



The Bishop–Phelps theorem in complete random normed modules endowed with the (ε, λ) -topology[☆]

Mingzhi Wu

LMIB and School of Mathematics and Systems Science, Beihang University, Beijing 100191, PR China

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ABSTRACT

In this paper, we adopt a new approach so that we can prove that the Bishop–Phelps theorem in complete random normed modules still holds under the (ε, λ) -topology, which solves an open problem posed in [T.X. Guo, Y.J. Yang, Ekeland's variational principle for an L^0 -valued function on a complete random metric space, J. Math. Anal. Appl. 389 (2012) 1–14].

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1. Introduction

In 1961, Bishop and Phelps [1] proved that every Banach space is subreflexive, then in 1963 their paper [2] proved a more general theorem, which says that a nonempty closed convex subset C of a Banach space admits “many” support points and also “many” support functionals if C is additionally bounded (“many” means “norm dense in the appropriate set”). Recently, some efforts have been made in Zhao and Guo [3] and Guo and Yang [4] to extend this Bishop–Phelps theorem to the context of a complete random normed (briefly RN) module.

The notion of RN modules, which was first introduced in [5] and subsequently elaborated in [6], is a random generalization of that of ordinary normed spaces. When an RN module is endowed with (ε, λ) -topology, it is not a locally convex space in general so that the theory of classical conjugate spaces universally fails to serve the further development of the theory of RN modules. The theory of random conjugate spaces for RN modules has been developed to overcome this obstacle. Up to now, the theory of random conjugate spaces has played an essential role in both the development of the theory of RN modules and their applications to various topics, see [7] for details. Under the framework of random conjugate spaces, Zhao and Guo [3] proved the random subreflexivity of a complete RN module under the two kinds of topologies – the (ε, λ) -topology and the locally L^0 -convex topology (denoted by $\mathcal{T}_{\varepsilon, \lambda}$ and \mathcal{T}_C , respectively). Following [3], Guo and Yang [4] further established the Bishop–Phelps theorem in a complete RN module under the locally L^0 -convex topology as an application of their precise form of the Ekeland's variational principle on a complete RN module. However, it is a delicate problem whether the Bishop–Phelps theorem in a complete RN module still holds for the (ε, λ) -topology. For this problem, Guo and Yang [4] only proved the following conclusion: let $(E, \|\cdot\|)$ be a $\mathcal{T}_{\varepsilon, \lambda}$ -complete RN module and G a $\mathcal{T}_{\varepsilon, \lambda}$ -closed L^0 -convex subset in E , then the set of almost surely (briefly, a.s.) bounded random linear functionals supporting G is $\mathcal{T}_{\varepsilon, \lambda}$ -dense in E^* (namely, the random conjugate space of E) if G is also a.s. bounded. Therefore, they posed in [4, p. 10] a natural open problem: when G is a $\mathcal{T}_{\varepsilon, \lambda}$ -closed L^0 -convex subset, then: is the set of support points of G $\mathcal{T}_{\varepsilon, \lambda}$ -dense in the $\mathcal{T}_{\varepsilon, \lambda}$ -boundary of G ? The aim of this paper is to solve this problem.

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E-mail address: wumz@smss.buaa.edu.cn.

As pointed out in [3] and [4], the set of attainable or supporting a.s. bounded random linear functionals has the countable concatenation property, so using the fact, which was given in Guo [8], that a set with the countable concatenation property has the same closure under the two topologies, they can only need to prove either of the two kinds of denseness under the two topologies because they may convert one kind of denseness problem into the other. However, the set of support points does not necessarily have the countable concatenation property, it is this reason that Guo and Yang [4] cannot solve the above-stated problem. In this paper, we adopt a new approach so that we can solve this problem, meanwhile we also give a new proof of Guo and Yang's $\mathcal{T}_{\varepsilon,\lambda}$ -denseness theorem of a.s. bounded random linear functionals supporting a $\mathcal{T}_{\varepsilon,\lambda}$ -closed and a.s. bounded L^0 -convex subset.

