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Estimates at infinity for positive solutions to a class of p-Laplacian problems in $\mathbb{R}^N$

David G. Costa[a,*], Hossein Tehrani[a], Ralph Thomas[b]

[a] Department of Mathematical Sciences, University of Nevada Las Vegas, Box 454020, Las Vegas, NV 89154-4020, United States
[b] Database Marketing, Seminole Hard Rock Gaming, 5223 Orient Rd, Tampa, FL 33610, United States

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ABSTRACT

In this paper we consider a class of *logistic-type* problems for the p-Laplacian in the whole space. Using minimization we prove existence of a positive solution and its behavior at infinity. We also consider questions of uniqueness and sharp estimates at infinity.

1. Introduction

There has been much study of finding positive solutions to various logistic problems involving the Laplacian and the p-Laplacian; problems which, loosely speaking, contain a nonlinear term that behaves like $\lambda u^{p-1}(1-u^\gamma)$ with $\gamma > 1$.

Also, there has been study of logistic problems with harvesting, where one subtracts a harvesting term of the form $ch(x)$. These arise from problems in fishery or hunting management [17], in which case one is interested in finding positive solutions. In [18], Oruganti, Shi and Shivaji looked for results in bounded domains, finding positive solutions to

$$\begin{cases} -\Delta u = au - bu^2 - ch(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b, c > 0$ are constants, Ω is a smooth bounded region with $\partial\Omega \in C^2$ and $h \in C^\alpha(\bar{\Omega})$ is positive in Ω and vanishes on $\partial\Omega$. They proved that if $a > \lambda_1$ then there exists a constant $c_2 = c_2(a, b)$ such that, for $0 < c < c_2$, (1.1) has a maximal positive solution and no positive solution for $c > c_2$. In addition, Oruganti, Shi and Shivaji in [19] were able to extend the above result to the p-Laplacian, finding positive solutions to

$$\begin{cases} -\Delta_p u = au^{p-1} - u^{\gamma-1} - ch(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

On the other hand, Du and Ma [8] studied a logistic problem for the Laplacian in $\mathbb{R}^N$, looking for positive solutions (λ, u) to

$$-\Delta u = \lambda a(x)u - b(x)u^\gamma \quad \text{in } \mathbb{R}^N$$

* Corresponding author.
E-mail address: costa@unlv.nevada.edu (D.G. Costa).

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where $\gamma > 1$, $0 < b(x) \in C^\infty(\mathbb{R}^N)$, $0 < a(x) \in C^1(\mathbb{R}^N)$ and $a(x) \leqslant P(x)$ for a radially symmetric $P(x)$ satisfying

$$\int_{\mathbb{R}^N} \frac{P(x)}{|x|^{N-2}} < \infty.$$

In addition, Costa, Drabek and Tehrani [4] and Girão and Tehrani [12] considered the logistic problem for the Laplacian in $\mathbb{R}^N$ with harvesting and extended the result of Oruganti, Shi and Shivaji for the Laplacian in bounded domains. In [4] the authors find positive solutions to

$$-\Delta u = a(x)\big(\lambda u - u^\gamma\big) - \mu h(x) \quad \text{in } \mathbb{R}^N,$$

where $\gamma > 1$, $\lambda > \lambda_1$, $0 < a(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $0 < h(x)$ is a rapidly decreasing function in $\mathbb{R}^N$. They showed that there exists $\hat{\mu} = \hat{\mu}(\lambda) > 0$ such that for all $0 < \mu < \hat{\mu}$ there exists a solution $u_\mu > 0$ in $\mathbb{R}^N$ satisfying

$$u_\mu \geqslant \frac{C}{|x|^{N-2}} \quad \text{for } |x| \text{ large.}$$

In this paper we will generalize the above results in the non-harvesting case by considering the p-Laplacian on the whole space. We will look for positive solutions to

$$-\Delta_p u = a(x)\big(\lambda |u|^{p-2}u - g(u)\big), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $g(s)$ behaves like $s^{\gamma-1}$, $\gamma > p$, for s large. We will borrow some ideas from [4] and, in doing so, will not only prove the existence of positive (weak) solutions, but will also have estimates for the behavior of these solutions at infinity. Namely, we will show existence of a positive solution u_0 satisfying

$$u_0(x) \geqslant C|x|^{-\frac{N-p}{p-1}} \quad \text{for } |x| \text{ large,}$$

and, by strengthening the assumptions on $g(u)$ and $a(x)$, we prove that u_0 is in fact the unique positive solution to (1.3) and that the above estimate at infinity is sharp.

We note that the method of sub and super solutions was used in [19], while the approach in the other works mentioned above (and in the present paper) uses minimization methods applied to the underlying functional.

2. Preliminaries and variational framework

2.1. Preliminaries

Since the needed preliminary results are scattered throughout the literature, and for the convenience of the reader, we collect in this section the known results on solutions to some quasilinear equations which have a bearing on the problems considered in this paper.

We start with a few comments on the notation. All integrals will be assumed to be taken over $\mathbb{R}^N$ unless otherwise stated. We let $D^{1,p} = D^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty = C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\| = (\int |\nabla u|^p)^{1/p}$. In addition, for a bounded domain $\Omega \subset \mathbb{R}^N$, $W^{1,p}(\Omega)$ denotes the completion of $C^\infty(\Omega)$ under the norm $\|u\|_{W^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p + |u|^p)^{1/p}$, and $W_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ under the norm $\|\nabla u\|_{p,\Omega} = (\int_\Omega |\nabla u|^p)^{1/p}$. We also denote the norm on $L^r = L^r(\mathbb{R}^N)$ by $\|u\|_r = (\int |u|^r)^{1/r}$, and define the norm on the weighted L^r space $L^r_{a(x)} = L^r_{a(x)}(\mathbb{R}^N)$ by $\|u\|_{r,a(x)} = (\int a(x)|u|^r)^{1/r}$. Finally, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator. Throughout the paper we will be assuming $1 < p < N$ and letting $p^* = \frac{Np}{N-p}$ denote the limiting exponent $q = p^*$ in the Sobolev embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$.

A basic result on existence and uniqueness of solution to the p-Laplacian under Dirichlet boundary condition is the following

Theorem 2.1. (i) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the Dirichlet problem*

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in L^{p^{*\prime}}(\Omega)$ and $p^{*\prime} = \frac{p^*}{p^*-1} = \frac{Np}{Np-(N-p)}$. Then (2.1) has a unique weak solution $u_f \in W_0^{1,p}(\Omega)$, i.e.

$$\int_\Omega |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla v = \int_\Omega f v \quad \forall v \in W_0^{1,p}(\Omega).$$

In addition, we have $\|\nabla u_f\|_{p,\Omega} \leqslant C\|f\|_{p^{*\prime}}^{1/(p-1)}$ for some $C > 0$ independent of f.

