J. Math. Anal. Appl. 398 (2013) 857–866

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Multiple solutions of sublinear elliptic equations with small perturbations

Ryuji Kajikiya

Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga, 840-8502, Japan

ARTICLE INFO

Article history:
Received 30 August 2011
Available online 29 September 2012
Submitted by Thomas P. Witelski

Keywords:
Semilinear elliptic equation
Sublinear equation
Variational method
Symmetric mountain pass lemma

ABSTRACT

We study the sublinear elliptic equation having two nonlinear terms, where the main term $f(x, u)$ is sublinear and odd with respect to u and the perturbation term is any continuous function with a small coefficient. Then we prove the existence of multiple small solutions.

1. Introduction and main results

In this paper we prove the existence of multiple solutions for the sublinear elliptic equation

$$\begin{cases} -\Delta u = f(x, u) + \varepsilon g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and ε is a small parameter. We study the problem (1.1) under the condition that $f(x, u)$ is odd on u, sublinear near $u = 0$ and $g(x, u)$ is any continuous function. Then we shall show that if $|\varepsilon|$ is small enough, (1.1) has many small solutions. We impose the next assumption.

Assumption (A). Let $f(x, u)$ and $g(x, u)$ be Hölder continuous functions defined on $\overline{\Omega} \times [-a, a]$ with some $a > 0$ and satisfy the conditions below.

(A1) $f(x, -u) = -f(x, u)$ for $x \in \overline{\Omega}$ and $|u| \le a$.
(A2) $uf(x, u) - 2F(x, u) < 0$ when $0 < |u| < a$ and $x \in \overline{\Omega}$. Here $F(x, u)$ is defined by

$$F(x, u) := \int_0^u f(x, s)ds.$$

(A3) $\lim_{u\to 0} \left(\min_{x\in\overline{\Omega}} u^{-2}F(x, u)\right) = \infty$.

Theorem 1.1. *Suppose that Assumption (A) holds. Then for any $k \in \mathbb{N}$ and any $\delta > 0$, there exists an $\varepsilon(k, \delta) > 0$ such that if $|\varepsilon| \le \varepsilon(k, \delta)$, then (1.1) has at least k distinct solutions whose $C^2(\overline{\Omega})$-norms are less than δ. When $\varepsilon = 0$, (1.1) has a sequence of solutions whose $C^2(\overline{\Omega})$-norm converges to zero.*

E-mail address: kajikiya@ms.saga-u.ac.jp.

0022-247X/\$ – see front matter
doi:10.1016/j.jmaa.2012.09.045

Example 1.2. We give some examples of $f(x, u)$ which satisfy Assumption (A). In the following, we suppose that $\alpha(x)$ and $\beta(x)$ are Hölder continuous and $\alpha(x) > 0$ on $\overline{\Omega}$.

(i) $f(x, u) = \alpha(x)|u|^p \operatorname{sgn} u$ with $0 < p < 1$.
(ii) $f(x, u) = -\alpha(x)u \log |u|$.
(iii) $f(x, u) = \alpha(x)|u|^p \operatorname{sgn} u + \beta(x)|u|^q \operatorname{sgn} u$ with $0 < p < \min(1, q)$.

In Case (iii), $\beta(x)$ may change its sign. Indeed, we have

$$uf(x, u) - 2F(x, u) = -\frac{1-p}{1+p}\alpha(x)|u|^{p+1} - \frac{1-q}{1+q}\beta(x)|u|^{q+1} < 0,$$

provided that $|u| > 0$ is small enough. For these nonlinear terms, (1.1) has sufficiently many small solutions if $|\varepsilon|$ is small enough.

For the sublinear elliptic problem with $\varepsilon = 0$, i.e., $f(u)$ is like $|u|^p \operatorname{sgn} u$ with $0 < p < 1$, we refer the readers to [1,2,5]. Ambrosetti–Badiale [1] has proved the existence of infinitely many solutions if $f(x, u)$ is sublinear with $\varepsilon = 0$. Ambrosetti et al. [2] has investigated $f(u) = \lambda|u|^q \operatorname{sgn} u + |u|^p \operatorname{sgn} u$ with $0 < q < 1 < p \le (n+2)/(n-2)$. Then they have obtained the detailed and important results on the structure of positive solutions, the existence of two positive solutions and the existence of infinitely many solutions. Under more general and weak assumptions on $f(x, u)$, we have proved in [5] that (1.1) has a sequence of solutions whose $C^2(\overline{\Omega})$-norm converges to zero.

On the other hand, we have considered the sublinear perturbation problem in our paper [6] under the condition that $f(u) = |u|^p \operatorname{sgn} u$ with $0 < p < 1$, $\varepsilon = 1$, $g(x, 0) = 0$, $g(x, u)$ is not odd on u and $g(x, u)$ converges rapidly to zero as $u \to 0$. Then we have obtained a sequence of solutions whose $C^2(\overline{\Omega})$-norm converges to zero.

Degiovanni and Rădulescu [3] have proved the existence of multiple solutions $(u, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ of the problem

$$-\Delta u = \lambda(f(x, u) + g(x, u)) \quad \text{in } \mathcal{D}'(\Omega),$$
$$\int_\Omega |\nabla u|^2 dx = r^2, \quad f(x, u),\ g(x, u) \in L^1_{\text{loc}}(\Omega),$$

under the assumptions that $sf(x, s) > 0$, $sg(x, s) > 0$ for $s \ne 0$ and

$$\sup_{|s|\le t} f(x, s), \quad \sup_{|s|\le t} g(x, s) \in L^1_{\text{loc}}(\Omega)$$

for every $t > 0$ and $g(x, s)$ has at most a polynomial growth as $s \to \infty$.

