J. Math. Anal. Appl. 387 (2012) 698–720

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lifespan of classical discontinuous solutions to general quasilinear hyperbolic systems of conservation laws with small BV initial data: Shocks and contact discontinuities ☆

Zhi-Qiang Shao

Department of Mathematics, Fuzhou University, Fuzhou 350002, China

ARTICLE INFO

Article history:
Received 24 June 2011
Available online 17 September 2011
Submitted by D. Wang

Keywords:
Generalized Riemann problem
Quasilinear hyperbolic system of conservation laws
Classical discontinuous solution
Shock wave
Contact discontinuity
Lifespan

ABSTRACT

This work is a continuation of our previous work [Z.Q. Shao, Global structure stability of Riemann solutions for linearly degenerate hyperbolic conservation laws under small BV perturbations of the initial data, Nonlinear Anal. Real World Appl. 11 (2010) 3791–3808]. In the present paper we investigate the global structure stability of Riemann solutions for general quasilinear hyperbolic systems of conservation laws under small BV perturbations of the initial data, where the Riemann solution only contains the shocks and contact discontinuities, and at least a shock wave. The perturbations are in BV but they are assumed to be C^1-smooth, with bounded and possibly large C^1-norms. We get a lower bound of the lifespan of the piecewise C^1 solution to a class of the generalized Riemann problem, which can be regarded as a small BV perturbation of the corresponding Riemann problem. Some applications to quasilinear hyperbolic systems of conservation laws arising in physics, particularly to one-dimensional Euler equations of gas dynamics for a compressible, inviscid, non-heat conducting gas in Eulerian coordinates, are also given.

1. Introduction and main result

Consider the following quasilinear hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector-valued function of (t, x), $f : \mathbf{R}^n \to \mathbf{R}^n$ is a given C^3 vector function of u.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given u on the domain under consideration, the Jacobian $A(u) = \nabla f(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \dots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \bigl(\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)\bigr). \tag{1.3}$$

☆ Supported by the National Natural Science Foundation of China (No. 70371025), the Scientific Research Foundation of the Ministry of Education of China (No. 02JA790014), the Natural Science Foundation of Fujian Province of China (No. 2009J01006) and the Science and Technology Developmental Foundation of Fuzhou University (No. 2004-XQ-16).

E-mail address: zqshao_fzu@yahoo.com.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.037

We have

$$\det\left|l_{ij}(u)\right| \neq 0 \quad \left(\text{equivalently, } \det\left|r_{ij}(u)\right| \neq 0\right). \tag{1.4}$$

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \dots, n), \tag{1.6}$$

where δ_{ij} stands for Kronecker's symbol.

Clearly, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \dots, n)$ have the same regularity as $A(u)$, i.e., C^2 regularity.

We also assume that on the domain under consideration, each characteristic field is either genuinely nonlinear in the sense of Lax (cf. [24]):

$$\nabla\lambda_i(u)r_i(u) \neq 0 \tag{1.7}$$

or linearly degenerate in the sense of Lax:

$$\nabla\lambda_i(u)r_i(u) \equiv 0. \tag{1.8}$$

We are interested in the generalized Riemann problem for system (1.1), which is a Cauchy problem with a piecewise C^1 initial data of the form:

$$t = 0 : u = \begin{cases} u_0^-(x), & x \leqslant 0, \\ u_0^+(x), & x \geqslant 0, \end{cases} \tag{1.9}$$

where $u_0^-(x)$ and $u_0^+(x)$ are C^1 vector functions defined for $x \leqslant 0$ and $x \geqslant 0$ respectively with

$$u_0^-(0) \neq u_0^+(0). \tag{1.10}$$

Problem (1.1) and (1.9) may be regarded as a perturbation of the corresponding Riemann problem (1.1) and

$$t = 0 : u = \begin{cases} \widehat{u}_-, & x \leqslant 0, \\ \widehat{u}_+, & x \geqslant 0, \end{cases} \tag{1.11}$$

in which

$$\widehat{u}_\pm = u_0^\pm(0). \tag{1.12}$$

Let

$$\theta = |\widehat{u}_- - \widehat{u}_+|. \tag{1.13}$$

When $\theta > 0$ is suitably small, by Lax [24], the Riemann problem (1.1) and (1.10) admits a unique self-similar solution composed of $n+1$ constant states $\widehat{u}^{(0)} = \widehat{u}_-, \widehat{u}^{(1)}, \dots, \widehat{u}^{(n-1)}, \widehat{u}^{(n)} = \widehat{u}_+$ separated by shocks, centered rarefaction waves (corresponding characteristics are genuinely nonlinear) or contact discontinuities (corresponding characteristics are linearly degenerate). As in [21], this kind of solution is simply called Lax's Riemann solution of the system (1.1).

For the self-similar solution of the Riemann problem of general quasilinear hyperbolic systems of conservation laws, the local nonlinear structure stability has been proved by Li and Yu [27] for one-dimensional case, and by Majda [38] for multi-dimensional case. If system (1.1) is strictly hyperbolic and linearly degenerate, Li and Kong [26] proved the global structure stability of the self-similar solution with small amplitude under perturbation (1.9) satisfying (1.12). In this case the self-similar solution contains only n contact discontinuities. If system (1.1) is strictly hyperbolic and genuinely nonlinear, Li and Zhao [28] proved the global structure stability of the self-similar solution containing only n shocks under perturbation (1.9) satisfying (1.12). Precisely speaking, under certain reasonable hypotheses they obtained the following well-known result.

Theorem 1.1. *Suppose that system* (1.1) *is strictly hyperbolic and genuinely nonlinear. Suppose furthermore that $u_0^-(x)$ and $u_0^+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$ respectively, $f(u)$ is a C^2 vector function and*

$$\theta \stackrel{\triangle}{=} |\widehat{u}_+ - \widehat{u}_-| = \left|u_0^+(0) - u_0^-(0)\right| > 0$$

is suitably small. Suppose finally that the self-similar solution $u = U(\frac{x}{t})$ of the Riemann problem (1.1) *and* (1.11) *is composed of $n+1$ constant states $\widehat{u}^{(0)} = \widehat{u}_-, \widehat{u}^{(1)}, \dots, \widehat{u}^{(n-1)}, \widehat{u}^{(n)} = \widehat{u}_+$ and n non-degenerate typical shocks $x = \widehat{F}^i t$ $(i = 1, \dots, n)$:*

$$u = U\left(\frac{x}{t}\right) = \begin{cases} \widehat{u}^{(0)}, & x \leqslant \widehat{F}^1 t, \\ \widehat{u}^{(i)}, & \widehat{F}^i t \leqslant x \leqslant \widehat{F}^{i+1} t \ (i = 1, \dots, n-1), \\ \widehat{u}^{(n)}, & x \geqslant \widehat{F}^n t. \end{cases}$$

Then there exists a positive constant ε so small that if

$$\left|u_0^-(x) - u_0^-(0)\right|, \left|u_0^{-\prime}(x)\right| \leqslant \frac{\varepsilon}{1+|x|}, \quad \forall x \leqslant 0,$$
$$\left|u_0^+(x) - u_0^+(0)\right|, \left|u_0^{+\prime}(x)\right| \leqslant \frac{\varepsilon}{1+|x|}, \quad \forall x \geqslant 0,$$

then problem (1.1) *and* (1.9) *admits a unique global classical discontinuous solution $u = u(t,x)$ only containing n shocks $x = x_i(t)(x_i(0) = 0)$ $(i = 1, \dots, n)$, such that $u(t,x)$ belongs to C^1 on each domain D^i $(i = 0, 1, \dots, n)$ and $x_i(t)$ $(i = 1, \dots, n)$ to C^2 on $t \geqslant 0$ with*

$$\left|u(t,x) - u(0,0)\right| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in D^i \ (i = 0, 1, \dots, n),$$
$$\left|\frac{\partial u}{\partial x}(t,x)\right|, \left|\frac{\partial u}{\partial t}(t,x)\right| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in D^i \ (i = 0, 1, \dots, n),$$
$$\left|x_i'(t) - x_i'(0)\right| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall t \geqslant 0 \ (i = 1, \dots, n),$$

where

$$D^0 = \left\{(t,x) \mid t \geqslant 0, \ x \leqslant x_1(t)\right\},$$
$$D^i = \left\{(t,x) \mid t \geqslant 0, \ x_i(t) \leqslant x \leqslant x_{i+1}(t)\right\} \quad (i = 1, \dots, n-1),$$
$$D^n = \left\{(t,x) \mid t \geqslant 0, \ x \geqslant x_n(t)\right\}$$

and K is a positive constant independent of t. Moreover, $u(0,0) = \widehat{u}^{(i)}$ on the domain D^i $(i = 0, 1, \dots, n)$ and $x_i'(0) = \widehat{F}^i$ $(i = 1, \dots, n)$. Therefore, as a global perturbation, $u(t,x)$ possesses a similar structure to that of the self-similar solution to Riemann problem (1.1) *and* (1.11) *on $t \geqslant 0$.*

Remark 1.1. Recently, under certain reasonable hypotheses Kong [21,22] proved that Lax's Riemann solution of general $n \times n$ quasilinear hyperbolic system of conservation laws is globally structurally stable if and only if it contains only non-degenerate shocks and contact discontinuities, but no rarefaction waves and other weak discontinuities. Shao [42,43] also studied that the global structure stability and instability of this kind of Lax's Riemann solution with small amplitude in a half space.

However, it is well known that the BV space is a suitable framework for one-dimensional Cauchy problem for the hyperbolic systems of conservation laws (see Bressan [2], Glimm [16]), the result in Bressan [3] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large. So the following question arises naturally: can we obtain the global existence and uniqueness of piecewise C^1 solution containing only shocks and contact discontinuities to a class of the generalized Riemann problem, which can be regarded as a small BV perturbation of the corresponding Riemann problem, for system (1.1) with the following piecewise C^1 initial data:

$$t = 0 : u = \begin{cases} \widehat{u}_- + u_-(x), & x \leqslant 0, \\ \widehat{u}_+ + u_+(x), & x \geqslant 0, \end{cases} \tag{1.14}$$

where $u_\pm(x) \in C^1$ with bounded and possibly large C^1 norm, but of small bounded variation, such that

$$\left\|u_-(x)\right\|_{C^1}, \left\|u_+(x)\right\|_{C^1} \leqslant M, \tag{1.15}$$

for some $M > 0$ bounded but possibly large, and also such that

$$\int_0^{+\infty} \left|u_+'(x)\right| dx, \int_{-\infty}^0 \left|u_-'(x)\right| dx \leqslant \varepsilon, \tag{1.16}$$

for some $\varepsilon > 0$ sufficiently small? Here, it is important to mention that the global existence of weak solutions to a strictly hyperbolic system of conservation laws in one space dimension when the initial data is a small BV perturbation of a solvable Riemann problem has been proved by Schochet [41], unfortunately his method is not useful to show that the solutions are still either contact discontinuities or shocks. An analogous result on stability of a strong shock wave under perturbations of small bounded variation is stated by Corli and Sable-Tougeron [12]. In this paper we exploit to some extent the ideas of Bressan [3], we will develop the method of using continuous Glimm's functional to provide a new, concise proof of an estimate on the lifespan of the piecewise C^1 solution to the generalized Riemann problem under consideration mentioned above. The basic idea we will use here is to combine the techniques employed by Li and Kong [26], especially both the decomposition of waves and the global behavior of waves on the discontinuity curves, with the method of using continuous Glimm's functional. However, we must modify Glimm's functional in order to take care of the presence of shock waves. This makes our new analysis more complicated than those for the C^1 solutions of the Cauchy problem for linearly degenerate quasilinear hyperbolic systems in Bressan [3], Dai and Kong [14], Zhou [48].