(ii) *Let $f \in L^{p^{*\prime}}(\mathbb{R}^N)$. Then there is a unique weak solution $u_f \in D^{1,p}(\mathbb{R}^N)$ to*

$$-\Delta_p u_f = f,$$

i.e.

$$\int |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla v = \int f v \quad \forall v \in D_{1,p}(\mathbb{R}^N).$$

Proof. Part (i) follows from Theorem 3.1 of Drabek and Simander [7] and (ii) follows from Theorem 4.1 of the same source. □

The next two results provide some estimates for sub and super solutions of equations involving the p-Laplacian:

Theorem 2.2. *Consider the equation*

$$-\Delta_p u = c(x)|u|^{p-2}u + d(x) \tag{2.2}$$

in a domain $\Omega \subset \mathbb{R}^N$, where $1 < p < N$ and $c(x), d(x) \in L^\infty(\Omega)$. Let $K = K(3\rho)$ denote a cube with side length $3\rho > 0$ and $K \subset \Omega$. Assume $u \in D^{1,p}$ is nonnegative in K.

(i) *If u is a weak supersolution of* (2.2) *in K, then*

$$\rho^{-N/\gamma}\|u\chi_{K(2\rho)}\|_\gamma \leqslant C \min_{K(\rho)} u(x)$$

for any $\gamma < \frac{N(p-1)}{N-p}$, where $C = C(p, N, \|c\|_\infty, \|d\|_\infty)$.

(ii) *If u is a weak subsolution of* (2.2) *in K, then*

$$\max_{K(\rho)} u(x) \leqslant C\rho^{-N/\gamma}\|u\chi_{K(2\rho)}\|_\gamma$$

for any $\gamma > p-1$, where $C = C(p, N, \|c\|_\infty, \|d\|_\infty)$.

Proof. Noting that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and setting

$$A(x,u,v) = |v|^{p-2}v, \qquad B(x,u,v) = c(x)|u|^{p-2}u + d(x),$$

the forms $A(x,u,v)$ and $B(x,u,v)$ satisfy the conditions of Theorems 1.2 and 1.3 of Trudinger [22] and the result follows. □

Theorem 2.3. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and let u be a bounded, nonnegative, p-superharmonic function in Ω such that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = \int_\Omega \phi \, d\mu$$

for some nonnegative Radon measure μ on Ω and all $\phi \in C_0^\infty(\Omega)$. Define

$$W^\mu_{1,p}(x,r) = \int_0^r \left(\frac{\mu(B_t(x))}{t^{N-p}}\right)^{q-1} \frac{dt}{t}.$$

If $B_{3r}(a) \subset \Omega$, then there exists constants A_1, A_2 and A_3 such that

$$A_1 W^\mu_{1,p}(a,r) \leqslant u(a) \leqslant A_2 \inf_{x\in B_r(a)} u(x) + A_3 W^\mu_{1,p}(a,2r).$$

Proof. This is Theorem 1.6 of Kilpelainen and Maly [14] for the case $A(x,h) = |h|^{p-2}h$. (See also Theorem 3.1 of [13].) □

The next preliminary result provides some maximum and comparison principles for the p-Laplacian operator:

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1+\alpha}$, $0 < \alpha < 1$ and let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *Let $u \in C^1(\Omega)$ satisfy $u \geqslant 0$ in Ω and $-\Delta_p u \geqslant 0$ a.e. in Ω. Then either $u \equiv 0$ or $u > 0$ on Ω. Moreover, if $u \in C^1(\Omega \cup \{x_0\})$ for any $x_0 \in \partial\Omega$ that satisfies an interior sphere condition and $u(x_0) = 0$, then $\frac{\partial u}{\partial \nu} > 0$ where ν is an interior normal at x_0.*

(ii) *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to*

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = f_1 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$ and $f_1 \in C^{1+\alpha}(\partial\Omega)$. Then there exists $0 < \beta < 1$ such that $u \in C^{1+\beta}(\bar{\Omega})$.

(iii) (*Maximum Principle*) *Assume that* $u \in W^{1,p}(\Omega)$ *satisfies*

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u \geqslant 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in W^{-1,q}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, $f \geqslant 0$. *Then either* $u > 0$ *in* Ω, *or* $u \equiv 0$ *in* Ω.

(iv) (*Weak Comparison Principle*) *For* $i = 1, 2$, *suppose* $u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *satisfy* $\Delta_p u_i \in L^\infty(\Omega)$, $u_i|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$ *together with the inequalities*

$$\begin{cases} -\Delta_p u_1 \leqslant -\Delta_p u_2 & \text{in } \Omega, \\ u_1 \leqslant u_2 & \text{on } \partial\Omega. \end{cases}$$

Assume in addition that $-\Delta_p u_2 \geqslant 0$ *in* Ω *and* $u_2 \geqslant 0$ *on* $\partial\Omega$. *Then*

$$u_1(x) \leqslant u_2(x) \quad \text{for each } x \in \Omega.$$

Proof. (i) follows from Theorem 5 of Vazquez [23], while (ii) and (iii) follow from results of Garcia-Melian and de Lis [11].

(iv) By part (ii) we have $u_i \in C^{1+\beta}(\bar{\Omega})$, and by part (i), $u_2 > 0$ in Ω and $\frac{\partial u_2}{\partial \nu} < 0$ at that part of $\partial\Omega$ where $u_2 = 0$. Therefore there exists $c > 1$ such that $u_1 < cu_2$ in Ω. Consider the problem

$$\begin{cases} -\Delta_p v = -\Delta_p u_2 & \text{in } \Omega, \\ v = u_2 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Then u_1 and cu_2 are sub and supersolutions, respectively, of (2.3). Thus, the method of sub and supersolutions (e.g., see Theorem 4.14 of Diaz [5]) yields existence of a solution $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (2.3), with $u_1 \leqslant v \leqslant cu_2$, which must be nonnegative.

We claim that (2.3) has a unique nonnegative solution in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Suppose we have two such solutions v_1 and v_2. Then parts (i) and (ii) imply that $v_1/v_2,\ v_2/v_1 \in L^\infty(\Omega)$. Then following a proof similar to that of Lemma 2.7 of the next section, we have that $v_1 = cv_2$ for some constant c. Since $v_1 = v_2$ on $\partial\Omega$ we have proved the claim.

