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Corona theorems for subalgebras of $H^\infty \cap Q_t^p$ ☆

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ABSTRACT

The main goal of this work is to give a unified proof of several corona theorems for a collection of algebras of pointwise multipliers of spaces of holomorphic functions on the unit disk $\mathbb{D}$. These spaces include Besov spaces, invariant Dirichlet type spaces, weighted Sobolev spaces and Carleson measure spaces for Besov spaces.

1. Introduction

In this work we give a method which permits to unify the proof of the corona problem in some algebras of pointwise multipliers of spaces of holomorphic functions on the unit disk $\mathbb{D}$. This set of algebras contains, among other, pointwise multipliers of Besov spaces, of invariant Besov spaces, of weighted Sobolev spaces, of spaces of Carleson measures for Besov spaces.

In order to precise these results we need some definitions. We begin recalling the definition of the holomorphic Besov space on $\mathbb{D}$.

Weighted L^p spaces and holomorphic Besov spaces B_s^p: Let $1 \leqslant p < \infty$ and $\delta > 0$. We let $d\nu$ denote the normalized Lebesgue measure on $\mathbb{D}$. Let $d\nu_\delta(z) := (1-|z|^2)^{\delta-1}\, d\nu(z)$ and let $L_\delta^p := L^p(d\nu_\delta)$. For $s \in \mathbb{R}$, the Besov space B_s^p consists of holomorphic functions f in $\mathbb{D}$ satisfying $\|\partial^k f\|_{L^p_{(k-s)p}} < \infty$ for some (any) non-negative integer $k > s$.

Observe that if $s < 0$, then we can choose $k = 0$ and thus $B_s^p = H \cap L^p_{-sp}$, where $H := H(\mathbb{D})$ denotes the space of holomorphic functions on $\mathbb{D}$.

For more details about these spaces, we refer the reader to [5].

We now introduce the invariant Besov space Q_t^p of holomorphic functions on $\mathbb{D}$, as a generalization of the well-known invariant Dirichlet space Q_t.

Y_t^p spaces and invariant Besov spaces Q_t^p: If $1 \leqslant p < \infty$ and $0 < t < 1$, then let Y_t^p be the space defined by

$$Y_t^p := \{\varphi \in L^p_{t+p-1}\colon\ |\varphi|^p\, d\nu_{t+p-1} \in Y_t\},$$

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where Y_t denotes the set of the t-Carleson measures on $\mathbb{D}$, which consists of complex Borel measures μ on $\mathbb{D}$ satisfying

$$\|\mu\|_{Y_t} = \sup_{\substack{\zeta\in\mathbb{T}\\ 0<r<2}} \frac{1}{r^t} \int_{\{z\in\mathbb{D}:\ |z-\zeta|<r\}} d|\mu|(z) < \infty.$$

As usual, if $\varphi\, d\nu \in Y_t$, then we just write $\varphi \in Y_t$.

We recall that the invariant Dirichlet space Q_t consists of the holomorphic functions f on $\mathbb{D}$ such that $\|f\|_{Q_t} := \|f\|_{Y_t^2} + \|\partial f\|_{Y_t^2} < \infty$. (See [18] and [19] for more details about these spaces.)

As a generalization of these spaces, for $1 \leqslant p < \infty$ and $0 < t < 1$, we define the space Q_t^p as

$$Q_t^p := \left\{ f \in H \cap Y_t^p \colon \|f\|_{Q_t^p} := \|f\|_{Y_t^p} + \|\partial f\|_{Y_t^p} < \infty \right\}.$$

Let $g = (g_1, \dots, g_m)$ be a corona data in an algebra $A \subset H^\infty \cap Q_t^p$, that is, let $g_1, \dots, g_m \in A$ satisfy the corona condition

$$\inf_{z\in\mathbb{D}} \left|g(z)\right|^2 := \inf_{z\in\mathbb{D}} \sum_{j=1}^m \left|g_j(z)\right|^2 > 0.$$

In [10], it was shown that the corona theorem holds for the algebra $A = H^\infty \cap Q_t^2$, that is, for any corona data g in A, there exists $h_g = (h_1, \dots, h_m)$, $h_j \in A$, such that $g \cdot h_g := \sum_{j=1}^m g_j h_j = 1$.

To prove this result, the authors used the solution h_g given by

$$h_j = G_j - \sum_{\substack{k=1\\ k\neq j}}^m g_k \mathcal{K}_{|\omega_{j,k}|}(\omega_{j,k}), \quad j = 1, \dots, m, \tag{1}$$

where $G_j := \frac{\bar{g}_j}{|g|^2}$, $\omega_{j,k} = G_j \bar\partial G_k - G_k \bar\partial G_j$ and $\mathcal{K}_{|\omega_{j,k}|}(\omega_{j,k})$ is the Peter Jones' solution of the equation $\bar\partial U = \omega_{j,k}$.

We recall that if $\mu \in Y_1$, then the Peter Jones' solution $\mathcal{K}_{|\mu|}(\mu)$ satisfies:

$$\bar\partial \mathcal{K}_{|\mu|}(\mu) = \mu \quad \text{(in the sense of distributions)} \quad \text{and} \quad \left\|\mathcal{K}_{|\mu|}(\mu)\right\|_{L^\infty} \leqslant C\|\mu\|_{Y_1}. \tag{2}$$

See [8,10] or [18, Section 7.2] for more details about this solution.

The function h_g was also used in [12] to solve the corona problem in $\mathrm{Mult}(Q_t^2)$, the algebra of the pointwise multipliers of Q_t^2.

In this work we prove that the function h_g also gives a solution of the Bezout equation $g \cdot h = 1$ in several subalgebras of $H^\infty \cap Q_t^p$, and in particular in $\mathrm{Mult}(Q_t^p)$.

To do so, we will need to obtain estimates of $|\partial h_g|$. In particular we prove that:

For each $\tau > 0$, there exists $C_{\tau,g}$ such that

$$|\partial h_g| \leqslant C_{\tau,g}\left(\left|\partial g(z)\right| + \mathcal{T}^{-\tau}\left(|\partial g|\right)\right), \tag{3}$$

where $\mathcal{T}^{-\tau}$ is the integral operator defined by

$$\mathcal{T}^{-\tau}(\varphi)(z) := \int_{\mathbb{D}} \varphi(w) \frac{(1-|z|^2)^{-\tau}(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{2-2\tau}}\, d\nu(w),$$

$|\partial h_g|^2 := \sum_{j=1}^m |\partial h_j|^2$ and $|\partial g|^2 := \sum_{j=1}^m |\partial g_j|^2$.

It is clear that combining this estimate with the boundedness of $\mathcal{T}^{-\tau}$ in a solid normed space X, we can obtain estimates of $\||\partial h|\|_X$ in terms of $\||\partial g|\|_X$. These results lead us to introduce the following class of normed spaces X.

Solid normed spaces $X \subset Y_t^q$ with some $\mathcal{T}^{-\tau}$ bounded on X: In this work we consider normed spaces X of measurable functions on $\mathbb{D}$ satisfying the following conditions:

$\mathbb{X}1$: There exist $1 \leqslant q < \infty$ and $0 < t < 1$ such that $\mathbb{C} \subset X \subset Y_t^q$.
$\mathbb{X}2$: The space X is solid in the following sense: if $\psi_0 \in X$, $\psi_1 \in L^1_\delta$ for some $\delta > 0$ and $|\psi_1| \leqslant |\psi_0|$ a.e., then $\psi_1 \in X$ and $\|\psi_1\|_X \leqslant \|\psi_0\|_X$.
$\mathbb{X}3$: There exists $\tau > 0$ such that the operator $\mathcal{T}^{-\tau}$ is bounded on X.

Given a normed space X of functions on $\mathbb{D}$, we denote the subspace $HX := H \cap X$ and the corresponding Sobolev space

$$HX_1 := \left\{ f \in HX \colon \|f\|_{HX_1} = \|f\|_X + \|\partial f\|_X < \infty \right\}.$$

We now state our main results.

Theorem 1.1. *Let X be a normed space satisfying conditions $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$. Then, the corona theorem holds for the algebras $H^\infty \cap HX_1$ and $\mathrm{Mult}(HX_1)$.*

It is easy to check that, if $1 < p < \infty$, then the space $X = L^p_{p-1}$ satisfies conditions $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$. We also prove that if $1 \leqslant p < \infty$ and $0 < t < 1$, then $X = Y^p_t$ also satisfies these conditions. Therefore, we have:

Corollary 1.2. *If $1 < p < \infty$, then the corona theorem holds for $H^\infty \cap B^p_{1/p}$, $\mathrm{Mult}(B^p_{1/p})$, $H^\infty \cap Q^p_t$ and $\mathrm{Mult}(Q^p_t)$.*

We recall that the first two cases were proved in [9] and [16] using non-explicit solutions. As we have already said, if $p = 2$, the two last cases were proved in [10] and in [12] respectively using the solution h_g. However, the key point used in their proofs, which is the characterizations of the functions in Q^2_t in terms of its admissible boundary values, cannot be used in the general situation we consider.