In the present paper, we do not assume the condition that $sf(x, s) > 0$ or $sg(x, s) > 0$ for $s \ne 0$ (see Example 1.2(ii), (iii)) and do not need any growth condition on $g(x, s)$ as $s \to \infty$. We assume the Hölder continuity of f and g for the $C^2(\overline{\Omega})$ regularity of solutions. Even if we do not assume this condition, Theorem 1.1 is still valid after replacing the $C^2(\overline{\Omega})$ norm by the $C^1(\overline{\Omega})$ norm. We emphasize that the nonlinear term $f(x, u)$ in our paper is more general than those of the papers above and our theorem does not need any growth or sign condition on $g(x, u)$. To prove Theorem 1.1, we develop a new variational method based on the symmetric mountain pass lemma under the next assumption.

Assumption (B). Let E be an infinite dimensional Banach space and $I \in C([0, 1]\times E, \mathbb{R})$. Suppose that $I(t, u)$ has a continuous partial derivative I_u and satisfies (B1)–(B5) below.

(B1) $\inf\{I(t, u) : t \in [0, 1],\ u \in E\} > -\infty$.
(B2) There exists a function $\psi \in C([0, 1], \mathbb{R})$ such that $\psi(0) = 0$ and

$$|I(t, u) - I(0, u)| \le \psi(t) \quad \text{for } (t, u) \in [0, 1] \times E.$$

(B3) $I(t, u)$ satisfies the Palais–Smale condition uniformly on t, i.e. if a sequence (t_k, u_k) in $[0, 1] \times E$ satisfies that $\sup_k |I(t_k, u_k)| < \infty$ and $I_u(t_k, u_k)$ converges to zero, then (t_k, u_k) has a convergent subsequence.
(B4) $I(0, u) = I(0, -u)$ for $u \in E$ and $I(0, 0) = 0$.
(B5) For any $u \in E \setminus \{0\}$ there exists a unique $s(u) > 0$ such that $I(0, tu) < 0$ if $0 < |t| < s(u)$ and $I(0, tu) \ge 0$ if $|t| \ge s(u)$.

We define a critical value a_k of $I(0, u)$ in the following definition.

Definition 1.3. We set

$$S^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\},$$
$$\mathcal{A}_k := \{h \in C(S^k, E) : h \text{ is odd}\},$$
$$a_k := \inf_{h\in\mathcal{A}_k} \max_{x\in S^k} I(0, h(x)). \tag{1.2}$$

In Section 2, we shall prove that $a_k \le a_{k+1} < 0$ for $k \in \mathbb{N}$ and $\{a_k\}$ converges to zero. Hence there exist infinitely many k's satisfying $a_k < a_{k+1}$, and so the next theorem makes sense.

Theorem 1.4. *Suppose that Assumption (B) holds. Let k be a positive integer satisfying $a_k < a_{k+1}$. Then there exist constants t_{k+1}, c_{k+1} such that $0 < t_{k+1} \le 1$, $a_{k+1} \le c_{k+1} < -\psi(t)$ for $t \in [0, t_{k+1}]$ and for each $t \in [0, t_{k+1}]$, $I(t, \cdot)$ has a critical value in the interval $[a_{k+1} - \psi(t), c_{k+1} + \psi(t)]$.*

Note that $c_{k+1} + \psi(t) < 0$, and hence the critical value in Theorem 1.4 is not zero. We organize this paper into four sections. In Section 2, we prove Theorem 1.4 by using Lemmas 2.3 and 2.7, which will be stated in Section 2. The proofs of these lemmas will be given in Section 3. In Section 4, we prove Theorem 1.1 by applying Theorem 1.4.

2. Proof of Theorem 1.4

The purpose of this section is to prove Theorem 1.4. To this end, we need the deformation lemma and a notion of genus.

Definition 2.1. Let E be a Banach space and $J \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$ and $\delta > 0$, we define

$$K_c := \{u \in E : J'(u) = 0,\ J(u) = c\},$$
$$N_\delta(K_c) := \{u \in E : \operatorname{dist}(u, K_c) \le \delta\},$$
$$\operatorname{dist}(u, K_c) := \inf\{\|u - v\| : v \in K_c\}.$$

If c is a regular value of $J(u)$, then K_c and $N_\delta(K_c)$ are empty.

Lemma 2.2 (*Deformation Lemma*)**.** *Let $J \in C^1(E, \mathbb{R})$ satisfy the Palais–Smale condition. If $c \in \mathbb{R}$, $\varepsilon_0 > 0$ and $\delta > 0$, then there exist an $\varepsilon \in (0, \varepsilon_0)$ and $\eta \in C(E, E)$ satisfying the conditions below.*

(i) *η is homeomorphic on E.*
(ii) *If $J(u) \le c - \varepsilon_0$, then $\eta(u) = u$.*
(iii) *If $J(u) \le c + \varepsilon$ and $u \notin N_\delta(K_c)$, then $J(\eta(u)) \le c - \varepsilon$.*
(iv) *If c is a regular value of J and if $J(u) \le c + \varepsilon$, then $J(\eta(u)) \le c - \varepsilon$.*
(v) *If J is even, then $\eta(\cdot)$ is odd.*

For the proof of Lemma 2.2, we refer the readers to [7, p. 82, Theorem A.4] or [8, p. 83, Theorem 3.4]. Throughout this section, we impose Assumptions (B1)–(B5).

Lemma 2.3. *For any $k \in \mathbb{N}$ there exists a $g_k \in \mathcal{A}_k$ such that*

$$\max_{S^k} I(0, g_k(x)) < 0.$$

This lemma will be proved in Section 3. To prove that $\{a_k\}$ defined by (1.2) converges to zero, we use a notion of genus.

