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A Newton iteration for differentiable set-valued maps

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ABSTRACT

We employ recent developments of generalized differentiation concepts for set-valued mappings and present a Newton-like iteration for solving generalized equations of the form $f(x) + F(x) \ni 0$ where f is a single-valued function while F stands for a set-valued map, both of them being smooth mappings acting between two general Banach spaces X and Y. The Newton iteration we propose is constructed on the basis of a linearization of both f and F; we prove that, under suitable assumptions on the "derivatives" of f and F, it converges Q-linearly to a solution to the generalized equation in question. When we strengthen our assumptions, we obtain the Q-quadratic convergence of the method.

1. Introduction

In 1994, Bonnans presented in [1] a Newton-type algorithm for variational inequalities, *i.e.*, inequalities of the form

$$\langle f(x), y - x\rangle \geq 0, \quad \forall y \in K,\ x \in K,$$

where f is a continuously differentiable mapping from $\mathbb{R}^q$ to itself and K is a nonempty closed convex subset of $\mathbb{R}^q$. It is well-known that such an inequality can be equivalently rewritten as

$$f(x) + N_K(x) \ni 0, \tag{1}$$

where N_K denotes the normal cone of K at x, defined by

$$N_K(x) = \begin{cases} \{v \in \mathbb{R}^q \mid \langle v, y - x\rangle \leq 0, \forall y \in K\} & \text{if } x \in K; \\ \emptyset & \text{otherwise.} \end{cases}$$

Bonnans proposed the following iterative scheme for solving (1):

$$f(x_k) + M_k(x_{k+1} - x_k) + N_K(x_{k+1}) \ni 0, \tag{2}$$

where M_k stands for a $q \times q$ matrix. Two key notions appear to be crucial in [1] for establishing convergence results of the above method. They read as follows.

Semistable solutions. A solution $\bar{x}$ of (1) is said to be semistable if there are positive constants c_1 and c_2 such that, for all $(x, y) \in \mathbb{R}^q \times \mathbb{R}^q$, such that $f(x) + N_K(x) \ni y$, and $x \in \mathbb{B}_{c_1}(\bar{x})$, then $\|x - \bar{x}\| \leq c_2\|y\|$.

Hemistable solutions. A solution $\bar{x}$ of (1) is said to be hemistable if, for all $\alpha > 0$, there exists $\varepsilon > 0$ such that, given $\hat{x} \in \mathbb{R}^q$, the variational inequality (in x)

$$f(\hat{x}) + M(x - \hat{x}) + N_K(x) \ni 0$$

has a solution x satisfying $\|x - \bar{x}\| \leq \alpha$, whenever $\|\hat{x} - \bar{x}\| + \|M - f'(\bar{x})\| < \varepsilon$.

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Bonnans proved the following two results by considering successively the case when the variational inequality (1) admits a semistable solution and a solution that is both semistable and hemistable.

Convergence under semistability (*Bonnans* [1]). *Let $\bar{x}$ be a semistable solution to* (1), *and let x_k be a sequence satisfying* (2) *and converging to $\bar{x}$. Then*:

(i) *If $(f'(\bar{x}) - M_k)(x_{k+1} - x_k) = o(x_{k+1} - x_k)$, then x_k converges superlinearly.*
(ii) *If $(f'(\bar{x}) - M_k)(x_{k+1} - x_k) = O(\|x_{k+1} - x_k\|^2)$ and f' is locally Lipschitz, then x_k converges quadratically.*

Convergence under semi stability and hemistability (*Bonnans* [1]). *If $\bar{x}$ is a semistable and hemistable solution of* (1), *there exists $\varepsilon > 0$ such that if $\|x_0 - \bar{x}\| \leq \varepsilon$, then*:

(i) *There is a sequence x_k satisfying* (2), *with $M_k = f'(x_k)$, such that $\|x_{k+1} - x_k\| \leq 2\varepsilon$.*
(ii) *The sequence x_k converges superlinearly* (*quadratically if f' is locally Lipschitz*) *to $\bar{x}$.*

Note that the semistability of the solution $\bar{x}$ in the first statement does not imply the existence of a Newton sequence x_k; to prove the existence of such a sequence, the author needed an additional assumption, namely, the hemistability of the solution $\bar{x}$.

Two years later, Dontchev considered in [2] a more general setting by presenting a Newton method for solving generalized equations of the form

$$f(x) + F(x) \ni 0, \tag{3}$$

with $f : X \to Y$ a function and $F : X \rightrightarrows Y$ a set-valued map, X and Y being Banach spaces. Dontchev showed the local quadratic convergence of a Newton-type iteration, based on a partial linearization of the mapping $f + F$, for solving (3). More precisely, by considering the iterative procedure

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}), \tag{4}$$

where ∇f is the Fréchet derivative of f, he proved the following result:

Local convergence of a Newton-type method (*Dontchev* [2]). *Let x^* be a solution of* (3), *let f be a function which is Fréchet differentiable in an open neighborhood O of x^*, and let its derivative ∇f be Lipschitz in O with constant L. Let F have closed graph and let the map $(f(x^*) + \nabla f(x^*)(\cdot - x^*) + F(\cdot))^{-1}$ be Aubin continuous at $(0, x^*)$ with modulus M. Then for every $c > (1/2)ML$ one can find $\delta > 0$ such that for every starting point $x_0 \in \mathbb{B}_\delta(x^*)$ there exists a Newton sequence x_k for* (3), *defined by* (4), *which satisfies*

$$\|x_{k+1} - x_k\| \leq c\|x_k - x^*\|^2.$$

The *Aubin continuity* assumption in the above statement refers to the *pseudo-Lipschitz continuity* of the mapping in question (see Section 2, Definition 2.3). Additional results in connection with this Newton iteration for generalized equations can be found in [3,4].

It was also in the mid-nineteen nineties that Azé and Chou [5] presented a Newton method for solving the inclusion

$$F(x) \ni 0, \tag{5}$$

involving a *strictly lower differentiable* set-valued map F acting between two Banach spaces X and Y. Under some assumptions on the derivative of F at a reference point in the graph of F they showed the (strong) convergence of a Newton-type method for solving (5). Lately, following the works of Aze and Chou, Dias and Smirnov [6] considered a Newton iteration, for solving (5) in finite dimension, involving a locally Lipschitz continuous and differentiable set-valued map. They proved the Q-quadratic convergence of their algorithm under some additional regularity properties on the mapping F. We will take a closer look at theses works of Aze and Chou and Dias and Smirnov in the last section of this paper.