The key idea of our approach is the precise connection between the random conjugate space E^* of an RN module E and the classical conjugate space $(L^p(E))'$ of the abstract normed space $L^p(E)$ generated from E , namely $(L^p(E))' \cong L^q(E^*)$ under the canonical embedding mapping ($1 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$), which unifies all the dual representation theorems of Lebesgue–Bochner function spaces (see [9]). This connection was established in Guo [10,9] and has become a powerful tool in the development of the theory of RN modules and their random conjugate spaces. For example, making use of this connection, Guo and Li [11] proved the James theorem in complete RN modules, Guo, Xiao and Chen [12] established a basic strict separation theorem in random locally convex modules and Zhang and Guo [13] established a mean ergodic theorem on random reflexive RN modules. Inspired by the very work in [11], in this paper we use this connection to build a bridge between the support points and a.s. bounded random linear support functionals of a $\mathcal{T}_{\varepsilon,\lambda}$ -closed L^0 -convex subset G in a $\mathcal{T}_{\varepsilon,\lambda}$ -complete RN module E and the ordinary support points and support functionals of a closed convex subset $L^1(G)$ in the Banach space $L^1(E)$, so that we can solve the above-stated problem by skillfully utilizing the classical Bishop–Phelps theorem.

The remainder of this paper is organized as follows: in Section 2 we briefly recall some necessary notions and facts and in Section 3 we present and prove our main results.

2. Preliminaries

Throughout this paper, (Ω, \mathcal{F}, P) denotes a given probability space, K the scalar field R of real numbers or C of complex numbers, N the set of positive integers and $L^0(\mathcal{F}, K)$ the algebra over K of equivalence classes of K -valued \mathcal{F} -measurable random variables on Ω under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes. Specifically, $L^0(\mathcal{F})$ briefly denotes $L^0(\mathcal{F}, R)$.

For any $A \in \mathcal{F}$, A^c denotes the complement of A , I_A the characteristic function of A , and \tilde{I}_A is used to denote the equivalence class of I_A ; for an arbitrary element ξ in $L^0(\mathcal{F}, K)$, $|\xi|$ stands for the equivalence class of $|\xi^0|$, where ξ^0 is an arbitrarily chosen representative of ξ .

$L^0(\mathcal{F})$ is partially ordered via $\xi \leq \eta$ iff $\xi^0(\omega) \leq \eta^0(\omega)$ P -a.s. (namely P -almost surly), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η in $L^0(\mathcal{F})$, respectively. It is well known from [14] that $L^0(\mathcal{F})$ is a complete lattice: every subset H with an upper (a lower) bound has a supremum (accordingly, an infimum), denoted by $\bigvee H$ (accordingly, $\bigwedge H$).

Let ξ and η be two elements in $L^0(\mathcal{F})$, then $\xi < \eta$ is understood as usual, namely $\xi \leq \eta$ and $\xi \neq \eta$. For $A \in \mathcal{F}$, $\xi > \eta$ on A means $\xi^0(\omega) > \eta^0(\omega)$ P -a.s. on A , where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η , respectively.

Specifically, $L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \geq 0\}$.

Definition 2.1. (See [6,8].) An ordered pair $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base (Ω, \mathcal{F}, P) if E is a left module over the algebra $L^0(\mathcal{F}, K)$ and $\|\cdot\|$ is a mapping from E to $L^0_+(\mathcal{F})$ such that the following three axioms are satisfied:

- (1) $\|x\| = 0$ if and only if $x = \theta$ (the null vector of E);
- (2) $\|\xi x\| = |\xi| \|x\|$, $\forall \xi \in L^0(\mathcal{F}, K)$ and $x \in E$;
- (3) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in E$.

Clearly, $(L^0(\mathcal{F}, K), |\cdot|)$ is an RN module over K with base (Ω, \mathcal{F}, P) .

In this paper, given an RN module E , θ always denotes its null element and E is always endowed with the (ε, λ) -topology, whose definition is given as follows.

