y(t) = y(0) + \int_0^t f\big(y(s),s\big)\,ds$$

admits a continuous solution, $y : [0,T] \to K$ such that $y(0) = y(T)$.

Proof. Consider (the tail of) the sequence of Y_N, whose terms are given by Lemma 3.1, and where $N \in \mathbb{N}$ satisfies $N > \frac{T}{\epsilon}$. The terms of this sequence are given by (2.2). We extend each Y_N to a continuous $\bar{Y}_N : [0,T] \to K$ by the interpolation formula:

$$\bar{Y}_N(t) = Y_N\big(n(t)\Delta t\big) + f\big(Y_N\big(n(t)\Delta t\big), n(t)\Delta t\big)\,\delta(t). \tag{3.1}$$

Here, $n(t)$ is the largest integer, m, satisfying $m\Delta t \leqslant t$ and $\delta(t) \in \mathbb{R}^+$ is given by $\delta(t) = t - n(t)\Delta t$. Notice that $0 \leqslant \delta(t) < \Delta t < \epsilon$.

By construction, each $\bar{Y}_N$ is a continuous function on $[0,T]$ verifying $\bar{Y}_N(T) = \bar{Y}_N(0)$. From Eq. (3.1) and E-invariance of K under f, $\bar{Y}_N(t) \in K$ for all $t \in \mathbb{R}$. In particular, the sequence $\bar{Y}_N$ is uniformly bounded.

As for equicontinuity, let $t_2, t_1 \in [0,T]$ and consider $n_i = n(t_i)$ for $i = 1,2$. Without loss of generality, $t_2 > t_1$, so $n_2 \geqslant n_1$. Let $M \geqslant 1$ be an upper bound for $|f|$ on $K \times [0,T]$.

If $n_2 = n_1$ then $|t_2 - t_1| < \Delta t$ and so:

$$\big|\bar{Y}_N(t_2) - \bar{Y}_N(t_1)\big| \leqslant M\big(\delta(t_2) - \delta(t_1)\big) = M|t_2 - t_1|.$$

If $n_2 > n_1$ and $t_2 - t_1 < \Delta t$ then $n_2 = n_1 + 1$. Using both (3.1) and its equivalent form

$$\bar{Y}_N(t) = Y_N\big(\big(n(t)+1\big)\Delta t\big) + f\big(Y_N\big(n(t)\Delta t\big), n(t)\Delta t\big)\big(\delta(t) - \Delta t\big),$$

leads to:

$$\begin{aligned}\big|\bar{Y}_N(t_2) - \bar{Y}_N(t_1)\big| &= \big|Y_N(n_2\Delta t) + f\big(Y_N(n_2\Delta t), n_2\Delta t\big)\delta(t_2) \\ &\quad - Y_N\big((n_1+1)\Delta t\big) - f\big(Y_N(n_1\Delta t), n_1\Delta t\big)\big(\delta(t_1) - \Delta t\big)\big| \\ &\leqslant M\delta(t_2) + M\big(\Delta t - \delta(t_1)\big) \\ &= M|t_2 - t_1|.\end{aligned}$$

If $t_2 - t_1 \geqslant \Delta t$ then $|\bar{Y}_N(t_2) - \bar{Y}_N(t_1)| \leqslant M|t_2 - t_1|$ by the previous cases and the triangle inequality. We conclude that $\bar{Y}_N$ is equicontinuous on $[0,T]$, as wanted.

By the Ascoli–Arzelà Theorem, there exists a subsequence, $\bar{Y}_{N_j}$ of $\bar{Y}_N$ converging uniformly on $[0,T]$ to a continuous limit function $y : [0,T] \to K$. Since for all N, $Y_N(T) = Y_N(0)$, it follows that $y(T) = y(0)$.

We prove that y is a solution of the integral equation:

$$y(t) = y(0) + \int_0^t f\big(y(s),s\big)\,ds \tag{3.2}$$

for $t \in [0,T]$. Here, $y(0) = \lim_{j\to\infty} Y_{N_j}(0)$.

