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Small data blow-up for a system of nonlinear Schrödinger equations

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ABSTRACT

We give examples of small data blow-up for a three-component system of quadratic nonlinear Schrödinger equations in one space dimension. Our construction of the blowing-up solution is based on the Hopf–Cole transformation, which allows us to reduce the problem to getting suitable growth estimates for a solution to the transformed system. Amplification in the reduced system is shown to have a close connection with the mass resonance.

1. Introduction

We consider the initial value problem for a system of nonlinear Schrödinger equations in the form

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2m_j}\partial_x^2\right) u_j = N_j(u, \partial_x u), & t > 0,\ x \in \mathbb{R},\ j = 1, 2, 3, \\ u_j(0, x) = \varphi_j(x), & x \in \mathbb{R},\ j = 1, 2, 3 \end{cases} \tag{1.1}$$

where $u = (u_j)_{j=1,2,3}$ is a $\mathbb{C}^3$-valued unknown, m_j is a positive constant and the nonlinear term N_j satisfies

$$N_j(u, q) = O((|u| + |q|)^2) \quad \text{as } (u, q) \to (0, 0).$$

We assume that $\varphi = (\varphi_j)_{j=1,2,3}$ belongs to the Sobolev space $H^s(\mathbb{R})$ with $s \geq 1$, which is defined by $H^s(\mathbb{R}) = \{\psi\ ;\ \partial_x^k \psi \in L^2(\mathbb{R})$ for all $k \leq s\}$ equipped with the norm

$$\|\psi\|_{H^s} = \sum_{k \leq s} \|\partial_x^k \psi\|_{L^2}.$$

A typical nonlinear Schrödinger system appearing in various physical settings is

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2m_1}\Delta\right) u_1 = \overline{u_1} u_2, \\ \left(i\partial_t + \dfrac{1}{2m_2}\Delta\right) u_2 = {u_1}^2, \end{cases} \quad t > 0,\ x \in \mathbb{R}^n \tag{1.2}$$

(see e.g., [1,2] for physical background). What is interesting in (1.2) is that the ratio of the masses can affect the large-time behavior of the solutions. In the case of $n = 2$, Hayashi–Li–Naumkin [3] obtained a small data global existence result for

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(1.2) under the relation $m_2 = 2m_1$. The non-existence of usual scattering state is also proved when $m_2 = 2m_1$. On the other hand, when $m_2 \neq 2m_1$, it is shown in [4] that there is a usual scattering state under some restriction of the data. Higher dimensional case ($n \geq 3$) is considered by Hayashi–Li–Ozawa [5] from the viewpoint of small data scattering. Remark that the relation $m_2 = 2m_1$ is often called the mass resonance relation, which was first discovered in the study of nonlinear Klein–Gordon systems (see [6–13], etc.). More recently, large data case is discussed by Hayashi–Ozawa–Tanaka [14]. In particular, their result includes finite time blow-up of the negative energy solutions for (1.2) under mass resonance in the case of $4 \leq n \leq 6$. However, their approach relies on the so-called virial identity which requires that the initial data of the blowing-up solutions must be suitably large (whence it should be distinguished from small data blow-up; see the Appendix below for more details). Also it seems difficult to generalize blow-up results of this type to the case where the nonlinearity involves the derivatives of the unknowns. Concerning small data blow-up for NLS, very few results are known so far and many interesting problems are left unsolved (even in the case of single equations without derivatives in the nonlinear terms). We refer the readers to [15–21] etc. for more information and the related topics.

The aim of this paper is to give examples of small data blow-up for (1.1). More precisely, we will show that there exist m_j, N_j and φ_j with $\|\varphi\|_{H^s} = \varepsilon$ such that the corresponding solution blows up in finite time no matter how small $\varepsilon > 0$ is. We will also specify the order of the lifespan with respect to ε. What we intend here is to illustrate, by using a simple model, how the interplay between the mass resonance and the nonlinear structure can affect global behavior of the solution. Although our examples below are somewhat artificial, they will help us to develop the understanding for possible mechanisms of singularity formation in more general nonlinear Schrödinger systems.

2. Main result

In what follows, we always assume that the nonlinearity in (1.1) is in the form

$$N_1 = 0, \qquad N_2 = {u_1}^2, \qquad N_3 = (\partial_x u_3)^2 + Q(u_1, u_2)\frac{\exp(2m_3u_3)}{2m_3}, \tag{2.1}$$

where $Q(u_1, u_2)$ is either ${u_2}^2$, u_1u_2, $\overline{u_1}u_2$ or $|u_2|u_2$. The main result is as follows.

Theorem 1. (1) *Let $Q = {u_2}^2$ and assume $m_1 : m_2 : m_3 = 1 : 2 : 4$. Then, for any $\varepsilon \in (0, 1]$ and $s \geq 1$, there exists $\varphi \in H^s(\mathbb{R})$ with $\|\varphi\|_{H^s} = \varepsilon$ such that the corresponding solution u for (1.1) satisfies*

$$\lim_{t \to T_\varepsilon - 0} \|u(t, \cdot)\|_{H^s} = \infty \tag{2.2}$$

with $T_\varepsilon \in (\kappa\varepsilon^{-4}, K\varepsilon^{-4})$, where κ and K are positive constants not depending on ε.