Note that Theorem 1.1 permits to obtain corona theorems in algebras of pointwise multipliers $\mathrm{Mult}(HX_1)$ for spaces X satisfying $\mathbb{X}1$. However, if $0 < s < 1/p$, then $X = L^p_{(1-s)p}$ does not satisfy this condition, and therefore in this case we cannot apply the result to prove the corona theorem in $\mathrm{Mult}(B^p_s)$.

In the next theorem we replace the condition $\mathbb{X}1$ on X by other weaker conditions which, in particular, are satisfied for all the spaces $L^p_{(1-s)p}$ with $1 \leqslant p < \infty$ and $0 < s < 1/p$.

Theorem 1.3. *Let $\mathbb{C} \subset E \subset L^1(d\nu)$ be a normed space satisfying conditions $\mathbb{X}2$ and $\mathbb{X}3$. If $X = \mathrm{Mult}(HE_1, E)$ satisfies $\mathbb{X}1$, then:*

(i) *X also satisfies $\mathbb{X}2$ and $\mathbb{X}3$.*
(ii) $\mathrm{Mult}(HE_1) = H^\infty \cap HX_1$.
(iii) *The corona theorem is true for* $\mathrm{Mult}(HE_1)$ *and* $\mathrm{Mult}(HX_1)$.

Observe that assertion (i) in the above theorem follows from (ii), (iii) and Theorem 1.1. Moreover, since $Y^p_t \subset L^1(d\nu)$ and $X = \mathrm{Mult}(HE_1, E) \subset E$, we have that if E satisfies $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$, then X also satisfies these properties.

Therefore, starting from a normed space $E := X^{(0)}$ satisfying the conditions in Theorem 1.3, we can construct the decreasing sequence of normed spaces

$$X^{(k)} := \mathrm{Mult}\bigl(HX^{(k-1)}_1, X^{(k-1)}\bigr) \subset X^{(k-1)}, \quad k \geqslant 1,$$

whose terms $X^{(k)}$ satisfy conditions $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$.

Thus, as a consequence of Theorem 1.3, we have:

Theorem 1.4. *If $E = X^{(0)}$ satisfies the hypothesis of Theorem 1.3, then the corona theorem holds for all the algebras $\mathrm{Mult}(HX^{(k)}_1) = H^\infty \cap HX^{(k+1)}_1$, $k \geqslant 0$.*

Let us give some applications of this theorem.

First, observe that the space $X^{(k)}$ consists of the functions $\varphi \in X^{(k-1)}$ satisfying

$$\|\varphi\|_{X^{(k)}} := \sup_{\|f_{k-1}\|_{HX^{(k-1)}_1}=1} \cdots \sup_{\|f_0\|_{HX^{(0)}_1}=1} \|\varphi f_{k-1} \cdots f_0\|_{X^{(0)}} < \infty.$$

It is easy to check that, if $1 \leqslant p < \infty$ and $0 < s \leqslant 1/p$, then $X^{(0)} = L^p_{(1-s)p}$ satisfies the hypotheses of Theorem 1.3. Thus, since $HX^{(0)}_1 = B^p_s$, then we have

$$HX^{(1)}_1 = CB^p_s := \Bigl\{g \in B^p_s\colon \|g\|_{CB^p_s} := \sup_{\|f\|_{B^p_s}=1} \bigl\{\bigl\|f\bigl(|g| + |\partial g|\bigr)\bigr\|_{L^p_{(1-s)p}}\bigr\} < \infty\Bigr\},$$

that is, CB^p_s consists of functions $g \in B^p_s$ such that

$$d\mu_g(z) = \bigl(\bigl|g(z)\bigr| + \bigl|\partial g(z)\bigr|\bigr)^p \bigl(1 - |z|^2\bigr)^{(1-s)p-1}\, d\nu(z) \in \mathrm{Car}\bigl(B^p_s\bigr),$$

where $\mathrm{Car}(B^p_s)$ denotes the space of Carleson measures for B^p_s.

In this case, $C^2B^p_s := HX^{(2)}_1$ consists of functions $g \in CB^p_s$ such that

$$\|g\|_{C^2B^p_s} := \sup\bigl\{\bigl\|f_0 f_1\bigl(|g| + |\partial g|\bigr)\bigr\|_{L^p_{(1-s)p}}\colon \|f_0\|_{B^p_s} = \|f_1\|_{CB^p_s} = 1\bigr\} < \infty.$$

Analogously we can define the spaces $C^kB^p_s := HX^{(k)}_1$, $k \geqslant 2$.

Then, we have:

Theorem 1.5. *If $1 \leqslant p < \infty$ and $0 < s \leqslant 1/p$, then the corona theorem is true for* $\mathrm{Mult}(B_s^p)$, $\mathrm{Mult}(CB_s^p)$ *and* $\mathrm{Mult}(C^k B_s^p)$, $k \geqslant 2$.

If $X^{(0)} = Y_t^p$, then $Q_s^p = HX_1^{(0)}$, and we can also repeat the above arguments to define the spaces $C^k Q_t^p := HX_1^{(k)}$, $k \geqslant 1$. If $p = 2$, an explicit characterization of CQ_t^2 in terms of logarithmic t-Carleson measures can be found in [13] and [20]. As in the above case, we have:

Theorem 1.6. *If $1 \leqslant p < \infty$ and $0 < t < 1$, then the corona theorem is true for* $\mathrm{Mult}(Q_t^p)$, $\mathrm{Mult}(CQ_t^p)$ *and* $\mathrm{Mult}(C^k Q_t^p)$, $k \geqslant 2$.

As a consequence of Theorems 1.1 and 1.3, we also obtain the following corona theorem in weighted Sobolev spaces with weights in the Békollé class $\mathcal{B}_p$ (see [6] or Section 5.4 below for more details about these weights).

Theorem 1.7. *Let $1 < p < \infty$ and $\theta \in L^1(d\nu)$ a weight in the Békollé class $\mathcal{B}_p$. Let $HW_1^p(\theta)$ be the weighted Sobolev space defined by*

$$HW_1^p(\theta) := \left\{ f \in H \cap L^p(\theta):\ \|f\|_{HW_1^p(\theta)} := \|f\|_{L^p(\theta)} + \|\partial f\|_{L^p(\theta)} \right\} < \infty.$$

Then, there exists a weight Θ in the Muckenhoupt class $\mathcal{A}_p$ such that:

(i) $HW_1^p(\Theta) = HW_1^p(\theta)$.
(ii) $X^{(0)} = L^p(\Theta)$ *satisfies the hypothesis of Theorem* 1.3, *and consequently the corona theorem holds for* $\mathrm{Mult}(HX_1^{(k)})$, $k \geqslant 0$.

In particular, the corona theorem holds for $\mathrm{Mult}(HW_1^p(\theta))$.

We recall that assertion (i) was proved in [7]. We also point out that, by analogy with the case $\theta = 1$, the space $HW_1^p(\theta)$ is denoted by many authors as $B_{1/p'}^p(\theta)$ (see for instance [1] and [7]).

The paper is organized as follows. In Sections 2 and 3 we prove some properties of the spaces Y_t^p and of the operator $\mathcal{T}^{-\tau}$ respectively. The corona theorem in $H^\infty \cap Q_t^p$ and some pointwise estimate of h_g needed to prove our main theorems will be proved in Section 4. Finally, in Section 5 we prove our main results.

Throughout the paper $F \lesssim G$ means that there exists a constant C, which does not depend of F and G, such that $F \leqslant CG$. We will use $F \approx G$ to denote that $G \lesssim F \lesssim G$.

2. The spaces Y_t^p, Q_t^p, $\mathrm{Car}(B_s^p)$ and CB_s^p

Let $\zeta \in \mathbb{T}$ and $r > 0$. The subarcs of the unit circle $\mathbb{T}$ will be denoted by $I = I(\zeta, r) := \{\eta \in \mathbb{T}:\ |\eta - \zeta| < r\}$, and the corresponding tents by $T(I) = T(I(\zeta, r)) := \{z \in \mathbb{D}:\ |z - \zeta| < r\}$.

In order to obtain norm-estimates of the integral operators which appear in the paper, we will need the following well-known lemma.

Lemma 2.1. *Let $M, L \geqslant 0$ and $0 < N < M + L$. Then for $w, z \in \mathbb{D}$ we have*

$$\int_{\mathbb{D}} \frac{(1-|u|^2)^{N-1}}{|1-u\bar{z}|^M |1-u\overline{w}|^{1+L}}\, d\nu(u) \lesssim \begin{cases} |1-z\overline{w}|^{N-M-L}, & \text{if } M-1, L < N, \\ \frac{(1-|w|^2)^{N-L}}{|1-z\overline{w}|^M}, & \text{if } M-1 < N < L. \end{cases}$$

The proof of this lemma can be done using standard techniques. See for instance Lemma 2.4 below or Lemma 3.4 in [11].

The following well-known estimate will be used frequently in the next sections.