Definition 2.4. Let E be an infinite dimensional Banach space and A a subset of E. A is said to be *symmetric* if $x \in A$ implies $-x \in A$. For a closed symmetric set A which does not contain the origin, we define a *genus* $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let $\mathcal{B}_k$ denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \ge k$. We define

$$b_k := \inf_{A \in \mathcal{B}_k} \sup_{u \in A} I(0, u).$$

Since $I(t, u)$ is bounded from below, it holds that $-\infty < b_k < \infty$. It follows from the Borsuk–Ulam theorem that $\gamma(S^k) = k + 1$. If $h \in \mathcal{A}_k$ and $0 \notin h(S^k)$, then

$$\gamma(h(S^k)) \ge \gamma(S^k) = k + 1,$$

hence $h(S^k) \in \mathcal{B}_{k+1}$. If $0 \in h(S^k)$, then $\max_{S^k} I(0, h(x)) \ge 0$. Since $a_k < 0$ by Lemma 2.3, we can assume that $0 \notin h(S^k)$ in (1.2) without loss of generality. Consequently, we have

$$b_{k+1} \le a_k < 0 \quad \text{for } k \in \mathbb{N}. \tag{2.1}$$

Lemma 2.5. *Each a_k is a critical value of $I(0, u)$ and satisfies*

$$a_k \le a_{k+1} < 0 \quad \textit{for } k \in \mathbb{N}, \qquad \lim_{k \to \infty} a_k = 0.$$

Proof. Suppose on the contrary that a_k is a regular value. We take an $\varepsilon > 0$ and an odd mapping η by Lemma 2.2. By the definition of a_k, there exists an $h \in \mathcal{A}_k$ such that the supremum of $I(0, h(x))$ on S^k is less than $a_k + \varepsilon$. Then the composite function $\eta \circ h$ belongs to $\mathcal{A}_k$. Moreover, Lemma 2.2(iv) implies

$$\sup_{S^k} I(0, \eta \circ h(x)) \le a_k - \varepsilon,$$

which contradicts the definition of a_k. Therefore a_k is a critical value.

We have already proved that $a_k < 0$ in (2.1). To show the convergence of $\{a_k\}$ to zero, it is enough to prove the convergence of $\{b_k\}$ to zero. We use the same method as in Rabinowitz's argument [7, Proposition 9.33]. Since $b_k \le b_{k+1}$ for $k \in \mathbb{N}$ by definition, it has a finite limit b_∞ (≤ 0). Suppose to the contrary that $b_\infty < 0$. We set

$$M := \{u \in E : I_u(0, u) = 0,\ I(0, u) \le b_\infty\}.$$

Note that $0 \notin M$ because $b_\infty < 0$. Since $I(0, u)$ is bounded from below, the Palais–Smale condition means that M is compact. Hence M has a finite genus, which is denoted by $m := \gamma(M) < \infty$. For $\delta > 0$, we set

$$N_\delta(M) := \{u \in E : \mathrm{dist}(u, M) \le \delta\},$$
$$\mathrm{dist}(u, M) = \inf\{\|u - v\| : v \in M\}.$$

It is known (see [7, p. 46, Proposition 7.5]) that if $\delta > 0$ is small enough, then

$$\gamma(N_\delta(M)) = \gamma(M) = m.$$

By Lemma 2.2 with $c = b_\infty$, there exist an $\varepsilon > 0$ and an odd mapping $\eta \in C(E, E)$ such that

$$I(0, \eta(u)) \le b_\infty - \varepsilon \quad \text{if } I(0, u) \le b_\infty + \varepsilon,\ u \notin N_\delta(M). \tag{2.2}$$

Fix an integer $n \in \mathbb{N}$ such that

$$b_\infty - \varepsilon < b_n. \tag{2.3}$$

We choose $A \in \mathcal{B}_{m+n}$ such that

$$\sup_{u \in A} I(0, u) < b_{m+n} + \varepsilon \le b_\infty + \varepsilon. \tag{2.4}$$

Set $B = \overline{A \setminus N_\delta(M)}$. Then (2.2) with (2.4) gives

$$I(0, \eta(u)) \le b_\infty - \varepsilon \quad \text{for } u \in B. \tag{2.5}$$

Since $\gamma(B) \ge \gamma(A) - \gamma(N_\delta(M)) \ge n$, it follows that $B \in \mathcal{B}_n$. Since η is odd and homeomorphic, $\eta(B)$ belongs to $\mathcal{B}_n$. By the definition of b_n with (2.3) and (2.5), we have

$$b_\infty - \varepsilon < b_n \le \sup_{u \in \eta(B)} I(0, u) \le b_\infty - \varepsilon.$$

A contradiction occurs. Thus we conclude that $b_\infty = 0$ and the proof is complete. □

Definition 2.6. Hereafter we fix the integer $k \in \mathbb{N}$ such that $a_k < a_{k+1}$. Determine the constant $r > 0$ so small that

$$a_k + r < a_{k+1} < 0. \tag{2.6}$$

Define

$$S_+^{k+1} := \left\{(x_1, \dots, x_{k+2}) : \sum_{i=1}^{k+2} x_i^2 = 1,\ x_{k+2} \ge 0\right\},$$
$$S^k := \left\{(x_1, \dots, x_{k+2}) : \sum_{i=1}^{k+2} x_i^2 = 1,\ x_{k+2} = 0\right\},$$
$$\mathcal{C}_{k+1} := \{h \in C(S_+^{k+1}, E) : h \text{ satisfies (H1), (H2)}\}.$$

(H1) $h(-x) = -h(x)$ for $x \in S^k$.
(H2) $I(0, h(x)) < a_k + r$ for $x \in S^k$.