Definition 2.2. (See [15,8,7].) Let $(E, \|\cdot\|)$ be an RN module over K with base (Ω, \mathcal{F}, P) . For any $\varepsilon > 0$, $0 < \lambda < 1$, denote $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$, then $\mathcal{U}_\theta = \{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ is a local base at θ of some Hausdorff linear topology, called the (ε, λ) -topology induced by $\|\cdot\|$.

Remark 2.3. The idea of introducing the (ε, λ) -topology for RN modules inherits from that of Schweizer and Sklar in 1961 for probabilistic metric spaces, see [16] for details. Since an RN module can be regarded as a special probabilistic normed (briefly, PN) space, we shall mention [17–19] for the studies related to the (ε, λ) -topology for a general PN space. For this paper, it suffices to notice that the (ε, λ) -topology for an RN module $(E, \|\cdot\|)$ is a metrizable linear topology and a sequence $\{x_n, n \in N\}$ in E converges in the (ε, λ) -topology to some $x \in E$ iff $\{\|x_n - x\|, n \in N\}$ converges in probability P

to 0. Specifically, the (ε, λ) -topology for $(L^0(\mathcal{F}, K), |\cdot|)$ is exactly the topology of convergence in probability. Actually, the (ε, λ) -topology for an RN module $(E, \|\cdot\|)$ is the same as the topology induced by the metric $d: E \times E \rightarrow [0, +\infty)$ defined by $d(x, y) = \int_{\Omega} \frac{\|x-y\|}{1+\|x-y\|} dP$, $\forall x, y \in E$. This metric will be used in the proofs of Lemma 3.4 and Theorem 3.1.

Definition 2.4. (See [6,8].) Let $(E, \|\cdot\|)$ be an RN module over K with base (Ω, \mathcal{F}, P) . A linear operator f from E to $L^0(\mathcal{F}, K)$ is called an a.s. bounded random linear functional on E if there exists some $\xi \in L^0_+(\mathcal{F})$ such that $f(x) \leq \xi \|x\|$, $\forall x \in E$. Let E^* be the linear space of a.s. bounded random linear functionals on E , further define the module multiplication $\cdot: L^0(\mathcal{F}, K) \times E^* \rightarrow L^0(\mathcal{F}, K)$ by $(\xi \cdot f)(x) = \xi(f(x))$ for all $\xi \in L^0(\mathcal{F}, K)$, $f \in E^*$ and $x \in E$, and the mapping $\|\cdot\|^*: E^* \rightarrow L^0_+(\mathcal{F})$ by $\|f\|^* = \bigwedge \{\xi \in L^0_+(\mathcal{F}) \mid f(x) \leq \xi \|x\|, \forall x \in E\}$ for all $f \in E^*$, then it is easy to see that $(E^*, \|\cdot\|^*)$ is an RN module over K with base (Ω, \mathcal{F}, P) , called the random conjugate space of E .

Let $(E, \|\cdot\|)$ be an RN module over K with base (Ω, \mathcal{F}, P) and $p \in [1, +\infty]$. Define $\|\cdot\|_p: E \rightarrow [0, +\infty]$ as follows for x in E :

$$\|x\|_p = \begin{cases} (\int_{\Omega} (\|x\|)^p dP)^{\frac{1}{p}}, & \text{when } 1 \leq p < +\infty; \\ \text{the } P\text{-essential supremum of } \|x\|, & p = +\infty. \end{cases}$$

Denote $\{x \in E \mid \|x\|_p < +\infty\}$ by $L^p(E)$, then $(L^p(E), \|\cdot\|_p)$ is a normed space over K , and a Banach space if E is complete. When $E = L^0(\mathcal{F}, K)$, it is easily seen that $L^p(E)$ is just the Banach space L^p of the equivalence classes of p -integrable (or essentially bounded if $p = +\infty$) random variables.