Both Bonnans and Dontchev's iterations rely on a partial linearization technique, in the sense that they only linearize the single-valued function f and leave the set-valued mapping (N_K or F) unchanged. In [5,6] the authors are interested in solving the general inclusion (5). In this paper, our approach is somewhat different. First of all, we are primarily interested in solving the generalized equation

$$f(x) + F(x) \ni 0; \tag{6}$$

and it is our belief that it is worth studying this specific type of inclusions in a separate manner rather than just considering them as a special case of inclusion (5); first, because of the important role played by the function f and second because they perfectly fit the framework of several problems in variational analysis such as complementarity problems, first order necessary conditions or feasibility problems. Taking advantage of some recent developments on generalized differentiation concepts for set-valued mappings, we propose here a Newton-type iteration for solving (6) where both the single-valued function f, which is supposed to be not identically zero, and the set-valued mapping F are "linearized".

In a seminal paper dealing with differential calculus of nondifferentiable mappings, Ioffe [7] introduced a general class of objects, for local approximation of nonsmooth single-valued functions, consisting of set-valued mappings which are positively homogeneous and closed-valued (see Definition 2.1). Such objects are called *prederivatives.* This concept extended the framework of differential calculus to more general classes of functions. Ever since, a growing literature on generalized differentiation of set-valued maps attests the importance of this topic, especially in variational analysis where this kind of tools happen to be crucial. For an overview of the State-of-the-Art, one may refer to the monograph by Mordukhovich [8]. Lately, Pang [9] adapted the work of Ioffe by proposing a notion of generalized differentiation for set-valued mappings, called T-differentiability, involving positively homogeneous maps (in this work, we chose to call it H-differentiability because of the positive homogeneity property of the mapping H). Actually, Pang introduced several concepts of generalized differentiability, all of them inspired by the works of Ioffe. The one we are interested in is called *strict H-differentiability* and reads as follows.

Strict H-differentiability (*Pang* [9]). *Let X and Y be Banach spaces. Let H be a positively homogeneous set-valued mapping from X into Y. We say that a set-valued mapping F from X into Y is* strictly H-differentiable *at $\bar{x}$ if for any $\delta > 0$, there exists a neighborhood U of $\bar{x}$ such that*

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \quad \text{for all } x, x' \in U. \tag{7}$$

For a comprehensive study on these concepts of generalized differentiability for set-valued mappings involving positively homogeneous maps one can refer to [9] and as well to [10,11] for recent advances on this topic.

In this paper, we work in general real Banach spaces X and Y, we assume that the single-valued function $f : X \to Y$ is Fréchet differentiable on a neighborhood of some solution $\bar{x}$ to (6) and that its derivative is Lipschitz continuous around this point. In addition, we suppose that the set-valued mapping $F : X \rightrightarrows Y$ is strictly H-differentiable at $\bar{x}$ in the sense of Pang for some suitable positively homogeneous mapping H. Then, we prove in our main result (Theorem 3.3) that, whenever the mapping H^{-1} is pseudo-Lipschitz continuous around $(0, 0)$, there is a neighborhood Ω of $\bar{x}$ such that for any initial guess $x_0 \in \Omega$, there exists a sequence x_n, the elements of which lie in Ω, generated by the following Newton iteration

$$f(x_n) + \nabla f(x_n)(x_{n+1} - x_n) + F(x_n) + H(x_{n+1} - x_n) \ni 0, \tag{8}$$

which converges Q-linearly to the solution $\bar{x}$ to (6).

In the same way $f(x_n) + \nabla f(x_n)(x_{n+1} - x_n)$ linearizes $f(x_{n+1})$, the expression $F(x_n) + H(x_{n+1} - x_n)$ can be viewed as a first-order approximation of $F(x_{n+1})$, hence the iteration (8) is nothing but the method (4) proposed by Dontchev where we have replaced $F(x_{n+1})$ with its first order approximation. When the set-valued mapping F does not enjoy "nice" properties (as being a cone, for instance), the set $F(x_{n+1})$ in Dontchev's method could be difficult to deal with in practical situations, in that case the iteration (8) could provide us with a useful alternative.

We end the present section by providing some notation, then, in Section 2, we collect some definitions and results that we will need in the sequel. In Section 3, we state and prove our theorems regarding the Q-linear (local) convergence of the Newton iteration (8) for solving the generalized equation (6) while we prove in Section 4 its Q-quadratic convergence by considering some additional assumptions. Finally, in the last section, we point out the major differences between the Newton method we presented and the other works we are aware of; by doing so, we try to emphasize the interest of our study.

Notation. Let (E, d) be a metric space. If $x \in E$ and $\rho > 0$, then the (closed) ball with center x and radius ρ is $\mathbb{B}_\rho(x) := \{z \in E \mid d(z, x) \le \rho\}$ and the (closed) unit ball is denoted by $\mathbb{B}$. The interior of a subset A in E is denoted by $\operatorname{int}(A)$. If A and B are two subsets of (E, d), the *excess* of A over B (with respect to d) is defined by the formula

$$e(A, B) = \sup_{a \in A} d(a, B).$$

It is clear that $e(A, B) = \inf\{\varepsilon > 0 \mid A \subset B + \varepsilon\mathbb{B}\}$, moreover, we adopt the convention that $e(\emptyset, B) = 0$ when $B \ne \emptyset$ and $e(\emptyset, B) = \infty$ if $B = \emptyset$.

Let F be a set-valued mapping from X into the subsets of Y, indicated by $F : X \rightrightarrows Y$. Then, $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F and the range of F is the set $\operatorname{rge} F = \{y \in Y \mid \exists x,\ F(x) \ni y\}$. The inverse of F, denoted by F^{-1}, is defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

2. Background material

The concept of generalized differentiation we are dealing with strongly relies on positively homogeneous set-valued mappings. This is the reason why we start this section by recalling a few facts about these particular mappings. First, we state their definition.