(2) *Let $Q = u_1u_2$ and assume $m_1 : m_2 : m_3 = 1 : 2 : 3$. Then, for any $\varepsilon \in (0, 1]$ and $s \geq 1$, there exists $\varphi \in H^s(\mathbb{R})$ with $\|\varphi\|_{H^s} = \varepsilon$ such that the corresponding solution u for (1.1) satisfies (2.2) with $T_\varepsilon \in (\kappa\varepsilon^{-6}, K\varepsilon^{-6})$, where κ and K are positive constants not depending on ε.*

(3) *Let $Q = \overline{u_1}u_2$ and assume $m_1 : m_2 : m_3 = 1 : 2 : 1$. Then, for any $\varepsilon \in (0, 1]$ and $s \geq 1$, there exists $\varphi \in H^s(\mathbb{R})$ with $\|\varphi\|_{H^s} = \varepsilon$ such that the corresponding solution u for (1.1) satisfies (2.2) with $T_\varepsilon \in (\kappa\varepsilon^{-6}, K\varepsilon^{-6})$, where κ and K are positive constants not depending on ε.*

(4) *Let $Q = |u_2|u_2$ and assume $m_1 : m_2 : m_3 = 1 : 2 : 2$. Then, for any $\varepsilon \in (0, 1]$, there exists $\varphi \in H^1(\mathbb{R})$ with $\|\varphi\|_{H^1} = \varepsilon$ such that the corresponding solution u for (1.1) satisfies*

$$\lim_{t \to T_\varepsilon - 0} \|u(t, \cdot)\|_{H^1} = \infty$$

with $T_\varepsilon \in (\kappa\varepsilon^{-4}, K\varepsilon^{-4})$, where κ and K are positive constants not depending on ε.

Remark 1. For general $\varphi \in H^s$ with $\|\varphi\|_{H^s} = \varepsilon$, it is not difficult to show a lower bound for T_ε of the same order in ε (that is to say, we can show $T_\varepsilon \geq \kappa\varepsilon^{-4}$ in the case of (1), for instance) if ε is small enough. The novelty of the above theorem is the upper bound for T_ε. In particular, this tells us that the order of the lifespan is actually influenced by the choice of Q and the ratio of the masses.

Remark 2. The relation between the choice of Q and the ratio of the masses in Theorem 1 is characterized by the following condition:

$$Q(e^{im_1\theta}z_1, e^{im_2\theta}z_2) = e^{im_3\theta}Q(z_1, z_2), \quad \theta \in \mathbb{R},\ z_1, z_2 \in \mathbb{C}. \tag{2.3}$$

Our approach does not work without this condition.

We close this section by explaining our strategy of the proof. By setting

$$\sigma(t, x) = 1 - \exp(-2m_3u_3(t, x)), \tag{2.4}$$

we can rewrite the system (1.1) with (2.1) as

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2m_1}\partial_x^2\right) u_1 = 0, \\ \left(i\partial_t + \dfrac{1}{2m_2}\partial_x^2\right) u_2 = {u_1}^2, \\ \left(i\partial_t + \dfrac{1}{2m_3}\partial_x^2\right) \sigma = Q(u_1, u_2). \end{cases}$$

This kind of transformation is first introduced by Hopf [22] and Cole [23] for the Burgers equation, and (2.4) is used effectively by Ozawa [24,25] in the study of the quadratic NLS in the form $i\partial_t u + \frac{1}{2}\Delta u = (\nabla u)^2$ (see also p.38 of [26]). Note that (2.4) can be rewritten as

$$u_3(t, x) = \frac{-1}{2m_3}\log(1 - \sigma(t, x))$$

if $|\sigma(t, x)| < 1$, where the branch of the logarithm is chosen so that $\log 1 = 0$. Our main task in the proof of Theorem 1 is to choose φ appropriately so that

$$\sigma(T_\varepsilon, x^*) = 1 \tag{2.5}$$

holds at some point $x^* \in \mathbb{R}$ (while $\|\sigma(t, \cdot)\|_{L^\infty} < 1$ for $t < T_\varepsilon$). The mass resonance condition (or, equivalently, (2.3)) will play a crucial role in the proof of this amplification. Once (2.5) is verified, we have

$$\|u_3(t, \cdot)\|_{H^s} \geq C|u_3(t, x^*)| = \frac{C}{2m_3}|\log(1 - \sigma(t, x^*))| \to \infty$$

as $t \to T_\varepsilon - 0$ (while $\|u_3(t, \cdot)\|_{H^s} < \infty$ for $t < T_\varepsilon$). Similar idea can be found in the paper by Yagdjian [27], where semilinear wave equations with time-periodic coefficients are considered (see also [28,29]). Remark that the amplification in [27] is due to parametric resonance and the proof is based on the Floquet theory.

3. Preliminaries

In this section, we collect several identities and estimates which are useful in the subsequent sections. In what follows, we denote several positive constants by the same letter C, which may vary from one line to another.