As in [44,49], the aim of this paper is to study the global structure stability of Lax's Riemann solution containing only shocks and contact discontinuities (particularly shocks are present). In this case, we shall first get a lower bound of the lifespan of the piecewise C^1 solution to the generalized Riemann problem.

To do so, we consider the generalized Riemann problem for the system (1.1) with the following piecewise C^1 initial data:

$$t=0: u=\begin{cases}\widehat{u}_- + \varepsilon u_-(x), & x\leqslant 0,\\ \widehat{u}_+ + \varepsilon u_+(x), & x\geqslant 0,\end{cases} \tag{1.17}$$

where ε $(0<\varepsilon \ll |\widehat{u}_+ - \widehat{u}_-|)$ is a small parameter, $u_-(x)$ and $u_+(x)$ are C^1 vector functions defined on $x\leqslant 0$ and $x\geqslant 0$ respectively, which satisfy

$$\|u_-(x)\|_{C^1}, \|u_+(x)\|_{C^1} \leqslant K_1 \tag{1.18}$$

and

$$\int_0^{+\infty} |u'_+(x)|\,dx, \int_{-\infty}^{0} |u'_-(x)|\,dx \leqslant K_2, \tag{1.19}$$

where K_1 and K_2 are positive constants independent of ε.

Introduce

$$J_S \stackrel{\triangle}{=} \left\{ j \;\middle|\; j\in\{1,\dots,n\},\ j\text{-wave in } u=U\left(\frac{x}{t}\right) \text{ is a shock wave}\right\}, \tag{1.20}$$

$$J \stackrel{\triangle}{=} \left\{ j \;\middle|\; j\in\{1,\dots,n\},\ \lambda_j(u) \text{ is genuinely nonlinear}\right\} \tag{1.21}$$

and

$$I \stackrel{\triangle}{=} \left\{ i \;\middle|\; i\in\{1,\dots,n\},\ \lambda_i(u) \text{ is linearly degenerate}\right\}. \tag{1.22}$$

Then, the assumption that each characteristic field is either genuinely nonlinear or linearly degenerate gives

$$I\cup J=\{1,\dots,n\}. \tag{1.23}$$

Our main results can be summarized as follows:

Theorem 1.2. *Suppose that system* (1.1) *is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. Suppose furthermore that $u_-(x)$ and $u_+(x)$ are all C^1 vector functions on $x\leqslant 0$ and on $x\geqslant 0$ respectively satisfying* (1.18) *and* (1.19) *as well as*

$$u_-(0)=u_+(0)=0, \tag{1.24}$$

and

$$\theta=|\widehat{u}_+ - \widehat{u}_-| = |u_0^+(0)-u_0^-(0)|>0 \tag{1.25}$$

is suitably small. Suppose finally that the self-similar solution $u=U(\frac{x}{t})$ of the Riemann problem (1.1) *and* (1.11) *consists of k shock waves and $n-k$ contact discontinuities for some integer k $(1\leqslant k\leqslant n)$. Then for small $\theta>0$, there exists a constant $\varepsilon_0>0$ so small that for any fixed $\varepsilon\in(0,\varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem* (1.1) *and* (1.17) *satisfies*

$$\widetilde{T}(\varepsilon)\geqslant K_3\varepsilon^{-1}, \tag{1.26}$$

where K_3 is a positive constant independent of ε. Moreover, when $u=u(t,x)$ blows up in a finite time, $u=u(t,x)$ itself is bounded on the domain $[0,\widetilde{T}(\varepsilon))\times\mathbf{R}$, while the first-order derivatives of $u=u(t,x)$ tend to be unbounded as $t\nearrow\widetilde{T}(\varepsilon)$.

Remark 1.2. Our result implies that classical discontinuous solutions to the generalized Riemann problem under consideration exists almost globally in time. We refer to Kong [23] for the definition of an almost global solution.

Remark 1.3. Suppose that (1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \quad (1 \leqslant p \leqslant n). \tag{1.27}$$

Then the conclusion of Theorem 1.2 still holds (cf. [14]).

Some of the results related to these topics are listed below. Chen et al. [8–11] investigated the asymptotic stability of Riemann waves for hyperbolic conservation laws. Hsiao and Tang [17] investigated the construction and qualitative behavior of the solution of the perturbated Riemann problem for the system of one-dimensional isentropic flow with damping. Xin et al. [46,19] proved the nonlinear stability of contact discontinuities in systems of conservation laws. Smoller et al. [45] investigated the instability of rarefaction shocks in systems of conservation laws. For the overcompressive shock waves, Liu [31] proved the nonlinear stability and instability. Bressan and LeFloch [6] investigated the structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws. Lions et al. [29] proved the existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. Recently, L^1 stability for hyperbolic systems of conservation laws was proved by Bressan, Liu and Yang [7], within the class of solutions with small total variation (see also [2,5,34,35]). Their results were extended in Lewicka [25] to the case where the initial data is a BV perturbation of a possibly large Riemann data. Liu and Xin [33] proved the nonlinear stability of discrete shocks for systems of conservation laws. Dafermos [13] studied the entropy and the stability of classical solutions of hyperbolic systems of conservation laws. For a relaxation system in several space dimensions, Luo and Xin [37] proved the nonlinear stability of shock fronts. Liu and Xin [32] investigated the nonlinear stability of rarefaction waves for compressible Navier–Stokes equations. Hsiao and Pan [18] investigated the nonlinear stability of rarefaction waves for a rate-type viscoelastic system. Moreover, the nonlinear stability of an undercompressive shock for complex Burgers equation was studied by Liu and Zumbrun [36]. For the viscous conservation laws, the theory of nonlinear stability of shock waves was established (see [30,47] and the references therein).

The rest of this paper is organized as follows. For the sake of completeness, in Section 2, we briefly recall John's formula on the decomposition of waves with some supplements and give a generalized Hörmander Lemma. In Section 3, we first review the definition of shock and contact discontinuity, and then analyze some properties of waves on discontinuous curves, which will play an important role in our proof. The main results, Theorem 1.2 is proved in Section 4. Some applications with physical interest will be given in Section 5.

2. John's formula, generalized Hörmander Lemma

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements, which will play an important role in our proof.

Let

$$v_i = l_i(u)u \quad (i = 1, \ldots, n) \tag{2.1}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \ldots, n), \tag{2.2}$$

where $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ denotes the ith left eigenvector.

By (1.5), it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.3}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

be the directional derivative along the ith characteristic. We have (cf. [20,21,26])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \ldots, n), \tag{2.6}$$

where

$$\beta_{ijk}(u) = \big(\lambda_k(u) - \lambda_i(u)\big) l_i(u) \nabla r_j(u) r_k(u). \tag{2.7}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall i, j. \tag{2.8}$$

On the other hand, we have (cf. [20,21,26])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{2.9}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\big\{\big(\lambda_j(u) - \lambda_k(u)\big) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_i(u) r_j(u) \delta_{ik} + (j|k)\big\}, \tag{2.10}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i \ (i, j = 1, \dots, n) \tag{2.11}$$

and

$$\gamma_{iii}(u) \equiv -\nabla \lambda_i(u) r_i(u) \quad (i = 1, \dots, n). \tag{2.12}$$

Noting (2.4), by (2.9) we have (cf. [14])

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \stackrel{\text{def}}{=} G_i(t, x), \tag{2.13}$$

equivalently,

$$d\big[w_i\big(dx - \lambda_i(u)\, dt\big)\big] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \, dt \wedge dx = G_i(t, x)\, dt \wedge dx, \tag{2.14}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}\big(\lambda_j(u) - \lambda_k(u)\big) l_i(u) \big[\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)\big]. \tag{2.15}$$

Hence, we have

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall i, j. \tag{2.16}$$

Lemma 2.1 *(Generalized Hörmander Lemma). Suppose that $u = u(t, x)$ is a piecewise C^1 solution to system (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the ith characteristic direction, and $\mathcal{D}$ is the domain bounded by τ_1, τ_2 and two ith characteristic curves L_i^- and L_i^+. Suppose furthermore that the domain $\mathcal{D}$ contains m C^1 curves of discontinuity of u, denoted by $\widehat{C}_j : x = x_j(t)$ $(j = 1, \dots, m)$, which are never tangent to the ith characteristic direction. Then we have*

$$\int_{\tau_1} \big|w_i\big(dx - \lambda_i(u)\, dt\big)\big| \leqslant \int_{\tau_2} \big|w_i\big(dx - \lambda_i(u)\, dt\big)\big| + \sum_{j=1}^{m} \int_{\widehat{C}_j} \big|[w_i]\, dx - \big[w_i \lambda_i(u)\big]\, dt\big| + \iint_{\mathcal{D}} \left|\sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k\right| dt\, dx, \tag{2.17}$$

where $\Gamma_{ijk}(u)$ is given by (2.15) and $[w_i] = w_i^+ - w_i^-$ denotes the jump of w_i over the curve of discontinuity $\widehat{C}_j$ $(j = 1, \dots, m)$, etc.

The proof can be found in Li and Kong [26].

3. Shock wave and contact discontinuity

In this section, we first review the definitions of shock and contact discontinuity, and then analyze some properties of waves on the discontinuous curves, which will play an important role in our proof.

Definition 3.1. A piecewise C^1 vector function $u = u(t,x)$ is called a piecewise C^1 solution containing a kth shock $x = x_k(t)$ $(x_k(0) = 0)$ for system (1.1), if $u = u(t,x)$ satisfies system (1.1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the following Rankine–Hugoniot condition:

$$f(u^+) - f(u^-) = s(u^+ - u^-) \tag{3.1}$$

and the Lax entropy condition:

$$\lambda_k(u^+) < s < \lambda_k(u^-), \qquad \lambda_{k+1}(u^+) > s > \lambda_{k-1}(u^-), \tag{3.2}$$

where $u^\pm = u^\pm(t, x_k(t)) \stackrel{\triangle}{=} u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$ (when $k = 1$ (resp. $k = n$), the term $\lambda_{k-1}(u^-)$ (resp. $\lambda_{k+1}(u^+)$) disappears in (3.2)).

Definition 3.2. A piecewise C^1 vector function $u = u(t,x)$ is called a piecewise C^1 solution containing a kth contact discontinuity $x = x_k(t)$ $(x_k(0) = 0)$ for system (1.1), if $u = u(t,x)$ satisfies system (1.1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the Rankine–Hugoniot condition (3.1) and

$$s = \lambda_k(u^+) = \lambda_k(u^-), \tag{3.3}$$

where $u^\pm = u^\pm(t, x_k(t)) \stackrel{\triangle}{=} u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$.