Let $t \in [0,T]$. Let $p = n(t)$. Then:

$$\begin{aligned}\bar{Y}_N(t) &= \bar{Y}_N(0) + \sum_{k=0}^{p-1} f\big(\bar{Y}_N(k\Delta t), k\Delta t\big)\Delta t + f\big(\bar{Y}_N(p\Delta t), p\Delta t\big)\delta(t) \\ &= y(0) + \sum_{k=0}^{p-1} f\big(y(k\Delta t), k\Delta t\big)\Delta t + f\big(y(p\Delta t), p\Delta t\big)\delta(t) + E_N(t),\end{aligned} \tag{3.3}$$

where:

$$E_N(t)=\sum_{k=0}^{p-1}\bigl(f\bigl(\bar{Y}_N(k\Delta t),k\Delta t\bigr)-f\bigl(y(k\Delta t),k\Delta t\bigr)\bigr)\Delta t+\bigl(f\bigl(\bar{Y}_N(p\Delta t),p\Delta t\bigr)-f\bigl(y(p\Delta t),p\Delta t\bigr)\bigr)\delta(t).$$

From the uniform convergence of $\bar{Y}_{N_j}$ and the uniform continuity of f on $K\times[0,T]$, the terms of the form $f(\bar{Y}_{N_j}(\theta),\theta)-f(y(\theta),\theta)$ have a uniform bound on $\theta\in[0,T]$ that converges to 0 along N_j. Hence, $E_{N_j}(t)\to 0$ as $j\to\infty$.

On the other hand, the term $\sum_{k=0}^{p-1}f(y(k\Delta t),k\Delta t)\Delta t+f(y(p\Delta t),p\Delta t)\delta(t)$ in Eq. (3.3), is a Riemann sum of the integral $\int_0^t f(y(s),s)\,ds$ on the partition $\{0,\Delta t,2\Delta t,\dots,p\Delta t,t\}$, with $f(y(\cdot),\cdot)$ continuous on $[0,t]$. Taking the limit on both sides of Eq. (3.3) along N_j, we obtain integral equation (3.2). $\square$

4. Existence of a *T*-periodic solution

In this section, we prove the main theorem.

Lemma 4.1. *Let $T>0$, $A\subset\mathbb{R}^\ell$ compact and $f:A\times[0,T]\to\mathbb{R}^\ell$ continuous, T-periodic with respect to t. If $y:[0,T]\to K$ satisfies $y(0)=y(T)$ and*

$$y(t)=y(0)+\int_0^t f\bigl(y(s),s\bigr)\,ds \tag{4.1}$$

for all $t\in[0,T]$, then the continuous periodic extension of y to $\mathbb{R}$ satisfies the integral equation (4.1) *for all $t\in\mathbb{R}$. Consequently, $y\in C^1(A,K)$ and satisfies $y'=f(y,t)$ in $\mathbb{R}$.*

Proof. Let $\bar{y}:\mathbb{R}\to K$ be the periodic extension of y to $\mathbb{R}$. First note that, from T-periodicity of $f(\bar{y}(\cdot),\cdot)$:

$$\int_0^{mT} f\bigl(\bar{y}(s),s\bigr)\,ds=m\int_0^T f\bigl(y(s),s\bigr)\,ds=m\bigl(y(T)-y(0)\bigr)=0,$$

for all $m\in\mathbb{Z}$. Given $t\in\mathbb{R}$, let m be the largest integer such that $mT<t$. Then:

$$\begin{aligned}\bar{y}(t)=y(t-mT)=y(0)+\int_0^{t-mT} f\bigl(y(s),s\bigr)\,ds\\ =\bar{y}(0)+\int_{mT}^{t} f\bigl(\bar{y}(s),s\bigr)\,ds\\ =\bar{y}(0)+\int_0^{t} f\bigl(\bar{y}(s),s\bigr)\,ds. \quad\square\end{aligned}$$

Given a compact set $B\subset\mathbb{R}^m$, with $m\in\mathbb{N}$, we consider the Banach space of continuous functions $\varphi:B\to\mathbb{R}^\ell$ endowed with the L^∞ norm.