Definition 2.1. Let $H : X \rightrightarrows Y$ be a set-valued mapping. It is called positively homogeneous if $H(0) \ni 0$ and $H(\lambda x) = \lambda H(x)$ for all $x \in X$ and $\lambda > 0$.

One can immediately note that a mapping is positively homogeneous if and only if its graph is a cone and that the inverse of a positively homogeneous mapping is another positively homogeneous mapping. Graphical derivatives of set-valued mappings, introduced by Aubin [12], are positively homogeneous set-valued mappings and so are sublinear mappings (*i.e.,* set-valued mappings such that their graph is a convex cone).

To be able to work efficaciously with positively homogeneous mappings we need the following tool known as the *outer norm*.

Definition 2.2. Let $H : X \rightrightarrows Y$ be a positively homogeneous mapping. The outer norm of H is

$$|H|^+ = \sup_{\|x\|\le 1} \sup_{y\in H(x)} \|y\|, \tag{9}$$

with the convention that $\sup_{y\in\emptyset} \|y\| = -\infty$.

Note that an equivalent (and useful) formulation of (9) is given by

$$|H|^+ = \inf\{\kappa > 0 \mid H(\mathbb{B}) \subset \kappa\mathbb{B}\}.$$

Let us now present a Lipschitz-like concept and a regularity property for set-valued mappings. The pseudo-Lipschitz continuity for set-valued maps (also known as the Aubin property or Aubin continuity) has been introduced by Aubin in [13,14] in the framework of the inverse function theorem in finite-dimensional spaces, and later, in [15] for Banach spaces. It definition reads as follows.

Definition 2.3 (*Pseudo-Lipschitz Continuity*)**.** A set-valued mapping $F : X \rightrightarrows Y$ is said to be pseudo-Lipschitz around the point $(\bar{x}, \bar{y})$ of its graph if one of the following equivalent assertions holds

(i) there is a positive constant κ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x') \cap V \subset F(x) + \kappa\|x' - x\|\mathbb{B}, \quad \text{for all } x, x' \in U. \tag{10}$$

(ii) there exist κ, U and V as described in(i), such that

$$e(F(x') \cap V, F(x)) \le \kappa\|x' - x\|, \quad \text{for all } x, x' \in U. \tag{11}$$

Note that, in general, relations (10) and (11) are not themselves equivalent, it is the case when the mapping F is closed-valued on U, *i.e.*, when $F(x)$ is a closed subset of Y for all $x \in U$.

Definition 2.4 (*Strong Metric Subregularity*)**.** A mapping $F : X \rightrightarrows Y$ is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exists $\kappa > 0$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\|x - \bar{x}\| \le \kappa d(\bar{y}, F(x)) \quad \text{for all } x \in U.$$

This property is equivalent to the "local Lipschitz property at a point" of the inverse mapping, a property first formally introduced in [16] where a stability result parallel to the Lyusternik–Graves theorem was proved. Finally, we state the following set-valued generalization of the Banach fixed point established by Dontchev and Hager in [17]. It will play an important role in proving our main convergence theorem.

Theorem 2.5 (*Set-Valued Banach Fixed Point Theorem*)**.** *Let (X, ρ) be a complete metric space, and consider a set-valued mapping $\Phi : X \rightrightarrows X$, a point $\bar{x} \in X$, and nonnegative scalars α and θ be such that $0 \le \theta < 1$, the sets $\Phi(x) \cap \mathbb{B}_\alpha(\bar{x})$ are closed for all $x \in \mathbb{B}_\alpha(\bar{x})$ and the following conditions hold:*

(i) *$d(\bar{x}, \Phi(\bar{x})) < \alpha(1 - \theta)$;*
(ii) *$e(\Phi(u) \bigcap \mathbb{B}_\alpha(\bar{x}), \Phi(v)) \le \theta\rho(u, v)$ for all $u, v \in \mathbb{B}_\alpha(\bar{x})$.*

Then Φ has a fixed point in $\mathbb{B}_\alpha(\bar{x})$. That is, there exists $x \in \mathbb{B}_\alpha(\bar{x})$ such that $x \in \Phi(x)$.

3. Local behavior of the Newton iteration

From now on we assume that the solution set of (6) is nonempty, *i.e.*, there exists an element $\bar{x} \in X$ such that

$$-f(\bar{x}) \in F(\bar{x}). \tag{12}$$

Before giving our main convergence result, we establish two technical lemmas which are useful in the sequel.

Lemma 3.1. *Consider a single-valued function $f : X \to Y$ and let $F : X \rightrightarrows Y$ be a set-valued map that is strictly H-differentiable at $\bar{x}$ for some positively homogeneous map $H : X \rightrightarrows Y$. If $\bar{x}$ is a solution to (6) then for all $\delta > 0$ there exists a neighborhood U of $\bar{x}$ so that for all $x \in U$ there are some elements $z \in F(x)$ and $u \in \mathbb{B}$ such that*

$$-f(\bar{x}) - z - \delta\|\bar{x} - x\|u \in H(\bar{x} - x). \tag{13}$$

Proof. Since the mapping F is strictly H-differentiable at $\bar{x}$, for any $\delta > 0$, there is a constant $a > 0$ such that

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \quad \text{for all } x, x' \in \mathbb{B}_a(\bar{x}). \tag{14}$$

Take any $x \in \mathbb{B}_a(\bar{x})$. From (12) and (14) we obtain

$$-f(\bar{x}) \in F(\bar{x}) \subset F(x) + H(\bar{x} - x) + \delta\|\bar{x} - x\|\mathbb{B}.$$