Given an RN module $(E, \|\cdot\|)$ and a fixed $p \in [1, +\infty]$, for any $x \in E$, let $\|x\|^0$ be an arbitrarily chosen representative of $\|x\|$, for each $n \in \mathbb{N}$, denote $A_n = \{\omega \mid \|x\|^0(\omega) \leq n\}$ and take $x_n = \tilde{I}_{A_n} x$, then $x_n \in L^p(E)$, and $\{x_n, n \in \mathbb{N}\}$ converges to x since $\{\|x_n - x\|, n \in \mathbb{N}\}$ converges to 0 in probability. Thus we have the following proposition:

Proposition 2.5. (See [10,11].) Let $(E, \|\cdot\|)$ be an arbitrary RN module and $1 \leq p \leq +\infty$, then $L^p(E)$ is dense in E .

In the same manner, for a subset $G \subset E$, denote $\{x \in G \mid \|x\|_p < +\infty\}$ by $L^p(G)$, it is easy to verify that if G is closed in E then $L^p(G)$ is $\|\cdot\|_p$ -closed in $L^p(E)$. For a general subset G of E , $L^p(G)$ may be empty, but when G contains θ and has the property that $\tilde{I}_{Ax} + \tilde{I}_{A^c} y \in G$ for all $x, y \in G$ and for all $A \in \mathcal{F}$, then the above argument used to prove Proposition 2.5 is also used to show that $L^p(G)$ is dense in G . We state this fact as a proposition.

Proposition 2.6. Let $(E, \|\cdot\|)$ be an RN module, G a subset of E such that G contains θ and has the property that $\tilde{I}_{Ax} + \tilde{I}_{A^c} y \in G$ for all $x, y \in G$ and for all $A \in \mathcal{F}$, then $L^p(G)$ is dense in G for each fixed $p \in [1, +\infty]$.

Proposition 2.7 below is of fundamental importance, which presents a precise connection between the random conjugate space E^* of an RN module E and the classical conjugate space $(L^p(E))^{'}$ of the abstract normed space $L^p(E)$ generated from E .

Proposition 2.7. (See [10,11,15].) Let $(E, \|\cdot\|)$ be an RN module over K with base (Ω, \mathcal{F}, P) , E^* its random conjugate space, $1 \leq p < +\infty$ and $1 < q \leq +\infty$ a pair of Hölder conjugate numbers. Then the canonical embedding $T: (L^q(E^*), \|\cdot\|_q) \rightarrow (L^p(E), \|\cdot\|_p)^{'}$ defined by $[Tf](x) = \int_{\Omega} f(x) dP$, $\forall f \in L^q(E^*)$ and $x \in L^p(E)$, is an isometric isomorphism, where $(L^p(E), \|\cdot\|_p)^{'}$ denotes the classical conjugate space of $(L^p(E), \|\cdot\|_p)$.

In the end of this section, let us recall the Bishop–Phelps theorem in Banach spaces. Suppose X is a real normed space and $C \subset X$, if there exist an $x \in C$ and a nonzero $f \in X^{'}$ such that $f(x) = \sup_{y \in C} f(y)$, where $X^{'}$ is the conjugate space of X , then x is called a support point of C and f a support functional of C .

Proposition 2.8. (See [2].) Let X be a real Banach space and C a closed convex subset of X , then the set of support points of C is dense in the boundary of C . If C is additionally bounded, then the set of support functionals of C is dense in $X^{'}$.

3. Main results and proofs

Let E be a left module over the algebra $L^0(\mathcal{F}, K)$, a nonempty subset M of E is said to be L^0 -convex if $\lambda x + (1 - \lambda)y \in M$, $\forall x, y \in M$ and $\forall \lambda \in L^0(\mathcal{F})$ such that $0 \leq \lambda \leq 1$.

Let E be an RN module over R with base (Ω, \mathcal{F}, P) and G a subset of E , if there exist an $x \in G$ and a nonzero $f \in E^*$ such that $f(x) \geq f(y)$, $\forall y \in G$, then x is called a support point of G and f an a.s. bounded random linear functional supporting G at x .