Lemma 2.2. *If $w, z \in \overline{\mathbb{D}}$, then*

$$1-|z|^2,\ 1-|w|^2 \leqslant 2\Re(1-z\overline{w}) = 1-|z|^2 + 1-|w|^2 + |w-z|^2 \leqslant 2|1-z\overline{w}|.$$

The next lemma states a well-known characterization of the space of t-Carleson measures (see [18, Lemma 1.4.1]).

Lemma 2.3. *If $0 \neq a \in \overline{\mathbb{D}}$, let $\zeta_a = a/|a|$ and if $a = 0$ let $\zeta_a = 1$. We denote by $T(I_a)$ the tent $T(I_a) = \{z \in \mathbb{D}:\ |1 - z\bar{\zeta}_a| < 2(1-|a|^2)\}$.*
For a Borel measure μ on $\mathbb{D}$, the following assertions are equivalent:

(i) $\mu \in Y_t$.
(ii) $\sup_{a\in\mathbb{D}} \frac{|\mu|(T(I_a))}{(1-|a|^2)^t} < \infty$.
(iii) *For some (any)* $\kappa > 0$, $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^\kappa}{|1-z\bar{a}|^{t+\kappa}}\, d|\mu|(z) < \infty$.

Moreover, the quantities in (ii) *and* (iii) *are equivalent to* $\|\mu\|_{Y_t}$.

The same techniques used to prove the above lemma give:

Lemma 2.4. *Let* $\mu \in Y_t$. *If* $0 < N < t$ *and* $M > 0$, *then*

$$\int_{\mathbb{D}} \frac{d|\mu|(w)}{|1-w\bar{z}|^{t-N}|1-w\bar{u}|^{t+M}} \lesssim \|\mu\|_{Y_t} \frac{(1-|u|^2)^{-M}}{|1-z\bar{u}|^{t-N}}.$$

Proof. Assume μ is positive. Let $\Omega_1 = \{w \in \mathbb{D}\colon\ |1-w\bar{u}| \leqslant |1-w\bar{z}|\}$ and let $\Omega_2 = \mathbb{D} \setminus \Omega_1$. We will prove that

$$\begin{aligned} A_1 &:= \int_{\Omega_1} \frac{d\mu(w)}{|1-w\bar{z}|^{t-N}|1-w\bar{u}|^{t+M}} \lesssim \|\mu\|_{Y_t} \frac{(1-|u|^2)^{-M}}{|1-z\bar{u}|^{t-N}}, \\ A_2 &:= \int_{\Omega_2} \frac{d\mu(w)}{|1-w\bar{z}|^{t-N}|1-w\bar{u}|^{t+M}} \lesssim \|\mu\|_{Y_t} \frac{1}{|1-z\bar{u}|^{t+M-N}}. \end{aligned} \tag{4}$$

These estimates together with the fact that $1-|u|^2 \leqslant 2|1-z\bar{u}|$ prove the lemma.
So we are led to show the estimates in (4).
If $w \in \Omega_1$, then $|1-z\bar{u}| \leqslant |1-w\bar{z}| + |1-w\bar{u}| \leqslant 2|1-w\bar{z}|$. Since $t-N>0$, Lemma 2.3 gives

$$A_1 \lesssim \frac{1}{|1-z\bar{u}|^{t-N}} \int_{\mathbb{D}} \frac{d\mu(w)}{|1-w\bar{u}|^{t+M}} \lesssim \|\mu\|_{Y_t} \frac{(1-|u|^2)^{-M}}{|1-z\bar{u}|^{t-N}}.$$

Analogously, if $w \in \Omega_2$, then $|1-z\bar{u}| \leqslant 2|1-w\bar{u}|$ and

$$A_2 \lesssim \int_{\mathbb{D}} \frac{d\mu(w)}{|1-w\bar{z}|^{t-N}(|1-z\bar{u}|+|1-w\bar{z}|)^{t+M}}.$$

Let $J = J(z)$ be the integer part of $-\log_2(1-|z|^2)$. We now consider the partition $\mathbb{D} = \bigcup_{j=1}^{J}(U_j \setminus U_{j-1})$, where

$$\begin{aligned} &U_j = U_j(z) := \{w \in \mathbb{D}\colon\ |1-w\bar{z}| \leqslant 2^j(1-|z|^2)\}, \quad j=1,\dots,J-1, \\ &U_0 = \emptyset \quad \text{and} \quad U_J = \mathbb{D} \setminus U_{J-1}. \end{aligned} \tag{5}$$

Because $\mu(U_j) \lesssim \|\mu\|_{Y_t} 2^{jt}(1-|z|^2)^t$, we have

$$A_2 \lesssim \|\mu\|_{Y_t} \sum_{j=1}^{J} \frac{2^{jN}(1-|z|^2)^N}{(|1-z\bar{u}|+2^j(1-|z|^2))^{t+M}} \lesssim \frac{1}{|1-z\bar{u}|^{t+M-N}}.$$

The last estimate can be checked decomposing the sum as a sum on the set $\{j\colon\ |1-z\bar{u}| < 2^j(1-|z|^2)\}$ and a sum on its complementary set. $\square$

Lemma 2.5. *If* $1 \leqslant p < \infty$ *and* $0 < t < 1$, *then* $Y_t^p \subset Y_1$.

Proof. Let $p=1$. If $z \in T(I)$, then $1-|z|^2 \leqslant |I|$. Since $0<t<1$, we have

$$\int_{T(I)} |\varphi(z)|\, d\nu(z) \leqslant |I|^{1-t} \int_{T(I)} |\varphi(z)| (1-|z|^2)^{t-1}\, d\nu(z) \lesssim \|\varphi\|_{Y_t^1} |I|.$$

If $1<p<\infty$, the result follows easily from Hölder's inequality. Indeed

$$\int_{T(I)} |\varphi(z)|\, d\nu(z) \leqslant \Phi_0(I)^{1/p} \Phi_1(I)^{1/p'},$$

where

$$\Phi_0(I) := \int_{T(I)} \left|\varphi(z)\right|^p \left(1-|z|^2\right)^{t+p-2} d\nu(z) \leqslant \|\varphi\|^p_{Y^p_t} |I|^t,$$

$$\Phi_1(I) := \int_{T(I)} \left(1-|z|^2\right)^{-(t+p-2)p'/p} d\nu(z) = \int_{T(I)} \left(1-|z|^2\right)^{(1-t)p'/p-1} d\nu(z) \lesssim |I|^{1+(1-t)p'/p} = |I|^{p'-tp'/p}. \qquad \square$$

To conclude this section we recall the following well-known result (see [17]):

Proposition 2.6. *Let $1 < p < \infty$ and $0 < s < 1/p$. If $|\varphi(z)|^p(1-|z|^2)^{(1-s)p-1}\,d\nu(z) \in \mathrm{Car}(B^p_s)$, then $\varphi \in Y^p_{1-sp}$, but, in general, the converse is not true.*

Corollary 2.7. *If $1 \leqslant p < \infty$ and $0 < s < 1/p$, then $CB^p_s \subset Q^p_{1-sp}$.*

3. The operator $\mathcal{T}^{-\tau}$

In the next proposition, we state the properties of the operator $\mathcal{T}^{-\tau}$, which we will need in the forthcoming sections to prove our main results.

Proposition 3.1. *Let $1 \leqslant p < \infty$, $0 < t < 1$, $0 < \delta < p$ and $0 \leqslant \tau < \min\{\frac{t}{p}, \frac{1-t}{p}, \frac{\delta}{p}, \frac{p-\delta}{p}\}$.*
Then, we have:

(i) *If $0 \leqslant \tau < \tau' < 1$ and $\varphi \geqslant 0$, then $\mathcal{T}^{-\tau}(\varphi) \leqslant \mathcal{T}^{-\tau'}(\varphi)$.*
(ii) *$\mathcal{T}^{-\tau}$ is bounded on L^p_δ.*
(iii) *$\mathcal{T}^{-\tau}$ is bounded on Y^p_t.*
(iv) *If $f \in B^1_{-1}$ and $\varphi \in Y^p_t$, then*

$$|f|\left|\mathcal{T}^{-\tau}(\varphi)\right| \lesssim \mathcal{T}^{-\tau}\left(|f\varphi|\right) + \|\varphi\|_{Y^p_t} \mathcal{T}^{-\tau}\left(|\partial f|\right).$$

Remark 3.2. Operators of type $\mathcal{T}^{-\tau}$ appear in different problems on the theory of spaces of holomorphic functions. Special cases of the results in Proposition 3.1 were used in related circumstances and they can be found for instance in [14,4,2] and [3].

Remark 3.3. The operator $\mathcal{T}^{-\tau}$ is not bounded on Y^1_1 for any $\tau \geqslant 0$, and therefore it is not possible to extend assertion (iii) to Y^1_1.

For instance, if $\varphi(r) = \frac{r^2 \log^{-2}(1-r)}{1-r} \in L^1[0,1]$, then by integration in polar coordinates, we have $\varphi(|z|) \in Y^1_1$ and

$$\left\|\mathcal{T}^0(\varphi)\right\|_{Y^1_1} \geqslant \int_{\mathbb{D}} \mathcal{T}^0(\varphi)(z)\,d\nu(z) \approx \int_0^1\int_0^1 \frac{rt\varphi(r)}{1-rt}\,dr\,dt = +\infty,$$

which proves the remark.