Therefore there exist $z \in F(x)$ and $u \in \mathbb{B}$ such that

$$-f(\bar{x}) - z - \delta\|\bar{x} - x\|u \in H(\bar{x} - x). \quad \square$$

Lemma 3.2. *Consider a single-valued function $f : X \to Y$, a positively homogeneous mapping $H : X \rightrightarrows Y$ such that $|H|^+ < \infty$, a constant $\delta > 0$ and let $\bar{x}$ be a solution to (6). Assume in addition that the function f is Fréchet differentiable at $\bar{x}$ and that its derivative ∇f is Lipschitz continuous on a neighborhood Ω of $\bar{x}$ with constant $\overline{\kappa}$ and is such that there exists a constant $\hat{\kappa}$ satisfying $\|\nabla f(x)\| \le \hat{\kappa}$, for all $x \in \Omega$. If there exist a constant $\tau > 0$ together with some elements $z \in Y$, $u \in \mathbb{B}$ such that*

$$-f(\bar{x}) - z - \delta\|\bar{x} - x\|u \in H(\bar{x} - x) \quad \text{for all } x \in \mathbb{B}_\tau(\bar{x}) \tag{15}$$

then for all $v \in \mathbb{B}_\tau(\bar{x})$

$$\|f(x) + z + \nabla f(x)(v - x)\| \le \frac{\overline{\kappa}}{2}\|\bar{x} - x\|^2 + (|H|^+ + \delta)\|\bar{x} - x\| + \hat{\kappa}\|v - \bar{x}\|. \tag{16}$$

Proof. Let $\tau > 0$, $z \in Y$ and $u \in \mathbb{B}$ such that relation (15) holds. Then, for each $x \in \mathbb{B}_\tau(\bar{x})$ there exists an element $h \in H(\bar{x} - x)$ such that

$$-f(\bar{x}) - z = \delta\|\bar{x} - x\|u + h,$$

therefore

$$\|f(\bar{x}) + z\| \le (\delta + |H|^+)\|\bar{x} - x\|.$$

Moreover, the function ∇f being Lipschitz continuous on a neighborhood Ω of $\bar{x}$ with constant $\overline{\kappa}$ we have

$$\|\nabla f(x) - \nabla f(x')\| \le \overline{\kappa}\|x - x'\|, \quad \text{for all } x, x' \in \Omega. \tag{17}$$

Now adjust τ if necessary so that $\mathbb{B}_\tau(\bar{x}) \subset \Omega$ and take any $x, v \in \mathbb{B}_\tau(\bar{x})$. We have

$$\begin{aligned}
\|f(x) + \nabla f(x)(v - x) + z\| &= \|f(\bar{x}) - f(x) - \nabla f(x)(v - x) - (z + f(\bar{x}))\| \\
&= \|f(\bar{x}) - f(x) - \nabla f(x)(\bar{x} - x) - (z + f(\bar{x})) - \nabla f(x)(v - \bar{x})\| \\
&\le \left\| \int_0^1 \left[\nabla f(t\bar{x} + (1 - t)x)(\bar{x} - x) - \nabla f(x)(\bar{x} - x)\right] dt \right\| \\
&\quad + \|z + f(\bar{x})\| + \|\nabla f(x)(v - \bar{x})\| \\
&\le \|\bar{x} - x\| \int_0^1 \|\nabla f(t\bar{x} + (1 - t)x) - \nabla f(x)\| dt \\
&\quad + (\delta + |H|^+)\|\bar{x} - x\| + \|\nabla f(x)\|\,\|v - \bar{x}\| \\
&\le \frac{\overline{k}}{2}\|\bar{x} - x\|^2 + (\delta + |H|^+)\|\bar{x} - x\| + \hat{\kappa}\|v - \bar{x}\|,
\end{aligned}$$

which gives us the desired conclusion. $\square$

The following theorem establishes the local convergence of the Newton iteration we consider in this paper.

Theorem 3.3. *Consider the generalized equation (6). Assume that there is a point $\bar{x} \in X$ satisfying (12) and that the function f is Fréchet differentiable on a neighborhood Ω of $\bar{x}$ while the set-valued mapping F is strictly H-differentiable at $\bar{x}$. In addition, we suppose that:*

- (i) *the derivative of f, ∇f, is Lipschitz continuous on Ω with a constant $\bar{\kappa}$ and is such that there exists a constant $\hat{\kappa}$ satisfying $\|\nabla f(x)\| \le \hat{\kappa}$, for all $x \in \Omega$;*
- (ii) *the graph of the set-valued mapping H^{-1} is locally closed at $(0, 0)$ and H^{-1} is pseudo-Lipschitz continuous around this point with a constant $\kappa > 0$, moreover, $|H|^+ < \infty$;*
- (iii) *$\kappa\tilde{\kappa} < 1/2$, where $\tilde{\kappa} := \max\{\bar{\kappa}, \hat{\kappa}\}$.*

Then, there is a positive constant r such that for any initial guess $x_0 \in \mathbb{B}_r(\bar{x})$ there exists a sequence x_n, the elements of which lie in $\mathbb{B}_r(\bar{x})$, generated by the Newton iteration (8) and converging Q-linearly to $\bar{x}$.

Proof. Fix $\delta \in (0, \tilde{\kappa}/2)$, since the mapping F is strictly H-differentiable at $\bar{x}$, there is a constant $a > 0$ such that

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \quad \text{for all } x, x' \in \mathbb{B}_a(\bar{x}). \tag{18}$$

Making a smaller if necessary we can assume that

(a) $a < 1$
(b) H^{-1} is pseudo-Lipschitz continuous around the point $(0, 0)$ with constant κ and with neighborhoods $\mathbb{B}_a(0)$ and $\mathbb{B}_b(0)$, *i.e.*,

$$H^{-1}(y) \cap \mathbb{B}_a(0) \subset H^{-1}(y') + \kappa\|y - y'\|\mathbb{B}, \quad \text{for all } y, y' \in \mathbb{B}_b(0); \tag{19}$$

(c) $\mathbb{B}_a(\bar{x}) \subset \Omega$;
(d) $(\frac{\tilde{\kappa}}{2}a + \frac{3\tilde{\kappa}}{2} + |H|^+)a \leq b$.