We can now state the main result of this paper.

Theorem 3.1. Let $(E, \|\cdot\|)$ be a complete RN module over R with base (Ω, \mathcal{F}, P) and G a closed and L^0 -convex subset of E . Then the set of support points of G is dense in the boundary of G (denoted by $\partial_{\varepsilon, \lambda} G$).

Theorem 3.1 solves the problem posed in Guo and Yang [4, p. 10]. Besides Theorem 3.1, using the same idea, in this paper we also give a new proof of Proposition 3.2 below. Noting that the proof in [4] is somewhat indirect since it is proved through its version in the locally L^0 -convex topology and by an argument of the countable concatenation property, whereas the proof in this paper is direct.

Proposition 3.2. (See [4].) Let $(E, \|\cdot\|)$ be a complete RN module over R with base (Ω, \mathcal{F}, P) and G a closed and L^0 -convex subset of E . If G is additionally a.s. bounded (namely, there exists a $\xi \in L^0_+(\mathcal{F})$ such that $\|x\| \leq \xi$ for all $x \in G$), then the set of a.s. bounded random linear functionals supporting G is dense in E^* .

To give our proofs of Proposition 3.2 and Theorem 3.1, we need some preparations.

In the sequel, according to the convention, for any $\xi \in L^1$, the notation $E[\xi]$ stands for $\int_{\Omega} \xi dP$. Besides, T specifically refers to the canonical embedding from $(L^\infty(E^*), \|\cdot\|_\infty)$ to $(L^1(E), \|\cdot\|_1)'$ defined by $[Tf](x) = E[f(x)]$ for all $f \in L^\infty(E^*)$ and $x \in L^1(E)$. By Proposition 2.7, T is an isometric isomorphism, and hence has an inverse T^{-1} .

Lemma 3.3. Let $(E, \|\cdot\|)$ be a complete RN module over R with base (Ω, \mathcal{F}, P) , G a closed and L^0 -convex subset of E and $\theta \in G$. If $x \in L^1(G)$ is an ordinary support point of $L^1(G)$ and $f \in (L^1(E), \|\cdot\|_1)'$ is an ordinary functional supporting $L^1(G)$ at x , then x is also a support point of G and $u = T^{-1}f \in L^\infty(E^*)$ an a.s. bounded random linear functional supporting G at x .

Proof. Since f is an ordinary functional supporting $L^1(G)$ at x , we have $f(x) = \sup\{f(y) \mid y \in L^1(G)\}$, which just means $E[u(x)] = \sup\{E[u(y)] \mid y \in L^1(G)\}$ according to the definition of T . We claim that $u(x) \geq u(y)$, $\forall y \in L^1(G)$. By way of contradiction, if there exist a $y \in L^1(G)$ and an $A \in \mathcal{F}$ with $P(A) > 0$ such that $u(y) > u(x)$ on A , then take $x' = \tilde{I}_A y + \tilde{I}_{A^c} x$, we have $x' \in L^1(G)$ and $u(x') > u(x)$ on A and $u(x') = u(x)$ on A^c , which is impossible since $E[u(x)] \geq E[u(x')]$. Finally, for any $y \in G$, by Proposition 2.6, there exists a sequence $\{y_n, n \in \mathbb{N}\}$ in $L^1(G)$ converging to y in the (ε, λ) -topology, therefore $\{u(y_n), n \in \mathbb{N}\}$ converges to $u(y)$ in probability. Since $u(y_n) \leq u(x)$ for all n , we have $u(y) \leq u(x)$. Thus $u(x) \geq u(y)$, $\forall y \in G$, which means that x is a support point of G and u an a.s. bounded random linear functional supporting G at x . \square

We can now give the proof of Proposition 3.2. We point out that the proof below indeed gives a little more than Proposition 3.2, for we actually prove that the set of functionals in $L^\infty(E^*)$ supporting G is dense in E^* .