The rest of the section is devoted to prove Proposition 3.1.

Proof of (i). Clearly (i) follows from $(1-|z|^2)(1-|w|^2) \leqslant |1-z\overline{w}|^2$. $\square$

Proof of (ii). See for instance [21, Lemma 4.2.3]. $\square$

Proof of (iii). We will prove that if $0 \leqslant \tau p < \min\{1-t, t+p-1\}$, then

$$\left\|\left|\mathcal{T}^{-\tau}(\varphi)(z)\right|^p\left(1-|z|^2\right)^{t+p-2}\right\|_{Y_t} \lesssim \left\|\left|\varphi(w)\right|^p\left(1-|w|^2\right)^{t+p-2}\right\|_{Y_t}.$$

Assume $1 < p < \infty$. Let $0 < \varepsilon p < \min\{1-t-\tau p, t+p-1-\tau p\}$. Since

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\varepsilon p'-1}}{|1-z\overline{w}|^{1+2\varepsilon p'}}\,d\nu(w) \lesssim \left(1-|z|^2\right)^{-\varepsilon p'},$$

Hölder's inequality gives

$$\left|\mathcal{T}^{-\tau}(\varphi)(z)\right|^p \lesssim \left(1-|z|^2\right)^{-\tau p-\varepsilon p}\int_{\mathbb{D}}\left|\varphi(w)\right|^p\frac{(1-|w|^2)^{(1-\tau-\varepsilon)p-1}}{|1-z\overline{w}|^{1+(1-2\tau-2\varepsilon)p}}\,d\nu(w).$$

By Lemma 2.3 and Fubini's theorem, we have

$$\begin{aligned}\left\|\mathcal{T}^{-\tau}(\varphi)\right\|^p_{Y^p_t} &\approx \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-z\bar a|^2}\right)^t\left|\mathcal{T}^{-\tau}(\varphi)(z)\right|^p\left(1-|z|^2\right)^{t+p-2}d\nu(z)\\ &\lesssim \sup_{a\in\mathbb{D}}\left(1-|a|^2\right)^t\int_{\mathbb{D}}\left|\varphi(w)\right|^p\left(1-|w|^2\right)^{(1-\tau-\varepsilon)p-1}\cdot\int_{\mathbb{D}}\frac{(1-|z|^2)^{t+p-2-(\tau+\varepsilon)p}}{|1-z\bar a|^{2t}|1-w\bar z|^{1+(1-2\tau-2\varepsilon)p}}\,d\nu(z)\,d\nu(w).\end{aligned}$$

If $p=1$, then Hölder's inequality is not needed and therefore the above estimate holds even for $\varepsilon=0$.

By Lemma 2.1 with

$$N=t+p-1-(\tau+\varepsilon)p,\qquad M=2t\quad\text{and}\quad L=(1-2\tau-2\varepsilon)p,$$

satisfying $M-1<N<L$, we have

$$\left\|\mathcal{T}^{-\tau}(\varphi)\right\|^p_{Y^p_t}\lesssim\sup_{a\in\mathbb{D}}\left(1-|a|^2\right)^t\int_{\mathbb{D}}\frac{|\varphi(w)|^p(1-|w|^2)^{t+p-2}}{|1-w\bar a|^{2t}}\,d\nu(w)\approx\left\|\left|\varphi(w)\right|^p\left(1-|w|^2\right)^{t+p-2}\right\|_{Y_t}=\|\varphi\|^p_{Y^p_t},$$

which ends the proof. □

In order to prove Proposition 3.1(iv) we need the following lemmas.

Lemma 3.4. *If $f\in B^1_{-1}$, then*

$$\left|f(z)-f(w)\right|\lesssim|z-w|\left(\int_{\mathbb{D}}\frac{|\partial f(u)|(1-|u|^2)}{|1-z\bar u|^2|1-w\bar u|}+\frac{|\partial f(u)|(1-|u|^2)}{|1-z\bar u||1-w\bar u|^2}\,d\nu(u)\right).$$

Proof. We can assume $f(0)=0$. Since

$$f(z)=\int_0^1 z(\partial f)(tz)\,dt=2\int_0^1\int_{\mathbb{D}}\frac{u(\partial f)(u)\,(1-|u|^2)}{(1-tz\bar u)^3}\,d\nu(u)\,dt,$$

we have

$$\left|f(z)-f(w)\right|\lesssim\int_{\mathbb{D}}\left|\partial f(u)\right|\left(1-|u|^2\right)\int_0^1\frac{|(1-tz\bar u)^3-(1-tw\bar u)^3|}{|1-tz\bar u|^3|1-tw\bar u|^3}\,dt\,d\nu(u).$$

Since $|1-tz\bar u|\approx 1-t+|1-z\bar u|\geqslant|1-z\bar u|$ and there is an analogous estimate for $|1-tw\bar u|$, it is easy to check that the integral on the variable t in the above inequality is bounded by

$$C\left(\frac{|z-w|}{|1-z\bar u|^2|1-w\bar u|}+\frac{|z-w|}{|1-z\bar u||1-w\bar u|^2}\right),$$

which concludes the proof. □

Lemma 3.5. *Let $1\leqslant p<\infty$, $0<t<1$, $0<\tau p<\min\{t,1-t\}$ and $0\leqslant\varphi\in Y^p_t$. Then*

$$\int_{\mathbb{D}}\frac{\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{1-2\tau}|1-u\overline{w}|}\,d\nu(w)\lesssim\|\varphi\|_{Y^p_t}\frac{(1-|u|^2)^{-\tau}}{|1-z\bar u|^{1-2\tau}}.$$

Proof. Let $F(z,u)$ denote the left-hand term in the above inequality. Assume $p=1$. As in the proof of Lemma 2.4, let

$$\Omega_1=\left\{w\in\mathbb{D}\colon\ |1-u\overline{w}|\leqslant|1-z\overline{w}|\right\}\quad\text{and}\quad\Omega_2=\mathbb{D}\setminus\Omega_1.$$

Since $1-t-\tau>0$ and $t-\tau>0$, we have

$$F(z,u)\lesssim\int_{\Omega_1}\frac{\varphi(w)(1-|w|^2)^{t-1}}{|1-z\overline{w}|^{1-2\tau}|1-u\overline{w}|^{t+\tau}}\,d\nu(w)+\int_{\Omega_2}\frac{\varphi(w)(1-|w|^2)^{t-1}}{|1-z\overline{w}|^{t-\tau}|1-u\overline{w}|}\,d\nu(w).$$

Thus, the fact that $|\varphi(w)|(1-|w|^2)^{t-1}\in Y_t$ and the same arguments used to prove (4) in Lemma 2.4, give

$$F(z,u)\lesssim\|\varphi\|_{Y_t^1}\left(\frac{(1-|u|^2)^{-\tau}}{|1-z\bar{u}|^{1-2\tau}}+\frac{1}{|1-z\bar{u}|^{1-\tau}}\right)\lesssim\|\varphi\|_{Y_t^1}\frac{(1-|u|^2)^{-\tau}}{|1-z\bar{u}|^{1-2\tau}},$$

which concludes the proof of the case $p=1$.

Let $1<p<\infty$. If $0<\varepsilon p<\min\{\tau p,1-t-\tau p\}$, then

$$F(z,u)\leqslant F_0(z,u)^{1/p}F_1(z,u)^{1/p'}$$

where

$$\begin{aligned}F_0(z,u)&=\int_{\mathbb{D}}\frac{\varphi^p(w)(1-|w|^2)^{(1+t/p-1/p)p-1}}{|1-z\overline{w}|^{1-2\tau p-(1/p-t/p-\tau-\varepsilon)p}|1-u\overline{w}|^{1-(1/p-t/p-\tau+\varepsilon)p}}\,d\nu(w)\\&=\int_{\mathbb{D}}\frac{\varphi^p(w)(1-|w|^2)^{t+p-2}}{|1-z\overline{w}|^{t-\tau p+\varepsilon p}|1-u\overline{w}|^{t+\tau p-\varepsilon p}}\,d\nu(w),\\F_1(z,u)&=\int_{\mathbb{D}}\frac{(1-|w|^2)^{(1/p-t/p-\tau)p'-1}}{|1-z\overline{w}|^{1+(1/p-t/p-\tau-\varepsilon)p'}|1-u\overline{w}|^{1+(1/p-t/p-\tau+\varepsilon)p'}}\,d\nu(w).\end{aligned}$$

Since $\varphi^p(w)(1-|w|^2)^{t+p-2}\in Y_t$, Lemma 2.4 gives

$$F_0(z,u)\lesssim\|\varphi\|_{Y_t^p}^p\frac{(1-|u|^2)^{-\tau p+\varepsilon p}}{|1-z\bar{u}|^{t-\tau p+\varepsilon p}},$$

and by Lemma 2.1 with $N=(1/p-t/p-\tau)p'>0$, $1\leqslant M=1+N-\varepsilon p'<N+1$ and $L=N+\varepsilon p'>N$, we obtain

$$F_1(z,u)\lesssim\frac{(1-|u|^2)^{-\varepsilon p'}}{|1-z\bar{u}|^{1+(1/p-t/p-\tau-\varepsilon)p'}}.$$