First we put $\mathcal{L}_m = i\partial_t + \frac{1}{2m}\partial_x^2$ and $\mathcal{J}_m(t) = x + \frac{it}{m}\partial_x$ for $m > 0$. Then we have $[\partial_x, \mathcal{J}_m(t)] = 1$ and $[\mathcal{L}_m, \partial_x] = [\mathcal{L}_m, \mathcal{J}_m(t)] = 0$, where $[\cdot, \cdot]$ denotes the commutator, i.e., $[\mathcal{P}, \mathcal{Q}] = \mathcal{P}\mathcal{Q} - \mathcal{Q}\mathcal{P}$ for linear operators $\mathcal{P}$ and $\mathcal{Q}$. Also we can easily check that

$$\mathcal{J}_{2m}(t)(\phi\psi) = \frac{1}{2}\left\{\left(\mathcal{J}_m(t)\phi\right)\psi + \phi\left(\mathcal{J}_m(t)\psi\right)\right\}, \tag{3.1}$$

$$\mathcal{J}_{3m}(t)(\phi\psi) = \frac{1}{3}\left\{\left(\mathcal{J}_m(t)\phi\right)\psi + 2\phi\left(\mathcal{J}_{2m}(t)\psi\right)\right\} \tag{3.2}$$

and

$$\mathcal{J}_m(t)(\overline{\phi}\psi) = -\left(\overline{\mathcal{J}_m(t)\phi}\right)\psi + 2\overline{\phi}\left(\mathcal{J}_{2m}(t)\psi\right) \tag{3.3}$$

for smooth functions ϕ and ψ. Next we put $\mathcal{A}_m(t) = \mathcal{F}_m\mathcal{U}_m(t)^{-1}$, where $\mathcal{F}_m$ and $\mathcal{U}_m(t)$ are defined by

$$\left(\mathcal{F}_m\phi\right)(\xi) = \sqrt{\frac{m}{2\pi}}\int_{\mathbb{R}} e^{-imy\xi}\phi(y)dy$$

and

$$\left(\mathcal{U}_m(t)\phi\right)(x) = \sqrt{\frac{m}{2\pi it}}\int_{\mathbb{R}} e^{im(x-y)^2/(2t)}\phi(y)dy,$$

respectively. Note that $w(t, x) = (\mathcal{U}_m(t)\phi)(x)$ solves

$$\mathcal{L}_m w = 0, \quad w(0, x) = \phi(x).$$

We also remark that $\partial_t\mathcal{A}_m(t)\phi = -i\mathcal{A}_m(t)\mathcal{L}_m\phi$.

Lemma 1. *For a smooth function $f(t, x)$, we have*

$$\|f(t)\|_{L^\infty} \geq t^{-1/2}\|\mathcal{A}_m(t)f(t)\|_{L^\infty} - Ct^{-3/4}\rho_m[f](t),$$

where

$$\rho_m[f](t) = \|f(t, \cdot)\|_{H^1} + \|\mathcal{J}_m(t)f(t, \cdot)\|_{L^2}. \tag{3.4}$$

Proof. By the relation $\mathcal{J}_m(t) = \mathcal{U}_m(t)x\mathcal{U}_m(t)^{-1}$, we have

$$\begin{aligned}\|\mathcal{A}_m(t)f\|_{H^1} &= \|\mathcal{F}_m\mathcal{U}_m(t)^{-1}f\|_{H^1}\\ &\le C\|(1+|x|)\mathcal{U}_m(t)^{-1}f\|_{L^2}\\ &\le C\rho_m[f](t).\end{aligned}$$

Next we observe that $\mathcal{U}_m(t)$ can be decomposed into $\mathcal{M}_m(t)\mathcal{D}(t)\mathcal{F}_m\mathcal{M}_m(t)$, where

$$\begin{aligned}&\big(\mathcal{M}_m(t)\phi\big)(x) = e^{imx^2/(2t)}\phi(x),\\ &\big(\mathcal{D}(t)\phi\big)(x) = \frac{1}{\sqrt{it}}\phi\left(\frac{x}{t}\right).\end{aligned}$$

We also set $\mathcal{W}_m(t) = \mathcal{F}_m\mathcal{M}_m(t)\mathcal{F}_m^{-1}$. Then we see that

$$\begin{aligned}f &= \mathcal{U}_m(t)\mathcal{U}_m(t)^{-1}f\\ &= \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{F}_m\mathcal{M}_m(t)\cdot\mathcal{F}_m^{-1}\mathcal{A}_m(t)f\\ &= \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{W}_m(t)\mathcal{A}_m(t)f.\end{aligned}$$

From the inequalities $\|f\|_{L^\infty} \le C\|f\|_{L^2}^{1/2}\|\partial_x f\|_{L^2}^{1/2}$ and $|e^{imx^2/(2t)} - 1| \le Ct^{-1/2}|x|$, it follows that

$$\begin{aligned}\|(\mathcal{W}_m(t)-1)f\|_{L^\infty} &\le C\|(\mathcal{W}_m(t)-1)f\|_{L^2}^{1/2}\|\partial_x(\mathcal{W}_m(t)-1)f\|_{L^2}^{1/2}\\ &\le C(Ct^{-1/2}\|f\|_{H^1})^{1/2}(C\|f\|_{H^1})^{1/2}\\ &= Ct^{-1/4}\|f\|_{H^1}.\end{aligned}\tag{3.5}$$

Consequently we have

$$\begin{aligned}\|f - \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{A}_m(t)f\|_{L^\infty} &= \|\mathcal{M}_m(t)\mathcal{D}(t)(\mathcal{W}_m(t)-1)\mathcal{A}_m(t)f\|_{L^\infty}\\ &\le t^{-1/2}\|(\mathcal{W}_m(t)-1)\mathcal{A}_m(t)f\|_{L^\infty}\\ &\le Ct^{-3/4}\|\mathcal{A}_m(t)f\|_{H^1}\\ &\le Ct^{-3/4}\rho_m[f](t),\end{aligned}$$

whence

$$\begin{aligned}\|f\|_{L^\infty} &\ge \|\mathcal{M}_m(t)\mathcal{D}(t)\mathcal{A}_m(t)f\|_{L^\infty} - \|f - \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{A}_m(t)f\|_{L^\infty}\\ &\ge t^{-1/2}\|\mathcal{A}_m(t)f\|_{L^\infty} - Ct^{-3/4}\rho_m[f](t)\end{aligned}$$