Theorem 4.1. *Let K be a star-shaped compact subset of $\mathbb{R}^\ell$, and $f:K\times\mathbb{R}\to\mathbb{R}^\ell$ be continuous, T-periodic with respect to t and such that K is invariant under f. Then the equation*

$$\frac{dy}{dt}=f(y,t)$$

admits a T-periodic solution, $y:\mathbb{R}\to K$.

Proof. Let f_n be the sequence of approximations of f given by the condition of invariance of K under f. It is a consequence of Lemma 3.2 that for each $n\in\mathbb{N}$ there is a function $y_n:[0,T]\to K$ satisfying

$$y_n(0)=y_n(T) \tag{4.2}$$

and

$$y_n(t) = y_n(0) + \int_0^t f_n\big(y_n(s), s\big)\, ds. \tag{4.3}$$

Since $\|f_n - f\| \to 0$, there exists $M_1 > 0$ such that $\|f_n\| \leqslant M_1$, for all $n \in \mathbb{N}$. Therefore, for all $t \in [0, T]$:

$$\big|y_n(t)\big| \leqslant \big|y_n(0)\big| + M_1 t \leqslant M_2 + M_1 T,$$

where $M_2 = \max\{|y|\colon\ y \in K\}$. This means that y_n is uniformly bounded.

Furthermore, if $0 \leqslant t_1 < t_2 \leqslant T$ then

$$\big|y_n(t_2) - y_n(t_1)\big| \leqslant \int_{t_1}^{t_2} \big|f_n\big(y_n(s), s\big)\big|\, ds \leqslant M_1|t_2 - t_1|.$$

We conclude that y_n is equicontinuous.

By the Ascoli–Arzelà Theorem, there exists a subsequence of y_n, y_{n_j}, converging uniformly on $[0, T]$ to some continuous $y : [0, T] \to K$. Taking the limit on both sides of Eqs. (4.2) and (4.3) and using the uniform convergence of both f_{n_j} and y_{n_j}, we get $y(0) = y(T)$ and

$$y(t) = y(0) + \int_0^t f\big(y(s), s\big)\, ds, \quad \text{for all } t \in [0, T].$$

The conclusion follows from Lemma 4.1. $\square$

5. Invariance tests

In this section, we derive sufficient conditions for the invariance of compact sets under a direction field. The purpose of these results is to make a large class of practical applications of Theorem 3.2 into problems of Calculus.

Let $A \subset \mathbb{R}^\ell$ be open and bounded. Consider a real function $H \in C^1(A) \cap C(\bar{A})$ with a minimum $b = H(\bar{y})$ at some $\bar{y} \in A$. Choose $a > b$ in $H(A)$ such that $a < H(y)$ for all $y \in \partial A$ and $\nabla H(y) \neq 0$ whenever $H(y) = a$. Consider the set:

$$K = \big\{y \in A\colon\ H(y) \leqslant a\big\}.$$

By construction, K is compact in the relative topology of A but, by continuity of H, the distance from K to ∂A is positive. So K is compact in the usual topology of $\mathbb{R}^\ell$, $\partial K = \{y \in A\colon\ H(y) = a\}$ and H has no stationary points on ∂K. If H is a convex function then K is convex. More generally, if for any $v \in \mathbb{R}^\ell$, $H(\bar{y} + tv)$ is an increasing function of $t \in \mathbb{R}$, then K is star-shaped.

Lemma 5.1. *With K and H as defined above, let $f : K \times \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic with respect to t. If for all $y \in \partial K$ and $t \in [0, T]$*

$$\nabla H(y) \cdot f(y, t) < 0$$

then K is E-invariant under f.