Therefore $v = u_2$ and we have $u_1 \leqslant u_2$, completing the proof of the theorem. □

Finally we present a general regularity result for solutions of quasilinear equations associated with the p-Laplacian:

Theorem 2.5. *(Cf. Theorem 1 of Tolksdorf [21].) Suppose* $\Omega \subset \mathbb{R}^N$ *is open and* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $1 < p < \infty$, *is a weak solution to*

$$-\Delta_p u = a(x, u, \nabla u)$$

in Ω, *where* $|a(x, u, \nabla u)| \leqslant \Gamma(1 + |\nabla u|)^p$ *for some constant* $\Gamma > 0$ *and all* $x \in \Omega$. *Then there exists* $0 < \alpha < 1$ *such that* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

2.2. Variational eigenvalues and eigenfunctions of the p-Laplacian

Next we collect some properties and results on the eigenvalue problem for a weighted p-Laplacian operator. Namely let $0 < a(x) \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and consider the eigenvalue problem:

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u, \tag{2.4}$$

in $D^{1,p}$, where $1 < p < N$. First we need to setup some variational framework. Let V be the completion of C_0^∞ with respect to the norm

$$\|u\|_V^p = \int |\nabla u|^p + \int \frac{|u|^p}{(1 + |x|)^p}.$$

Let $G = \{u \in V \mid \int a(x)|u|^p = 1\}$, and define

$$\Gamma_k = \{A \subset G \mid A \text{ is symmetric, compact, and } \gamma(A) \geqslant k\},$$

where $\gamma(A)$ is the genus of A, i.e. the smallest integer k such that there exists an odd continuous map from A to $\mathbb{R}^k \backslash \{0\}$.

Define $I(u) = \frac{1}{p}\int |\nabla u|^p$ and $\Psi(u) = \frac{1}{p}\int a(x)|u|^p$. Clearly I is well-defined on V. Furthermore, I is bounded below on G as a simple application of Holder and Sobolev inequalities imply

$$\int a(x)|u|^p \leqslant C\|a\|_{N/p}\|u\|^p \quad \forall u \in D^{1,p}. \tag{2.5}$$

Finally, the functional I satisfies the Palais–Smale condition on G, i.e. for $\{u_n\} \subset G$, if $I(u_n)$ is bounded and $I'(u_n) \to 0$, then $\{u_n\}$ has a convergent subsequence in V (cf. Allegretto and Huang [2]).

The various items in the following theorem as well as the accompanying lemma follow in a straightforward manner from Ljusternik–Schnirelmann theory and by adapting the standard techniques of bounded domain case (cf. [3,16,15]) to $\mathbb{R}^N$. See also [9,10].

Theorem 2.6. (i) *The eigenvalue problem* (2.4) *has a sequence of solutions* (λ_k, Φ_k) *with* $\int a(x)|\Phi_k|^p = 1$, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$. *Furthermore,*

$$\lambda_1 = \inf_{u \in D^{1,p}} \frac{\int |\nabla u|^p}{\int a(x)|u|^p} \quad \text{and} \quad \lambda_k = \inf_{A \in \Gamma_k} \sup_{u \in A} \int |\nabla u|^p \quad (k \geqslant 2).$$

As such, the solutions (λ_k, Φ_k) *are called* minimax *eigenvalues and eigenvectors.*

(ii) *There exists a first eigenfunction* Φ_1 *such that* $\Phi_1 > 0$ *on* $\mathbb{R}^N$.

(iii) λ_1 *is simple, i.e. the positive eigenfunction corresponding to* λ_1 *is unique up to a constant multiple.*

(iv) λ_1 *is unique, i.e. if* $v \geqslant 0$ *is an eigenfunction associated with an eigenvalue* λ *with* $\int a(x)|v|^p = 1$, *then* $\lambda = \lambda_1$.

(v) *If* $\mu > \lambda_1$ *is an eigenvalue with eigenfunction* v, *then* v *must change signs in* $\mathbb{R}^N$.

Lemma 2.7. *Suppose* $u, v \in C^1 \cap D^{1,p}$, $u, v > 0$ *on* $\mathbb{R}^N$, $\frac{u}{v}, \frac{v}{u} \in L^\infty$, *and let*

$$K(u, v) = \left(-\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\right) - \left(-\Delta_p v, \frac{u^p - v^p}{v^{p-1}}\right).$$

Then $\frac{u^p - v^p}{u^{p-1}}, \frac{u^p - v^p}{v^{p-1}} \in D^{1,p}$, $K(u, v) \geqslant 0$ *and* $K(u, v) = 0$ *if and only if there exists* $\alpha > 0$ *such that* $u = \alpha v$.

We conclude this section with the following result on the first eigenfunction:

Theorem 2.8. *Let* Φ_1 *denote a first eigenfunction of* (2.4) *satisfying* $\Phi_1 > 0$. *Then* (i) $\Phi_1 \in L^r$ *for all* $p^* \leqslant r < \infty$; (ii) $\Phi_1 \in D^{1,p} \cap L^\gamma_{a(x)}$ *for all* $\gamma \geqslant p$.

Proof. The proof of part (i) follows by employing an iteration argument similar to that used in Appendix B of Struwe [20]. For part (ii), we have

$$\begin{aligned}\int a(x)\Phi_1^\gamma &\leqslant \left(\int a(x)^{\frac{N}{p}}\right)^{\frac{p}{N}} \left(\int \Phi_1^{\frac{\gamma N}{N-p}}\right)^{\frac{N-p}{N}} \\ &= \|a\|_{N/p} \|\Phi_1\|^\gamma_{\gamma N/(N-p)} < \infty\end{aligned}$$

where we used part (i) since $\frac{\gamma N}{N-p} \geqslant \frac{pN}{N-p} = p^*$. $\square$

3. Main results

In this chapter we prove the first of our main results. Unless stated otherwise, we assume throughout this section that the following conditions hold:

(A_0) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous,
(A_1) $\lim_{s \to 0^+} \frac{g(s)}{s^{p-1}} = 0$,
(A_2) $0 < \liminf_{s \to \infty} \frac{g(s)}{s^{\gamma-1}} \leqslant \limsup_{s \to \infty} \frac{g(s)}{s^{\gamma-1}} < \infty$ with $\gamma > p$,
(A_3) $\frac{g(s)}{s^{p-1}}$ is nondecreasing,
(B_1) $0 < a(x) \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,
(B_2) $\lambda > \lambda_1$,

where $\lambda_1 = \lambda_1(a(x), p)$ denotes the first eigenvalue of the p-Laplacian with weight $a(x)$ as defined in Theorem 2.6.