as required. □

Lemma 2. *Let $f(t,x)$ and $g(t,x)$ be smooth functions satisfying $\mathcal{L}_{2m}g = f^2$. We have*

$$\rho_{2m,s}[g](t) \le \rho_{2m,s}[g](0) + C\int_0^t \rho_{m,s}[f](\tau)^2\frac{d\tau}{\tau^{1/2}},\tag{3.6}$$

where $\rho_{m,s}[\,\cdot\,]$ is defined by

$$\rho_{m,s}[f](t) = \|f(t,\cdot)\|_{H^s} + \|\mathcal{J}_m(t)f(t,\cdot)\|_{H^{s-1}}$$

for $s \ge 1$. Also we have

$$\left\|\partial_t\big(\mathcal{A}_{2m}(t)g(t)\big) - e^{-i3\pi/4}t^{-1/2}\big(\mathcal{A}_m(t)f(t)\big)^2\right\|_{L^\infty} \le Ct^{-3/4}\rho_m[f](t)^2,\tag{3.7}$$

where $\rho_m[f] = \rho_{m,1}[f]$, as defined in (3.4).

Proof. First we note that $\mathcal{J}_m(t) = \frac{it}{m}\mathcal{M}_m(t)\partial_x\mathcal{M}_m(t)^{-1}$, which implies

$$\begin{aligned}\|f\|_{L^\infty} &= \|\mathcal{M}_m(t)^{-1}f\|_{L^\infty}\\ &\le C\|\mathcal{M}_m(t)^{-1}f\|_{L^2}^{1/2}\|\partial_x\mathcal{M}_m(t)^{-1}f\|_{L^2}^{1/2}\\ &\le C\|f\|_{L^2}^{1/2}(t^{-1}\|\mathcal{J}_m(t)f\|_{L^2})^{1/2}\\ &\le Ct^{-1/2}\rho_m[f](t).\end{aligned}\tag{3.8}$$

Since $[\mathcal{L}_{2m}, \partial_x^j \mathcal{J}_{2m}(t)^k] = 0$, the standard energy method yields

$$\frac{d}{dt}\|\partial_x^j \mathcal{J}_{2m}(t)^k g(t,\cdot)\|_{L^2} \le \|\partial_x^j \mathcal{J}_{2m}(t)^k (f^2)\|_{L^2} \tag{3.9}$$

for $k = 0, 1$ and $j \le s-k$. From (3.1), (3.8) and (3.9) it follows that

$$\begin{aligned}
\frac{d}{dt}\rho_{2m,s}[g](t) &= \sum_{k=0}^{1}\sum_{j=0}^{s-k}\frac{d}{dt}\|\partial_x^j \mathcal{J}_{2m}(t)^k g(t,\cdot)\|_{L^2} \\
&\le \sum_{k=0}^{1}\sum_{j=0}^{s-k}\|\partial_x^j \mathcal{J}_{2m}(t)^k (f^2)\|_{L^2} \\
&\le C\left(\sum_{j'=0}^{s-1}\|\partial_x^{j'} f\|_{L^\infty}\right)\left(\sum_{k=0}^{1}\sum_{j''=0}^{s-k}\|\partial_x^{j''}\mathcal{J}_m(t)^k f\|_{L^2}\right) \\
&\le Ct^{-1/2}\rho_{m,s}[f](t)^2.
\end{aligned}$$

By integrating with respect to t, we obtain the desired estimate (3.6). To prove (3.7), we put $\alpha(t,\xi) = \big(\mathcal{A}_m(t)f(t,\cdot)\big)(\xi)$, $\beta(t,\xi) = \big(\mathcal{A}_{2m}(t)g(t,\cdot)\big)(\xi)$ and

$$R(t,\xi) = i\partial_t\beta(t,\xi) - e^{-i\pi/4}t^{-1/2}\alpha(t,\xi)^2.$$

Note that $\|\alpha(t,\cdot)\|_{H^1} \le C\rho_m[f](t)$ and

$$\begin{aligned}
i\partial_t\beta &= \mathcal{A}_{2m}(t)\mathcal{L}_{2m}g \\
&= \mathcal{F}_{2m}\mathcal{U}_{2m}(t)^{-1}(f^2) \\
&= \mathcal{W}_{2m}(t)^{-1}\mathcal{D}(t)^{-1}\mathcal{M}_{2m}(t)^{-1}\big(\mathcal{M}_m(t)\mathcal{D}(t)\mathcal{W}_m(t)\alpha\big)^2 \\
&= \mathcal{W}_{2m}(t)^{-1}\mathcal{D}(t)^{-1}\big(\mathcal{D}(t)\mathcal{W}_m(t)\alpha\big)^2 \\
&= e^{-i\pi/4}t^{-1/2}\mathcal{W}_{2m}(t)^{-1}\big(\mathcal{W}_m(t)\alpha\big)^2.
\end{aligned}$$