Definitions 3.1 and 3.2 can be found in [24] or [27].

The following lemmas give some properties of waves on the shock and contact discontinuity.

Lemma 3.1. *On the kth shock or contact discontinuity $x = x_k(t)$, it holds that*

$$v_i^+ = v_i^- + O(|v^\pm|^2) \quad (i = 1, \dots, k-1, k+1, \dots, n), \tag{3.4}$$

provided that $|u^\pm|$ is sufficiently small, where v_i is defined by (2.1) and $v_i^\pm \stackrel{\triangle}{=} v_i(t, x_k(t) \pm 0)$, etc.

Lemma 3.2. *On the kth contact discontinuity $x = x_k(t)$, it holds that*

$$w_i^- = w_i^+ + O\Big(|u^+ - u^-| \cdot \sum_{j \neq k} |w_j^\pm|\Big) \quad (i = 1, \dots, k-1, k+1, \dots, n), \tag{3.5}$$

provided that $|u^\pm|$ is sufficiently small, where w_i are defined by (2.1) and $w_i^\pm \stackrel{\triangle}{=} w_i(t, x_k(t) \pm 0)$, etc.

Lemma 3.3. *On the kth shock $x = x_k(t)$, it holds that*

$$\begin{aligned} w_i^- = {} & w_i^+ + O\Big(|u^+ - u^-| \cdot \sum_{j \neq k} |w_j^\pm|\Big) + O\big(|u^+ - u^-| \cdot \big|\big(\lambda_k(u^-, u^+) - \lambda_k(u^+)\big) w_k^+\big|\big) \\ & + O\big(|u^+ - u^-| \cdot \big|\big(\lambda_k(u^-, u^+) - \lambda_k(u^-)\big) w_k^-\big|\big) \quad (i = 1, \dots, k-1, k+1, \dots, n), \end{aligned} \tag{3.6}$$

provided that $|u^\pm|$ is sufficiently small, where $\lambda_k(u^-, u^+)$ is the kth eigenvalue of the matrix

$$A(u^-, u^+) \stackrel{\triangle}{=} \int_0^1 \nabla f\big(u^- + \varsigma(u^+ - u^-)\big)\, d\varsigma. \tag{3.7}$$

Remark 3.1. By (1.2), if $|u^+ - u^-|$ is sufficiently small, then the matrix $A(u^-, u^+)$ has n distinct real eigenvalues:

$$\lambda_1(u^-, u^+) < \lambda_2(u^-, u^+) < \cdots < \lambda_n(u^-, u^+). \tag{3.8}$$

The proofs of Lemmas 3.1–3.3 can be found in Kong [21,22].

Corollary 3.1. *On the kth contact discontinuity $x = x_k(t)$, it holds that*

$$\big(w_i\lambda_i(u)\big)^+ = \big(w_i\lambda_i(u)\big)^- + O\Big(\big|u^+ - u^-\big| \cdot \sum_{j\neq k}\big|w_j^{\pm}\big|\Big) \quad (i = 1, \dots, k-1, k+1, \dots, n), \tag{3.9}$$

provided that $|u^{\pm}|$ is sufficiently small.

Proof. Noting

$$\big(w_i\lambda_i(u)\big)^+ - \big(w_i\lambda_i(u)\big)^- = \big[w_i^+ - w_i^-\big]\big(\lambda_i(u)\big)^+ + w_i^-\big[\big(\lambda_i(u)\big)^+ - \big(\lambda_i(u)\big)^-\big], \tag{3.10}$$

from (3.5), we immediately get (3.9). □

4. Proof of Theorem 1.2

For the sake of simplicity and without loss of generality, we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0) \tag{4.1}$$

and

$$|\widehat{u}_{\pm}| \leqslant \theta. \tag{4.2}$$

By the existence and uniqueness of local classical discontinuous solutions of quasilinear hyperbolic systems of conservation laws (see [27]), when $\theta > 0$ is suitably small, the generalized Riemann problem (1.1) and (1.17) admits a unique piecewise C^1 solution $u = u(t,x)$ containing only shocks and (or) contact discontinuities (denoted by $x = x_i(t)$ $(i = 1, \dots, n)$) on the strip $[0,h] \times \mathbf{R}$, where $h > 0$ is a small number; moreover, this solution has a local structure similar to the one of the self-similar solution to the corresponding Riemann problem. In order to prove Theorem 1.2, we first establish some uniform a priori estimates on u and u_x on the domain of existence of the piecewise C^1 solution $u = u(t,x)$.

By (4.1), there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geqslant \delta_0, \quad \forall |u|, |v| \leqslant \delta \ (i = 1, \dots, n-1). \tag{4.3}$$

For the time being it is supposed that on the domain of existence of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (1.1) and (1.17), we have

$$\big|u(t,x)\big| \leqslant \delta. \tag{4.4}$$

At the end of the proof of Lemma 4.5, we will explain that this hypothesis is reasonable.

For any fixed $T > 0$, let

$$U_\infty(T) = \sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}}\big|u(t,x)\big|, \tag{4.5}$$

$$V_\infty(T) = \sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}}\big|v(t,x)\big|, \tag{4.6}$$

$$W_\infty(T) = \sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}}\big|w(t,x)\big|, \tag{4.7}$$

$$\widetilde{W}_1(T) = \max_{i=1,\dots,n}\max_{j\neq i}\sup_{\widetilde{C}_j}\int_{\widetilde{C}_j}\big|w_i(t,x)\big|\,dt, \tag{4.8}$$

$$W_1(T) = \max_{j\in J_S}\int_0^T\Big|\big(x_j'(t) - \lambda_j\big(u\big(t,x_j(t)\pm 0\big)\big)\big)w_j\big(t,x_j(t)\pm 0\big)\Big|\,dt, \tag{4.9}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $v = (v_1, \dots, v_n)^T$ and $w = (w_1, \dots, w_n)^T$ in which v_i and w_i are defined by (2.1) and (2.2) respectively, while $\widetilde{C}_j$ stands for any given jth characteristic on the domain $[0,T] \times \mathbf{R}$. In (4.4)–(4.7), on any contact discontinuity or shock $x = x_k(t)$ the values of $u(t,x)$, $v(t,x)$ and $w(t,x)$ are taken to be $u^{\pm}(t,x) = u(t, x_k(t) \pm 0)$, $v^{\pm}(t,x) = v(t, x_k(t) \pm 0)$ and $w^{\pm}(t,x) = w(t, x_k(t) \pm 0)$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

First we recall some basic L^1 estimates. They are essentially due to Schatzman [39,40] and Zhou [48].

Lemma 4.1. *Let $\phi = \phi(t,x) \in C^1$ satisfies*

$$\phi_t + \big(\lambda(t,x)\phi\big)_x = F(t,x), \quad 0 \leqslant t \leqslant T,\ x \in R, \qquad \phi(0,x) = g(x),$$

where $\lambda \in C^1$. Then

$$\int_{-\infty}^{+\infty} \big|\phi(t,x)\big|\,dx \leqslant \int_{-\infty}^{+\infty} \big|g(x)\big|\,dx + \int_0^T \int_{-\infty}^{+\infty} \big|F(t,x)\big|\,dx\,dt, \quad \forall t \leqslant T, \tag{4.10}$$

provided that the right-hand side of the inequality is bounded.

Lemma 4.2. *Let $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ be C^1 functions satisfying*

$$\phi_t + \big(\lambda(t,x)\phi\big)_x = F_1(t,x), \quad 0 \leqslant t \leqslant T,\ x \in R, \qquad \phi(0,x) = g_1(x),$$

and

$$\psi_t + \big(\mu(t,x)\psi\big)_x = F_2(t,x), \quad 0 \leqslant t \leqslant T,\ x \in R, \qquad \psi(0,x) = g_2(x),$$

respectively, where $\lambda, \mu \in C^1$ such that there exists a positive constants δ_0 independent of T verifying

$$\mu(t,x) - \lambda(t,x) \geqslant \delta_0, \quad 0 \leqslant t \leqslant T,\ x \in R.$$

Then

$$\begin{aligned}\int_0^T \int_{-\infty}^{+\infty} \big|\phi(t,x)\big|\big|\psi(t,x)\big|\,dx\,dt \leqslant{}& C\left(\int_{-\infty}^{+\infty} \big|g_1(x)\big|\,dx + \int_0^T \int_{-\infty}^{+\infty} \big|F_1(t,x)\big|\,dx\,dt\right) \\ &\times \left(\int_{-\infty}^{+\infty} \big|g_2(x)\big|\,dx + \int_0^T \int_{-\infty}^{+\infty} \big|F_2(t,x)\big|\,dx\,dt\right),\end{aligned} \tag{4.11}$$

provided that the two factors on the right-hand side of the inequality is bounded.

In the present situation, similar to the above basic L^1 estimates (4.10)–(4.11), we have

Lemma 4.3. *Under the assumptions of Theorem 1.2, on any given domain of existence $[0,T] \times \mathbf{R}$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (1.1) and (1.17), there exists a positive constant k_1 independent of θ, ε and T such that*

$$\int_{-\infty}^{+\infty} \big|w_i(t,x)\big|\,dx \leqslant k_1 \left\{\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_i(t,x)\big|\,dx\,dt\right\}, \quad \forall t \leqslant T, \tag{4.12}$$

provided that the right-hand side of the inequality is bounded.

Proof. To estimate $\int_{-\infty}^{+\infty} |w_i(t,x)|\,dx$, we need only to estimate

$$\int_{-a}^{a} \big|w_i(t,x)\big|\,dx \tag{4.13}$$

for any given $a > 0$ and then let $a \to +\infty$.

For $i = 1, \dots, n$, for any given t with $0 \leqslant t \leqslant T$, passing through point $A(t,a)$ $(a > x_n(t))$ (resp. $B(t,-a)$), we draw the ith backward characteristic which intersects the x-axis at a point $D(0,x_D)$ (resp. $C(0,x_C)$), see Fig. 1.