Our main results in this section concern existence of a positive solution and its asymptotic behavior for

$$-\Delta_p u = \lambda a(x)\big(|u|^{p-2}u - g(u)\big), \quad x \in \mathbb{R}^N. \tag{3.1}$$

We say that $u \in D^{1,p}$ is a (weak) solution to (3.1) if

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda \int a(x)|u|^{p-2}uv + \int a(x)g(u)v = 0 \tag{3.2}$$

holds for all $v \in D^{1,p} \cap L^\gamma_{a(x)}$. Note that the condition $v \in L^\gamma_{a(x)}$ arises from

$$\int a(x)|u|^{\gamma-1}v \leqslant \left(\int a(x)|u|^\gamma\right)^{\frac{\gamma}{\gamma-1}} \left(\int a(x)|v|^\gamma\right)^{\frac{1}{\gamma}},$$

and, as our construction below of a weak solution shows, we have $u \in L^\gamma_{a(x)}$. Furthermore we do not require $v \in L^p_{a(x)}$, since $D^{1,p} \subset L^p_{a(x)}$ by (2.5).

Our main results in this paper are the following theorems.

Theorem 3.1. *If* (A_0)-(A_3) *and* (B_1), (B_2) *hold then* (3.1) *has a solution* $u_0 > 0$ *in* $\mathbb{R}^N$ *satisfying*

$$u_0(x) \geqslant \frac{C}{|x|^{\frac{N-p}{p-1}}} \quad \text{for } |x| \text{ large.}$$

Theorem 3.2. *Assume the hypotheses of Theorem* 3.1. *If, in addition,*

(A_3') $\frac{g(s)}{s^{p-1}}$ *is increasing,*
(B_3') *there exist* $C > 0$ *and* $P > \frac{N}{p}$, *with* $\frac{1}{P} + \frac{1}{Q} = 1$, *such that*

$$|x|^{\frac{N}{Q(p-1)}} \|a\|_{L^P(\mathbb{R}^N \setminus B_{|x|}(0))} \leqslant C \quad \forall x \in \mathbb{R}^N,$$

then u_0 *is the unique positive solution of Eq.* (3.1) *and*

$$u_0(x) = \frac{d(x)}{|x|^{\frac{N-p}{p-1}}} \quad \text{for } |x| \text{ large,} \tag{3.3}$$

where $C_1 \leqslant d(x) \leqslant C_2$ *for all* $x \in \mathbb{R}^N$ *and some constants* $C_1, C_2 > 0$.

3.1. Existence of solution

We first show that the hypothesis (B_2) is in fact necessary for existence of a positive solution.

Theorem 3.3. *If* $u \in D^{1,p}$ *is a positive weak solution to* (3.1) *then* $\lambda > \lambda_1$.

Proof. By Theorem 2.6 we have that $\lambda_1 \leqslant \frac{\int |\nabla v|^p}{\int a(x)|v|^p}$ for all $v \in D^{1,p}$, hence

$$\lambda_1 \int a(x)u^p \leqslant \lambda \int a(x)u^p - \int a(x)g(u)u,$$

so that $(\lambda_1 - \lambda) \int a(x)u^p \leqslant -\int a(x)g(u)u < 0$, as $a(x) > 0$, $g(u)u > 0$. □

Next, in order to prove Theorem 3.1, we extend the definition of g to all of $\mathbb{R}$ by taking $g(s) = 0$, for all $s \leqslant 0$. Also let $G(s) = \int_0^s g(t)\,dt$ and consider the functionals I and J defined by

$$J : D^{1,p} \to \mathbb{R} \cup \{\infty\}, \quad J(u) = \int a(x)G(u),$$

$$I(u) = \begin{cases} \frac{1}{p}\int |\nabla u|^p - \frac{\lambda}{p}\int a(x)(u^+)^p + J(u), & J(u) < \infty, \\ \infty, & J(u) = \infty. \end{cases}$$

Our goal is to find a solution to (3.1) by minimizing I. We will need the following preliminary results:

Lemma 3.4. *Define* $T : D^{1,p} \to (D^{1,p})^*$ *by* $\langle T(u), v\rangle = \int a(x)|u|^{p-2}uv$, *where* $a(x) \in L^{N/p} \cap L^\infty$. *Then* T *is compact.*

Proof. See Lemma 2.2ii in [6]. □

Lemma 3.5. *For any* $\epsilon > 0$ *there exists constants* $C_1 = C_1(\epsilon)$ *and* $C_2 = C_2(\epsilon)$ *such that*

$$-\epsilon (s^+)^{p-1} + C_1 (s^+)^{\gamma-1} \leqslant g(s) \leqslant \epsilon (s^+)^{p-1} + C_2 (s^+)^{\gamma-1},$$
$$-\epsilon (s^+)^{p} + C_1 (s^+)^{\gamma} \leqslant G(s) \leqslant \epsilon (s^+)^{p} + C_2 (s^+)^{\gamma}.$$

Proof. This follows from our conditions on g and Holder's Inequality. □

In the next result we provide basic properties of the functional I which allows its minimization in $D^{1,p}$.

Lemma 3.6. (a) *I is coercive, i.e.* $I(u) \to \infty$ *as* $\|u\| \to \infty$.
(b) *I is a weakly lower semi-continuous functional.*

Proof. (a) We use some ideas of Du and Ma [8] and argue by contradiction.

Assume that there exists $\{u_n\} \subset D^{1,p}$ such that $\{I(u_n)\}$ is bounded above and $\|u_n\| \to \infty$. Let $d_n = (\int a(x)(u_n^+)^p)^{1/p}$. Then $\|u_n\| \to \infty$ and $I(u_n) \geqslant \frac{1}{p}\|u_n\|^p - \frac{\lambda}{p} d_n^p$ implies that $d_n \to \infty$.