With the help of (3.5), we have

$$\begin{aligned}
\|R(t,\cdot)\|_{L^\infty} &= t^{-1/2}\left\|\mathcal{W}_{2m}(t)^{-1}\big(\mathcal{W}_m(t)\alpha\big)^2 - \alpha^2\right\|_{L^\infty} \\
&\le t^{-1/2}\left(\left\|(\mathcal{W}_{2m}(t)^{-1}-1)\big(\mathcal{W}_m(t)\alpha\big)^2\right\|_{L^\infty} + \left\|\big(\mathcal{W}_m(t)\alpha\big)^2 - \alpha^2\right\|_{L^\infty}\right) \\
&\le t^{-1/2}\left(Ct^{-1/4}\left\|\big(\mathcal{W}_m(t)\alpha\big)^2\right\|_{H^1} + \|(\mathcal{W}_m(t)+1)\alpha\|_{L^\infty}\|(\mathcal{W}_m(t)-1)\alpha\|_{L^\infty}\right) \\
&\le t^{-1/2}\left(Ct^{-1/4}\|\mathcal{W}_m(t)\alpha\|_{L^\infty}\|\mathcal{W}_m(t)\alpha\|_{H^1} + C\|\alpha\|_{H^1}\cdot Ct^{-1/4}\|\alpha\|_{H^1}\right) \\
&\le Ct^{-3/4}\|\alpha(t,\cdot)\|_{H^1}^2 \\
&\le Ct^{-3/4}\rho_m[f](t)^2,
\end{aligned}$$

which yields (3.7). □

Remark 3. The above argument can be generalized as follows: let v_1, v_2, v_3 be smooth functions of (t,x) satisfying $\mathcal{L}_{m_3}v_3 = Q(v_1, v_2)$, where $Q : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ satisfies (2.3) and

$$Q(\lambda z_1, \lambda z_2) = \lambda^2 Q(z_1, z_2), \quad \lambda > 0,\ z_1, z_2 \in \mathbb{C}.$$

Then we have

$$i\partial_t\alpha_3 = t^{-1/2}\tilde{Q}(\alpha_1,\alpha_2) + R,$$

where

$$\begin{aligned}
&\alpha_j(t,\xi) = \big(\mathcal{A}_{m_j}(t)v_j(t,\cdot)\big)(\xi), \quad j = 1,2,3, \\
&\tilde{Q}(\alpha_1,\alpha_2) = e^{i\pi/4}Q(e^{-i\pi/4}\alpha_1, e^{-i\pi/4}\alpha_2), \\
&R(t,\xi) = t^{-1/2}\big\{\mathcal{W}_{m_3}(t)^{-1}\tilde{Q}(\mathcal{W}_{m_1}(t)\alpha_1, \mathcal{W}_{m_2}(t)\alpha_2) - \tilde{Q}(\alpha_1,\alpha_2)\big\}.
\end{aligned}$$

4. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Since the essential idea is the same, we consider the case of (1) in detail and only the outline of the proof will be given for the other cases.

In what follows, we fix $s \geq 1$ and $\psi : \mathbb{R} \to \mathbb{C}$ which satisfies $\|\mathcal{F}_{m_1}^{-1}\psi\|_{H^s} = 1$ and $\|(1+|x|)^{s-1}\partial_x \psi\|_{L^2} < \infty$. Let $v = (v_j(t,x))_{j=1,2,3}$ be the solution to

$$\begin{cases} \mathcal{L}_{m_1} v_1 = 0, \\ \mathcal{L}_{m_2} v_2 = {v_1}^2, \quad t > 0,\ x \in \mathbb{R} \\ \mathcal{L}_{m_3} v_3 = {v_2}^2, \end{cases}$$

with the initial condition

$$\begin{cases} v_1(0,x) = \varepsilon(\mathcal{F}_{m_1}^{-1}\psi)(x), \\ v_2(0,x) = 0, \\ v_3(0,x) = 0. \end{cases} \tag{4.1}$$

We have the following.

Lemma 3. *Let v be as above. Under the assumption $m_1 : m_2 : m_3 = 1 : 2 : 4$, there exist positive constants κ and K, independent of ε, such that*

$$\sup_{0\leq t\leq \kappa\varepsilon^{-4}} \|v_3(t,\cdot)\|_{L^\infty} < 1 \tag{4.2}$$

and

$$\|v_3(K\varepsilon^{-4},\cdot)\|_{L^\infty} > 1. \tag{4.3}$$

Before turning to the proof of Lemma 3, we show that (1) of Theorem 1 is derived from this lemma: we set $T_\varepsilon = \sup\{T > 0;\ \|v_3(t,\cdot)\|_{L^\infty} < 1 \text{ for } 0 \leq t < T\}$. Then (4.2) and (4.3) imply $\kappa\varepsilon^{-4} < T_\varepsilon < K\varepsilon^{-4}$. Also, since the function $\mathbb{R} \ni x \mapsto |v_3(T_\varepsilon, x)|$ is continuous, we can choose $x^* \in \mathbb{R}$ such that

$$|v_3(T_\varepsilon, x^*)| = \|v_3(T_\varepsilon,\cdot)\|_{L^\infty} = 1.$$

Now we take $\theta \in \mathbb{R}$ so that $v_3(T_\varepsilon, x^*) = e^{im_3\theta}$, and we set

$$\varphi_1(x) = \varepsilon e^{-im_1\theta}(\mathcal{F}_{m_1}^{-1}\psi)(x), \quad \varphi_2(x) = \varphi_3(x) = 0.$$