By definition of a_k, there exists an $h \in \mathcal{A}_k$ satisfying (H1) and (H2). We extend h onto S_+^{k+1} as a continuous function, which belongs to $\mathcal{C}_{k+1}$. Hence $\mathcal{C}_{k+1}$ is nonempty. Moreover, we have the next lemma.

Lemma 2.7. *There exists an $f_{k+1} \in \mathcal{A}_{k+1} \cap \mathcal{C}_{k+1}$ such that*

$$\max_{x \in S^{k+1}} I(0, f_{k+1}(x)) < 0. \tag{2.7}$$

Lemma 2.7 will be proved in Section 3. Using this lemma, we prove Theorem 1.4.

Proof of Theorem 1.4. Condition (B2) means

$$I(0,u)-\psi(t)\le I(t,u)\le I(0,u)+\psi(t)\quad \text{for } (t,u)\in[0,1]\times E. \tag{2.8}$$

Let f_{k+1} be as in Lemma 2.7. We define c_{k+1} by

$$c_{k+1}:=\max_{S_+^{k+1}} I(0,f_{k+1}(x))=\max_{S^{k+1}} I(0,f_{k+1}(x))<0. \tag{2.9}$$

By definition, it follows that $a_{k+1}\le c_{k+1}$. We choose $t_{k+1}\in(0,1]$ so small that

$$a_k+r+2\psi(t)<a_{k+1},\qquad c_{k+1}+\psi(t)<0\quad \text{for } t\in[0,t_{k+1}]. \tag{2.10}$$

Fix $t\in[0,t_{k+1}]$ arbitrarily and define

$$d_{k+1}(t):=\inf_{h\in\mathcal{C}_{k+1}}\max_{x\in S_+^{k+1}} I(t,h(x)). \tag{2.11}$$

Combining (2.11) with (2.8), we have

$$d_{k+1}(t)\le\max_{S_+^{k+1}} I(0,f_{k+1}(x))+\psi(t)=c_{k+1}+\psi(t)<0. \tag{2.12}$$

Give $h\in\mathcal{C}_{k+1}$ arbitrarily. Denote the odd extension of h on S^{k+1} by $\bar{h}$, i.e., $\bar{h}(x)=h(x)$ for $x\in S_+^{k+1}$ and $\bar{h}(x)=-h(-x)$ for $x\in S_-^{k+1}$, where $S_-^{k+1}:=-S_+^{k+1}$. Then $\bar{h}\in\mathcal{A}_{k+1}$. Since $I(0,u)$ is even, it holds that

$$\max_{S_+^{k+1}} I(0,h(x))=\max_{S^{k+1}} I(0,\bar{h}(x)). \tag{2.13}$$

By (2.8) and (2.13), we have

$$\begin{aligned}\max_{S_+^{k+1}} I(t,h(x)) &\ge \max_{S_+^{k+1}} I(0,h(x))-\psi(t)\\ &=\max_{S^{k+1}} I(0,\bar{h}(x))-\psi(t)\\ &\ge a_{k+1}-\psi(t).\end{aligned} \tag{2.14}$$

The last inequality follows directly from the definition of a_{k+1}. Taking the infimum on $h\in\mathcal{C}_{k+1}$ in (2.14) and using (2.10), we obtain

$$d_{k+1}(t)\ge a_{k+1}-\psi(t)>a_k+r+\psi(t). \tag{2.15}$$

We shall show that $d_{k+1}(t)$ is a critical value of $I(t,\cdot)$. Suppose on the contrary that $d_{k+1}(t)$ is a regular value. Observing (2.15), we use Lemma 2.2 with $c=d_{k+1}(t)$ and $c-\varepsilon_0=a_k+r+\psi(t)$. Then we have an $\varepsilon>0$ and $\eta\in C(E,E)$ satisfying the conditions below.

(E1) If $I(t,u)\le d_{k+1}(t)+\varepsilon$, then $I(t,\eta(u))\le d_{k+1}(t)-\varepsilon$.
(E2) If $I(t,u)\le a_k+r+\psi(t)$, then $\eta(u)=u$.

By the definition (2.11) of $d_{k+1}(t)$, there exists an $h_0\in\mathcal{C}_{k+1}$ such that

$$\max_{S_+^{k+1}} I(t,h_0(x))<d_{k+1}(t)+\varepsilon. \tag{2.16}$$

Combining (E1) with (2.16), we have

$$\max_{S_+^{k+1}} I(t,\eta(h_0(x)))\le d_{k+1}(t)-\varepsilon. \tag{2.17}$$

We define $h_1(x)$ by

$$h_1(x):=\eta(h_0(x))\quad \text{for } x\in S_+^{k+1}. \tag{2.18}$$

Then (2.17) is rewritten as

$$I(t,h_1(x))\le d_{k+1}(t)-\varepsilon\quad \text{for } x\in S_+^{k+1}. \tag{2.19}$$

Since $h_0\in\mathcal{C}_{k+1}$, (H2) means

$$I(t,h_0(x))\le I(0,h_0(x))+\psi(t)<a_k+r+\psi(t)\quad \text{for } x\in S^k.$$

The inequality above with (E2) implies that $\eta(h_0(x)) = h_0(x)$ for $x \in S^k$, i.e., $h_1(x) = h_0(x)$ on S^k. Since $h_0(x)$ satisfies (H1) and (H2), so does $h_1(x)$. Thus $h_1 \in \mathcal{C}_{k+1}$. However, (2.19) contradicts the definition of $d_{k+1}(t)$. Consequently, $d_{k+1}(t)$ is a critical value. Moreover, (2.12) and (2.15) are combined into the inequality,

$$a_{k+1} - \psi(t) \le d_{k+1}(t) \le c_{k+1} + \psi(t) < 0.$$

This completes the proof. □

3. Proofs of Lemmas 2.3 and 2.7

In this section, we shall prove Lemmas 2.3 and 2.7. To this end, we use the same idea as in our paper [6]. Throughout this section, we always suppose that E is an infinite dimensional Banach space and (B4), (B5) hold.