Then, applying (2.17) on the domain $ABCD$, we have

$$\int_B^A \big|w_i(t,x)\big|\,dx \leqslant \int_{x_C}^{x_D} \big|w_i(0,x)\big|\,dx + \sum_{k=1}^n \int_{\widehat{C}_k} \big|\big([w_i]x_k'(t) - \big[w_i\lambda_i(u)\big]\big)\,dt\big| + \int\!\!\int_{ABCD} |G_i|\,dx\,dt, \tag{4.14}$$

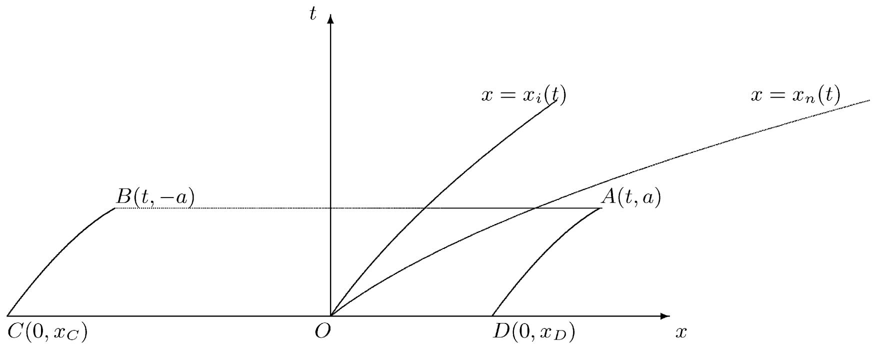

Fig. 1. The domain ABCD in (t, x)-plane.

where $\widehat{C}_k : x = x_k(t)$ stands for the kth discontinuous curve (shock or contact discontinuity) passing through the origin, which is contained in the region $ABCD$. Thus, we get

$$\int_{-a}^{a} \left|w_i(t,x)\right| dx \leqslant \int_{-\infty}^{+\infty} \left|w_i(0,x)\right| dx + \int_0^T \left|\left(x_i'(t) - \lambda_i\left(u\left(t, x_i(t) \pm 0\right)\right)\right) w_i\left(t, x_i(t) \pm 0\right)\right| dt$$
$$+ \sum_{k=1, k\neq i}^{n} \int_{\widehat{C}_k} \left|\left([w_i]x_k'(t) - \left[w_i \lambda_i(u)\right]\right) dt\right| + \int_0^T \int_{-\infty}^{+\infty} |G_i| \, dx\, dt. \tag{4.15}$$

Using (3.3), (3.5), (3.6), (3.9) and (4.4), and noting (4.9), it is easy to see that

$$\int_{-a}^{a} \left|w_i(t,x)\right| dx \leqslant \int_{-\infty}^{+\infty} \left|w_i(0,x)\right| dx + W_1(T) + c_1 V_\infty(T)\left(\widetilde{W}_1(T) + W_1(T)\right) + \int_0^T \int_{-\infty}^{+\infty} |G_i| \, dx\, dt, \tag{4.16}$$

where here and henceforth, c_i $(i = 1, 2, \ldots)$ will denote positive constants independent of θ, ε and T.

Noting (1.19), we have

$$\int_{-a}^{a} \left|w_i(t,x)\right| dx \leqslant c_2 \left\{ \varepsilon + W_1(T) + V_\infty(T)\left(\widetilde{W}_1(T) + W_1(T)\right) + \int_0^T \int_{-\infty}^{+\infty} |G_i| \, dx\, dt \right\}. \tag{4.17}$$

Letting $a \to +\infty$, we immediately get the assertion in (4.12). The proof of Lemma 4.3 is finished. □

Lemma 4.4. *Under the assumptions of Theorem* 1.2, *on any given domain of existence* $[0, T] \times \mathbf{R}$ *of the piecewise* C^1 *solution* $u = u(t, x)$ *to the generalized Riemann problem* (1.1) *and* (1.17), *there exists a positive constant* k_2 *independent of* θ, ε *and* T *such that*

$$\int_0^T \int_{-\infty}^{+\infty} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\, dt \leqslant k_2 \left(\varepsilon + W_1(T) + V_\infty(T)\left(\widetilde{W}_1(T) + W_1(T)\right) + \int_0^T \int_{-\infty}^{+\infty} \left|G_i(t,x)\right| dx\, dt \right)$$
$$\times \left(\varepsilon + W_1(T) + V_\infty(T)\left(\widetilde{W}_1(T) + W_1(T)\right) + \int_0^T \int_{-\infty}^{+\infty} \left|G_j(t,x)\right| dx\, dt \right),$$
$$\forall i \neq j \ (i, j = 1, \ldots, n), \tag{4.18}$$

provided that the right-hand side of the inequality is bounded.

Proof. To estimate

$$\int_0^T \int_{-\infty}^{+\infty} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\, dt, \tag{4.19}$$

it is enough to estimate

$$\int_0^T \int_{-L}^{L} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\, dt \tag{4.20}$$

for any given $L > 0$ and then let $L \to +\infty$.

For $i, j \in \{1, \dots, n\}$ and $i \neq j$, without loss of generality, we suppose that $i < j$. Let $x = x_i(t, L)$ $(0 \leqslant t \leqslant T)$ be the ith forward characteristic passing through point $(0, L)$ $(L > x_n(T))$. Then, we draw the ith backward characteristic $x = s_i(t)$ $(0 \leqslant t \leqslant T)$ passing through point (T, a) $(a > x_i(T, L))$. In the meantime, passing through the point $(T, -L)$, we draw the jth backward characteristic $x = s_j(t)$ $(0 \leqslant t \leqslant T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional" (cf. [3,4,48])

$$Q(t) = \int\int_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right|\left|w_i(t,y)\right| dx\, dy. \tag{4.21}$$

Because of the piecewise C^1 solution $u = u(t,x)$ containing only n shocks or contact discontinuities $x = x_k(t)$ $(x_k(0) = 0)$ $(k = 1, \dots, n)$, we divide the bounded domain $\widetilde{\Omega} \stackrel{\triangle}{=} \{(x, y) \mid s_j(t) < x < y < s_i(t)\}$ by the straight lines $y = x_k(t)$ $(k = 1, \dots, n)$ into some parts. Then, the straightforward calculations on all parts of the domain $\widetilde{\Omega}$ reveal that

$$\begin{aligned}
\frac{dQ(t)}{dt} ={}& s_i'(t)\left|w_i\big(t, s_i(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_j(t,x)\right| dx - s_j'(t)\left|w_j\big(t, s_j(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right| dx \\
&+ \sum_{k=1}^{n} x_k'(t)\left\{\left|w_i\big(t, x_k(t)-0\big)\right| - \left|w_i\big(t, x_k(t)+0\big)\right|\right\} \int_{s_j(t)}^{x_k(t)} \left|w_j(t,x)\right| dx \\
&+ \int\int_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial t}\big(\left|w_j(t,x)\right|\big)\left|w_i(t,y)\right| dx\, dy \\
&+ \int\int_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right| \frac{\partial}{\partial t}\big(\left|w_i(t,y)\right|\big) dx\, dy \\
={}& s_i'(t)\left|w_i\big(t, s_i(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_j(t,x)\right| dx - s_j'(t)\left|w_j\big(t, s_j(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right| dx \\
&+ \sum_{k=1}^{n} x_k'(t)\left\{\left|w_i\big(t, x_k(t)-0\big)\right| - \left|w_i\big(t, x_k(t)+0\big)\right|\right\} \int_{s_j(t)}^{x_k(t)} \left|w_j(t,x)\right| dx \\
&- \int\int_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial x}\big(\lambda_j(u)\left|w_j(t,x)\right|\big)\left|w_i(t,y)\right| dx\, dy \\
&- \int\int_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right| \frac{\partial}{\partial y}\big(\lambda_i(u)\left|w_i(t,y)\right|\big) dx\, dy \\
&+ \int\int_{s_j(t)<x<y<s_i(t)} \operatorname{sgn}(w_j) G_j(t,x)\left|w_i(t,y)\right| dx\, dy \\
&+ \int\int_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right| \operatorname{sgn}(w_i) G_i(t,y)\, dx\, dy
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{s_j(t)}^{s_i(t)} \big(\lambda_j\big(u(t,x)\big)-\lambda_i\big(u(t,x)\big)\big)\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\\
&\quad + \big(s_i'(t)-\lambda_i\big(u\big(t,s_i(t)\big)\big)\big)\big|w_i\big(t,s_i(t)\big)\big|\int_{s_j(t)}^{s_i(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \big(\lambda_j\big(u\big(t,s_j(t)\big)\big)-s_j'(t)\big)\big|w_j\big(t,s_j(t)\big)\big|\int_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\,dx\\
&\quad + \big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)-0\big)\big)\big)\big|w_i\big(t,x_i(t)-0\big)\big|\int_{s_j(t)}^{x_i(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \big(\lambda_i\big(u\big(t,x_i(t)+0\big)\big)-x_i'(t)\big)\big|w_i\big(t,x_i(t)+0\big)\big|\int_{s_j(t)}^{x_i(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \sum_{k=1,k\neq i}^{n} x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)\big|-\big|w_i\big(t,x_k(t)+0\big)\big|\big\}\int_{s_j(t)}^{x_k(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \sum_{k=1,k\neq i}^{n} \big\{\lambda_i\big(u\big(t,x_k(t)+0\big)\big)\big|w_i\big(t,x_k(t)+0\big)\big|-\lambda_i\big(u\big(t,x_k(t)-0\big)\big)\big|w_i\big(t,x_k(t)-0\big)\big|\big\}\int_{s_j(t)}^{x_k(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \int\int_{s_j(t)<x<y<s_i(t)} \operatorname{sgn}(w_j)G_j(t,x)\big|w_i(t,y)\big|\,dx\,dy\\
&\quad + \int\int_{s_j(t)<x<y<s_i(t)} \big|w_j(t,x)\big|\operatorname{sgn}(w_i)G_i(t,y)\,dx\,dy.
\end{aligned}
\tag{4.22}
$$

Noting (3.2)–(3.3) and (4.1) and using (4.3), we get from (4.22) that

$$
\begin{aligned}
\frac{dQ(t)}{dt} &\leqslant -\delta_0\int_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx+\big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)\pm 0\big)\big)\big)w_i\big(t,x_i(t)\pm 0\big)\big|\int_{s_j(t)}^{x_i(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \sum_{k\neq i} x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)-w_i\big(t,x_k(t)+0\big)\big|\big\}\int_{s_j(t)}^{x_k(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \sum_{k\neq i}\big\{\big|\lambda_i\big(u\big(t,x_k(t)+0\big)\big)w_i\big(t,x_k(t)+0\big)-\lambda_i\big(u\big(t,x_k(t)-0\big)\big)w_i\big(t,x_k(t)-0\big)\big|\big\}\int_{s_j(t)}^{x_k(t)}\big|w_j(t,x)\big|\,dx\\
&\quad + \int_{s_j(t)}^{s_i(t)}\big|G_j(t,x)\big|\,dx\int_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\,dx+\int_{s_j(t)}^{s_i(t)}\big|G_i(t,x)\big|\,dx\int_{s_j(t)}^{s_i(t)}\big|w_j(t,x)\big|\,dx\\
&\leqslant -\delta_0\int_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx+\big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)\pm 0\big)\big)\big)w_i\big(t,x_i(t)\pm 0\big)\big|\int_{-\infty}^{+\infty}\big|w_j(t,x)\big|\,dx\\
&\quad + \sum_{k\neq i} x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)-w_i\big(t,x_k(t)+0\big)\big|\big\}\int_{-\infty}^{+\infty}\big|w_j(t,x)\big|\,dx
\end{aligned}
$$

$$+ \sum_{k \neq i} \left\{ \left| \lambda_i\big(u\big(t, x_k(t)+0\big)\big) w_i\big(t, x_k(t)+0\big) - \lambda_i\big(u\big(t, x_k(t)-0\big)\big) w_i\big(t, x_k(t)-0\big) \right| \right\} \int_{-\infty}^{+\infty} \left| w_j(t,x) \right| dx$$

$$+ \int_{-\infty}^{+\infty} \left| G_j(t,x) \right| dx \int_{-\infty}^{+\infty} \left| w_i(t,x) \right| dx + \int_{-\infty}^{+\infty} \left| G_i(t,x) \right| dx \int_{-\infty}^{+\infty} \left| w_j(t,x) \right| dx. \tag{4.23}$$