Then, by the uniqueness of solutions to (1.1), we have

$$\begin{aligned} u_1(t,x) &= e^{-im_1\theta} v_1(t,x), \\ u_2(t,x) &= e^{-im_2\theta} v_2(t,x), \\ u_3(t,x) &= \frac{-1}{2m_3}\log\bigl(1 - e^{-im_3\theta} v_3(t,x)\bigr), \end{aligned}$$

which is a desired blowing-up solution. $\square$

Now we are going to prove Lemma 3. First we show (4.2). We put $\rho_{j,s}(t) = \rho_{m_j,s}[v_j](t)$ for $j = 1, 2, 3$. By (3.6), we have

$$\rho_{1,s}(t) \leq \rho_{1,s}(0) = C\varepsilon, \tag{4.4}$$

$$\rho_{2,s}(t) \leq 0 + C\int_0^t \rho_{1,s}(\tau)^2 \frac{d\tau}{\tau^{1/2}} \leq C\varepsilon^2 t^{1/2}, \tag{4.5}$$

and

$$\rho_{3,s}(t) \leq 0 + C\int_0^t \rho_{2,s}(\tau)^2 \frac{d\tau}{\tau^{1/2}} \leq C\varepsilon^4 t^{3/2}. \tag{4.6}$$

From (3.8) and (4.6) it follows that

$$\|v_3(t,\cdot)\|_{L^\infty} \leq Ct^{-1/2}\rho_{3,s}(t) \leq C\varepsilon^4 t \leq C\kappa$$

for $t \leq \kappa\varepsilon^{-4}$. By choosing κ so small that $C\kappa < 1$, we arrive at (4.2). Next we turn to the proof of (4.3). We put $\alpha_j(t,\xi) = (A_{m_j}(t)v_j(t,\cdot))(\xi)$ and $\rho_j(t) = \rho_{m_j,1}[v_j](t)$ for $j = 1, 2, 3$. Since $\partial_t\alpha_1 = -i\mathcal{A}_{m_1}(t)\mathcal{L}_{m_1}v_1 = 0$, we have

$$\alpha_1(t,\xi) = \alpha_1(0,\xi) = \varepsilon\psi(\xi). \tag{4.7}$$

Also, it follows from (3.7) that

$$\left|\partial_t \alpha_{j+1}(t,\xi) - e^{-i3\pi/4} t^{-1/2} \alpha_j(t,\xi)^2\right| \le C t^{-3/4} \rho_j(t)^2$$

for $j = 1, 2$. By (4.4) and (4.7), we have

$$\begin{aligned} |\alpha_2(t,\xi) - 2e^{-i3\pi/4}\varepsilon^2\psi(\xi)^2 t^{1/2}| &\le |\alpha_2(1,\xi) - 2e^{-i3\pi/4}\varepsilon^2\psi(\xi)^2| + C\int_1^t \rho_1(\tau)^2 \frac{d\tau}{\tau^{3/4}} \\ &\le C\varepsilon^2 + C\varepsilon^2 \int_1^t \frac{d\tau}{\tau^{3/4}} \\ &\le C\varepsilon^2 t^{1/4} \end{aligned} \tag{4.8}$$

for $t \ge 1$. As for α_3, it follows from (4.5) that

$$\begin{aligned} \left|\alpha_3(t,\xi) - e^{-i3\pi/4}\int_1^t (\alpha_2(\tau,\xi))^2 \frac{d\tau}{\tau^{1/2}}\right| &\le |\alpha_3(1,\xi)| + C\int_1^t \rho_2(\tau)^2 \frac{d\tau}{\tau^{3/4}} \\ &\le C\varepsilon^4 + C\varepsilon^4 \int_1^t \tau^{1/4} d\tau \\ &\le C\varepsilon^4 t^{5/4}. \end{aligned}$$

On the other hand, (4.8) yields

$$\begin{aligned} \left|\int_1^t (\alpha_2(\tau,\xi))^2 - (2e^{-i3\pi/4}\varepsilon^2\psi(\xi)^2\tau^{1/2})^2 \frac{d\tau}{\tau^{1/2}}\right| &\le \int_1^t C\varepsilon^2\tau^{1/4}\cdot C\varepsilon^2\tau^{1/2}\frac{d\tau}{\tau^{1/2}} \\ &\le C\varepsilon^4 t^{5/4}. \end{aligned}$$

Summing up, we deduce that

$$\left|\alpha_3(t,\xi) - \frac{8}{3}e^{-i9\pi/4}\varepsilon^4\psi(\xi)^4 t^{3/2}\right| \le C\varepsilon^4 t^{5/4}$$

for $t \ge 1$. In particular, we obtain

$$\|\alpha_3(t,\cdot)\|_{L^\infty} \ge C^*\varepsilon^4 t^{3/2} - C\varepsilon^4 t^{5/4}, \tag{4.9}$$

where $C^* = \frac{8}{3}\|\psi\|_{L^\infty}^4 > 0$. From (4.6), (4.9) and Lemma 1 it follows that

$$\begin{aligned} \|v_3(t,\cdot)\|_{L^\infty} &\ge t^{-1/2}\|\alpha_3(t,\cdot)\|_{L^\infty} - Ct^{-3/4}\rho_3(t) \\ &\ge C^*\varepsilon^4 t - C\varepsilon^4 t^{3/4}. \end{aligned}$$