Set $\bar{u}_n = \frac{u_n}{d_n}$. Then $\int a(x)(\bar{u}_n^+)^p = 1$ and

$$I(u_n) = \frac{d_n^p}{p} \int |\nabla \bar{u}_n|^p - \frac{\lambda}{p} d_n^p \int a(x)\big(\bar{u}_n^+\big)^p + \int a(x) G(d_n \bar{u}_n)$$

so that

$$\frac{pI(u_n)}{d_n^p} = \int |\nabla \bar{u}_n|^p - \lambda + \frac{p}{d_n^p} \int a(x) G(d_n \bar{u}_n) \geqslant \int |\nabla \bar{u}_n|^p - \lambda.$$

Therefore, since $\frac{I(u_n)}{d_n^p} \to 0$, we have that $\{\|\bar{u}_n\|\}$ is bounded. This implies, passing to a subsequence if necessary, that $\bar{u}_n \rightharpoonup \bar{u}$ in $D^{1,p}$, $\bar{u}_n \to \bar{u}$ a.e. in $\mathbb{R}^N$, and $\bar{u}_n \to \bar{u}$ in $L^p_{a(x)}$ (by Lemma 3.4). In addition, we have

$$\begin{aligned}\frac{pI(u_n)}{d_n^p} &\geqslant \int |\nabla \bar{u}_n|^p - \lambda - \frac{\epsilon p}{d_n^p} \int a(x)\big(d_n \bar{u}_n^+\big)^p + \frac{C_1 p}{d_n^p} \int a(x)\big(d_n \bar{u}_n^+\big)^\gamma \\ &\geqslant -\lambda - \epsilon p + C_1 p d_n^{\gamma - p} \int a(x)\big(\bar{u}_n^+\big)^\gamma\end{aligned}$$

where we used Lemma 3.5.

Therefore, $\gamma > p$ and $d_n \to \infty$ imply that $\int a(x)(\bar{u}_n^+)^\gamma \to 0$. Hence, by Fatou's Lemma we have

$$\int a(x)\big(\bar{u}^+\big)^\gamma = 0.$$

Since $a(x) > 0$ on $\mathbb{R}^N$, we obtain $\bar{u} \leqslant 0$ on $\mathbb{R}^N$. However this contradicts the fact that $\bar{u}_n \to \bar{u}$ in $L^p_{a(x)}$ and $\int a(x)(\bar{u}_n^+)^p = 1$ for all n. This completes the proof of part (a).

(b) Assume $u_n \rightharpoonup u$ in $D^{1,p}$. Then Lemma 3.4 implies

$$\int a(x) u_n^p \to \int a(x) u^p.$$

Furthermore, since $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, we have by Lebesgue Dominated Convergence Theorem and Fatou's Lemma that

$$\begin{aligned}&\int a(x)\big|u_n^+\big|^p \to \int a(x)\big|u^+\big|^p, \\ &\int a(x) G(u) \leqslant \liminf_{n\to\infty} \int a(x) G(u_n).\end{aligned}$$

Finally, $I(u) \leqslant \liminf_{n\to\infty} I(u_n)$ since $\|\cdot\|^p$ is weakly lower semi-continuous. $\square$

We have proved:

Theorem 3.7. *The minimization problem below has a solution $u_0 \in D^{1,p}$ with*

$$\inf_{u \in D^{1,p}} I(u) = I(u_0). \tag{3.4}$$

Next we have

Theorem 3.8. *Suppose $\bar{u}$ is a minimum point of I, i.e. a solution to* (3.4). *Assume that $\{t_n\} \in \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$. Then, if $v \in D^{1,p} \cap L^\gamma_{a(x)}$, we have*

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{I(\bar{u} + t_n v) - I(\bar{u})}{t_n} \\ &= \int |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla v - \lambda \int a(x)\big(\bar{u}^+\big)^{p-1} v + \int a(x) g(\bar{u}) v.\end{aligned}$$

Proof. It is enough to show that for $v \in D^{1,p} \cap L^{\gamma}_{a(x)}$, we have

$$\lim_{n\to\infty} \frac{J(\bar{u}+t_n v) - J(\bar{u})}{t_n} = \int a(x) g(\bar{u}) v,$$

as this implies the second equality above. Then using the fact that $\bar{u}$ is a minimum of I and v can be replaced by $-v$, the conclusion follows.

To prove the above equation we need to show that

$$\int a(x)\left(\frac{1}{t_n}\int_{\bar{u}}^{\bar{u}+t_n v} g(s)\,ds\right)dx \to \int a(x) g(\bar{u}) v\,dx.$$

Define $F_n(x) = \frac{1}{t_n}\int_{\bar{u}}^{\bar{u}+t_n v} g(s)\,ds$. Since g is continuous, $F_n(x) \to g(\bar{u}(x))v(x)$ for a.e. $x \in \mathbb{R}^N$. In addition, using the estimates of Lemma 3.5, we have (for some $0 \leqslant \bar{t}_n \leqslant t_n$ and assuming without loss of generality that $t_n \leqslant 1$)

$$\begin{aligned}
\left|F_n(x)\right| &\leqslant \frac{1}{t_n}\big(t_n|v|\big) g(\bar{u}+\bar{t}_n v) \\
&\leqslant \frac{1}{t_n}\big(t_n|v|\big)\big(\epsilon\big(\bar{u}^+ + v^+\big)^{p-1} + C_2\big(\bar{u}^+ + v^+\big)^{\gamma-1}\big) \\
&\leqslant \epsilon C\big(\big(\bar{u}^+\big)^{p-1}|v| + \big(v^+\big)^p\big) + C\big(\big(\bar{u}^+\big)^{\gamma-1}|v| + \big(v^+\big)^{\gamma}\big).
\end{aligned}$$

Therefore, for any domain $\Omega \subset \mathbb{R}^N$,

$$\begin{aligned}
\left|\int_\Omega a(x)F_n(x)\right| \leqslant{}& \epsilon C\left(\int_\Omega a(x)\left|\bar{u}^+\right|^p\right)^{\frac{p-1}{p}} \cdot \left(\int_\Omega a(x)|v|^p\right)^{\frac{1}{p}} + \epsilon C \int_\Omega a(x)|v|^p \\
&+ C\left(\int_\Omega a(x)\left|\bar{u}^+\right|^\gamma\right)^{\frac{\gamma-1}{\gamma}} \cdot \left(\int_\Omega a(x)|v|^\gamma\right)^{\frac{1}{\gamma}} + C\int_\Omega a(x)|v|^\gamma.
\end{aligned}$$

Now, we have $v \in D^{1,p} \cap L^{\gamma}_{a(x)} \subset L^{p}_{a(x)} \cap L^{\gamma}_{a(x)}$, $\bar{u} \subset D^{1,p} \subset L^{p}_{a(x)}$ and $\int a(x)|\bar{u}|^\gamma < \infty$ (because $\bar{u}$ minimizing I implies $\int a(x)G(\bar{u}) < \infty$). The result now follows from an application of Vitali's Convergence Theorem. $\square$

Corollary 3.9. *A solution u_0 to the minimization problem* (3.4) *is a weak solution of the problem*

$$-\Delta_p u = \lambda a(x)\big(u^+\big)^{p-1} - a(x)g(u), \quad x \in \mathbb{R}^N, \tag{3.5}$$

i.e.

$$\int |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla v = \int \big(\lambda a(x)\big(u_0^+\big)^{p-1} - a(x)g(u_0)\big)v \quad \forall v \in L^{\gamma}_{a(x)} \cap D^{1,p}. \tag{3.6}$$

In addition, $I(u_0) < 0$, so that u_0 is nontrivial.