Lemma 3.1. *Let K be a compact subset of E such that $0 \notin K$. We put*

$$\mathbb{R}K := \{\lambda u : \lambda \in \mathbb{R},\ u \in K\}, \qquad S^\infty := \{u \in E : \|u\| = 1\}.$$

Then $S^\infty \setminus \mathbb{R}K \neq \emptyset$.

Proof. To the contrary, suppose that $S^\infty \subset \mathbb{R}K$. Since S^∞ is non-compact, we choose a sequence $\{u_n\}$ in S^∞ whose any subsequence does not converge. Since $u_n \in S^\infty \subset \mathbb{R}K$, it is represented as $u_n = \lambda_n y_n$ with $\lambda_n \in \mathbb{R}$ and $y_n \in K$. Choose a subsequence of $\{y_n\}$ which converges to a limit $y_\infty \in K$. Note that $y_\infty \neq 0$ because $0 \notin K$. Since $\|u_n\| = 1$, $\{\lambda_n\}$ is bounded and has a convergent subsequence. Consequently, $\{u_n\}$ has a convergent subsequence. This is a contradiction. The proof is complete. □

Observing Assumption (B5), we define

$$U := \{u \in E : I(0, u) < 0\} = \{tu : u \in E \setminus \{0\},\ 0 < |t| < s(u)\}.$$

Lemma 3.2. *Let K be a compact subset of E such that $0 \notin K$. For $\delta > 0$, we set*

$$\delta K := \{\delta u : u \in K\}.$$

Then there exists a $\delta_0 > 0$ such that $\delta K \subset U$ for $0 < \delta \le \delta_0$.

Proof. Suppose that the assertion is false. Then there exist sequences $\delta_n > 0$ and $u_n \in K$ such that $\{\delta_n\}$ converges to zero and $\delta_n u_n \notin U$ for $n \in \mathbb{N}$. Choose a subsequence of $\{u_n\}$ which converges to a limit $u_\infty \in K$. Fix $t > 0$ arbitrarily. Since $t > \delta_n > 0$ for n large enough and $\delta_n u_n \notin U$, the definition of U implies that $tu_n \notin U$. Since U is open, it follows that $tu_\infty \notin U$ for any $t > 0$. This contradicts (B5). The proof is complete. □

We show Lemma 2.3 by using the lemma above.

Proof of Lemma 2.3. Fix $k \in \mathbb{N}$. Let $e_1, \dots, e_{k+1}$ be linearly independent in E and set

$$K = \left\{ \sum_{i=1}^{k+1} t_i e_i : \sum_{i=1}^{k+1} t_i^2 = 1 \right\}.$$

Then Lemma 3.2 gives a $\delta_0 > 0$ such that $\delta K \subset U$ for $0 < \delta \le \delta_0$. In other words, we have

$$I(0, tu) < 0 \quad \text{if } u \in K \text{ and } 0 < |t| \le \delta_0. \tag{3.1}$$

We define

$$g_k(x) = \delta_0 \sum_{i=1}^{k+1} x_i e_i \quad \text{for } x = (x_1, \dots, x_{k+1}) \in S^k.$$

This satisfies Lemma 2.3. □

To prove Lemma 2.7, we need the next lemma.

Lemma 3.3. *Let K be a compact subset of E such that $K \subset U$, where U has been defined before Lemma 3.2. Let v_0 be determined by Lemma 3.1 such that $\|v_0\| = 1$ and $v_0 \notin \mathbb{R}K$. Then there exists an $\varepsilon_0 > 0$ such that*

$$tu + (1-t)\varepsilon v_0 \in U \quad \text{for } u \in K,\ 0 \le t \le 1 \text{ and } 0 < \varepsilon < \varepsilon_0. \tag{3.2}$$

Proof. Note that $0 \notin K$ because $0 \notin U$. Suppose that the assertion of the lemma is false. Then there exist sequences $\{\varepsilon_n\}$, $\{u_n\}$ and $\{t_n\}$ such that $u_n \in K$, $t_n \in [0, 1]$, $\varepsilon_n > 0$, $\{\varepsilon_n\}$ converges to zero and

$$t_n u_n + (1-t_n)\varepsilon_n v_0 \notin U \quad \text{for } n \in \mathbb{N}. \tag{3.3}$$

Since K is compact, there are subsequences (denoted by $\{u_n\}$ and $\{t_n\}$ again) of $\{u_n\}$ and $\{t_n\}$ which converge to $u_\infty \in K$ and $t_\infty \in [0, 1]$, respectively. Letting $n \to \infty$ in (3.3) yields that $t_\infty u_\infty \notin U$ because U is open. Note that by (B5), if $u \in U$, then $tu \in U$ for $0 < |t| \le 1$. Since $u_\infty \in K \subset U$ and $t_\infty u_\infty \notin U$, we deduce that $t_\infty = 0$. We divide the proof into two cases below.

(i) There is a subsequence (denoted by $\{t_n/\varepsilon_n\}$ again) of $\{t_n/\varepsilon_n\}$ which converges to a finite limit $\alpha \ge 0$.
(ii) $\{t_n/\varepsilon_n\}$ diverges to ∞.