By taking K large enough, we have

$$\begin{aligned} \|v_3(K\varepsilon^{-4},\cdot)\|_{L^\infty} &\ge C^*K - C\varepsilon K^{3/4} \\ &\ge (C^*K^{1/4} - C)K^{3/4} \\ &> 1, \end{aligned}$$

which completes the proof of (4.3). □

Finally, we give an outline of the proof of (2), (3), (4) of Theorem 1. In the case of (2), the problem is reduced to getting growth estimates for the solution $(v_j)_{j=1,2,3}$ to

$$\begin{cases} \mathcal{L}_{m_1} v_1 = 0, \\ \mathcal{L}_{m_2} v_2 = {v_1}^2, \\ \mathcal{L}_{m_3} v_3 = v_1 v_2 \end{cases}$$

with the initial condition (4.1). Along the same line as the preceding argument, we can show that

$$\rho_{3,s}(t) \le C\varepsilon^3 t \tag{4.10}$$

and

$$\|\alpha_3(t,\cdot)\|_{L^\infty} \ge 2\varepsilon^3\|\psi\|_{L^\infty}^3 t - C\varepsilon^3 t^{3/4}.$$

Note that the identity (3.2), instead of (3.1), plays the key role in the proof of (4.10). By virtue of (3.8) and Lemma 1, we have

$$\sup_{0\le t\le K'\varepsilon^{-6}} \|v_3(t,\cdot)\|_{L^\infty} < 1 \quad \text{and} \quad \|v_3(K'\varepsilon^{-6},\cdot)\|_{L^\infty} > 1$$

with some positive constants κ' and K'. It follows from these estimates that $T_\varepsilon \in (\kappa'\varepsilon^{-6}, K'\varepsilon^{-6})$, which yields the desired conclusion. As for the proof of (3), we just have to replace (3.2) with (3.3) to obtain (4.10). The proof of (4) is also similar: just use

$$\|\mathcal{J}_{m_2}(t)(|v_2|v_2)\|_{L^2} \le C\|v_2\|_{L^\infty}\|\mathcal{J}_{m_2}(t)v_2\|_{L^2} \tag{4.11}$$

in order to get the growth bound for $\rho_{3,1}(t)$. $\square$

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Appendix. A quick review on blow-up of negative energy solutions

To make the difference between typical blow-up results and ours clearer, we will give a quick review on the proof of finite time blow-up for the 3-component NLS system

$$\begin{cases} \left(i\partial_t + \dfrac{1}{2m_1}\Delta\right)u_1 = \overline{u_2}u_3, \\ \left(i\partial_t + \dfrac{1}{2m_2}\Delta\right)u_2 = \overline{u_1}u_3, \quad t > 0,\ x \in \mathbb{R}^n, \\ \left(i\partial_t + \dfrac{1}{2m_3}\Delta\right)u_3 = u_1u_2, \end{cases} \tag{A.1}$$

under the assumptions $E[u(0)] < 0$, $m_3 = m_1 + m_2$ and $4 \le n \le 6$, where the energy $E[\,\cdot\,]$ is defined by

$$E[\psi] = \sum_{j=1}^{3} \frac{1}{2m_j}\|\nabla\psi_j\|_{L^2}^2 + 2\mathrm{Re}\int_{\mathbb{R}^n} \psi_1(x)\psi_2(x)\overline{\psi_3(x)}dx$$

for $\psi = (\psi_j)_{j=1,2,3}$. Note that the 2-component system (1.2) can be regarded as a degenerate case of (A.1), and the relation $m_3 = m_1 + m_2$ should be interpreted as the mass resonance relation associated with (A.1).

The core of the proof is that the following three identities hold (cf. [30,31], etc.):

$$\frac{d}{dt}E[u(t)] = 0,$$

$$\frac{d}{dt}\sum_{j=1}^{3} m_j\|xu_j(t)\|_{L^2}^2 = 2V[u(t)] - 2(m_1 + m_2 - m_3)\mathrm{Im}\int_{\mathbb{R}^n} |x|^2 u_1(t,x)u_2(t,x)\overline{u_3(t,x)}dx,$$

$$\frac{d}{dt}V[u(t)] = \frac{n}{2}E[u(t)] + \frac{4-n}{2}\sum_{j=1}^{3}\frac{1}{2m_j}\|\nabla u_j(t)\|_{L^2}^2,$$

where $V[\,\cdot\,]$ is defined by

$$V[\psi] = \sum_{j=1}^{3} \mathrm{Im}\int_{\mathbb{R}^n} \overline{\psi_j(x)}x\cdot\nabla\psi_j(x)dx.$$

Once these identities are obtained, we can easily see that

$$\sum_{j=1}^{3} m_j\|xu_j(t)\|_{L^2}^2 \le \sum_{j=1}^{3} m_j\|xu_j(0)\|_{L^2}^2 + 2V[u(0)]t + \frac{n}{2}E[u(0)]t^2 < 0$$

for sufficiently large t. This contradiction implies the non-existence of global solutions to (A.1) in $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2dx)$ when $E[u(0)] < 0$, $m_3 = m_1 + m_2$ and $n \ge 4$, while the local existence for (A.1) can be shown when $n \le 6$, which comes from $p + 1 \le \frac{2n}{n-2}$ with $p = 2$ (see [14] for the details).

We remark that $E[u(0)] < 0$ implies $u(0)$ cannot be arbitrarily small, because

$$E[\varepsilon\psi] = \varepsilon^2\left(\sum_{j=1}^{3}\frac{1}{2m_j}\|\nabla\psi_j\|_{L^2}^2 + 2\varepsilon\mathrm{Re}\int_{\mathbb{R}^n}\psi_1(x)\psi_2(x)\overline{\psi_3(x)}dx\right) > 0$$

if $\psi \ne 0$ and $\varepsilon > 0$ is small enough. In fact, we can show the global existence of solutions to (A.1) if the data are suitably small in $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2dx)$ when $n = 4$ (see [14]). In this sense, our small data blow-up result presented in Theorem 1 should be distinguished from this kind of "large data" blow-up.