Proof. The first part follows immediately from Theorem 3.8. For the second part, let $\Phi_1 \geqslant 0$ be a first eigenfunction (as in Theorem 2.6), normalized so that $\|\Phi_1\| = 1$. Then we have

$$\begin{aligned}
I(t\Phi_1) &\leqslant \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)t^p + \int a(x)\big(\epsilon t^p \Phi_1^p + C_2 t^\gamma \Phi_1^\gamma\big) \\
&= \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1} + \frac{p\epsilon}{\lambda_1}\right)t^p + C_2 t^\gamma \int a(x)\Phi_1^\gamma.
\end{aligned}$$

The result now follows since we are assuming that $\lambda > \lambda_1$, by choosing ϵ and t sufficiently small and using the facts that $\gamma > p$ and $\Phi_1 \in L^{\gamma}_{a(x)}$. $\square$

3.2. Properties of nonnegative solutions

We next consider the properties of our minimizer u_0 found above. First, note that since G is a function of u^+ and $\int |\nabla u_0| \geqslant \int |\nabla u_0^+|$ we have $I(u_0) \geqslant I(u_0^+)$ and, therefore, we may assume that the minimizer $u_0(x)$ is nonnegative in all of $\mathbb{R}^N$.

We now consider the question of regularity of *any* nonnegative solution of (3.6) such as $u_0 \geqslant 0$.

Theorem 3.10. *If $u_0 \geqslant 0$ is a solution of* (3.6), *then*

$$u_0 \in L^\infty \cap L^\gamma_{a(x)} \cap C^{1,\alpha}_{\mathrm{loc}} \quad \text{and} \quad \lim_{|x|\to\infty} u_0(x) = 0.$$

Proof. We note that u_0 is a weak solution to the variational inequality

$$-\Delta_p u \leqslant \lambda a(x)\chi_{\{x|u_0(x)>0\}}|u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

Therefore, since $u_0 \in L^{p^*}$, an application of Theorem 2.2 part (ii) (with $\gamma = p^*$ and moving the center of the cube $K(3\rho)$ to any $x \in \mathbb{R}^N$) yields that $\sup u_0^+ \leqslant C'\|u_0^+\|_{p^*} \leqslant C\|u_0^+\|$ and $\lim_{|x|\to\infty} u_0^+(x) = 0$ for some constant $C = C(\lambda, \|a\|_\infty)$. Therefore

$$\sup u_0^+ \leqslant \tilde{C}, \tag{3.7}$$

where $\tilde{C} = \tilde{C}(\lambda, \|a\|_\infty, \|u_0^+\|)$. Also since $u_0(x) \geqslant 0$, we have $u_0 \in L^\infty$ and $\lim_{|x|\to\infty} u_0(x) = 0$.

Next, since $u_0 \in L^\infty \cap L^p_{a(x)}$ and $\gamma > p$, we have that $u_0 \in L^\gamma_{a(x)}$. Finally an application of Theorem 2.5 implies that $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. □

Our next goal is to study the asymptotic behavior of u_0 at infinity, proving in the process that $u_0 > 0$. For that, we will need the following two preliminary lemmas, whose proofs are slight modifications of those given in [4].

Lemma 3.11. *Set $f(x) = \lambda a(x)(u_0^+(x))^{p-1} - a(x)g(u_0(x))$. Then $f(x) \geqslant 0$ for all $x \in \mathbb{R}^N$.*

Proof. Let $S = \max\{s \mid \frac{g(s)}{s^{p-1}} = \lambda\}$. Then, by assumption ($A_3$) in our conditions on g, we have that $a(x)(\lambda(u_0^+(x))^{p-1} - g(u_0(x))) < 0$ if and only if $u_0(x) > S$. Now define $v = (u_0 - S)^+$. Then since $0 \leqslant v \leqslant u_0^+ \leqslant |u_0|$ and $u_0 \in L^\gamma_{a(x)} \cap D^{1,p}$, v is an admissible test function in (3.6). Therefore, if $\{u_0 > S\}$ is nonempty then

$$\begin{aligned} 0 &\leqslant \int_{\{u_0>S\}} |\nabla u_0|^p \\ &= \int |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla v \\ &= \int_{\{u_0>S\}} a(x)\big(\lambda\big(u_0^+\big)^{p-1} - g(u_0)\big)(u_0 - S)^+ < 0, \end{aligned}$$

a contradiction, so that $u_0 \leqslant S$ in $\mathbb{R}^N$, proving the lemma. □

At this point we could prove that $u_0 > 0$ by applying the Maximum Principle of Theorem 2.4. However we employ a different method, which in addition provides estimates for the behavior of u_0 at infinity.

Lemma 3.12. *Given $\epsilon > 0$, set*

$$V_\epsilon = \big\{x \bigm| u_0(x) > \epsilon a(x)^{\frac{N-p}{p^2}},\ f(x) > \epsilon a(x)\big(u_0(x)\big)^{p-1}\big\}.$$

Then, there exists positive constants ϵ_0, L_0 and $R_1 \geqslant 1$ such that

$$\|a\chi_\Omega\|_{N/p} \geqslant L_0,$$

for all $0 < \epsilon \leqslant \epsilon_0$, where $\Omega = V_\epsilon \cap B_{R_1}(0) = V_\epsilon \cap \{x \mid |x| \leqslant R_1\}$. Here our constants ϵ_0 and L_0 may depend on λ, p, $\|a\|_{N/p}$ and $\|u_0\|$.