It then follows from Lemma 4.3 that

$$\begin{aligned}
&\frac{dQ(t)}{dt} + \delta_0 \int_{s_j(t)}^{s_i(t)} \left| w_i(t,x) \right| \left| w_j(t,x) \right| dx \\
&\leqslant k_1 \int_{-\infty}^{+\infty} \left| G_j(t,x) \right| dx \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left| G_i(t,x) \right| dx\, dt \right) \\
&\quad + k_1 \Bigg(\left| \big(x_i'(t) - \lambda_i\big(u\big(t, x_i(t) \pm 0\big)\big)\big) w_i\big(t, x_i(t) \pm 0\big) \right| + \sum_{k \neq i} x_k'(t) \left\{ \left| w_i\big(t, x_k(t)-0\big) - w_i\big(t, x_k(t)+0\big) \right| \right\} \\
&\quad + \sum_{k \neq i} \left\{ \left| \lambda_i\big(u\big(t, x_k(t)+0\big)\big) w_i\big(t, x_k(t)+0\big) - \lambda_i\big(u\big(t, x_k(t)-0\big)\big) w_i\big(t, x_k(t)-0\big) \right| \right\} + \int_{-\infty}^{+\infty} \left| G_i(t,x) \right| dx \Bigg) \\
&\quad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left| G_j(t,x) \right| dx\, dt \right).
\end{aligned} \tag{4.24}$$

Therefore

$$\begin{aligned}
&\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} \left| w_i(t,x) \right| \left| w_j(t,x) \right| dx\, dt \\
&\leqslant Q(0) + k_1 \int_0^T \int_{-\infty}^{+\infty} \left| G_j(t,x) \right| dx\, dt \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left| G_i(t,x) \right| dx\, dt \right) \\
&\quad + k_1 \Bigg(\int_0^T \left| \big(x_i'(t) - \lambda_i\big(u\big(t, x_i(t) \pm 0\big)\big)\big) w_i\big(t, x_i(t) \pm 0\big) \right| dt + \sum_{k \neq i} \int_{\widehat{C}_k} \left| [w_i] \right| \lambda_k\big(u^{\pm}\big)\, dt \\
&\quad + \sum_{k \neq i} \int_{\widehat{C}_k} \left| \big[w_i \lambda_i(u) \big] \right| dt + \int_0^T \int_{-\infty}^{+\infty} \left| G_i(t,x) \right| dx\, dt \Bigg) \\
&\quad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left| G_j(t,x) \right| dx\, dt \right).
\end{aligned} \tag{4.25}$$

Using (3.5), (3.6), (3.9) and noting (4.4), we obtain

$$\begin{aligned}
\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} \left| w_i(t,x) \right| \left| w_j(t,x) \right| dx\, dt &\leqslant Q(0) + c_3 \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left| G_i(t,x) \right| dx\, dt \right) \\
&\quad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left| G_j(t,x) \right| dx\, dt \right).
\end{aligned} \tag{4.26}$$

Noting

$$Q(0) \leqslant \int_{-\infty}^{+\infty} |w_i(0,x)|\,dx \int_{-\infty}^{+\infty} |w_j(0,x)|\,dx, \tag{4.27}$$

we get

$$\begin{aligned}
&\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|\,dx\,dt \\
&\quad \leqslant c_4\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|\,dx\,dt\right) \\
&\qquad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|\,dx\,dt\right).
\end{aligned} \tag{4.28}$$

It then follows

$$\begin{aligned}
&\int_0^T \int_{s_j(t)}^{s_i(t)} |w_i(t,x)||w_j(t,x)|\,dx\,dt \\
&\quad \leqslant \frac{c_4}{\delta_0}\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|\,dx\,dt\right) \\
&\qquad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|\,dx\,dt\right).
\end{aligned} \tag{4.29}$$

Therefore

$$\begin{aligned}
&\int_0^T \int_{-L}^{L} |w_i(t,x)||w_j(t,x)|\,dx\,dt \\
&\quad \leqslant \frac{c_4}{\delta_0}\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_i(t,x)|\,dx\,dt\right) \\
&\qquad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_j(t,x)|\,dx\,dt\right)
\end{aligned} \tag{4.30}$$

and the desired conclusion follows by taking $L \to +\infty$. The proof of Lemma 4.4 is finished. □

Lemma 4.5. *Under the assumptions of Theorem* 1.2, *for small $\theta > 0$ there exists a constant $\varepsilon > 0$ so small that on any given domain of existence $[0, T] \times \mathbf{R}$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem* (1.1) *and* (1.17), *there exist positive constants k_i $(i = 3, \ldots, 7)$ independent of θ, ε and T, such that the following uniform a priori estimates hold*:

$$W_1(T) \leqslant k_3\varepsilon, \tag{4.31}$$

$$\widetilde{W}_1(T) \leqslant k_4\varepsilon, \tag{4.32}$$

$$U_\infty(T), V_\infty(T) \leqslant k_5\theta \tag{4.33}$$

and

$$W_\infty(T) \leqslant k_6\varepsilon, \tag{4.34}$$

where T satisfies

$$T\varepsilon \leqslant k_7. \tag{4.35}$$

Proof. We introduce

$$Q_W(T)=\sum_{j=1}^{n}\sum_{i\neq j}\int_0^T\int_{-\infty}^{+\infty}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt. \tag{4.36}$$

By (2.13), it follows from Lemma 4.4 that

$$Q_W(T)\leqslant c_5\left(\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+\int_0^T\int_{-\infty}^{+\infty}\big|G(t,x)\big|\,dx\,dt\right)^2, \tag{4.37}$$

where $G=(G_1,G_2,\dots,G_n)$.

Noting (2.16), we have

$$\int_0^T\int_{-\infty}^{+\infty}\big|G(t,x)\big|\,dx\,dt\leqslant c_6 Q_W(T). \tag{4.38}$$

Substituting (4.38) into (4.37), we obtain

$$Q_W(T)\leqslant c_7\big(\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big)^2. \tag{4.39}$$

We next estimate $\widetilde{W}_1(T)$.

Let

$$\widetilde{C}_j: x=x_j(t)\quad (0\leqslant t_1\leqslant t\leqslant t_2\leqslant T) \tag{4.40}$$

be any given jth characteristic on the domain $[0,T]\times\mathbf{R}$. Then, passing through the point $P_1(t_1,x_j(t_1))$ (resp. $P_2(t_2,x_j(t_2))$) we draw the ith characteristic which intersects the x-axis at a point $A_1(0,y_1)$ (resp. $A_2(0,y_2)$). Without loss of generality, we assume that the ith contact discontinuity or shock $x=x_i(t)$ passing through $O(0,0)$ is partly contained in the domain $P_1A_1A_2P_2$. Then, applying (2.17) on the domain $P_1A_1A_2P_2$ and noting (2.16), it is easy to see that

$$\begin{aligned}
&\int_{t_1}^{t_2}\big|w_i\big(t,x_j(t)\big)\big|\big|\lambda_j\big(u\big(t,x_j(t)\big)\big)-\lambda_i\big(u\big(t,x_j(t)\big)\big)\big|\,dt\\
&\quad\leqslant\int_{y_1}^{y_2}\big|w_i(0,x)\big|\,dx+\sum_{k\in S_1}\int_{\widehat{C}_k}\big|\big([w_i]x_k'(t)-\big[w_i\lambda_i(u)\big]\big)\,dt\big|+\int\int_{P_1A_1A_2P_2}\sum_{j\neq k}\big|\Gamma_{ijk}(u)w_jw_k\big|\,dt\,dx\\
&\quad\leqslant\int_{y_1}^{y_2}\big|w_i(0,x)\big|\,dx+\int_0^T\big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)\pm0\big)\big)\big)w_i\big(t,x_i(t)\pm0\big)\big|\,dt\\
&\qquad+\sum_{k\neq i,k\in S_1}\int_{\widehat{C}_k}\big|\big([w_i]x_k'(t)-\big[w_i\lambda_i(u)\big]\big)\,dt\big|+\int\int_{P_1A_1A_2P_2}\sum_{j\neq k}\big|\Gamma_{ijk}(u)w_jw_k\big|\,dt\,dx,
\end{aligned} \tag{4.41}$$

where S_1 stands for the set of all indices k such that the kth discontinuous curve $\widehat{C}_k: x=x_k(t)$ is partly contained in the domain $P_1A_1A_2P_2$. Using (1.19), (3.3), (3.5), (3.6), (3.9), (4.3) and (4.4), we have

$$\int_{t_1}^{t_2}\big|w_i\big(t,x_j(t)\big)\big|\,dt\leqslant c_8\big\{\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}. \tag{4.42}$$

Thus, we get

$$\widetilde{W}_1(T)\leqslant c_8\big\{\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}. \tag{4.43}$$

We next estimate $W_1(T)$.

(i) For $i=n$, passing through any fixed point $A(T,a)$ $(a>x_n(T))$, we draw the nth backward characteristic which intersects the x-axis at a point $B(0,x_B)$.

We rewrite (2.14) as

$$d\big(\big|w_i(t,x)\big|\big(dx-\lambda_i(u)\,dt\big)\big)=\operatorname{sgn}(w_i)G_i\,dx\,dt. \tag{4.44}$$

Without loss of generality, we assume that the nth discontinuous curve $x = x_n(t)$ passing through the origin is the shock curve. Let D denotes the point $(T, x_n(T))$. Then, integrating (4.44) (in which we take $i = n$) on the domain $ABOD$ gives

$$\int_0^T \bigl(x_n'(t) - \lambda_n\bigl(u\bigl(t, x_n(t)+0\bigr)\bigr)\bigr)\bigl|w_n\bigl(t, x_n(t)+0\bigr)\bigr|\,dt + \int_D^A \bigl|w_n(T, x)\bigr|\,dx$$
$$\leqslant \int_0^{x_B} \bigl|w_n(0, x)\bigr|\,dx + \iint_{ABOD} |G_n|\,dx\,dt. \tag{4.45}$$

Using (1.19) and (2.16), it is easy to see that

$$\int_0^T \bigl(x_n'(t) - \lambda_n\bigl(u\bigl(t, x_n(t)+0\bigr)\bigr)\bigr)\bigl|w_n\bigl(t, x_n(t)+0\bigr)\bigr|\,dt$$
$$\leqslant \int_0^{+\infty} \bigl|w_n(0, x)\bigr|\,dx + \int_0^T \int_{-\infty}^{+\infty} |G_n|\,dx\,dt \leqslant c_9\bigl\{\varepsilon + Q_W(T)\bigr\}. \tag{4.46}$$

Noting (3.2), we have

$$\int_0^T \bigl|\bigl(x_n'(t) - \lambda_n\bigl(u\bigl(t, x_n(t)+0\bigr)\bigr)\bigr)w_n\bigl(t, x_n(t)+0\bigr)\bigr|\,dt \leqslant c_9\bigl\{\varepsilon + Q_W(T)\bigr\}. \tag{4.47}$$