Proof of Proposition 3.2. Since translation does not change the set of a.s. bounded random linear support functionals of a subset of E , without loss of generality we assume $\theta \in G$. Moreover, we can also assume that G is $\|\cdot\|_1$ -bounded, otherwise, let $\xi = \bigvee\{\|x\| \mid x \in G\}$, we can define a probability measure Q on (Ω, \mathcal{F}) by $\frac{dQ}{dP} = \frac{1}{c(1+\xi)}$, where $c = E[\frac{1}{1+\xi}]$. Then Q is equivalent to P and $\int_{\Omega} \|x\| dQ \leq \int_{\Omega} \xi dQ = E[\frac{\xi}{c(1+\xi)}] < +\infty$, $\forall x \in G$, which means that G is $\|\cdot\|_1$ -bounded under the probability measure Q . Noting that replacing the probability measure P of the base space (Ω, \mathcal{F}, P) with a probability measure Q which is equivalent to P changes neither the (ε, λ) -topologies of E and E^* nor the set of a.s. bounded random linear functionals supporting G , our assumption that G is $\|\cdot\|_1$ -bounded is therefore justified.

Making these assumptions, it is easily seen that $L^1(G) = G$ is a nonempty bounded convex closed subset of the Banach space $(L^1(E), \|\cdot\|_1)$, thus by Proposition 2.8, the set of bounded linear functionals supporting $L^1(G)$ is dense in $(L^1(E), \|\cdot\|_1)'$. According to Lemma 3.3, Proposition 2.8 and Proposition 2.7, we conclude that the set of all functionals in $L^\infty(E^*)$ supporting G is $\|\cdot\|_\infty$ -dense in the Banach space $(L^\infty(E^*), \|\cdot\|_\infty)$, and hence also dense in E^* under the (ε, λ) -topology by Proposition 2.5, which completes the proof. \square

To prove Theorem 3.1, we need another lemma.

In the following, given an RN module $(E, \|\cdot\|)$, d denotes the metric on E as defined in Remark 2.3. Besides, for a subset A of E , $\partial_{\varepsilon, \lambda} A$ and $\bar{A}_{\varepsilon, \lambda}$ denote the boundary and the closure of A under the (ε, λ) -topology, respectively, and given $p \in [1, +\infty]$ and a subset A in $L^p(E)$, $\partial_p A$ stands for the boundary of A under the $\|\cdot\|_p$ -topology.

Lemma 3.4. Let $(E, \|\cdot\|)$ be a complete RN module over R with base (Ω, \mathcal{F}, P) , G a closed and L^0 -convex subset of E and $\theta \in G$. Assume $\partial_{\varepsilon, \lambda} G \neq \emptyset$, then we have: $\partial_1(L^1(G)) \neq \emptyset$ and $\overline{\partial_1(L^1(G))}_{\varepsilon, \lambda} \supset \partial_{\varepsilon, \lambda} G$.

Proof. Under the assumptions, $L^1(G)$ is a nonempty closed convex subset of the Banach space $(L^1(E), \|\cdot\|_1)$. We assume, by way of contradiction, $\partial_1(L^1(G)) = \emptyset$, then $L^1(G)$ is open in $L^1(E)$. Since $L^1(E)$ is connected, we have $L^1(G) = L^1(E)$, therefore $G = \overline{L^1(G)}_{\varepsilon, \lambda} = E$, which contradicts to $\partial_{\varepsilon, \lambda} G \neq \emptyset$.