Let $r < a/2$ and take any $x_0 \in \mathbb{B}_r(\bar{x})$. Thanks to Lemma 3.1 (and its proof) there exist $z_0 \in F(x_0)$ and $u_0 \in \mathbb{B}$ such that

$$-f(\bar{x}) - z_0 - \delta\|\bar{x} - x_0\|u_0 \in H(\bar{x} - x_0). \tag{20}$$

Consider now the set-valued mapping $\Phi_0 : X \rightrightarrows X$ defined by

$$\Phi_0 : x \mapsto H^{-1}(-f(x_0) - z_0 - \nabla f(x_0)(x - x_0)) + x_0.$$

One can immediately see that if x_1 is a fixed point of Φ_0 then one has

$$-f(x_0) - \nabla f(x_0)(x_1 - x_0) \in z_0 + H(x_1 - x_0) \subset F(x_0) + H(x_1 - x_0),$$

which implies $f(x_0) + \nabla f(x_0)(x_1 - x_0) + F(x_0) + H(x_1 - x_0) \ni 0$, that is, the Newton iteration (8) is satisfied for $n = 0$. To prove the existence of such an element x_1, we apply the set-valued Banach fixed point theorem (Theorem 2.5) to the mapping Φ_0. First, we show that $w_0 := -f(x_0) - z_0 - \nabla f(x_0)(\bar{x} - x_0) \in \mathbb{B}_b(0)$. Applying Lemma 3.2 with $v = \bar{x}$, $\tau = r$ together with the choice of the constants $\tilde{\kappa}$, δ and a we get

$$\begin{aligned}\|w_0\| &\leq \frac{\tilde{\kappa}}{2}\|\bar{x} - x_0\|^2 + (|H|^+ + \delta)\|\bar{x} - x_0\| \\ &\leq \left(\frac{\tilde{\kappa}}{2}a + \frac{\tilde{\kappa}}{2} + |H|^+\right)a \leq b.\end{aligned}$$

Therefore,

$$-f(x_0) - z_0 - \nabla f(x_0)(\bar{x} - x_0) \in \mathbb{B}_b(0). \tag{21}$$

And, using (20) again, it follows that $\| - z_0 - f(\bar{x}) - \delta\|\bar{x} - x_0\|u_0\| \leq |H|^+a \leq b$, *i.e.*,

$$-z_0 - f(\bar{x}) - \delta\|\bar{x} - x_0\|u_0 \in \mathbb{B}_b(0). \tag{22}$$

By hypothesis, the constants κ and $\tilde{\kappa}$ are such that $\kappa\tilde{\kappa} < 1/2$, therefore there exists a number γ such that $\frac{\kappa\tilde{\kappa}}{1-\kappa\tilde{\kappa}} < \gamma < 1$. Let us set $\alpha_0 := \gamma\|x_0 - \bar{x}\|$. Note that, since $\gamma < 1$, we have $\alpha_0 < r$. Thanks to the pseudo-Lipschitz continuity of H^{-1} around $(0, 0)$ together with relations (20)–(22) we obtain

$$\begin{aligned}d(\bar{x}, \Phi_0(\bar{x})) &= d(\bar{x} - x_0, H^{-1}(-f(x_0) - z_0 - \nabla f(x_0)(\bar{x} - x_0))) \\ &\leq eH^{-1}(-f(\bar{x}) - z_0 - \delta\|\bar{x} - x_0\|u_0) \cap \mathbb{B}_a(0), H^{-1}(-f(x_0) - z_0 - \nabla f(x_0)(\bar{x} - x_0)) \\ &\leq \kappa\|f(\bar{x}) - f(x_0) - \nabla f(x_0)(\bar{x} - x_0)\| + \kappa\delta\|\bar{x} - x_0\| \\ &\leq \frac{\kappa\tilde{\kappa}}{2}\|\bar{x} - x_0\|^2 + \kappa\delta\|\bar{x} - x_0\| \leq \frac{\kappa\tilde{\kappa}}{2}r\|\bar{x} - x_0\| + \frac{\kappa\tilde{\kappa}}{2}\|\bar{x} - x_0\| \\ &\leq \kappa\tilde{\kappa}\|\bar{x} - x_0\| < \alpha_0(1 - \kappa\tilde{\kappa}).\end{aligned}$$

Consequently, the first assumption in Theorem 2.5 is satisfied. Now, we show that the second one holds too.

Take arbitrary points u and v in $\mathbb{B}_{\alpha_0}(\bar{x}) \subset \mathbb{B}_r(\bar{x})$. Let $\Psi^0_{u,v} := e\big(\Phi_0(u) \cap \mathbb{B}_{\alpha_0}(\bar{x}), \Phi_0(v)\big)$, and define the points z^0_u and z^0_v of Y by

$$z^0_u := -f(x_0) - z_0 - \nabla f(x_0)(u - x_0) \quad \text{and} \quad z^0_v := -f(x_0) - z_0 - \nabla f(x_0)(v - x_0).$$

Let us prove that z^0_u and z^0_v are in $\mathbb{B}_b(0)$:

Using Lemma 3.2 (with $v = u$ and $\tau = r$) together with the choice of the constants $\tilde{\kappa}$, δ and a, we obtain

$$\begin{aligned}\|z^0_u\| &\leq \frac{\tilde{\kappa}}{2}\|x_0 - \bar{x}\|^2 + \|z_0 + f(\bar{x})\| + \tilde{\kappa}\|u - \bar{x}\| \\ &\leq \left(\frac{\tilde{\kappa}}{2}a + \delta + |H|^+ + \tilde{\kappa}\right)a \leq \left(\frac{\tilde{\kappa}}{2}a + \frac{3\tilde{\kappa}}{2} + |H|^+\right)a \leq b.\end{aligned}$$

Likewise it is easy to see that $z^0_v \in \mathbb{B}_b(0)$. Moreover, we have

$$\Psi^0_{u,v} = \sup\{d(\zeta^0_u - x_0, H^{-1}(-f(x_0) - z_0 - \nabla f(x_0)(v - x_0))) \mid \zeta^0_u \in \Phi_0(u) \cap \mathbb{B}_{\alpha_0}(\bar{x})\}.$$

Noting that, for all $\zeta_u^0 \in \Phi_0(u)\cap\mathbb{B}_{\alpha_0}(\bar{x})$, one has $\zeta_u^0 - x_0 \in H^{-1}(z_u^0)\cap\mathbb{B}_a(0)$ we get, thanks to the pseudo-Lipschitz continuity of the mapping H^{-1} around $(0, 0)$,

$$\begin{aligned}\Psi_{u,v}^0 &\le e(H^{-1}(z_u^0)\cap\mathbb{B}_a(0), H^{-1}(z_v^0))\\ &\le \kappa\|z_u^0 - z_v^0\|\\ &= \kappa\|\nabla f(x_0)(u-x_0) - \nabla f(x_0)(v-x_0)\|\\ &\le \kappa\tilde{\kappa}\|u-v\|.\end{aligned}$$