(ii) For $i = 1, \dots, n-1$, passing through point $A(T, a)$ $(a > x_n(T))$, we draw the ith backward characteristic which intersects the x-axis at a point $B(0, x_B)$. Let D denotes the point $(T, x_i(T))$. Then, we divide the bounded domain $ABOD$ by the discontinuous curves (shocks or contact discontinuities) $x = x_k(t)$ $(x_k(0) = 0)$ $(k = i+1, \dots, n)$ into some parts. Thus, integrating (4.44) on all parts of the domain $ABOD$ gives

$$\int_0^T \bigl(x_i'(t) - \lambda_i\bigl(u\bigl(t, x_i(t)+0\bigr)\bigr)\bigr)\bigl|w_i\bigl(t, x_i(t)+0\bigr)\bigr|\,dt + \int_D^A \bigl|w_i(T, x)\bigr|\,dx$$
$$\leqslant \int_0^{x_B} \bigl|w_i(0, x)\bigr|\,dx + \sum_{k=i+1}^{n} \int_{\widehat{C}_k} \bigl|[w_i]x_k'(t) - \bigl[w_i\lambda_i(u)\bigr]\bigr|\,dt + \iint_{ABOD} |G_i|\,dx\,dt. \tag{4.48}$$

where $\widehat{C}_k : x = x_k(t)$ stands for the kth discontinuous curve (shock or contact discontinuity) passing through the origin, which is contained in the domain $ABOD$. Without loss of generality, we assume that the ith discontinuous curve $x = x_i(t)$ passing through the origin is the shock curve. Then, using (1.19), (2.16), (3.5), (3.6) and (3.9), we obtain

$$\int_0^T \bigl(x_i'(t) - \lambda_i\bigl(u\bigl(t, x_i(t)+0\bigr)\bigr)\bigr)\bigl|w_i\bigl(t, x_i(t)+0\bigr)\bigr|\,dt$$
$$\leqslant \int_0^{+\infty} \bigl|w_i(0, x)\bigr|\,dx + c_{10}V_\infty(T)\bigl(\widetilde{W}_1(T) + W_1(T)\bigr) + \int_0^T \int_{-\infty}^{+\infty} |G_i|\,dx\,dt$$
$$\leqslant c_{11}\bigl\{\varepsilon + V_\infty(T)\bigl(\widetilde{W}_1(T) + W_1(T)\bigr) + Q_W(T)\bigr\}. \tag{4.49}$$

Noting (3.2), it is easy to see that

$$\int_0^T \bigl|\bigl(x_i'(t) - \lambda_i\bigl(u\bigl(t, x_i(t)+0\bigr)\bigr)\bigr)w_i\bigl(t, x_i(t)+0\bigr)\bigr|\,dt \leqslant c_{11}\bigl\{\varepsilon + V_\infty(T)\bigl(\widetilde{W}_1(T) + W_1(T)\bigr) + Q_W(T)\bigr\}. \tag{4.50}$$

(iii) For $i = 1$, passing through any fixed point $A(T, a)$ $(a < x_1(T))$, we draw the 1th backward characteristic which intersects the x-axis at a point $B(0, x_B)$. Without loss of generality, we assume that the 1th discontinuous curve $x = x_1(t)$ passing through the origin is the shock curve.

Let D denotes the point $(T, x_1(T))$. Then, integrating (4.44) (in which we take $i = 1$) on the domain $ABOD$ gives

$$\int_A^D |w_1(T,x)|\,dx + \int_T^0 \big(x_1'(t) - \lambda_1\big(u\big(t, x_1(t) - 0\big)\big)\big)\big|w_1\big(t, x_1(t) - 0\big)\big|\,dt$$
$$\leqslant \int_{x_B}^0 |w_1(0,x)|\,dx + \int\int_{ABOD} |G_1|\,dx\,dt. \tag{4.51}$$

Using (1.19) and (2.16), it is easy to see that

$$\int_0^T \big(\lambda_1\big(u\big(t, x_1(t) - 0\big)\big) - x_1'(t)\big)\big|w_1\big(t, x_1(t) - 0\big)\big|\,dt$$
$$\leqslant \int_{-\infty}^0 |w_1(0,x)|\,dx + \int_0^T \int_{-\infty}^{+\infty} |G_1|\,dx\,dt \leqslant c_{12}\big\{\varepsilon + Q_W(T)\big\}. \tag{4.52}$$

Noting (3.2), we have

$$\int_0^T \big|\big(x_1'(t) - \lambda_1\big(u\big(t, x_1(t) - 0\big)\big)\big)w_1\big(t, x_1(t) - 0\big)\big|\,dt \leqslant c_{12}\big\{\varepsilon + Q_W(T)\big\}. \tag{4.53}$$

(iv) For $i = 2, \ldots, n$, passing through point $A(T, a)$ $(a < x_i(T))$, we draw the ith backward characteristic which intersects the x-axis at a point $B(0, x_B)$. Let D denotes the point $(T, x_i(T))$. Then, we divide the bounded domain $ABOD$ by the discontinuous curves (shocks or contact discontinuities) $x = x_k(t)$ $(x_k(0) = 0)$ $(k = 1, \ldots, i - 1)$ into some parts. Thus, integrating (4.44) on all parts of the domain $ABOD$ gives

$$\int_0^T \big(\lambda_i\big(u\big(t, x_i(t) - 0\big)\big) - x_i'(t)\big)\big|w_i\big(t, x_i(t) - 0\big)\big|\,dt + \int_A^D |w_i(T,x)|\,dx$$
$$\leqslant \int_{x_B}^0 |w_i(0,x)|\,dx + \sum_{k=1}^{i-1} \int_{\widehat{C}_k} \big|[w_i]x_k'(t) - \big[w_i\lambda_i(u)\big]\big|\,dt + \int\int_{ABOD} |G_i|\,dx\,dt. \tag{4.54}$$

where $\widehat{C}_k : x = x_k(t)$ stands for the kth discontinuous curve (shock or contact discontinuity) passing through the origin, which is contained in the domain $ABOD$. Without loss of generality, we assume that the ith discontinuous curve $x = x_i(t)$ passing through the origin is the shock curve. Then, using (1.19), (2.16), (3.5), (3.6) and (3.9), we obtain

$$\int_0^T \big(\lambda_i\big(u\big(t, x_i(t) - 0\big)\big) - x_i'(t)\big)\big|w_i\big(t, x_i(t) - 0\big)\big|\,dt$$
$$\leqslant \int_{-\infty}^0 |w_i(0,x)|\,dx + c_{13}V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_i|\,dx\,dt$$
$$\leqslant c_{14}\big\{\varepsilon + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + Q_W(T)\big\}. \tag{4.55}$$

Noting (3.2), it is easy to see that

$$\int_0^T \big|\big(x_i'(t) - \lambda_i\big(u\big(t, x_i(t) - 0\big)\big)\big)w_i\big(t, x_i(t) - 0\big)\big|\,dt \leqslant c_{14}\big\{\varepsilon + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + Q_W(T)\big\}. \tag{4.56}$$

Combining (4.47), (4.50), (4.53) and (4.56) all together, we have

$$W_1(T) \leqslant c_{15}\big\{\varepsilon + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + Q_W(T)\big\}. \tag{4.57}$$

We next estimate $U_\infty(T)$ and $V_\infty(T)$.

Passing through any fixed point $(t,x) \in [0,T] \times \mathbf{R}$, we draw the ith backward characteristic C_i which intersects the x-axis at a point $(0,y)$. Integrating (2.6) along this characteristic C_i and noting (2.8) yields

$$v_i(t,x) = v_i(0,y) + \sum_{k \in S_2} [v_i]_k + \int_{C_i} \sum_{j,k=1,k \neq i}^{n} \beta_{ijk}(u) v_j w_k \, dt, \tag{4.58}$$

where S_2 denotes the set of all indices k such that this characteristic C_i intersects the kth discontinuous curve (shock or contact discontinuity) $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[v_i]_k = v_i(t_k, x_k(t_k)+0) - v_i(t_k, x_k(t_k)-0)$. Noting (1.19) and using (1.24), we have

$$\left|u_+(x)\right| \leqslant \int_0^{+\infty} \left|u_+'(x)\right| dx \leqslant K_2, \quad \forall x \in \mathbf{R}^+ \tag{4.59}$$

and

$$\left|u_-(x)\right| \leqslant \int_{-\infty}^{0} \left|u_-'(x)\right| dx \leqslant K_2, \quad \forall x \in \mathbf{R}^-. \tag{4.60}$$

Therefore, noting the fact that $i \notin S_2$, and using (1.17), (2.1), (3.4), (4.2) and (4.4), we get from (4.58)–(4.60) that

$$V_\infty(T) \leqslant c_{16}\left\{\theta + \varepsilon + V_\infty(T)\left(V_\infty(T) + \widetilde{W}_1(T)\right)\right\}. \tag{4.61}$$

We now prove (4.31)–(4.33) and

$$Q_W(T) \leqslant k_8 \varepsilon^2, \tag{4.62}$$

where k_8 is a positive constant independent of θ, ε and T.

Recalling (4.2), (4.59) and (4.60), evidently we have

$$U_\infty(0), V_\infty(0) \leqslant c_{17}\theta \tag{4.63}$$

and

$$Q_W(0) = W_1(0) = \widetilde{W}_1(0) = 0, \tag{4.64}$$

provided that $\varepsilon \ll \theta$. Thus, by continuity there exist positive constants k_3, k_4, k_5 and k_8 independent of θ, ε and T such that (4.31)–(4.33) and (4.62) hold at least for $0 \leqslant T \leqslant \tau_0$, where τ_0 is a small positive number. Hence, in order to prove (4.31)–(4.33) and (4.62) it suffices to show that we can choose k_3, k_4, k_5 and k_8 in such a way that for any fixed T_0 $(0 < T_0 \leqslant T)$ such that

$$W_1(T_0) \leqslant 2k_3\varepsilon, \tag{4.65}$$

$$\widetilde{W}_1(T_0) \leqslant 2k_4\varepsilon, \tag{4.66}$$

$$V_\infty(T_0) \leqslant 2k_5\theta, \tag{4.67}$$

$$Q_W(T_0) \leqslant 2k_8\varepsilon^2, \tag{4.68}$$

we have

$$W_1(T_0) \leqslant k_3\varepsilon, \tag{4.69}$$

$$\widetilde{W}_1(T_0) \leqslant k_4\varepsilon, \tag{4.70}$$

$$V_\infty(T_0) \leqslant k_5\theta, \tag{4.71}$$

$$Q_W(T_0) \leqslant k_8\varepsilon^2. \tag{4.72}$$

To this end, substituting (4.65)–(4.68) into the right-hand side of (4.39), (4.43), (4.57) and (4.61) (in which we take $T = T_0$), it is easy to see that, when $\theta > 0$ is suitably small, we have

$$Q_W(T_0) \leqslant 4(1+k_3)^2 c_7 \varepsilon^2, \tag{4.73}$$

$$\widetilde{W}_1(T_0) \leqslant 2(1+k_3)c_8\varepsilon, \tag{4.74}$$

$$W_1(T_0) \leqslant 2c_{15}\varepsilon, \tag{4.75}$$

$$V_\infty(T_0) \leqslant 3c_{16}\theta, \tag{4.76}$$

provided that $\varepsilon \ll \theta$.