Proof. Let $\epsilon > 0$. To simplify notation we write $V = V_\epsilon$. Then, letting $v = u_0$ in (3.6), we have

$$\begin{aligned} \|u_0\|^p &= \int \lambda a(x)\big(u_0^+\big)^{p-1}u_0 - \int a(x)g(u_0)u_0 \\ &= \int_V \lambda a(x)\big(u_0^+\big)^{p-1}u_0 - \int_V a(x)g(u_0)u_0 + \int_{\mathbb{R}^N\setminus V} f(x)u_0 \\ &\leqslant \lambda C\|a\chi_V\|_{N/p}\|u_0\|^p + \int_{\mathbb{R}^N\setminus V} f(x)u_0. \end{aligned}$$

Consider the decomposition $\mathbb{R}^N\setminus V = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, where

$$A_1 = \{x \mid u_0(x) \leqslant \epsilon a(x)^{\frac{N-p}{p^2}}\},$$
$$A_2 = \{x \mid f(x) \leqslant \epsilon a(x)\big(u_0^+(x)\big)^{p-1},\ u_0(x) > \epsilon a(x)^{\frac{N-p}{p^2}}\}.$$

Then we obtain

$$\begin{aligned}\int_{A_1} f(x)u_0 &= \lambda \int_{A_1} a(x)\big(u_0^+\big)^p - \int_{A_1} a(x)g(u_0)u_0 \\ &\leqslant \lambda\epsilon^p \int_{A_1} a(x)\big(a(x)^{\frac{N-p}{p^2}}\big)^p \\ &= \lambda\epsilon^p \|a\|_{N/p}^{N/p}.\end{aligned}$$

Furthermore,

$$\int_{A_2} f(x)u_0 \leqslant \epsilon \int_{A_2} a(x)|u_0|^p \leqslant \epsilon C\|a\|_{N/p}\|u_0\|^p.$$

Therefore, combining the above estimates, we obtain

$$\|u_0\|^p \leqslant \lambda C\|a\chi_V\|_{N/p}\|u_0\|^p + \lambda\epsilon^p\|a\|_{N/p}^{N/p} + \epsilon C\|u_0\|^p\|a\|_{N/p}.$$

Therefore we can find $\epsilon_0 > 0$ (depending only on λ, $\|u_0\|$ and $\|a\|_{N/p}$) such that

$$\lambda\epsilon^p\|a\|_{N/p}^{N/p} + \epsilon C\|u_0\|^p\|a\|_{N/p} \leqslant \frac{1}{2}\|u_0\|^p,$$

for $0 < \epsilon \leqslant \epsilon_0$ and, hence,

$$\|a\chi_V\|_{N/p} \geqslant \frac{1}{2\lambda C}.$$

Next, we let $L_0 = \frac{1}{4\lambda C}$. Since $a \in L^{N/p}(\mathbb{R}^N)$ there exists $R_1 \geqslant 1$ such that

$$\|a\chi_{\mathbb{R}^N \setminus B_{R_1}(0)}\|_{N/p} < L_0.$$

Therefore, considering that $V_{\epsilon_0} \subset V_\epsilon$ for $0 < \epsilon < \epsilon_0$, it follows that

$$\|a\chi_{V_\epsilon \cap B_{R_1}(0)}\|_{N/p} \geqslant L_0,$$

completing the proof of the lemma. $\square$

Theorem 3.13. *There exists $C > 0$ such that a nontrivial solution $u_0 \geqslant 0$ of* (3.6) *is a positive weak solution to problem* (3.1) *and satisfies*

$$u_0(x) \geqslant \frac{C}{|x|^{\frac{N-p}{p-1}}} \quad \textit{for } |x| \textit{ large.}$$

Proof. Let $u_0 \geqslant 0$ be a nontrivial solution of (3.5). Using the notation of Lemma 3.12 and letting $V = V_{\epsilon_0}$, we have

$$-\Delta_p u_0 = \lambda a(x)\big(u_0^+\big)^{p-1} - a(x)g(u_0) = f(x) \geqslant \epsilon_0^p\big(a(x)\big)^{\frac{Np-N+p}{p^2}} \quad \text{on } V \cap B_{R_1}(0).$$

For $R > R_1$, consider $z = z_R$, the solution to the Dirichlet problem

$$\begin{cases} -\Delta_p z = \epsilon_0^p\big(a(x)\big)^{\frac{Np-N+p}{p^2}}\chi_{V\cap B_{R_1}(0)} & \text{in } B_R(0), \\ z = 0 & \text{on } \partial B_R(0). \end{cases}$$

The solution z exists by Theorem 2.1, is continuous (and hence bounded) by Theorem 2.5, and is p-superharmonic by part (iv) of Theorem 2.4. Since $f(x) \geqslant \epsilon_0^p(a(x))^{\frac{Np-N+p}{p^2}}$ in $V \cap B_{R_1}(0)$ and $f(x) \geqslant 0$ in $\mathbb{R}^N$ (by Lemma 3.11), we have

$$-\Delta_p u_0 \geqslant -\Delta_p z \quad \text{in } B_R(0).$$

Furthermore, since $u_0 \geqslant 0$ in $\mathbb{R}^N$, we have that

$$u_0 \geqslant z \quad \text{on } \partial B_R(0).$$

Therefore, by the Weak Comparison Principle of Theorem 2.4, we conclude that

$$u_0 \geqslant z \quad \text{in } B_R(0).$$

Now, choose $R \geqslant 24R_1$. Then for $x \in B_{R/24}(0)$,

$$B_{R_1}(0) \subset B_{R/12}(x) \subset B_{R/6}(x), \quad \text{and} \quad B_{R/2}(x) \subset B_R(0).$$

Since $-\Delta_p z = \epsilon_0^p (a(x))^{\frac{Np-N+p}{Np}} \chi_{V\cap B_{R_1}(0)}$ in $B_{R/2}(x) \subset B_R(0)$, we can apply Theorem 2.3 to

$$\mu(\Omega) = \int_\Omega \epsilon_0^p \big(a(x)\big)^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)}\, dx$$

and get

$$\begin{aligned}
z(x) &\geqslant A_1 \int_0^{R/6} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} \epsilon_0^p \big(a(y)\big)^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)}\, dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\geqslant A_1 \int_{R/12}^{R/6} \left(\frac{1}{t^{N-p}} \int_{B_{R_1(0)}} \epsilon_0^p \big(a(y)\big)^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)}\, dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= C\epsilon_0^{\frac{p}{p-1}} \left(\frac{1}{R}\right)^{\frac{N-p}{p-1}} \left(\int_{V\cap B_{R_1}(0)} a^{\frac{Np-N+p}{p^2}} \right)^{\frac{1}{p-1}}
\end{aligned}$$

for $|x| \leqslant \frac{R}{24}$. Now, by Theorem 3.12, we have

$$L_0^{N/p} \leqslant \int_{V\cap B_{R_1}(0)} a^{N/p} \leqslant \|a\|_\infty^\alpha \int_{V\cap B_{R_1(0)}} a^{\frac{Np-N+p}{p^2}}$$

where $\alpha = \frac{N}{p} - \frac{Np-N+p}{p^2} = \frac{N-p}{p^2}$. Therefore, taking $|x| = \frac{R}{24}$, and using the fact that $R \geqslant 24R_1$ is arbitrary, we obtain the existence of $C_1 = C_1(\epsilon_0, \|a\|_\infty, L_0)$ such that

$$u_0(x) \geqslant \frac{C_1}{|x|^{\frac{N-p}{p-1}}}, \quad \text{for } |x| \geqslant R_1.$$