Then, Theorem 2.5 provides us with the existence of a fixed point $x_1 \in \mathbb{B}_{\alpha_0}(\bar{x}) \subset \mathbb{B}_r(\bar{x})$ of Φ_0, *i.e.*, the existence of a point x_1 such that

$$\|x_1 - \bar{x}\| \le \gamma\|x_0 - \bar{x}\|,$$

and satisfying

$$f(x_0) + \nabla f(x_0)(x_1 - x_0) + F(x_0) + H(x_1 - x_0) \ni 0.$$

We now proceed by induction to complete the proof. Assume that there are elements $x_1, x_2, \dots, x_n$ in X, all of them lying in the ball $\mathbb{B}_r(\bar{x})$, such that, for $i = 0, 1, \dots, n-1$, relation (8) holds and $\|x_{i+1} - \bar{x}\| \le \gamma\|x_i - \bar{x}\|$.

Since $x_n \in \mathbb{B}_a(\bar{x})$, using Lemma 3.1, and similar arguments as in (21) and (22), we show that there exist $z_n \in F(x_n)$ and $u_n \in \mathbb{B}$ such that

$$-f(\bar{x}) - z_n - \delta\|\bar{x} - x_n\|u_n \in H(\bar{x} - x_n)\cap\mathbb{B}_b(0) \quad \text{and} \quad -f(x_n) - z_n - \nabla f(x_n)(\bar{x} - x_n) \in \mathbb{B}_b(0).$$

Then, we consider the set-valued mapping $\Phi_n : X \rightrightarrows X$ defined by

$$\Phi_n : x \mapsto H^{-1}(-f(x_n) - z_n - \nabla f(x_n)(x - x_n)) + x_n.$$

Let $\alpha_n := \gamma\|\bar{x} - x_n\|$, clearly $\alpha_n < r$. We get, thanks to the pseudo-Lipschitz continuity of H^{-1} around the point $(0, 0)$,

$$\begin{aligned}d(\bar{x}, \Phi_n(\bar{x})) &\le e\big(H^{-1}(-f(\bar{x}) - z_n - \delta\|\bar{x} - x_n\|u_n)\cap\mathbb{B}_a(0), H^{-1}(-f(x_n) - z_n - \nabla f(x_n)(\bar{x} - x_n))\big)\\ &\le \kappa\|f(\bar{x}) - f(x_n) - \nabla f(x_n)(\bar{x} - x_n)\| + \kappa\delta\|\bar{x} - x_n\|\\ &\le \frac{\kappa\tilde{\kappa}}{2}\|x_n - \bar{x}\|^2 + \frac{\kappa\tilde{\kappa}}{2}\|\bar{x} - x_n\| < \alpha_n(1 - \kappa\tilde{\kappa}).\end{aligned}$$

Moreover for any u and v in $\mathbb{B}_{\alpha_n}(\bar{x})$, let $\Psi_{u,v}^n := e\big(\Phi_n(u)\cap\mathbb{B}_{\alpha_n}(\bar{x}), \Phi_n(v)\big)$, and set

$$z_u^n := -f(x_n) - z_n - \nabla f(x_n)(u - x_n) \quad \text{and} \quad z_v^n := -f(x_n) - z_n - \nabla f(x_n)(v - x_n).$$

Using again Lemma 3.2 and the same argument as we did for z_u^0 and z_v^0 we obtain that z_u^n and z_v^n are in $\mathbb{B}_b(0)$. Then, from the pseudo-Lipschitz continuity of the mapping H^{-1} around $(0, 0)$ we have

$$\begin{aligned}\Psi_{u,v}^n &\le e(H^{-1}(z_u^n)\cap\mathbb{B}_a(0), H^{-1}(z_v^n))\\ &\le \kappa\|z_u^n - z_v^n\|\\ &= \kappa\|\nabla f(x_n)(u-x_n) - \nabla f(x_n)(v-x_n)\|\\ &\le \kappa\tilde{\kappa}\|u-v\|.\end{aligned}$$

Finally, from the set-valued Banach fixed point theorem, we obtain that the mapping Φ_n has a fixed point x_{n+1} such that

$$f(x_n) + \nabla f(x_n)(x_{n+1} - x_n) + F(x_n) + H(x_{n+1} - x_n) \ni 0,$$

and

$$\|x_{n+1} - \bar{x}\| \le \gamma\|x_n - \bar{x}\|,$$

which gives us the desired conclusion. □

The following result gives a sufficient condition, in terms of conditioning, ensuring the finite termination of the Newton iteration (8). More precisely, we prove that under constant conditioning of the set-valued mapping $f + F$, for n large enough, the iterate x_n is a solution to the generalized equation (6). Before going further, we recall the concept of constant conditioning we use here.

We say that $T : X \rightrightarrows Y$ is ψ-conditioned (see, *e.g.*, [18]) if and only if there exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ with $\psi(0) = 0$ such that

$$\psi(\, d(x, T^{-1}(0))\,) \le d(0, T(x)), \quad \text{for all } x \in X. \tag{23}$$

When ψ is a linear mapping on $\mathbb{R}^+$ then relation (23) can be seen as a global version of metric subregularity of the mapping T at any point $\bar{x}$ for 0 where $(\bar{x}, 0)$ lies in the graph of T. The set-valued mapping T is said to be δ-constant conditioned if it is ψ-conditioned with $\psi(t) = \delta$ whenever $t > 0$ and $\psi(0) = 0$. Equivalently, T is δ-constant conditioned if and only if for all (x, y) in the graph of T, $\|y\| < \delta$ implies $x \in T^{-1}(0)$.