We then show that $\overline{\partial_1(L^1(G))}_{\varepsilon, \lambda} \supset \partial_{\varepsilon, \lambda} G$. For a point p in E and a nonempty subset A of E , let $d(p, A) = \inf\{d(p, q) \mid q \in A\}$, then according to Remark 2.3, we have $p \in \bar{A}_{\varepsilon, \lambda}$ if $d(p, A) = 0$. Thus we only need to show that $d(x, \partial_1(L^1(G))) = 0$,

$\forall x \in \partial_{\varepsilon, \lambda} G$. By Proposition 2.6, $L^1(G)$ is dense in G under the (ε, λ) -topology, which implies that $d(x, L^1(G)) = 0$, $\forall x \in \partial_{\varepsilon, \lambda} G$. The desired equality follows the next lemma. \square

Lemma 3.5. *Let E and G be the same as in the above lemma and $\partial_{\varepsilon, \lambda} G \neq \emptyset$, then we have:*

$$d(x, \partial_1(L^1(G))) = d(x, L^1(G)), \quad \forall x \in \partial_{\varepsilon, \lambda} G.$$

Proof. If the $\|\cdot\|_1$ -interior of $L^1(G)$ (denoted by $\text{int}(L^1(G))$) is empty, the conclusion is obvious. If $\text{int}(L^1(G)) \neq \emptyset$, for any given $x \in \partial_{\varepsilon, \lambda} G$ and $y \in \text{int}(L^1(G))$, we will show $d(x, \partial_1(L^1(G))) \leq d(x, y)$, which implies our conclusion. To this end, since $x \in \partial_{\varepsilon, \lambda} G$, we can find a sequence $\{x'_n, n \in \mathbb{N}\}$ in $E \setminus G$ converging to x , namely $\lim_{n \rightarrow \infty} d(x'_n, x) = 0$. It follows from the closedness of G that $d(x'_n, G) > 0$, $\forall n \in \mathbb{N}$. Noting that $L^1(E)$ is dense in E under the (ε, λ) -topology by Proposition 2.5, for all n we can find $x_n \in L^1(E)$ such that $d(x_n, x'_n) < \frac{1}{n} d(x'_n, G) \leq \frac{1}{n}$. Thus $\lim_{n \rightarrow \infty} d(x, x_n) = 0$, furthermore, every x_n is in $L^1(E) \setminus L^1(G)$ since it is not in G . For each n , consider the set $A_n = \{t \in [0, 1] \mid tx_n + (1-t)y \in L^1(G)\}$, let $\lambda_n = \sup A_n$ and take $y_n = \lambda_n x_n + (1 - \lambda_n)y$, then $y_n \in \partial_1(L^1(G))$ and $d(x_n, y_n) = E[\frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|}] = E[\frac{(1 - \lambda_n)\|x_n - y\|}{1 + (1 - \lambda_n)\|x_n - y\|}] \leq E[\frac{\|x_n - y\|}{1 + \|x_n - y\|}] = d(x_n, y)$. Thus $d(x_n, \partial_1(L^1(G))) \leq d(x_n, y_n) \leq d(x_n, y)$, then letting n tend to ∞ , we obtain $d(x, \partial_1(L^1(G))) \leq d(x, y)$, which completes the proof. \square

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. We suppose $\partial_{\varepsilon, \lambda} G \neq \emptyset$, and, without loss of generality, assume $\theta \in G$. To prove the theorem, for any given $x \in \partial_{\varepsilon, \lambda} G$ and for every $\epsilon \in (0, 1)$, we need to find a support point x_ϵ of G such that $d(x, x_\epsilon) < \epsilon$. To this end, according to Lemma 3.4, we can find x_1 in $\partial_1(L^1(G))$ such that $d(x, x_1) < \frac{\epsilon}{2}$. Under the assumptions, $L^1(G)$ is a nonempty closed convex subset of the Banach space $(L^1(E), \|\cdot\|_1)$, thus by Proposition 2.8, the set of ordinary support points of $L^1(G)$ is $\|\cdot\|_1$ -dense in $\partial_1(L^1(G))$, hence we can find an ordinary support points x_2 of $L^1(G)$ such that $\|x_2 - x_1\|_1 < \frac{\epsilon}{2}$. We have that x_2 is also a support point of G according to Lemma 3.3, moreover $d(x, x_2) \leq d(x, x_1) + d(x_1, x_2) \leq d(x, x_1) + \|x_1 - x_2\|_1 < \epsilon$. \square

Remark 3.6. Our proofs of Proposition 3.2 and Theorem 3.1 have taken advantage of two properties of the (ε, λ) -topology for an RN module E :

- (1) $L^p(E)$ is dense in E under the (ε, λ) -topology;
- (2) the $\|\cdot\|_p$ -topology for a subset A in $L^p(E)$ is stronger than the (ε, λ) -topology for A as a subset of E .