Combining these estimates we conclude the proof. □

Proof of (iv). Without loss of generality we may assume $\varphi\geqslant 0$. Then

$$\begin{aligned}\left|f(z)\right|\left|\mathcal{T}^{-\tau}(\varphi)(z)\right|\lesssim\left(1-|z|^2\right)^{-\tau}\int_{\mathbb{D}}\frac{|f(w)|\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{2-2\tau}}\,d\nu(w)+\left(1-|z|^2\right)^{-\tau}\\\times\int_{\mathbb{D}}\frac{|f(z)-f(w)|\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{2-2\tau}}\,d\nu(w).\end{aligned}\tag{6}$$

Therefore, Lemma 3.4 and $|z-w|\leqslant|1-z\overline{w}|$ give

$$\begin{aligned}\int_{\mathbb{D}}\frac{|f(z)-f(w)|\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{2-2\tau}}\,d\nu(w)&\lesssim\int_{\mathbb{D}}\frac{|\partial f(u)|(1-|u|^2)}{|1-z\bar{u}|^2}\int_{\mathbb{D}}\frac{\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{1-2\tau}|1-w\bar{u}|}\,d\nu(w)\,d\nu(u)\\&\quad+\int_{\mathbb{D}}\frac{|\partial f(u)|(1-|u|^2)}{|1-z\bar{u}|}\int_{\mathbb{D}}\frac{\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{1-2\tau}|1-w\bar{u}|^2}\,d\nu(w)\,d\nu(u).\end{aligned}$$

By Lemma 3.5

$$\int_{\mathbb{D}}\frac{\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{1-2\tau}|1-w\bar{u}|}\,d\nu(w)\lesssim\|\varphi\|_{Y_t^p}\frac{(1-|u|^2)^{-\tau}}{|1-z\bar{u}|^{1-2\tau}}.$$

Since $1-|u|^2\leqslant 2|1-w\bar{u}|$, the last estimate gives

$$\int_{\mathbb{D}}\frac{\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{1-2\tau}|1-w\bar{u}|^2}\,d\nu(w)\lesssim\|\varphi\|_{Y_t^p}\frac{(1-|u|^2)^{-1-\tau}}{|1-z\bar{u}|^{1-2\tau}}.$$

Therefore,

$$\int_{\mathbb{D}} \frac{|f(z)-f(w)|\varphi(w)(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{2-2\tau}}\,d\nu(w) \lesssim \|\varphi\|_{Y_t^p} \int_{\mathbb{D}} \frac{|\partial f(u)|(1-|u|^2)^{-\tau}}{|1-z\overline{u}|^{2-2\tau}}\,d\nu(u).$$

This estimate together with (6) complete the proof. □

4. The corona problem for $H^\infty \cap Q_t^p$

Theorem 4.1. *Let $1 \leqslant p < \infty$, $0 < t < 1$ and let $g = (g_1, \dots, g_m)$ be a corona data in $H^\infty \cap Q_t^p$. Then, the function $h_g = (h_1, \dots, h_m)$ defined by* (1) *satisfies*:

(i) *If $\tau > 0$, then there exists $c_{\tau,g} > 0$ such that for all $f \in B^1_{-1}$*

$$|f||\partial h_j| \leqslant c_{\tau,g}\big(\mathcal{T}^{-\tau}\big(|f||\partial g|\big) + \|\partial g\|_{Y_t^p}\mathcal{T}^{-\tau}\big(|\partial f|\big)\big).$$

(ii) $g \cdot h_g = 1$ *and* $h_j \in H^\infty \cap Q_t^p$.

Proof. Since $|\omega_{j,k}| \leqslant C_g|\partial g| \in Y_t^p \subset Y_1$, by (2), we have $h_j \in H^\infty$. Therefore, in order to prove (ii) we only need to show that $\|\partial h_j\|_{Y_t^p} < \infty$, and this will be a consequence of (i) with $f = 1$ and Proposition 3.1.

Let us prove (i). In the proof of [10, Theorem 3.1] (see also [18, Section 7.2]) it is shown that there exist functions $\mathcal{L}_{|\omega_{j,k}|}(\omega_{j,k})(z) \in L^\infty(d\nu) \cap C^1(\mathbb{D})$, whose boundary values coincide with the ones of $z\mathcal{K}_{|\omega_{j,k}|}(\omega_{j,k})(z)$ respectively, and which satisfy

$$\big|\nabla \mathcal{L}_{|\omega_{j,k}|}(\omega_{j,k})\big| \leqslant C_g \mathcal{T}^0\big(|\partial g|\big).$$

Therefore, replacing in the definition of h_j the functions $\mathcal{K}_{|\omega_{j,k}|}(\omega_{j,k})(z)$ by $\bar{z}\mathcal{L}_{|\omega_{j,k}|}(\omega_{j,k})(z)$, we obtain a new function $\tilde{h}_j \in L^\infty(d\nu) \cap C^1(\mathbb{D})$ whose admissible boundary values coincide with the ones of h_j, and which satisfies

$$\big|\partial \tilde{h}_j(w)\big| \leqslant C_g\big(\big|\partial g(w)\big| + \mathcal{T}^0\big(|\partial g|\big)(w)\big). \tag{7}$$

Since $|\partial g| \in Y_t^p$, Proposition 3.1(iii) gives $\mathcal{T}^0(|\partial g|) \in Y_t^p$ and thus $\partial \tilde{h}_j \in Y_t^p$. Moreover, we have

$$\partial h_j(z) = \partial \int_{\mathbb{T}} \frac{\tilde{h}_j(\zeta)}{1-z\bar{\zeta}}\,d\sigma(\zeta) = -\int_{\mathbb{T}} \frac{\tilde{h}_j(\zeta)}{(1-z\bar{\zeta})^2}\frac{d\bar{\zeta}}{2\pi i} = \int_{\mathbb{D}} \frac{\partial \tilde{h}_j(w)}{(1-z\overline{w})^2}\,d\nu(w).$$

By Lemmas 2.2 and 2.1, we have

$$\int_{\mathbb{D}} \frac{d\nu(w)}{|1-z\overline{w}|^2|1-w\overline{u}|^2} \leqslant c_\tau \frac{(1-|z|^2)^{-\tau}(1-|u|^2)^{-\tau}}{|1-z\overline{u}|^{2-2\tau}}, \quad \text{for any } \tau > 0.$$

Thus, Fubini's theorem and (7) give

$$|\partial h_j| \leqslant c_{\tau,g}\mathcal{T}^{-\tau}\big(|\partial g|\big). \tag{8}$$

Clearly, assertion (i) follows from (8) and Proposition 3.1(iv).

Finally, let us prove that $\partial h_j \in Y_t^p$. By Proposition 3.1, if $0 < \tau < \min\{t/p, (1-t)/p\}$, then $\mathcal{T}^{-\tau}$ is bounded on Y_t^p. Thus, by (8),

$$\|\partial h_j\|_{Y_t^p} \leqslant c_{\tau,g}\big\|\mathcal{T}^{-\tau}\big(|\partial g|\big)\big\|_{Y_t^p} \leqslant \big\|\mathcal{T}^{-\tau}\big\|\,\big\||\partial g|\big\|_{Y_t^p},$$

which concludes the proof of (ii). □

Remark 4.2. Observe that conditions $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$ are the properties of the space $X = Y_t^p$ that we have needed to prove the above result. Therefore, assuming that a normed space X satisfies these conditions, we can follow the proof of the above theorem to solve the corona problem in $H^\infty \cap HX_1$. We will formulate this result in the next section.

5. The corona theorem for algebras of pointwise multipliers

5.1. The main results

The following lemma will be needed to prove our results.

Lemma 5.1. *Let E be a normed space of functions satisfying $\mathbb{C} \subset E \subset L^1_\delta$, for some $\delta > 0$, and the condition $\mathbb{X}2$.*
If $\varphi \in E$, $\psi \in L^\infty$ and $f \in H^\infty \cap HE_1$, then we have:

(i) $|\varphi| \in E$ *and* $\||\varphi|\|_E = \|\varphi\|_E$.
(ii) $L^\infty(\mathbb{D}) \subset E$ *and* $\|\psi\varphi\|_E \lesssim \|\psi\|_\infty \|\varphi\|_E$.
(iii) $\|f\|_{H^\infty \cap HE_1} := \|f\|_{H^\infty} + \|f\|_{HE} + \|\partial f\|_{HE} \approx \|f\|_{H^\infty} + \|\partial f\|_{HE}$.

Proof. Assertion (i) follows applying $\mathbb{X}2$ to $\psi_0 = \psi \in E$ and $\psi_1 = |\psi| \in L^1_\delta$. In order to prove (ii) apply $\mathbb{X}2$ to $\psi_0 = \|\varphi\|_\infty \in E$ and $\psi_1 = \varphi \in L^\infty \subset L^1_\delta$. Assertion (iii) is a consequence of (ii). □

Theorem 5.2. *Let X be a normed space satisfying $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$. Then the corona theorem holds for $H^\infty \cap HX_1$.*

Proof. Given a corona data g in $H^\infty \cap HX_1$, we prove that the functions h_j defined in (1) are also in $H^\infty \cap HX_1$, that is $\|h_j\|_{H^\infty} + \|\partial h_j\|_X < \infty$.