Proposition 3.4 (*Finite Termination of the Newton Iteration*). *Under the same hypotheses as in Theorem 3.3, if the set-valued mapping $f + F$ is β-constant conditioned (for some positive constant β) then the Newton iteration (8) has finite termination, i.e., there is an integer n such that x_n is a solution to (6).*

Proof. Thanks to the proof of Theorem 3.3, there is a Newton sequence x_n satisfying (8) and such that for $n = 0, 1, \ldots,$

$$\|x_{n+1} - \bar{x}\| \le \gamma \|x_n - \bar{x}\|,$$

with $\frac{\kappa\tilde{\kappa}}{1-\kappa\tilde{\kappa}} < \gamma < 1$. After a straightforward computation, we get from the above inequality, the following estimation of the error in the nth iterate x_n:

$$\|x_n - \bar{x}\| \le \gamma^n \|x_0 - \bar{x}\| \le \gamma^n r, \tag{24}$$

where the constant r is as defined in the statement of Theorem 3.3. Moreover, the proof of this same theorem provides us with the existence of an element $z_n \in F(x_n)$, for $n = 0, 1, \ldots,$ such that

$$\| - f(x_n) - z_n - \nabla f(x_n)(\bar{x} - x_n)\| \le \frac{\tilde{\kappa}}{2}\|x_n - \bar{x}\|^2 + (|H|^+ + \delta)\|x_n - \bar{x}\|.$$

From which, together with (24), we infer,

$$\begin{aligned}
\|f(x_n) + z_n\| &\le \|\nabla f(x_n)(\bar{x} - x_n)\| + \frac{\tilde{\kappa}}{2}\|x_n - \bar{x}\|^2 + |H|^+\|x_n - \bar{x}\| + \frac{\tilde{\kappa}}{2}\|\bar{x} - x_n\| \\
&\le \tilde{\kappa}\|x_n - \bar{x}\| + \frac{\tilde{\kappa}}{2}\gamma^n r\|x_n - \bar{x}\| + |H|^+\|x_n - \bar{x}\,\| + \frac{\tilde{\kappa}}{2}\|\bar{x} - x_n\| \\
&\le \left(\frac{\tilde{\kappa}}{2}\gamma^n r + \frac{3\tilde{\kappa}}{2} + |H|^+\right)\|x_n - \bar{x}\| \\
&\le \left(\frac{\tilde{\kappa}}{2}\gamma^n r + \frac{3\tilde{\kappa}}{2} + |H|^+\right)\gamma^n r.
\end{aligned}$$

Since $\gamma < 1$ we have

$$\left(\frac{\tilde{\kappa}}{2}\gamma^n r + \frac{3\tilde{\kappa}}{2} + |H|^+\right)\gamma^n r < \left(\frac{\tilde{\kappa}}{2} r + \frac{3\tilde{\kappa}}{2} + |H|^+\right)\gamma^n r < \beta, \text{ eventually .}$$

It follows that there exists an integer n such that $\|f(x_n) + z_n\| < \beta$. Moreover, since $(x_n, f(x_n) + z_n) \in \operatorname{gph}(f + F)$ for all integer n, the set-valued mapping $f + F$ being β-constant conditioned we get $x_n \in (f + F)^{-1}(0)$, *i.e.*, x_n is a solution to the generalized equation (6) and the Newton iteration (8) has finite termination. □

4. Quadratic convergence of the Newton iteration

By strengthening our hypotheses we can show the quadratic convergence of our Newton iteration (8). Throughout this section, Y stands now for a Hilbert space. Before stating our second convergence theorem we need to establish two lemmas. The first one reads as follows.

Lemma 4.1. *Consider a Hilbert space H and let C be a closed convex subset of H. Let $x \in H$ and let ε be a positive real number. If $x + \varepsilon\mathbb{B} \subset C + \varepsilon\mathbb{B}$ then $x \in C$.*

Proof. Let $x \in H$ and take $\varepsilon > 0$. We show that if $x \notin C$ then $x + \varepsilon\mathbb{B} \not\subset C + \varepsilon\mathbb{B}$. Assume that $x \notin C$, then we shall find an element $y \in x + \varepsilon\mathbb{B}$ such that $y \notin C + \varepsilon\mathbb{B}$. Since C is a closed subset of H and $x \notin C$ there is a positive number α such that $d(x, C) = \alpha$. Moreover if x_C denotes the projection of the point x onto the closed convex subset C we have

$$d(x, C) = \|x - x_C\| = \alpha. \tag{25}$$

Now consider the point

$$y := x_C + \left(\frac{\varepsilon}{\alpha} + 1\right)(x - x_C). \tag{26}$$

Clearly, y belongs to the line through x and x_C, moreover, $\|y - x\| = \varepsilon$ that is $y \in x + \varepsilon\mathbb{B}$. To complete the proof we show that $y \notin C + \varepsilon\mathbb{B}$, to this end, we intend to prove that $d(y, C) > \varepsilon$.

We denote by y_C the projection of y onto C and we claim that $y_C = x_C$. Since $x_C \in C$, to prove that claim it is sufficient to show that

$$\langle y - x_C, c - x_C\rangle \leq 0, \quad \forall c \in C,$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product on the Hilbert space H. From (26), we get

$$\langle y - x_C, c - x_C\rangle = \left(\frac{\varepsilon}{\alpha} + 1\right)\langle x - x_C, c - x_C\rangle.$$

Since x_C is the projection of x onto C, $\langle x - x_C, c - x_C\rangle \leq 0$ for all $c \in C$ and thus we have proved that $y_C = x_C$. Therefore,

$$d(y, C) = \|y - y_C\| = \left(\frac{\varepsilon}{\alpha} + 1\right)\|x - x_C\|.$$

From (25) we infer that $d(y, C) = \varepsilon + \alpha > \varepsilon$ then $y \notin C + \varepsilon\mathbb{B}$ and the proof is complete. $\square$

From now on we make additional assumptions on the mappings F and H to ensure the quadratic convergence of the Newton iteration we consider. First, we assume that there is a solution $\bar{x}$ to the inclusion (6) such that

$$-f(\bar{x}) \in \operatorname{int}(F(\bar{x})). \tag{27}$$

Assumption (27) may occur in several situations, for instance, consider the case when $f = 0$ and $F = \partial g$ where $g : \mathbb{R} \ni x \to |x|$ is the absolute value function and ∂g denotes the subdifferential of g. Recall that the subdifferential (of convex analysis) of g at a point $x_0 \in \mathbb{R}$ is the set

$$\partial g(x_0) = \{u \in \mathbb{R} \mid u \cdot (x - x_0) \leq g(x) - g(x_0), \forall x \in \mathbb{R}\}.$$