Proof. For $y \in K$, let $d(y, \partial K)$ be the distance from y to ∂K, that is, $d(y, \partial K) = \min_{w\in\partial K} |y - w|$. Consider the function $g : K \times K \times [0, T] \to \mathbb{R}^\ell$, given by $g(\xi, y, t) = \nabla H(\xi) \cdot f(y, t)$ (in its domain). By hypothesis, $g(y, y, t) < 0$ for $(y, t) \in \partial K \times [0, T]$. From uniform continuity of g, there exists a $\delta > 0$ such that, for all $\xi, y \in K$ and $t \in [0, T]$:

$$d(y, \partial K) < \delta \wedge |\xi - y| < \delta \quad \Rightarrow \quad \nabla H(\xi) \cdot f(y, t) < 0. \tag{5.1}$$

Using uniform continuity of f, we can define:

$$\epsilon = \frac{\delta}{M}, \quad \text{where } M = \max\big\{\big|f(y, t)\big|\colon\ y \in K \text{ and } t \in [0, T]\big\}.$$

Fix any $\Delta t \in (0, \epsilon)$, and consider $y \in K$. If $d(y, \partial K) < \delta$, then

$$H\big(y + f(y, t)\Delta t\big) = H(y) + \Delta t \nabla H(\xi) \cdot f(y, t) \tag{5.2}$$

where $\xi \in K$ is some point in the line segment joining y with $y + f(y, t)\Delta t$. Since

$$|\xi - y| < \left|f(y,t)\right|\Delta t \leqslant M\epsilon = \delta,$$

from (5.1) and Eq. (5.2), it follows that

$$H\big(y + f(y,t)\Delta t\big) < H(y) \leqslant a,$$

and so $y + f(y,t)\Delta t \in K$, as wanted. In the case $d(y, \partial K) \geqslant \delta$, $|f(y,t)\Delta t| < M\epsilon < \delta$, hence $y + f(y,t)\Delta t$ is inside K. □

Next, we generalize the above lemma to the invariant case.

Proposition 5.1 *(Invariance test). With H and K as previously defined, let $f : K \times \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic with respect to t. If for all $y \in \partial K$ and $t \in [0, T]$*

$$\nabla H(y) \cdot f(y,t) \leqslant 0$$

then K is invariant under f.

Proof. Let $n \in \mathbb{N}$, and consider the sequence of T-periodic (with respect to t) and uniformly continuous functions on $K \times [0, T]$ given by:

$$f_n(y,t) = f(y,t) - \frac{1}{n}\nabla H(y).$$

The function ∇H is (uniformly) continuous in K, and so:

$$\|f_n - f\| \leqslant \frac{1}{n}\|\nabla H\| \to 0 \quad \text{as } n \to \infty.$$

On the other hand, since H has no stationary points on ∂K,

$$\nabla H(y) \cdot f_n(y,t) \leqslant -\frac{1}{n}\left|\nabla H(y)\right|^2 < 0 \quad \text{on } \partial K \times [0, T].$$

By the preceding lemma, we conclude that K is E-invariant under f_n. □

In the one-dimensional case, the test for invariance of a compact interval under a direction field becomes a statement of elementary Calculus.

Corollary 5.1. *Let $f : [a,b] \times [0,T] \to \mathbb{R}$ be continuous and T-periodic with respect to t. If, for all $t \in [0,T]$, $f(a,t) \geqslant 0$ and $f(b,t) \leqslant 0$ then $[a,b]$ is invariant under f.*

Proof. Let $H(y) = \frac{1}{2}(y - \frac{a+b}{2})^2$. Since $H'(y)f(y,t) = (y - \frac{a+b}{2})f(y,t)$, both $H'(a)f(a,t)$ and $H'(b)f(b,t)$ are nonpositive. □

6. Periodic solutions of the Solow equation

As an example, we prove the existence of strictly periodic solutions of the Solow equation. This example, with preliminary work in [6], was the main motivation for this work.

Let $\varphi : [0,\infty) \to \mathbb{R}$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and the following property holds: for any $\theta > 0$, there exists a unique $\kappa > 0$ satisfying:

(a) $\varphi(\kappa) - \theta\kappa = 0$;
(b) if $0 < k < \kappa$ then $\varphi(k) - \theta k > 0$;
(c) if $k > \kappa$ then $\varphi(k) - \theta k < 0$.