Since X satisfies $\mathbb{X}1$, we have $H^\infty \cap HX_1 \subset H^\infty \cap Q^q_t$, and thus Theorem 4.1 gives $h_j \in H^\infty$.

The estimate $\|\partial h_j\|_X \lesssim \|g\|_{HX_1}$ follows from (8) and properties $\mathbb{X}2$ and $\mathbb{X}3$. Indeed

$$\|\partial h_j\|_X \lesssim \left\|\mathcal{T}^{-\tau}\big(|\partial g|\big)\right\|_X \lesssim \left\|\mathcal{T}^{-\tau}\right\| \left\||\partial g|\right\|_X,$$

which concludes the proof. □

As a corollary we obtain an alternative proof of the following well-known result (see [16] and [10]).

Theorem 5.3. *The corona theorem holds for $H^\infty \cap B^p_{1/p}$, $p > 1$.*

Proof. First observe that if $X = L^p_{p-1}$, then $HX_1 = B^p_{1/p}$. Clearly L^p_{p-1} contains the constants, satisfies $\mathbb{X}2$ and by Proposition 3.1 also satisfies $\mathbb{X}3$.

Therefore, in order to apply Theorem 5.2, it is enough to prove that $L^p_{p-1} \subset Y^q_t$, for some $1 < q < \infty$ and $0 < t < 1$.

Let $1 < q < p$, $p_1 = (p/q)' = p/(p-q)$ and $1/p_1 < t < 1$. Then, by Hölder's inequality we have

$$\int_{T(I)} \left|\varphi(z)\right|^q \left(1-|z|^2\right)^{t+q-2} d\nu(z)$$
$$\leqslant \left(\int_{T(I)} \left|\varphi(z)\right|^p \left(1-|z|^2\right)^{p-2} d\nu(z)\right)^{q/p} \left(\int_{T(I)} \left(1-|z|^2\right)^{p_1 t-2} d\nu(z)\right)^{1/p_1} \lesssim \|\varphi\|^q_{L^p_{p-1}} |I|^t,$$

which proves that $X \subset Y^q_t$ for any $1 < q < p$ and $(p-q)/p < t < 1$. □

We now apply Theorem 5.2 to solve corona problems in some algebras of pointwise multipliers of Banach spaces $F \subset H$. In order to do so, we write the corresponding space of multipliers as $H^\infty \cap HX_1$ with X satisfying the hypothesis of Theorem 5.2.

The next results are needed to prove Theorem 1.3.

Proposition 5.4. *Let $\mathbb{C} \subset E \subset L^1(d\nu)$ be a normed space satisfying condition $\mathbb{X}2$. If $X = \mathrm{Mult}(HE_1, E)$, then $\mathrm{Mult}(HE_1) = H^\infty \cap HX_1$.*

Proof. Let us prove the embedding $H^\infty \cap HX_1 \subset \mathrm{Mult}(HE_1)$. If $f \in HE_1$ and $g \in H^\infty \cap HX_1$, then

$$\|gf\|_{HE_1} = \|gf\|_E + \left\|\partial(gf)\right\|_E \leqslant \|g\|_\infty \big(\|f\|_E + \|\partial f\|_E\big) + \|\partial g\|_X \|f\|_{HE_1},$$

which proves that $\|g\|_{\mathrm{Mult}(HE_1)} \leqslant \|g\|_\infty + \|g\|_{HX_1}$.

Let us prove the converse. If $g \in \mathrm{Mult}(HE_1)$, then for any positive integer k we have

$$\left\|g^k\right\|^{1/k}_{HE_1} \leqslant \|g\|_{\mathrm{Mult}(HE_1)} \|1\|^{1/k}_{HE_1}.$$

Since, $HE_1 \subset E \subset L^1(d\nu)$, we obtain

$$\|g\|_{H^\infty} \leqslant \sup_{k\in\mathbb{N}} \left\|g^k\right\|^{1/k}_{L^1} \lesssim \sup_{k\in\mathbb{N}} \left\|g^k\right\|^{1/k}_{HE_1} \lesssim \|g\|_{\mathrm{Mult}(HE_1)}. \tag{9}$$

To complete the proof, we show that

$$\|g\|_{HX_1} = \|g\|_X + \|\partial g\|_X \lesssim \|g\|_{\mathrm{Mult}(HE_1)},$$

which follows from

$$\|fg\|_{HE} \leqslant \|g\|_\infty \|f\|_{HE} \lesssim \|g\|_{\mathrm{Mult}(HE_1)} \|f\|_{HE_1},$$
$$\|f\partial g\|_{HE} \leqslant \big\|\partial(fg)\big\|_{HE} + \|g\partial f\|_{HE} \leqslant \big(\|g\|_{\mathrm{Mult}(HE_1)} + \|g\|_\infty\big)\|f\|_{HE_1}$$

and (9). □

In order to apply Theorem 5.2 to $X = \mathrm{Mult}(HE_1, E)$, we will need to check that this space satisfies conditions $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$.

Proposition 5.5. *Let $\mathbb{C} \subset E \subset L^1(\mathbb{D})$ be a normed space satisfying conditions $\mathbb{X}2$ and $\mathbb{X}3$.*
If $X = \mathrm{Mult}(HE_1, E)$ satisfies $\mathbb{X}1$, then it also satisfies $\mathbb{X}2$ and $\mathbb{X}3$.

Proof. Let us prove that X satisfies $\mathbb{X}2$. Assume that $\psi_0 \in X$, $\psi_1 \in L^1_\delta$ and $|\psi_1| \leqslant |\psi_0|$. We want to prove that $\psi_1 \in X$.

Observe that if $E \subset L^1$, then $HE_1 \subset B^1_0 \subset B^\infty_{-1}$, that is, if $f \in HE_1$, then $\|(1-|z|^2)f(z)\|_\infty < \infty$. Therefore, $f\psi_1 \in L^1_{\delta+1}$, $f\psi_0 \in E$ and $|f\psi_1| \leqslant |f\psi_0|$. Since E satisfies property $\mathbb{X}2$, $f\psi_1 \in E$ and

$$\|f\psi_1\|_E \leqslant \|f\psi_0\|_E \leqslant \|\psi_0\|_X \|f\|_{HE_1},$$

which proves that X also satisfies $\mathbb{X}2$.

Let us prove that X satisfies $\mathbb{X}3$. If $\varphi \in X \subset Y^q_t$ and $f \in HE_1$, then, by Proposition 3.1(iii) and the hypothesis that E satisfy $\mathbb{X}2$ and $\mathbb{X}3$, we have

$$\big\|f\mathcal{T}^{-\tau}(\varphi)\big\|_E \lesssim \big\|\mathcal{T}^{-\tau}\big(|f\varphi|\big)\big\|_E + \|\varphi\|_{Y^q_t}\big\|\mathcal{T}^{-\tau}\big(|\partial f|\big)\big\|_E \lesssim \big\|\mathcal{T}^{-\tau}\big\|\big(\|\varphi\|_{\mathrm{Mult}(HE_1,E)} + \|\varphi\|_{Y^q_t}\big)\|f\|_{HE_1},$$

which proves that $\|\mathcal{T}^{-\tau}(\varphi)\|_X \lesssim \|\varphi\|_X$. □

Corollary 5.6. *If E satisfies $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$, then $X = \mathrm{Mult}(HE_1, E)$ also satisfies these conditions.*

Proof. The result follows from Proposition 5.5 and the fact that for some $1 \leqslant q < \infty$ and $0 < t < 1$ we have $\mathrm{Mult}(HE_1, E) \subset E \subset Y^q_t$. □

Remark 5.7. Observe that if $E = X^{(0)}$ satisfies conditions in Proposition 5.5, then $X^{(1)} := \mathrm{Mult}(HE_1, E)$ satisfy conditions $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$, and, by Corollary 5.6, $X^{(k)} := \mathrm{Mult}(HX_1^{(k-1)}, X^{(k-1)})$, $k \geqslant 2$, also satisfy these conditions.

We now prove the following theorem.

Theorem 5.8. *Let $\mathbb{C} \subset X^{(0)} \subset L^1(\mathbb{D})$ be a normed space satisfying properties $\mathbb{X}2$ and $\mathbb{X}3$. If for some $1 < q < \infty$ and $0 < t < 1$ we have $\mathrm{Mult}(HX_1^{(0)}, X^{(0)}) \subset Y^q_t$, then the corona theorem holds for $\mathrm{Mult}(HX_1^{(k)})$, $k \geqslant 0$.*

Proof. Proposition 5.4 gives $\mathrm{Mult}(HX_1^{(k)}) = H^\infty \cap HX_1^{(k+1)}$, and by Proposition 5.5 the spaces $X^{(k+1)}$ satisfy properties $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$. Therefore, by Theorem 5.2, the corona theorem holds for $\mathrm{Mult}(HX_1^{(k)})$. □

Corollary 5.9. *Let $X^{(0)}$ be a normed space satisfying properties $\mathbb{X}1$, $\mathbb{X}2$ and $\mathbb{X}3$. Then the corona theorem holds for $\mathrm{Mult}(HX_1^{(k)})$, $k \geqslant 0$.*

Now we apply Theorem 5.8 and Corollary 5.9 to prove corona theorems for algebras of pointwise multipliers of some classical spaces of holomorphic functions on $\mathbb{D}$.