References

[1] M. Colin, T. Colin, On a quasilinear Zakharov system describing laser-plasma interactions, Differential Integral Equations 17 (3–4) (2004) 297–330.
[2] M. Colin, T. Colin, M. Ohta, Stability of solitary waves for a system of nonlinear Schrödinger equations with three wave interaction, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (6) (2009) 2211–2226.
[3] N. Hayashi, C. Li, P.I. Naumkin, On a system of nonlinear Schrödinger equations in 2d, Differential Integral Equations 24 (5–6) (2011) 417–434.
[4] N. Hayashi, C. Li, P.I. Naumkin, Modified wave operator for a system of nonlinear Schrödinger equations in 2d, Comm. Partial Differential Equations 37 (6) (2012) 947–968.
[5] N. Hayashi, C. Li, T. Ozawa, Small data scattering for a system of nonlinear Schrödinger equations, J. Difference Equ. Appl. 3 (3) (2011) 415–426.
[6] J.-M. Delort, D. Fang, R. Xue, Global existence of small solutions for quadratic quasilinear Klein–Gordon systems in two space dimensions, J. Funct. Anal. 211 (2) (2004) 288–323.
[7] S. Katayama, T. Ozawa, H. Sunagawa, A note on the null condition for quadratic nonlinear Klein–Gordon systems in two space dimensions, Comm. Pure Appl. Math. 65 (9) (2012) 1285–1302.
[8] Y. Kawahara, H. Sunagawa, Global small amplitude solutions for two-dimensional nonlinear Klein–Gordon systems in the presence of mass resonance, J. Differential Equations 251 (9) (2011) 2549–2567.
[9] H. Sunagawa, On global small amplitude solutions to systems of cubic nonlinear Klein–Gordon equations with different mass terms in one space dimension, J. Differential Equations 192 (2) (2003) 308–325.
[10] H. Sunagawa, A note on the large time asymptotics for a system of Klein–Gordon equations, Hokkaido Math. J. 33 (2) (2004) 457–472.
[11] H. Sunagawa, Large time asymptotics of solutions to nonlinear Klein–Gordon systems, Osaka J. Math. 42 (1) (2005) 65–83.
[12] E. Taflin, Simple non-linear Klein–Gordon equations in two space dimensions, with long-range scattering, Lett. Math. Phys. 79 (2) (2007) 175–192.
[13] Y. Tsutsumi, Stability of constant equilibrium for the Maxwell–Higgs equations, Funkcial. Ekvac. 46 (1) (2003) 41–62.
[14] N. Hayashi, T. Ozawa, K. Tanaka, On a system of nonlinear Schrödinger equations with quadratic interaction, preprint, 2012.
[15] N. Hayashi, P.I. Naumkin, H. Sunagawa, On the Schrödinger equation with dissipative nonlinearities of derivative type, SIAM J. Math. Anal. 40 (1) (2008) 278–291.
[16] M. Ikeda, Y. Wakasugi, Nonexistence of a non-trivial global weak solution for the nonlinear Schrödinger equation with a nongauge invariant power nonlinearity, arXiv:1111.0178 [math.AP].
[17] T. Oh, A blowup result for the periodic NLS without gauge invariance, C. R. Acad. Sci. Paris 350 (2012) 389–392.
[18] T. Ozawa, Y. Yamazaki, Life-span of smooth solutions to the complex Ginzburg–Landau type equation on a torus, Nonlinearity 16 (6) (2003) 2029–2034.
[19] H. Sasaki, On the life span of the Schrödinger equation with sub-critical power nonlinearity, Adv. Differential Equations 14 (11–12) (2009) 1021–1039.
[20] H. Sunagawa, Lower bounds of the lifespan of small data solutions to the nonlinear Schrödinger equations, Osaka J. Math. 43 (4) (2006) 771–789.
[21] H. Sunagawa, The lifespan of solutions to nonlinear Schrödinger and Klein–Gordon equations, Hokkaido Math. J. 37 (4) (2008) 825–838.
[22] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950) 201–230.
[23] J.D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, Quart. Appl. Math. 9 (1951) 225–236.
[24] T. Ozawa, Remarks on quadratic nonlinear Schrödinger equations, Funkcial. Ekvac. 38 (2) (1995) 217–232.
[25] T. Ozawa, Finite energy solutions for the Schrödinger equations with quadratic nonlinearity in one space dimension, Funkcial. Ekvac. 41 (3) (1998) 451–468.
[26] W.A. Strauss, Nonlinear wave equations, in: CBMS Regional Conference Series in Mathematics, vol. 73, American Mathematical Society, Providence, RI, 1989.
[27] K. Yagdjian, Parametric resonance and nonexistence of the global solution to nonlinear wave equations, J. Math. Anal. Appl. 260 (1) (2001) 251–268.
[28] K. Kajitani, K. Yagdjian, Quasilinear hyperbolic operators with the characteristics of variable multiplicity, Tsukuba J. Math. 22 (1) (1998) 49–85.
[29] M. Ohta, H. Takamura, Remarks on the blow-up boundaries and rates for nonlinear wave equations, Nonlinear Anal. 33 (7) (1998) 693–698.
[30] R.T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. 18 (9) (1977) 1794–1797.
[31] M. Tsutsumi, Nonexistence of global solutions to the Cauchy problem for the damped nonlinear Schrödinger equations, SIAM J. Math. Anal. 15 (2) (1984) 357–366.