We call these the generalized Inada conditions (in short, gI-conditions). If $s > 0$ and φ satisfies the gI-conditions, then $s\varphi$ also satisfies them and it follows that $s\varphi$ is a strictly concave function.

Let $\delta > 0$, $n : \mathbb{R} \to \mathbb{R}$ be a continuous strictly T-periodic function such that $\delta + n(t)$ is always positive and $s > 0$. The Solow equation is:

$$\frac{dk}{dt} = s\varphi(k) - \big(\delta + n(t)\big)k \quad \text{for } t \in \mathbb{R}. \tag{6.1}$$

For details on the economical significance of the equation and its parameters, see [9] and [6] for an interpretation of the meaning of its periodic solutions.

Theorem 6.1. *Under the gI-conditions on φ and given n, s and δ as defined above, the Solow equation,* (6.1)*, admits a strictly T-periodic solution $k \in C^1(\mathbb{R}, \mathbb{R})$.*

Proof. Let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be given by $f(k,t) = s\varphi(k) - (\delta + n(t))k$. Note that f is continuous and T-periodic in the variable t.

Our goal is to find a compact interval $K \subset (0,\infty)$ such that K is invariant under f. Let $J = [0,T]$ and define:

$$\underline{n} = \min_{t\in J} n(t); \qquad \bar{n} = \max_{t\in J} n(t).$$

By hypothesis, $0 < \delta + \underline{n} < \delta + \bar{n}$. Let $\underline{\kappa}$ and $\bar{\kappa}$ be the constants given by the gI-condition (a):

$$s\varphi(\underline{\kappa}) - (\delta + \bar{n})\underline{\kappa} = 0,$$
$$s\varphi(\bar{\kappa}) - (\delta + \underline{n})\bar{\kappa} = 0.$$

Since

$$s\varphi(\underline{\kappa}) - (\delta + \underline{n})\underline{\kappa} > s\varphi(\underline{\kappa}) - (\delta + \bar{n})\underline{\kappa} = 0,$$

using the gI-conditions (b) and (c) we conclude that $0 < \underline{\kappa} < \bar{\kappa}$.

Choose our compact interval to be $K = [\underline{\kappa}, \bar{\kappa}]$. From the gI-condition (b):

$$f(\underline{\kappa}, t) = s\varphi(\underline{\kappa}) - \big(\delta + n(t)\big)\underline{\kappa} \geqslant s\varphi(\underline{\kappa}) - (\delta + \bar{n})\underline{\kappa} = 0$$

and

$$f(\bar{\kappa}, t) = s\varphi(\bar{\kappa}) - \big(\delta + n(t)\big)\bar{\kappa} \leqslant s\varphi(\bar{\kappa}) - (\delta + \underline{n})\bar{\kappa} = 0.$$

By Proposition 5.1, it follows that K is invariant under f. Using Theorem 4.1, we can conclude that Eq. (6.1) has a T-periodic solution, $k : \mathbb{R} \to K$. Since there are no constant positive solutions of (6.1), k is not constant.

We remark that, by hypothesis, the function $n(t)$ is strictly T-periodic and, as a consequence, $k(t)$ is also T-periodic. If that were not true, there would be an $m \in \mathbb{N}$ for which $k(t)$, and consequently $k'(t)$, would be $\frac{T}{m}$-periodic. Hence, $\delta + n(t) = \frac{1}{k(t)}(s\varphi(k(t)) - k'(t))$ would necessarily be $\frac{T}{m}$-periodic, contradicting our hypothesis. $\square$

7. Periodic solutions of the forced nonlinear dissipative pendulum

The forced dissipative nonlinear pendulum has the equation of motion:

$$\ddot{q} + b\dot{q} + \sin q = f(q,t). \tag{7.1}$$

Here $q : \mathbb{R} \to \mathbb{R}$ is the angular position, $b > 0$ is a damping parameter, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the driving force. For our example, we assume that there exists some $\alpha \in (0,\pi)$ such that:

($f1$) f is continuous;
($f2$) For some $T > 0$, f is T-periodic with respect to t;
($f3$) $|f(q,t)| \leqslant b\sqrt{2(\cos q - \cos\alpha)}$, for all $(q,t) \in [-\alpha,\alpha] \times [0,T]$;
($f4$) For all $q \in [-\alpha,\alpha]$ there exists $t \in [0,T]$ such that $-\sin q + f(q,t) \neq 0$.