5.2. The corona theorem for subalgebras of $\mathrm{Mult}(B^p_s)$

Let $X^{(0)} = L^p_{(1-s)p}$, $1 \leqslant p < \infty$, $0 < s < 1/p$.

By Hölder's inequality, it is clear that $X^{(0)} \subset L^1(d\nu)$ and that $X^{(0)}$ satisfies $\mathbb{X}2$. By Proposition 3.1(ii), it also satisfies $\mathbb{X}3$.

In this case $HX^{(0)} = B^p_{s-1}$, $HX^{(0)}_1 = B^p_s$,

$$X^{(1)} = \mathrm{Mult}\big(HX^{(0)}_1, X^{(0)}\big) = \big\{\varphi \in L^p_{(1-s)p}\colon |\varphi|^p\, d\nu_{(1-s)p} \in \mathrm{Car}\big(B^p_s\big)\big\},$$
$$HX^{(1)}_1 = CB^p_s = \big\{g \in B^p_{s-1}\colon \big(\big|g(z)\big| + \big|\partial g(z)\big|\big)^p d\nu_{(1-s)p}(z) \in \mathrm{Car}\big(B^p_s\big)\big\}$$

and by Proposition 2.6, $X^{(1)} \subset Y^p_{1-sp}$.

Therefore, $X^{(0)}$ satisfies the hypothesis of Theorem 5.8, and then we have:

Theorem 5.10. *If $1 \leqslant p < \infty$, $0 < s \leqslant 1/p$, $s < 1$ and $X^{(0)} = L^p_{(1-s)p}$, then the corona theorem holds for the algebras* $\mathrm{Mult}(HX^{(k)}_1)$.
In particular, the corona theorem holds for $\mathrm{Mult}(B^p_s)$ *and* $\mathrm{Mult}(CB^p_s)$.

Remark 5.11. We recall that the corona theorem in $\mathrm{Mult}(B^p_s)$ was proved for all $1 \leqslant p < \infty$ and $s > 0$ in [16] using other methods. The cases $1 < p < \infty$, $s > 1/p$ and $p = 1$, $s \geqslant 1$ not considered in the above theorem correspond to the regular cases where $\mathrm{Mult}(B^p_s) = B^p_s$.

5.3. The corona theorem for subalgebras of $\mathrm{Mult}(Q^q_t)$

Let $X^{(0)} = Y^p_t$, $1 < p < \infty$, $0 < t < 1$. In this case $HX^{(0)} = H \cap Y^p_t$ and $HX^{(0)}_1 = Q^p_t$. It is clear that $X^{(0)}$ satisfies $\mathbb{X}1$ and $\mathbb{X}2$. By Proposition 3.1(iii) it also satisfies $\mathbb{X}3$. Therefore, by Corollary 5.9, we have:

Theorem 5.12. *If $1 \leqslant p < \infty$, $0 < t < 1$ and $X^{(0)} = Y^p_t$, then the corona theorem holds for the algebras* $\mathrm{Mult}(HX^{(k)}_1)$, $k \geqslant 0$.
In particular, the corona theorem holds for $\mathrm{Mult}(Q^p_t)$.

This theorem generalizes the corona theorem for $\mathrm{Mult}(Q^2_t)$ proved in [12].

5.4. The corona theorem for subalgebras of $\mathrm{Mult}(HW^p_1(\theta))$, $\theta \in \mathcal{B}_p$

In this section we prove Theorem 1.7. To do so we need to recall the definitions of the $\mathcal{B}_{p,\kappa}$ class given in [6] and of the Muckenhoupt class $\mathcal{A}_{p,\kappa}$.

Definition 5.13. Let $1 < p < \infty$ and $\kappa > 0$.

The set $\mathcal{B}_{p,\kappa}$ denotes the Békollé class of positive weights $\theta \in L^1(d\nu_\kappa)$ satisfying $\theta^{\frac{-1}{p-1}} \in L^1(d\nu_\kappa)$ and

$$\mathcal{B}_{p,\kappa}(\theta) := \sup_{I\subset\mathbb{T}}\Bigg(\frac{1}{|I|^{1+\kappa}}\int_{T(I)} \theta\, d\nu_\kappa\Bigg)^{1/p}\Bigg(\frac{1}{|I|^{1+\kappa}}\int_{T(I)} \theta^{\frac{-1}{p-1}}\, d\nu_\kappa\Bigg)^{1/p'} < \infty.$$

By $\mathcal{A}_{p,\kappa}$ we denote the Muckenhoupt class of positive weights $\theta \in L^1(d\nu_\kappa)$ satisfying $\theta^{\frac{-1}{p-1}} \in L^1(d\nu_\kappa)$ and

$$\mathcal{A}_{p,\kappa}(\theta) := \sup_{B}\Bigg(\frac{1}{r^{1+\kappa}}\int_{B\cap\mathbb{D}} \theta\, d\nu_\kappa\Bigg)^{1/p}\Bigg(\frac{1}{r^{1+\kappa}}\int_{B\cap\mathbb{D}} \theta^{\frac{-1}{p-1}}\, d\nu_\kappa\Bigg)^{1/p'} < \infty,$$

where the supremum is over all the balls $B = B(w,r)$ with $w \in \overline{\mathbb{D}}$ and $0 < r < 2$.

The classes $\mathcal{A}_p$ and $\mathcal{B}_p$ correspond to the case $\kappa = 1$, that is $\mathcal{A}_p := \mathcal{A}_{p,1}$ and $\mathcal{B}_p := \mathcal{B}_{p,1}$.

We recall that if $1 < p < \infty$, then the Bergman projection is bounded on $L^p(\theta)$ if and only if $\theta \in \mathcal{B}_p$ (see [6, Theorem 1]). Since for any $\zeta \in \mathbb{T}$, then we have that $T(I(\zeta,r)) = B(\zeta,r) \cap \mathbb{D}$, it is clear that $\mathcal{A}_{p,\kappa} \subset \mathcal{B}_{p,\kappa}$.

The next lemma states some properties of the $\mathcal{B}_{p,\kappa}$ weights. We will give a sketch of the proof for completeness.

Lemma 5.14. *Let $1 < p < \infty$, $0 < \kappa$ and $\theta \in \mathcal{B}_{p,\kappa}$. Then*

(i) *There exists $C > 0$, such that for all $\zeta \in \mathbb{T}$ and $0 < r < R$*

$$\int_{T(I(\zeta,R))} \theta\, d\nu_\kappa \leqslant C\mathcal{B}_{p,\kappa}(\theta)\left(\frac{R}{r}\right)^{(1+\kappa)p} \int_{T(I(\zeta,r))} \theta d\nu_\kappa.$$

(ii) *If $\kappa_0 < \kappa$, then $\mathcal{B}_{p,\kappa_0} \subset \mathcal{B}_{p,\kappa}$.*

Proof. In order to prove these results, let

$$\Theta_\kappa(\zeta,R):=\Bigl(\int_{T(I(\zeta,R))}\theta\,d\nu_\kappa\Bigr)^{1/p},\qquad \Theta'_\kappa(\zeta,R):=\Bigl(\int_{T(I(\zeta,R))}\theta^{\frac{-1}{p-1}}\,d\nu_\kappa\Bigr)^{1/p'}.$$

Since $\nu_\kappa(T(I(\zeta,R)))\approx R^{1+\kappa}$, Hölder's inequality gives

$$R^{1+\kappa}\approx\nu_\kappa\bigl(T\bigl(I(\zeta,R)\bigr)\bigr)=\int_{T(I(\zeta,R))}d\nu_\kappa\leqslant\Theta_\kappa(\zeta,R)\Theta'_\kappa(\zeta,R)\lesssim\mathcal{B}_{p,\kappa}(\theta)R^{1+\kappa}.$$

Therefore, (i) follows from

$$\Theta_\kappa(\zeta,R)\lesssim\frac{\mathcal{B}_{p,\kappa}(\theta)R^{1+\kappa}}{\Theta'_\kappa(\zeta,R)}\leqslant\frac{\mathcal{B}_{p,\kappa}(\theta)R^{1+\kappa}}{\Theta'_\kappa(\zeta,r)}\lesssim\mathcal{B}_{p,\kappa}(\theta)\frac{R^{1+\kappa}}{r^{1+\kappa}}\Theta_\kappa(\zeta,r).$$

If $z\in T(I(\zeta,R))$, then $1-|z|^2\leqslant 2R$ and thus

$$\Theta_\kappa(\zeta,R)\lesssim R^{(\kappa-\kappa_0)/p}\Theta_{\kappa_0}(\zeta,R),\qquad \Theta'_\kappa(\zeta,R)\lesssim R^{(\kappa-\kappa_0)/p'}\Theta'_{\kappa_0}(\zeta,R),$$

which proves $\mathcal{B}_{p,\kappa}(\theta)\lesssim\mathcal{B}_{p,\kappa_0}(\theta)$ and also (ii). □

Theorem 5.15. *Let $1<p<\infty$, $0<\kappa<1$ and $\theta\in\mathcal{B}_{p,\kappa}$. If $X^{(0)}=L^p(\theta)$, then the corona theorem holds for the algebras* $\mathrm{Mult}(HX_1^{(k)})$, *$k\geqslant 0$. In particular the corona theorem holds for* $\mathrm{Mult}(HW_1^p(\theta))$.