Hence, if $k_3 \geqslant 2c_{15}$, $k_4 \geqslant 2(1+k_3)c_8$, $k_5 \geqslant 3c_{16}$ and $k_8 \geqslant 4(1+k_3)^2 c_7$, then we get (4.69)–(4.72), provided that θ is suitably small. This proves (4.31)–(4.33) and (4.62).

We finally estimate $W_\infty(T)$.

For any fixed point $(t,x) \in [0,T] \times \mathbf{R}$, we draw the ith backward characteristic C_i passing through the point (t,x), which intersects the x-axis at a point $(0,y)$. Integrating (2.9) along this characteristic C_i and noting (2.11) yields

$$w_i(t,x) = w_i(0,y) + \sum_{k \in S_3} [w_i]_k + \int_{C_i} \left[\sum_{j,k=1,\, j \neq k}^{n} \gamma_{ijk}(u) w_j w_k + \gamma_{iii}(u) w_i^2 \right] dt, \tag{4.77}$$

where S_3 denotes the set of all indices k such that this characteristic C_i intersects the kth discontinuous curve (shock or contact discontinuity) $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[w_i]_k = w_i(t_k, x_k(t_k)+0) - w_i(t_k, x_k(t_k)-0)$. Using (3.5), (3.6) and (4.4) and noting the fact that $i \notin S_3$, we have

$$W_\infty(T) \leqslant c_{18} \left\{ \varepsilon + V_\infty(T) W_\infty(T) + W_\infty(T) \widetilde{W}_1(T) + T \big(W_\infty(T) \big)^2 \right\}. \tag{4.78}$$

Noting (1.18), by continuity there exists a positive constant k_6 independent of θ, ε and T such that (4.34) holds at least for $T > 0$ suitably small. Thus, in order to prove (4.34) it suffices to show that we can choose k_6 and k_7 in such a way that for any fixed T_0 $(0 < T_0 \leqslant T)$ with $T_0 \varepsilon \leqslant k_7$ such that

$$W_\infty(T_0) \leqslant 2k_6 \varepsilon, \tag{4.79}$$

we have

$$W_\infty(T_0) \leqslant k_6 \varepsilon. \tag{4.80}$$

Substituting (4.79) into the right-hand side of (4.78) (in which we take $T = T_0$) and noting (4.32)–(4.33), it is easy to see that, when $\theta > 0$ is suitably small, we have

$$W_\infty(T_0) \leqslant 2c_{18} \big(1 + 2k_6^2 k_7 \big) \varepsilon, \tag{4.81}$$

Hence, if $k_6 \geqslant 6c_{18}$ and $k_6^2 k_7 = 1$, then we have (4.80), provided that θ is suitably small. Therefore (4.34) is proved.

Finally, we observe that when $\theta > 0$ is suitably small, by (4.33) we have

$$U_\infty(T) \leqslant k_5 \theta \leqslant \frac{1}{2} \delta. \tag{4.82}$$

This implies the validity of hypothesis (4.4). The proof of Lemma 4.5 is finished. $\square$

Proof of Theorem 1.2. By (4.33)–(4.34), we know that for small $\theta > 0$ there exists $\varepsilon > 0$ suitably small such that the generalized Riemann problem (1.1) and (1.17) admits a unique piecewise C^1 solution $u = u(t,x)$ containing shocks and contact discontinuities on the strip $[0,T] \times \mathbf{R}$, where T satisfies (4.35). Therefore, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution satisfies

$$\widetilde{T}(\varepsilon) \geqslant K_3 \varepsilon^{-1}, \tag{4.83}$$

where $K_3 (= k_7)$ is a positive constant independent of ε. Moreover, by Lemma 4.5, when the piecewise C^1 solution $u = u(t,x)$ blows up in a finite time, $u = u(t,x)$ itself must be bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$. Hence, the first-order derivative u_x of $u = u(t,x)$ should tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$. The proof of Theorem 1.2 is finished. $\square$

5. Applications

5.1. System of one-dimensional gas dynamics

Consider the following Cauchy problem for the system of one-dimensional gas dynamics in Eulerian coordinates (cf. [9]):

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, \\ \partial_t (\rho v) + \partial_x \big(\rho v^2 + p \big) = 0, \\ \partial_t \left(\rho \left(\frac{1}{2} v^2 + e \right) \right) + \partial_x \left(\rho v \left(\frac{1}{2} v^2 + e \right) + p v \right) = 0, \end{cases} \tag{5.1}$$

$$t = 0 : (\rho, v, e) = \begin{cases} (\rho_0 + \varepsilon \rho_-(x), v_0 + \varepsilon v_-(x), e_0 + \varepsilon e_-(x)), & x \leqslant 0, \\ (\rho_0 + \varepsilon \rho_+(x), v_0 + \varepsilon v_+(x), e_0 + \varepsilon e_+(x)), & x \geqslant 0, \end{cases} \tag{5.2}$$

where ρ, v, p and e are the density, velocity, pressure and internal energy of the gas, respectively; the equations of state $p = p(\rho, S)$ and $e = e(\rho, S)$ are suitably smooth functions with respect to their arguments, which satisfy

$$p_\rho(\rho, S) > 0 \quad \text{and} \quad e_S(\rho, S) > 0, \quad \forall \rho > 0, \tag{5.3}$$

in which S is the entropy; moreover, $\varepsilon > 0$ is a small parameter, $\rho_0 > 0$, $e_0 > 0$ and v_0 are constants, $\rho_\pm(x)$, $v_\pm(x)$ and $e_\pm(x) \in C^1$ with

$$\|\rho_\pm(x)\|_{C^1}, \|v_\pm(x)\|_{C^1}, \|e_\pm(x)\|_{C^1} \leqslant K_4 \tag{5.4}$$

and

$$\int_0^{+\infty} |\rho'_+(x)|\,dx,\ \int_0^{+\infty} |v'_+(x)|\,dx,\ \int_0^{+\infty} |e'_+(x)|\,dx,\ \int_{-\infty}^{0} |\rho'_-(x)|\,dx,\ \int_{-\infty}^{0} |v'_-(x)|\,dx,\ \int_{-\infty}^{0} |e'_-(x)|\,dx \leqslant K_5, \tag{5.5}$$

where K_4 and K_5 are positive constants independent of ε.

Let

$$u = (\rho, v, e)^T. \tag{5.6}$$

Then from the basic law of thermodynamics

$$de = \theta\,dS + \frac{p}{\rho^2}\,d\rho, \tag{5.7}$$

where θ is the temperature of the gas, we rewrite system (5.1) as

$$u_t + A(u)u_x = 0, \tag{5.8}$$

where

$$A(u) = \begin{pmatrix} v & \rho & 0 \\ \frac{1}{\rho}(p_\rho - \frac{pp_S}{\rho^2 e_S}) & v & \frac{p_S}{\rho e_S} \\ 0 & \frac{p}{\rho} & v \end{pmatrix}. \tag{5.9}$$

By (5.3), it is easy to see that system (5.1) is strictly hyperbolic and has the following three distinct real eigenvalues:

$$\lambda_1(u) = v - c < \lambda_2(u) = v < \lambda_3(u) = v + c, \tag{5.10}$$

where

$$c = \sqrt{p_\rho(\rho.S)}.$$

The corresponding right eigenvectors are

$$r_1(u) // \left(1, -\frac{c}{\rho}, \frac{p}{\rho^2}\right)^T,$$

$$r_2(u) // \left(\frac{p_S}{\rho e_S}, 0, \frac{1}{\rho}\left(\frac{pp_S}{\rho^2 e_S} - c^2\right)\right)^T,$$

$$r_3(u) // \left(1, \frac{c}{\rho}, \frac{p}{\rho^2}\right)^T.$$

It is easy to see that the second characteristic field is linearly degenerate, i.e.,

$$\nabla\lambda_2(u) r_2(u) \equiv 0,$$

and if

$$\frac{\rho_0}{2}\frac{\partial^2 p}{\partial \rho^2}(\rho_o, S_0) + c^2(\rho_0, S_0) \neq 0, \tag{5.11}$$

where

$$S_0 = S(\rho_0, e_0),$$

then in a neighborhood of $u = (\rho_0, v_0, e_0)^T$, the first characteristic field and the third characteristic field are genuinely nonlinear, i.e.,

$$\nabla\lambda_j(u)r_j(u) \neq 0, \quad j = 1, 3.$$

By Theorem 1.2 we get

Theorem 5.1. *Suppose that* (5.11) *holds. Suppose furthermore that the corresponding Riemann problem has a self-similar solution consisting of only non-degenerate shocks and contact discontinuities but no centered rarefaction waves. Suppose finally that $\rho_-(x)$, $v_-(x)$, $e_-(x)$, $\rho_+(x)$, $v_+(x)$ and $e_+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$, respectively, satisfying* (5.4)–(5.5) *and*

$$\theta \overset{\triangle}{=} \left|\left(\rho_+(0), v_+(0), e_+(0)\right) - \left(\rho_-(0), v_-(0), e_-(0)\right)\right| > 0 \quad \text{is suitably small.} \tag{5.12}$$

Then for small $\theta > 0$, there exists a constant $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (5.1)–(5.2) *satisfies*

$$\widetilde{T}(\varepsilon) \geqslant K_6\varepsilon^{-1}, \tag{5.13}$$

where K_6 is a positive constant independent of ε.

5.2. System of traffic flow on a road network

Consider the following Cauchy problem for the system of traffic flow on a road network using the Aw–Rascle model (cf. [1,15]):

$$\begin{cases} \partial_t\rho + \partial_x\left(y - \rho^{\gamma+1}\right) = 0, \\ \partial_t y + \partial_x\left(\dfrac{y^2}{\rho} - y\rho^\gamma\right) = 0, \end{cases} \tag{5.14}$$

$$t = 0 : (\rho, y) = \begin{cases} (\widetilde{\rho}_0 + \varepsilon\rho_-(x), \widetilde{y}_0 + \varepsilon y_-(x)), & x \leqslant 0, \\ (\widetilde{\rho}_0 + \varepsilon\rho_+(x), \quad \widetilde{y}_0 + \varepsilon y_+(x)), & x \geqslant 0, \end{cases} \tag{5.15}$$

where $\gamma > 0$, $\rho > 0$ is the density of the cars and $y = \rho v + \rho^{\gamma+1}$ is the momentum, v is the velocity of the cars, $\widetilde{\rho}_0 > 0$ and $\widetilde{y}_0$ are constants, $\rho_\pm(x)$ and $y_\pm(x) \in C^1$ with

$$\left\|\rho_\pm(x)\right\|_{C^1}, \left\|y_\pm(x)\right\|_{C^1} \leqslant K_7 \tag{5.16}$$

and

$$\int_0^{+\infty} \left|\rho'_+(x)\right| dx, \int_0^{+\infty} \left|y'_+(x)\right| dx, \int_{-\infty}^0 \left|\rho'_-(x)\right| dx, \int_{-\infty}^0 \left|y'_-(x)\right| dx \leqslant K_8, \tag{5.17}$$

where K_7 and K_8 are positive constants independent of ε.