Writing Eq. (7.1) as a first-order system of ordinary differential equations yields:

$$\begin{cases} \dot{p} = -\sin q - bp + f(q,t), \\ \dot{q} = p. \end{cases} \tag{7.2}$$

Here, $p = \dot{q}$ is the angular velocity of the pendulum. Consider the Hamiltonian function of the (non-forced) conservative pendulum.

$$\mathcal{H}(p,q) = \frac{p^2}{2} + 1 - \cos q.$$

Theorem 7.1. *With b, f, α and $\mathcal{H}$ as defined above, let*:

$$K = \big\{(p,q) \in \mathbb{R}^2 \colon \mathcal{H}(p,q) \leqslant 1 - \cos\alpha \text{ and } |q| \leqslant \alpha\big\}.$$

Then system (7.2) *has a strictly periodic solution, $(p,q) \in C^1(\mathbb{R}, K)$ and, consequently, q is a strictly periodic C^2 solution of Eq.* (7.1)*.*

Proof. If $(p,q) \in K$ then

$$\frac{p^2}{2} \leqslant \cos q - \cos\alpha.$$

Since $\cos q - \cos\alpha \geqslant 0$, we have $|p| \leqslant \sqrt{2(\cos q - \cos\alpha)} < 2$. Take H to be the restriction of $\mathcal{H}$ to $A = (-2,2) \times (-\pi,\pi)$; then $K = H^{-1}([0, 1-\cos\alpha])$ is compact. Fix $(p_0,q_0) \in K$, and let $0 \leqslant t \leqslant 1$; then

$$\frac{d}{dt} H(tp_0, tq_0) = tp_0^2 + q_0 \sin(tq_0) \geqslant 0.$$

Hence K is star-shaped.

The gradient of H in A is given by

$$\nabla H(p,q) = (p, \sin q).$$

If $p \neq 0$, it is obvious that $\nabla H(p,q) \neq 0$. On the other hand, $(0,q) \in \partial K$ if and only if $q = \pm\alpha$, in which case $\nabla H(0,\pm\alpha)$ is also nonzero. We conclude that $\nabla H(p,q) \neq 0$ on ∂K.

The direction field, $\phi : \mathbb{R}^3 \to \mathbb{R}^2$, of system (7.2) is given by

$$\phi(p,q,t) = \big(-\sin q - bp + f(q,t), p\big).$$

For $|q| \leqslant \alpha$, $(p,q) \in \partial K$ is equivalent to $p^2 = 2(\cos q - \cos\alpha)$. From condition $(f3)$, for any $t \in [0,T]$,

$$pf(q,t) \leqslant \big|pf(q,t)\big| \leqslant |p| b \sqrt{2(\cos q - \cos\alpha)} = bp^2,$$

and so:

$$\phi(p,q,t) \cdot \nabla H(p,q) = -bp^2 + pf(q,t) \leqslant 0.$$

By Proposition 5.1, K is invariant under ϕ. Also, by $(f1)$ and $(f2)$ ϕ is continuous and T-periodic with respect to t. Applying Theorem 4.1, we conclude that there exists a periodic solution $(p,q) : \mathbb{R} \to K$ of system (7.2). By $(f4)$ and (7.2), q is not a constant solution. $\square$

It follows from the proof of Theorem 7.1 that condition $(f3)$ is equivalent to $|f(q,t)| \leqslant b|p|$, where p is such that $(p,q) \in \partial K$. We recall that $-bp$ equals the dissipative force at any point in the phase space.

If we take $\alpha = \pi$ then $(f3)$–$(f4)$ become inconsistent. When α reaches π, two unstable equilibrium points, $(0,\pm\pi)$, of the non-forced pendulum enter K.

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