Proof. We prove that the space $L^p(\theta)$ satisfies the hypothesis of Theorem 5.8.

It is clear that $\mathbb{C}\subset L^p(\theta)$ and that this space satisfies $\mathbb{X}2$. The proof of the fact that $L^p(\theta)\subset L^1(d\nu)$ follows from Hölder's inequality. Indeed, if $\psi\in L^p(\theta)$, then

$$\int_{\mathbb{D}}|\psi|\,d\nu\leqslant\|\psi\|_{L^p(\theta)}\Bigl(\int_{\mathbb{D}}\theta^{\frac{-1}{p-1}}\,d\nu\Bigr)^{1/p'}\leqslant\|\psi\|_{L^p(\theta)}\Bigl(\int_{\mathbb{D}}\theta^{\frac{-1}{p-1}}\,d\nu_\kappa\Bigr)^{1/p'}\lesssim\|\psi\|_{L^p(\theta)}.$$

In order to prove that $L^p(\theta)$ satisfies $\mathbb{X}3$ we use the fact that if $\theta\in\mathcal{B}_{p,\kappa}$, then the integral operator with kernel

$$\mathbb{P}^\kappa(w,z)=\frac{(1-|w|^2)^{\kappa-1}}{|1-z\overline{w}|^{1+\kappa}},$$

is bounded on $L^p(\theta d\nu_\kappa)$ (see [6, Propositions 3, 5]).

Using this result and the estimate $1-|w|^2, 1-|z|^2\leqslant 2|1-z\overline{w}|$, if $0<\tau<\min\{\frac{1-\kappa}{p},\frac{1-\kappa}{p'}\}$ and $0\leqslant\psi\in L^p(\theta)$, then we have

$$\begin{aligned}\bigl\|\mathcal{T}^{-\tau}(\psi)\bigr\|^p_{L^p(\theta)}&=\int_{\mathbb{D}}\Bigl(\int_{\mathbb{D}}\psi(w)\frac{(1-|w|^2)^{-\tau}}{|1-z\overline{w}|^{2-2\tau}}\,d\nu(w)\Bigr)^p\bigl(1-|z|^2\bigr)^{-\tau p}\theta(z)\,d\nu(z)\\&\lesssim\int_{\mathbb{D}}\bigl(\mathbb{P}^\kappa\bigl(\psi(w)\bigl(1-|w|^2\bigr)^{(1-\kappa)/p}\bigr)(z)\bigr)^p\theta(z)\,d\nu_\kappa(z)\\&\lesssim\bigl\|\psi(w)\bigl(1-|w|^2\bigr)^{(1-\kappa)/p}\bigr\|^p_{L^p(\theta\,d\nu_\kappa)}=\|\psi\|^p_{L^p(\theta)},\end{aligned}$$

which proves that $L^p(\theta)$ satisfies $\mathbb{X}3$.

We now prove that if t satisfies $\kappa-1<(t-1)p'<0$, then $X=\mathrm{Mult}(HW_1^p(\theta),L^p(\theta))\subset Y_t^1$.

Let $\varphi\in X$. We want to show that

$$\int_{T(I_a)}|\varphi|\,d\nu_t\lesssim\Bigl(\int_{T(I_a)}|\varphi|^p\theta\,d\nu\Bigr)^{1/p}\Bigl(\int_{T(I_a)}\theta^{\frac{-1}{p-1}}\,d\nu_{(t-1)p'+1}\Bigr)^{1/p'}\lesssim C_{\varphi,\theta}|I_a|^t.$$

For $z\in T(I_a)$, we have $|1-z\bar a|\approx 1-|a|^2$ and

$$\int_{T(I_a)}|\varphi|^p\theta\,d\nu\lesssim\int_{\mathbb{D}}\Bigl|\frac{(1-|a|^2)^2}{(1-z\bar a)^2}\Bigr|^p\bigl|\varphi(z)\bigr|^p\theta(z)\,d\nu(z)\lesssim\|\varphi\|^p_X\int_{\mathbb{D}}\frac{(1-|a|^2)^{2p}}{|1-z\bar a|^{3p}}\theta(z)\,d\nu(z).$$

Using the partition $\mathbb{D} = \bigcup_{j=1}^{J(a)} U_j(a) \setminus U_{j-1}(a)$ defined in (5) and Lemma 5.14, we have

$$\int_{T(I_a)} |\varphi|^p \theta \, d\nu \lesssim \|\varphi\|_X^p \sum_{j=0}^{J_a} \frac{(1-|a|^2)^{-p}}{2^{3pj}} \int_{U_j(a)} \theta \, d\nu \lesssim \|\varphi\|_X^p \sum_{j=0}^{\infty} \frac{(1-|a|^2)^{-p}}{2^{jp}} \int_{T(I_a)} \theta \, d\nu$$
$$\lesssim \|\varphi\|_X^p \big(1-|a|^2\big)^{-p} \int_{T(I_a)} \theta \, d\nu \lesssim \|\varphi\|_X^p \big(1-|a|^2\big)^{-p+1-\kappa} \int_{T(I_a)} \theta \, d\nu_\kappa.$$

Since $\kappa - 1 < (t-1)p' < 0$, we have

$$\int_{T(I_a)} \theta^{\frac{-1}{p-1}} \, d\nu_{(t-1)p'+1} \lesssim \big(1-|a|^2\big)^{(t-1)p'+1-\kappa} \int_{T(I_a)} \theta^{\frac{-1}{p-1}} \, d\nu_\kappa.$$

Combining these estimates, we obtain $\int_{T(I_a)} |\varphi| \, d\nu_t \lesssim \|\varphi\|_X^p \mathcal{B}_{p,\kappa}(\theta) |I_a|^t$, which proves the result. □

We now consider the case $\kappa = 1$.

Recall that if $\theta \in \mathcal{A}_p$, $p > 1$, then there exists $1 < q < p$ such that $\theta \in \mathcal{A}_q$ (see [15, Chapter 5, Section 3]). As it is pointed in [6], this result is not true for weights $\theta \in \mathcal{B}_p$. The next result for $\mathcal{A}_p$ weights is similar to the one above mentioned.

Lemma 5.16. *Let $1 < p < \infty$. If $\theta \in \mathcal{A}_p$, then there exists $0 < \kappa = \kappa(\theta) < 1$ such that $\theta \in \mathcal{A}_{p,\kappa}$.*

Proof. It is well known that if $\theta \in \mathcal{A}_p$, then there exists $q = q(\theta) > 1$ such that $\theta^q \in \mathcal{A}_p$ (see [15, Chapter 5, Section 6.1]). We prove that if $1/q < \kappa < 1$, then $\theta \in \mathcal{A}_{p,\kappa}$. Since $0 > (\kappa - 1)q' > -1$, we have

$$\left(\int_{B(w,r)} \big(1-|z|^2\big)^{(\kappa-1)q'} \, d\nu(z) \right)^{1/q'} \lesssim r^{2/q'+\kappa-1} = r^{1+\kappa-2/q}.$$

Thus, applying Hölder's inequality with exponent q to the next integrals

$$\frac{1}{r^{1+\kappa}} \left(\int_{B(w,r)} \theta \, d\nu_\kappa \right)^{1/p} \left(\int_{B(w,r)} \theta^{\frac{-1}{p-1}} d\nu_\kappa \right)^{1/p'},$$

we obtain $\mathcal{A}_{p,\kappa}(\theta) \lesssim \mathcal{A}_p(\theta^q)^{1/q}$ which proves the result. □

Proposition 5.17. *Let $1 < p < \infty$ and $\theta \in \mathcal{B}_p$. Then, there exists $0 < \kappa < 1$ and $\Theta \in \mathcal{A}_{p,\kappa}$ such that $HW_1^p(\Theta) = HW_1^p(\theta)$.*

Proof. By [7, Theorem 2.19] and [7, Proposition 3.9], given $\theta \in \mathcal{B}_p$ there exists a weight $\Theta \in \mathcal{A}_p$ such that $HW_1^p(\Theta) = HW_1^p(\theta)$. Therefore, the result follows from Lemma 5.16. □

As a consequence of Proposition 5.17 and Theorem 5.15 we have:

Theorem 5.18. *If $1 < p < \infty$ and $\theta \in \mathcal{B}_p$, then the corona theorem holds for* $\mathrm{Mult}(HW_1^p(\theta))$.

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