In Case (i), we set

$$M := \{tu + sv_0 : 0 \le t \le \alpha + 1,\ s \in [1/2, 1],\ u \in K\}.$$

Then M is compact and $0 \notin M$ because $v_0 \notin \mathbb{R}K$. By Lemma 3.2, there is a $\delta_0 > 0$ such that

$$\delta M \subset U \quad \text{for } 0 < \delta \le \delta_0. \tag{3.4}$$

Hence

$$t_n u_n + (1 - t_n)\varepsilon_n v_0 = \varepsilon_n\{(t_n/\varepsilon_n)u_n + (1 - t_n)v_0\} \in \varepsilon_n M \subset U,$$

for n large enough. This contradicts (3.3).

In Case (ii), we put

$$M := \{u + sv_0 : s \in [0, 1],\ u \in K\}.$$

In the same argument as in Case (i), we have (3.4). Since $\{\varepsilon_n/t_n\}$ converges to zero in Case (ii), (3.4) means

$$t_n u_n + (1 - t_n)\varepsilon_n v_0 = t_n\{u_n + (1 - t_n)(\varepsilon_n/t_n)v_0\} \in t_n M \subset U,$$

for n large enough. This contradicts (3.3). In both Cases (i) and (ii), a contradiction occurs. This completes the proof. $\square$

We are now in a position to prove Lemma 2.7.

Proof of Lemma 2.7. By the definition (1.2) of a_k, there exists an $f \in \mathcal{A}_k$ such that

$$I(0, f(x)) < a_k + r < 0 \quad \text{for } x \in S^k. \tag{3.5}$$

Put $K := f(S^k)$. Then K is compact and $0 \notin K$ because of (3.5). By Lemma 3.3, we choose $v_0 \in E$ and $\varepsilon > 0$ such that

$$tf(x) + (1 - t)\varepsilon v_0 \in U,$$

or equivalently,

$$I(0, tf(x) + (1 - t)\varepsilon v_0) < 0 \quad \text{for } x \in S^k \text{ and } 0 \le t \le 1. \tag{3.6}$$

We use notation,

$$x = (x_1, \ldots, x_{k+1}, x_{k+2}) = (x', x_{k+2}),$$
$$x' = (x_1, \ldots, x_{k+1}), \qquad |x'| = \left(\sum_{i=1}^{k+1} x_i^2\right)^{1/2}.$$

For $x \in S^{k+1}_+$, i.e., $\sum_{i=1}^{k+2} x_i^2 = 1$ and $x_{k+2} \ge 0$, we define

$$f_{k+1}(x) = \begin{cases} |x'|f(x'/|x'|) + \varepsilon(1 - |x'|)v_0, & \text{if } x' \ne 0, \\ \varepsilon v_0, & \text{if } x' = 0. \end{cases}$$

Then f_{k+1} is continuous on S^{k+1}_+. Furthermore, (3.6) is rewritten as

$$I(0, f_{k+1}(x)) < 0 \quad \text{for } x \in S^{k+1}_+. \tag{3.7}$$

Observe that $x = (x', x_{k+2})$ belongs to S^k if and only if $x = (x', 0)$ with $|x'| = 1$. Then $f_{k+1}(x) = f(x)$ on S^k and hence $f_{k+1}(x)$ is odd on S^k. We extend $f_{k+1}(x)$ onto S^{k+1} as an odd mapping. Then $f_{k+1} \in \mathcal{A}_{k+1} \cap \mathcal{C}_{k+1}$ and (2.7) follows from (3.7). The proof is complete. $\square$

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Instead of Assumption (A), we consider the next assumption.

Assumption (C). $f(x,u)$ and $g(x,u)$ are Hölder continuous functions defined on $\overline{\Omega}\times\mathbb{R}$ and satisfy the following conditions.

(C1) $f(x,-u)=-f(x,u)$ for $(x,u)\in\overline{\Omega}\times\mathbb{R}$.
(C2) There exists an $a>0$ such that

$$uf(x,u)-2F(x,u)<0 \quad \text{when } 0<|u|<a \text{ and } x\in\overline{\Omega},$$
$$f(x,u)=g(x,u)=F(x,u)=0 \quad \text{when } |u|\geq a \text{ and } x\in\overline{\Omega},$$
$$F(x,u)>0 \quad \text{when } 0<|u|<a,\ x\in\overline{\Omega}.$$

(C3) $\lim_{u\to 0}\left(\min_{x\in\overline{\Omega}} u^{-2}F(x,u)\right)=\infty$.

We shall explain that Assumption (C) can be supposed without loss of generality. In view of (A3), we replace $a>0$ by a smaller constant such that

$$F(x,u)>0 \quad (0<|u|<a,\ x\in\overline{\Omega}).$$

We choose a function $\phi\in C_0^\infty(\mathbb{R},\mathbb{R})$ such that $0\leq\phi(t)\leq 1$ for $t\in\mathbb{R}$, $\phi(t)=1$ for $|t|\leq a/2$, $\phi(t)>0$ for $|t|<a$, $\phi(t)=0$ for $|t|\geq a$, $\phi(t)$ is even in $\mathbb{R}$ and strictly decreasing in $(a/2,a)$. We define $\widetilde{f}(x,u)$, $\widetilde{F}(x,u)$ and $\widetilde{g}(x,u)$ by

$$\widetilde{f}(x,u):=\frac{\partial}{\partial u}(\phi(u)F(x,u)), \qquad \widetilde{g}(x,u):=\phi(u)g(x,u),$$
$$\widetilde{F}(x,u):=\int_0^u \widetilde{f}(x,s)ds=\phi(u)F(x,u). \tag{4.1}$$