Since the locally L^0 -convex topology is so strong that generally neither (1) nor (2) holds under it, our approach cannot apply directly to the corresponding cases under the locally L^0 -convex topology, and in particular our approach is not used to prove Theorem 4.2 in [4], either. Thus in the very sense, Theorem 4.2 in [4] and Theorem 3.1 of this paper are independent.

References

- [1] E. Bishop, R.R. Phelps, A proof that every Banach space is subreflexive, *Bull. Amer. Math. Soc.* 67 (1961) 97–98.
- [2] E. Bishop, R.R. Phelps, The support functionals of a convex set, in: V. Klee (Ed.), *Convexity*, in: *Proc. Sympos. Pure Math.*, vol. VII, Amer. Math. Soc., 1963, pp. 27–35.
- [3] S.E. Zhao, T.X. Guo, The random subreflexivity of complete random normed modules, *Internat. J. Math.*, doi:10.1142/S0129167X12500474, in press.
- [4] T.X. Guo, Y.J. Yang, Ekeland's variational principle for an \tilde{L}^0 -valued function on a complete random metric space, *J. Math. Anal. Appl.* 389 (2012) 1–14.
- [5] T.X. Guo, Extension theorems of continuous random linear operators on random domains, *J. Math. Anal. Appl.* 193 (1) (1995) 15–27.
- [6] T.X. Guo, Some basic theories of random normed linear spaces and random inner product spaces, *Acta Anal. Funct. Appl.* 1 (2) (1999) 160–184.
- [7] T.X. Guo, Recent progress in random metric theory and its applications to conditional risk measures, *Sci. China Ser. A* 54 (2011) 633–660.
- [8] T.X. Guo, Relations between some basic results derived from two kinds of topologies for a random locally convex module, *J. Funct. Anal.* 258 (2010) 3024–3047.
- [9] T.X. Guo, Representation theorems of the dual of Lebesgue–Bochner function spaces, *Sci. China Ser. A* 43 (2000) 234–243.
- [10] T.X. Guo, A characterization for a complete random normed module to be random reflexive, *J. Xiamen Univ. Natur. Sci.* 36 (1997) 499–502 (in Chinese).
- [11] T.X. Guo, S.B. Li, The James theorem in complete random normed modules, *J. Math. Anal. Appl.* 308 (2005) 257–265.
- [12] T.X. Guo, H.X. Xiao, X.X. Chen, A basic strict separation theorem in random locally convex modules, *Nonlinear Anal.* 71 (2009) 3794–3804.
- [13] X. Zhang, T.X. Guo, The mean ergodic theorem on random reflexive random normed modules, *Adv. Math. Sinica* 41 (2012) 21–30.
- [14] N. Dunford, J.T. Schwartz, *Linear Operators (I)*, Interscience, New York, 1957.
- [15] T.X. Guo, Survey of recent developments of random metric theory and its applications in China (II), *Acta Anal. Funct. Appl.* 3 (2001) 208–230.
- [16] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier, New York, 1983, Dover Publications, New York, 2005.
- [17] B. Lafuerza-Guillén, J.A. Rodríguez-Lallena, C. Sempí, A study of boundedness in probabilistic normed spaces, *J. Math. Anal. Appl.* 232 (1999) 183–196.
- [18] B. Lafuerza-Guillén, C. Sempí, Probabilistic norms and convergence of random variables, *J. Math. Anal. Appl.* 280 (2003) 9–16.
- [19] C. Sempí, A short and partial history of probabilistic normed spaces, *Mediterr. J. Math.* 3 (2006) 283–300.