Let

$$u = \begin{pmatrix} \rho \\ y \end{pmatrix}. \tag{5.18}$$

It is easy to see that in a neighborhood of $u = \binom{\widetilde{\rho}_0}{\widetilde{y}_0}$, system (5.14) is strictly hyperbolic and has the following two distinct real eigenvalues:

$$\lambda_1(u) = \frac{y}{\rho} - (\gamma+1)\rho^\gamma < \lambda_2(u) = \frac{y}{\rho} - \rho^\gamma, \tag{5.19}$$

i.e.,

$$\lambda_1(u) = v - \gamma\rho^\gamma < \lambda_2(u) = v. \tag{5.20}$$

The corresponding right eigenvectors are

$$r_1(u) // \begin{pmatrix} 1 \\ \frac{y}{\rho} \end{pmatrix}, \qquad r_2(u) // \begin{pmatrix} 1 \\ \frac{y}{\rho} + \gamma\rho^\gamma \end{pmatrix}. \tag{5.21}$$

It is easy to see that the first characteristic field is genuinely nonlinear, i.e.,

$$\nabla\lambda_1(u)r_1(u) \neq 0, \tag{5.22}$$

while the second characteristic field is linearly degenerate, i.e.,

$$\nabla\lambda_2(u)r_2(u) \equiv 0. \tag{5.23}$$

By Theorem 1.2 we get

Theorem 5.2. *Suppose that the corresponding Riemann problem has a self-similar solution consisting of only non-degenerate shocks and contact discontinuities but no centered rarefaction waves. Suppose furthermore that $\rho_-(x)$, $y_-(x)$, $\rho_+(x)$ and $y_+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$, respectively, satisfying* (5.16)–(5.17)*. Suppose finally that*

$$\theta \triangleq \left|\left(\rho_+(0), y_+(0)\right) - \left(\rho_-(0), y_-(0)\right)\right| > 0 \quad \text{is suitably small.} \tag{5.24}$$

Then for small $\theta > 0$, there exists a constant $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (5.14)–(5.15) *satisfies*

$$\widetilde{T}(\varepsilon) \geqslant K_9\varepsilon^{-1}, \tag{5.25}$$

where K_9 is a positive constant independent of ε.

Acknowledgment

The author would like to express his gratitude to the anonymous referee for careful reading, valuable comments and suggestions.

References

[1] A. Aw, M. Rascle, Resurrection of "second order" models of traffic flow, SIAM J. Appl. Math. 60 (2000) 916–938.
[2] A. Bressan, Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem, Oxford University Press, 2000.
[3] A. Bressan, Contractive metrics for nonlinear hyperbolic systems, Indiana Univ. Math. J. 37 (1988) 409–421.
[4] A. Bressan, A locally contractive metric for systems of conservation laws, Ann. Sc. Norm. Super. Pisa IV 22 (1995) 109–135.
[5] A. Bressan, G. Crasta, B. Piccoli, Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws, Mem. Amer. Math. Soc. 146 (694) (2000) 1–134.
[6] A. Bressan, P.G. LeFloch, Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws, Indiana Univ. Math. J. 48 (1999) 43–84.
[7] A. Bressan, T.P. Liu, T. Yang, L^1 stability estimates for $n \times n$ conservation laws, Arch. Ration. Mech. Anal. 149 (1999) 1–22.
[8] G.Q. Chen, H. Frid, Asymptotic stability of Riemann waves for conservation laws, Z. Angew. Math. Phys. 48 (1997) 30–44.
[9] G.Q. Chen, H. Frid, Uniqueness and asymptotic stability of Riemann solutions for the compressible Euler equations, Trans. Amer. Math. Soc. 353 (2001) 1103–1117.
[10] G.Q. Chen, H. Frid, Y.C. Li, Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics, Comm. Math. Phys. 228 (2002) 201–217.
[11] G.Q. Chen, Y.C. Li, Stability of Riemann solutions with large oscillation for the relativistic Euler equations, J. Differential Equations 202 (2004) 332–353.
[12] A. Corli, M. Sable-Tougeron, Perturbations of bounded variation of a strong shock wave, J. Differential Equations 138 (1997) 195–228.
[13] C.M. Dafermos, Entropy and the stability of classical solutions of hyperbolic systems of conservation laws, in: T. Ruggeri (Ed.), Recent Mathematical Methods in Nonlinear Wave Propagation, Montecatini Terme, 1994, in: Lecture Notes in Math., vol. 1640, Springer, Berlin, 1996, pp. 48–69.
[14] W.R. Dai, D.X. Kong, Global existence and asymptotic behavior of classical solutions of quasilinear hyperbolic systems with linearly degenerate characteristic fields, J. Differential Equations 235 (2007) 127–165.
[15] M. Garavello, B. Piccoli, Traffic flow on a road network using the Aw–Rascle model, Comm. Partial Differential Equations 31 (2006) 243–275.
[16] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math. 18 (1965) 697–715.
[17] L. Hsiao, S.Q. Tang, Construction and qualitative behavior of the solution of the perturbated Riemann problem for the system of one-dimensional isentropic flow with damping, J. Differential Equations 123 (1995) 480–503.
[18] L. Hsiao, R.H. Pan, Nonlinear stability of rarefaction waves for a rate-type viscoelastic system, Chin. Ann. Math. Ser. B 20 (1999) 223–232.
[19] F.M. Huang, Z.P. Xin, T. Yang, Contact discontinuity with general perturbations for gas motions, Adv. Math. 219 (2008) 1246–1297.
[20] F. John, Formation of singularities in one-dimensional nonlinear wave propagation, Comm. Pure Appl. Math. 27 (1974) 377–405.
[21] D.X. Kong, Global structure stability of Riemann solutions of quasilinear hyperbolic systems of conservation laws: shocks and contact discontinuities, J. Differential Equations 188 (2003) 242–271.
[22] D.X. Kong, Global structure instability of Riemann solutions of quasilinear hyperbolic systems of conservation laws: rarefaction waves, J. Differential Equations 219 (2005) 421–450.
[23] D.X. Kong, Life-span of classical solutions to quasilinear hyperbolic systems with slow decay initial data, Chin. Ann. Math. Ser. B 21 (2000) 413–440.
[24] P.D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math. 10 (1957) 537–556.
[25] M. Lewicka, Well-posedness for hyperbolic systems of conservation laws with large BV data, Arch. Ration. Mech. Anal. 173 (2004) 415–445.
[26] T.T. Li, D.X. Kong, Global classical discontinuous solutions to a class of generalized Riemann problem for general quasilinear hyperbolic systems of conservations laws, Comm. Partial Differential Equations 24 (1999) 801–820.
[27] T.T. Li, W.C. Yu, Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke Univ. Math. Ser. V, Duke University, Durham, 1985.
[28] T.T. Li, Y.C. Zhao, Global existence of classical solutions to the typical free boundary problem for general quasilinear hyperbolic systems and its applications, Chin. Ann. Math. Ser. B 11 (1990) 15–32.
[29] P.L. Lions, B. Perthame, P.E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math. 49 (1996) 599–638.
[30] T.P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Amer. Math. Soc. 56 (328) (1985) 1–108.
[31] T.P. Liu, Nonlinear stability and instability of overcompressive shock waves, in: J.E. Dunn, R. Posdick, M. Slemrod (Eds.), Shock Induced Transitions and Phase Structures in General Media, in: IMA Vol. Math. Appl., vol. 52, Springer, New York, 1993, pp. 159–167.
[32] T.P. Liu, Z.P. Xin, Nonlinear stability of rarefaction waves for compressible Navier–Stokes equations, Comm. Math. Phys. 118 (1988) 451–465.
[33] J.G. Liu, Z.P. Xin, Nonlinear stability of discrete shocks for systems of conservation laws, Arch. Ration. Mech. Anal. 125 (1993) 217–256.

[34] T.P. Liu, T. Yang, L^1 stability of conservation laws with coinciding Hugoniot and characteristic curves, Indiana Univ. Math. J. 48 (1999) 237–247.
[35] T.P. Liu, T. Yang, L^1 stability for 2×2 systems of hyperbolic conservation laws, J. Amer. Math. Soc. 12 (1999) 729–774.
[36] T.P. Liu, K. Zumbrun, Nonlinear stability of an undercompressive shock for complex Burgers equation, Comm. Math. Phys. 168 (1995) 163–186.
[37] T. Luo, Z.P. Xin, Nonlinear stability of shock fronts for a relaxation system in several space dimensions, J. Differential Equations 139 (1997) 365–408.
[38] A. Majda, The stability of multidimensional shock fronts, Mem. Amer. Math. Soc. 41 (275) (1983) 1–95.
[39] M. Schatzman, Continuous Glimm functional and uniqueness of the solution of Riemann problem, Indiana Univ. Math. J. 34 (1985) 533–589.
[40] M. Schatzman, The Geometry of Continuous Glimm Functionals, Lectures Appl. Math., vol. 23, American Mathematical Society, Providence, RI, 1986, pp. 417–439.
[41] S. Schochet, Sufficient condition for local existence via Glimm's scheme for large BV data, J. Differential Equations 89 (1991) 317–354.
[42] Z.Q. Shao, Global structure stability of Riemann solutions for general hyperbolic systems of conservation laws in the presence of a boundary, Nonlinear Anal. 69 (2008) 2651–2676.
[43] Z.Q. Shao, Global structure instability of Riemann solutions for general quasilinear hyperbolic systems of conservation laws in the presence of a boundary, J. Math. Anal. Appl. 330 (2007) 511–540.
[44] Z.Q. Shao, Global structure stability of Riemann solutions for linearly degenerate hyperbolic conservation laws under small BV perturbations of the initial data, Nonlinear Anal. Real World Appl. 11 (2010) 3791–3808.
[45] J.A. Smoller, J.B. Temple, Z.P. Xin, Instability of rarefaction shocks in systems of conservation laws, Arch. Ration. Mech. Anal. 112 (1990) 63–81.
[46] Z.P. Xin, On nonlinear stability of contact discontinuities, in: Hyperbolic Problems: Theory, Numerics, Applications, Stony Brook, NY, 1994, World Scientific Publishing, River Edge, NJ, 1996, pp. 249–257.
[47] Z.P. Xin, Theory of viscous conservation laws, in: L. Hsiao, Z. Xin (Eds.), Some Current Topics on Nonlinear Conservation Laws, in: AMS/IP Stud. Adv. Math., vol. 15, American Mathematical Society, Providence, RI, 2000, pp. 141–193.
[48] Y. Zhou, Global classical solutions to quasilinear hyperbolic systems with weak linear degeneracy, Chin. Ann. Math. Ser. B 25 (2004) 37–56.
[49] Z.Q. Shao, The generalized nonlinear initial-boundary Riemann problem for linearly degenerate quasilinear hyperbolic systems of conservation laws, J. Math. Anal. Appl. 379 (2011) 589–615.