It is clear that $\widetilde{f}$ satisfies (C1) and (C3). We verify (C2). By definition, $\widetilde{f}(x,u)$, $\widetilde{g}(x,u)$ and $\widetilde{F}(x,u)$ vanish when $|u|\geq a$ and $x\in\overline{\Omega}$. Moreover, $\widetilde{F}(x,u)>0$ when $0<|u|<a$ and $x\in\overline{\Omega}$. Observe the relation,

$$\frac{\partial}{\partial u}\left(u^{-2}\widetilde{F}(x,u)\right)=u^{-3}(u\widetilde{f}(x,u)-2\widetilde{F}(x,u)). \tag{4.2}$$

Using (4.1) with (A2), we get

$$\begin{aligned}\frac{\partial}{\partial u}\left(u^{-2}\widetilde{F}(x,u)\right)&=\phi'(u)u^{-2}F(x,u)+\phi(u)\frac{\partial}{\partial u}\left(u^{-2}F(x,u)\right)\\&=\phi'(u)u^{-2}F(x,u)+\phi(u)u^{-3}(uf(x,u)-2F(x,u))<0,\end{aligned}$$

provided that $0<u<a$ and $x\in\overline{\Omega}$. This inequality with (4.2) means that $\widetilde{f}(x,u)$ satisfies (C2). Accordingly, $\widetilde{f}(x,u)$ satisfies (C1)–(C3).

It is enough to prove Theorem 1.1 with f and g replaced by $\widetilde{f}$ and $\widetilde{g}$, respectively, because $\widetilde{f}(x,u)=f(x,u)$ and $\widetilde{g}(x,u)=g(x,u)$ for $|u|$ sufficiently small. Consequently, we shall prove Theorem 1.1 under Assumption (C) instead of (A).

We set $E:=H_0^1(\Omega)$ and define

$$I(t,u):=\int_\Omega\left(\frac{1}{2}|\nabla u|^2-F(x,u)-tG(x,u)\right)dx, \tag{4.3}$$
$$F(x,u):=\int_0^u f(x,s)ds, \qquad G(x,u):=\int_0^u g(x,s)ds.$$

Lemma 4.1. *There exists a constant $C>0$ such that if $I_u(t,u)=0$ with $|t|\leq 1$, then $\|u\|_{C^2(\overline{\Omega})}\leq C$.*

Proof. The critical point u of $I(t,\cdot)$ satisfies

$$-\Delta u=f(x,u)+tg(x,u), \quad \text{in } \Omega, \qquad u=0, \quad \text{in } \partial\Omega. \tag{4.4}$$

Since the right hand side of the first equation in (4.4) is bounded and Hölder continuous, the elliptic regularity theorem (see [4]) gives an a priori bound for the $C^2(\overline{\Omega})$-norm of solutions. □

Lemma 4.2. *For any $\varepsilon>0$ there exists a $\delta>0$ such that if $|t|\leq\delta$, $I_u(t,u)=0$ and $|I(t,u)|\leq\delta$, then $\|u\|_{C^2(\overline{\Omega})}\leq\varepsilon$.*

Proof. Suppose on the contrary that there exist sequences $\{u_k\}$ and $\{t_k\}$ such that $\{t_k\}$ converges to zero, u_k satisfies (4.4) with $t=t_k$, $I(t_k,u_k)$ converges to zero and moreover $\|u_k\|_{C^2(\overline{\Omega})}\geq\varepsilon_0>0$. Here $\varepsilon_0>0$ is independent of k. The Ascoli–Arzelà theorem with Lemma 4.1 yields a subsequence of $\{u_k\}$ which converges to a certain limit u_0 in the $C^1(\overline{\Omega})$-space. Moreover, the elliptic regularity theorem guarantees that the convergence is valid in the $C^2(\overline{\Omega})$-sense. Consequently, we have

$$-\Delta u_0 = f(x, u_0), \quad x \in \Omega, \qquad u_0 = 0, \quad x \in \partial\Omega, \tag{4.5}$$

$$I(0, u_0) = \int_\Omega \left(\frac{1}{2}|\nabla u_0|^2 - F(x, u_0)\right) dx = 0. \tag{4.6}$$

Furthermore, $\|u_0\|_{C^2(\overline{\Omega})} \geq \varepsilon_0 > 0$. Multiplying (4.5) by u_0 and integrating it over Ω, we get

$$\int_\Omega |\nabla u_0|^2 dx = \int_\Omega u_0 f(x, u_0) dx.$$

Substituting the relation above into (4.6), we obtain

$$\int_\Omega \left(\frac{1}{2}u_0 f(x, u_0) - F(x, u_0)\right) dx = 0.$$

Since $u_0 \in H_0^1(\Omega)$, (C2) means that $u_0 \equiv 0$. This contradicts that $\|u_0\|_{C^2} \geq \varepsilon_0 > 0$. The proof is complete. □

We conclude this paper by proving Theorem 1.1.