Furthermore, choosing $R = 24R_1$, we have that there exists $C_2 = C_2(\epsilon_0, \|a\|_\infty, L_0)$ such that

$$u_0(x) \geqslant C_2, \quad \text{for } |x| \leqslant \frac{R}{24} = R_1.$$

Therefore $u_0 > 0$ in $\mathbb{R}^N$ and

$$u_0(x) \geqslant \frac{C}{|x|^{\frac{N-p}{p-1}}}$$

for some $C > 0$ and $|x|$ sufficiently large. This completes the proof. □

3.3. Uniqueness and sharp estimate at infinity

We finish this section by providing a proof for Theorem 3.2. In fact we address the question of uniqueness and sharp estimate at infinity for u_0. First we have

Lemma 3.14. *Suppose $0 \leqslant h \in L^1 \cap L^\infty$ and, for all $x \in \mathbb{R}^N$,*

$$|x|^{\frac{N}{Q(p-1)}} \|h\|_{L^P(\mathbb{R}^N \setminus B_{|x|}(0))} \leqslant C \tag{3.8}$$

for some constant C and some $P > \frac{N}{p}$, with $\frac{1}{P} + \frac{1}{Q} = 1$. Then there exists a unique weak solution w to

$$-\Delta_p w = h$$

with $w \in D^{1,p} \cap C^1 \cap L^\infty$, $\lim_{|x|\to\infty} w(x) = 0$, *and*

$$w(x) \leqslant \frac{d}{|x|^{\frac{N-p}{p-1}}} \quad \forall x \in \mathbb{R}^N$$

for some $d > 0$.

Proof. We use techniques developed in Lemma 4 of Allegretto and Odiobala [1]. The solution $w \in D^{1,p}$ exists and is unique by Theorem 2.1 and is p-superharmonic on bounded domains by part (iv) of Theorem 2.4. In addition, $w \in C^1 \cap L^\infty$ and $\lim_{|x|\to\infty} w(x) = 0$ by an application of Theorem 2.5 and Theorem 2.2.

Now, let $r > 0$. Then by Theorem 2.3 we have $A_2, A_3 > 0$ such that

$$w(x) \leqslant A_2 \inf_{a \in B(x,r)} w(a) + A_3 \int_0^{2r} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} h(y)\,dy \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Letting $r \to \infty$ and using the fact that $\lim_{|x|\to\infty} w(x) = 0$ we get

$$\begin{aligned} w(x) &\leqslant 0 + A_3 \int_0^{\infty} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} h(y)\,dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= A_3 \int_0^{|x|/2} \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt + A_3 \int_{|x|/2}^{\infty} \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt. \end{aligned}$$

For the second term on the right we have

$$\begin{aligned} \int_{|x|/2}^{\infty} \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt &\leqslant \|h\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p-1}} \int_{|x|/2}^{\infty} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt \\ &= C \|h\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p-1}} \left(\frac{1}{|x|}\right)^{\frac{N-p}{p-1}}. \end{aligned}$$

Now, we also have, with $h_x(y) = h(y+x)$ and $\frac{1}{P} + \frac{1}{Q} = 1$,

$$\begin{aligned} \|h\|_{L^1(B_t(x))} &= c_1 \int_{B_1(0)} h_x(ty) t^N\,dy \\ &\leqslant c_2 t^N \left(\int_{B_1(0)} \big(h_x(ty)\big)^P\,dy \right)^{1/P} \\ &= c_3 t^{N/Q} \|h\|_{L^P(B_t(x))}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^{|x|/2} \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt &= c_3^{\frac{1}{p-1}} \int_0^{|x|/2} \|h\|_{L^P(B_t(x))}^{\frac{1}{p-1}} t^{\frac{1-N}{p-1} + \frac{N}{Q(p-1)}}\,dt \\ &\leqslant c_5 \|h\|_{L^P(\mathbb{R}^N \setminus B_{|x|/2}(0))}^{\frac{1}{p-1}} \left(\frac{1}{|x|}\right)^{\frac{N-p}{p-1}} |x|^{\frac{N}{Q(p-1)}} \end{aligned}$$

where we used the fact that $P > \frac{N}{p}$ implies that $\frac{p-N}{p-1} + \frac{N}{Q(p-1)} > 0$. The lemma now follows from our condition (3.8) on h. $\square$

Proof of Theorem 3.2. By our conditions on g, there exists an $S > 0$ such that

$$S = \sup_{s \in \mathbb{R}^+} \lambda s^{p-1} - g(s).$$

Therefore $a(x)(\lambda u_0^{p-1}(x) - g(u_0(x))) \leqslant Sa(x)$, and so the sharp estimate (3.3) follows by using condition (B_1'), Theorem 3.13 and Lemma 3.14.

Finally, suppose we have two positive solutions $u = u_0$ and v to (3.1). Then, since our results in section 3.2 hold for any nonnegative solution to (3.6), we get sharp estimates for both u and v at infinity, proving that $\frac{u}{v}, \frac{v}{u} \in L^\infty$. In addition, we have $u, v \in C^1$ by Theorem 3.10. Therefore, by Lemma 2.7, we may use test functions $\frac{u^p - v^p}{u^{p-1}}$ and $\frac{u^p - v^p}{v^{p-1}}$ in (3.2) to get

$$\begin{aligned}
0 &\leqslant K(u, v) \\
&= \lambda \int a(x) u^{p-1} \frac{u^p - v^p}{u^{p-1}} - \int a(x) g(u) \frac{u^p - v^p}{u^{p-1}} \\
&\quad - \lambda \int a(x) v^{p-1} \frac{u^p - v^p}{v^{p-1}} + \int a(x) g(v) \frac{u^p - v^p}{v^{p-1}} \\
&= \int a(x) \left(\frac{g(v)}{v^{p-1}} - \frac{g(u)}{u^{p-1}} \right) \left(u^p - v^p \right).
\end{aligned}$$

Therefore assuming (A_3') on g (i.e. $\frac{g(s)}{s^{p-1}}$ is increasing) we have that $0 \leqslant K(u, v) < 0$ if u and v are not identical, a contradiction. Therefore $u \equiv v$, and hence we have uniqueness for positive solutions to (3.1). □

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