Proof of Theorem 1.1. We suppose $t \geq 0$ because the case $t < 0$ is similarly treated by replacing $g(x, u)$ by $-g(x, u)$. Let us verify that $I(t, u)$ defined by (4.3) satisfies Assumption (B). Since $F(x, u)$ and $G(x, u)$ are bounded on $\overline{\Omega} \times \mathbb{R}$, (B1) holds. Assumption (B2) follows from

$$|I(t, u) - I(0, u)| \leq |t| \int_\Omega |G(x, u)| dx \leq C|t| \equiv \psi(t),$$

with a constant $C > 0$ independent of u and t. We shall verify the Palais–Smale condition. Let $(t_n, u_n) \in [0, 1] \times H_0^1(\Omega)$ be any sequence such that $I(t_n, u_n)$ is bounded and $I_u(t_n, u_n)$ converges to zero. Since $F(x, u)$ and $G(x, u)$ are bounded in $[0, 1] \times \mathbb{R}$ by (C2), $\|\nabla u_n\|_2$ is also bounded. Here $\|\cdot\|_2$ denotes the L^2-norm. Therefore a subsequence of u_n converges weakly in $H_0^1(\Omega)$. Moreover, this convergence becomes a strong one, which can be proved in the standard method (see [7] or [8]). Thus (B3) holds. It is clear that (B4) is fulfilled.

We verify (B5). Fix $u \in H_0^1(\Omega) \setminus \{0\}$ arbitrarily. For $s > 0$, we define

$$J(s) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - s^{-2}F(x, su)\right) dx.$$

Then it follows that

$$I(0, su) = \int_\Omega \left(\frac{s^2}{2}|\nabla u|^2 - F(x, su)\right) dx = s^2 J(s).$$

Since $u \in H_0^1(\Omega) \setminus \{0\}$, we use (C2) to get

$$J'(s) = -s^{-3} \int_\Omega (suf(x, su) - 2F(x, su))\, dx > 0 \quad \text{for } s > 0.$$

Thus $J(s)$ is strictly increasing. Choose $\delta > 0$ so small that

$$\mu(D) > 0 \quad \text{and} \quad D := \{x \in \Omega : \delta < |u(x)| < 1/\delta\},$$

where μ denotes the Lebesgue measure of $\mathbb{R}^N$. Since $F(x, u) \geq 0$, the function $J(s)$ is estimated as

$$\begin{aligned} J(s) &\leq \frac{1}{2}\|\nabla u\|_2^2 - \int_D s^{-2}F(x, su)dx \\ &\leq \frac{1}{2}\|\nabla u\|_2^2 - \delta^2 \mu(D) \inf_{x \in D}\left((su(x))^{-2}F(x, su(x))\right). \end{aligned}$$

From (C3) it follows that $\lim_{s \to 0+} J(s) = -\infty$. Therefore $J(s) < 0$ for $s > 0$ small enough. Since F is bounded, $J(s) > 0$ for $s > 0$ sufficiently large. Accordingly, $J(s)$ has a unique zero $s(u)$ such that $J(s) < 0$ for $0 < s < s(u)$ and $J(s) > 0$ for $s > s(u)$. Thus (B5) holds.

In view of Lemma 4.2, it is enough to prove that for any $k \in \mathbb{N}$ and $\delta > 0$, if $|t|$ is sufficiently small, then $I(t, \cdot)$ has at least k distinct critical values whose absolute values are less than δ. Let a_k be defined by (1.2). We choose a subsequence of a_k which is strictly increasing. By Theorem 1.4, there exist sequences $\{t_{k+1}\}$ and $\{d_{k+1}(t)\}$ such that $t_{k+1} > 0$, $d_{k+1}(t)$ is a critical value of $I(t, \cdot)$ for $t \in [0, t_{k+1}]$ and

$$a_{k+1} - \psi(t) \leq d_{k+1}(t) \leq c_{k+1} + \psi(t) < 0.$$

Give $k \in \mathbb{N}$ and $\delta > 0$ arbitrarily. Choose increasing positive integers $p(i)$ with $1 \le i \le k$ such that $-\delta < a_{p(1)}$ and $c_{p(i)} < a_{p(i+1)}$ for $1 \le i \le k$. We choose $\varepsilon > 0$ so small that $d_{p(i)}(t)$ with $1 \le i \le k$ are defined for $t \in [0, \varepsilon]$ and

$$-\delta < a_{p(1)} - \psi(t), \qquad c_{p(i)} + \psi(t) < a_{p(i+1)} - \psi(t) \quad \text{on } [0, \varepsilon].$$

This means that $d_{p(i)}(t) < d_{p(i+1)}(t)$. Then for $t \in [0, \varepsilon]$, $I(t, \cdot)$ has at least k critical values

$$-\delta < d_{p(1)}(t) < d_{p(2)}(t) < \cdots < d_{p(k)}(t) < 0.$$

Therefore (1.1) has at least k solutions whose $C^2(\overline{\Omega})$-norms are small enough.

Let $\varepsilon = 0$. By Lemma 2.5, $\{a_k\}$ is a sequence of critical values of $I(0, \cdot)$ which converges to zero. Hence the corresponding critical points are solutions of (1.1) with $\varepsilon = 0$, which converges to zero in $C^2(\overline{\Omega})$ by Lemma 4.2. The proof is complete. □

Acknowledgment

The author was supported in part by the Grant-in-Aid for Scientific Research (C) (No. 24540179), Japan Society for the Promotion of Science.

References

[1] A. Ambrosetti, M. Badiale, The dual variational principle and elliptic problems with discontinuous nonlinearities, J. Math. Anal. Appl. 140 (1989) 363–373.
[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.
[3] M. Degiovanni, V. Rădulescu, Perturbations of nonsmooth symmetric nonlinear eigenvalue problems, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999) 281–286.
[4] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1983.
[5] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, J. Funct. Anal. 225 (2005) 352–370.
[6] R. Kajikiya, Multiple solutions of sublinear Lane–Emden elliptic equations, Calc. Var. Partial Differential Equations 26 (2006) 29–48.
[7] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Providence, 1986.
[8] M. Struwe, Variational Methods, second ed., Springer, Berlin, 1996.