Since the absolute value function is a convex continuous mapping, searching critical points of g (actually, its minimizers) reduces to solving the inclusion $\partial g(x) \ni 0$. A straightforward computation gives

$$\partial g(x) = \begin{cases} -1 & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Hence, the unique solution to the inclusion $\partial g(x) \ni 0$ is 0 and one has

$$-f(0) = 0 \in \operatorname{int}(\partial g(0)) = [-1, 1].$$

Second, we assume that the mapping F is both closed-valued and convex-valued in a neighborhood of $\bar{x}$, *i.e.*,

$$\exists\alpha > 0,\ \forall x \in \mathbb{B}_\alpha(\bar{x}), \quad F(x) \text{ is a closed convex subset of } Y, \tag{28}$$

while the mapping H is both convex-valued and compact-valued in a neighborhood of 0, *i.e.*,

$$\exists\alpha' > 0,\ \forall x \in \mathbb{B}_{\alpha'}(0), \quad H(x) \text{ is a convex compact subset of } Y. \tag{29}$$

The second lemma we need provides us with a technical result that turns out to be a cornerstone of our forthcoming proof, it reads as follows.

Lemma 4.2. *Consider a Banach space X along with a Hilbert space Y. Let $f : X \to Y$ be a single-valued function and let $F : X \rightrightarrows Y$ be a set-valued map that is strictly H-differentiable at $\bar{x}$ for some positively homogeneous map $H : X \rightrightarrows Y$. Let $\bar{x}$ be a solution to (6). If assumptions (27)–(29) are satisfied then there exists a neighborhood U of $\bar{x}$ such that for all $x \in U$, there is an element $z \in F(x)$ such that $-f(\bar{x}) - z \in H(\bar{x} - x)$.*

Proof. Since $-f(\bar{x}) \in \operatorname{int}(F(\bar{x}))$ there exists a positive number ε such that

$$-f(\bar{x}) + \varepsilon\mathbb{B} \subset F(\bar{x}). \tag{30}$$

Fix $\delta > 0$, the mapping F being strictly H-differentiable at $\bar{x}$, there is a constant $a > 0$ such that

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \quad \text{for all } x, x' \in \mathbb{B}_a(\bar{x}). \tag{31}$$

Adjust a if necessary so that $\delta a < \varepsilon$. Take any $x \in \mathbb{B}_a(\bar{x})$, we get from (30) and (31)

$$-f(\bar{x}) + \varepsilon\mathbb{B} \subset F(\bar{x}) \subset F(x) + H(\bar{x} - x) + \delta\|\bar{x} - x\|\mathbb{B};$$
$$-f(\bar{x}) + \varepsilon\mathbb{B} \subset F(x) + H(\bar{x} - x) + \varepsilon\mathbb{B}.$$

Adjusting a if necessary, thanks to assumptions (28) and (29) we get that $F(x) + H(\bar{x} - x)$ is a closed convex subset of the Hilbert space Y. Thus, we can apply Lemma 4.1 and we obtain that $-f(\bar{x}) \in F(x) + H(\bar{x} - x)$, hence, there is an element $z \in F(x)$ such that

$$-f(\bar{x}) - z \in H(\bar{x} - x), \tag{32}$$

which is the desired conclusion. $\square$

We can now state and prove the main result of this section regarding the quadratic convergence of the Newton iteration (8).

Theorem 4.3. *Consider the generalized equation (6). Assume that there is a point $\bar{x} \in X$ satisfying (27) and that the function f is Fréchet differentiable on a neighborhood Ω of $\bar{x}$ while the set-valued mapping F is strictly H-differentiable at $\bar{x}$. In addition, we suppose that:*

(i) *the derivative of f, ∇f, is Lipschitz continuous on Ω with a constant $\bar{\kappa}$ and is such that there exists a constant $\hat{\kappa}$ satisfying $\|\nabla f(x)\| \le \hat{\kappa}$, for all $x \in \Omega$;*
(ii) *the graph of the set-valued mapping H^{-1} is locally closed at $(0, 0)$ and H^{-1} is pseudo-Lipschitz continuous around this point with a constant $\kappa > 0$, moreover, $|H|^+ < \infty$;*
(iii) *$\kappa\tilde{\kappa} < 2/3$, where $\tilde{\kappa} := \max\{\bar{\kappa}, \hat{\kappa}\}$;*
(iv) *assumptions (28) and (29) are satisfied.*

Then, there is a positive constant r such that for any initial guess $x_0 \in \mathbb{B}_r(\bar{x})$ there exists a sequence x_n, the elements of which lie in $\mathbb{B}_r(\bar{x})$, generated by the Newton iteration (8) and converging Q-quadratically to $\bar{x}$.

Since Theorem 4.3 can be proved in much the same way as Theorem 3.3 we only give here a sketch of its proof mentioning the main arguments we need.

Proof. Fix $\delta > 0$, since the mapping F is strictly H-differentiable at $\bar{x}$, there is a constant $a > 0$ such that

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \quad \text{for all } x, x' \in \mathbb{B}_a(\bar{x}). \tag{33}$$

Making a smaller if necessary we can assume that $\delta a < \varepsilon$ and that a satisfies assertions (a) to (d) in the proof of Theorem 3.3 where assertion (d) becomes

$$\text{(d)} \quad \left(\frac{\tilde{\kappa}}{2}a + \tilde{\kappa} + |H|^+\right) a \le b.$$

Let $r < a/2$ and take any $x_0 \in \mathbb{B}_r(\bar{x})$. Thanks to Lemma 4.2 (and its proof) there exists an element $z_0 \in F(x_0)$ such that

$$-f(\bar{x}) - z_0 \in H(\bar{x} - x_0). \tag{34}$$

We consider here the same mapping Φ_0 defined in the proof of Theorem 3.3. and using the same arguments we prove that

$$-f(x_0) - z_0 - \nabla f(x_0)(\bar{x} - x_0) \in \mathbb{B}_b(0) \quad \text{and} \quad -z_0 - f(\bar{x}) \in \mathbb{B}_b(0). \tag{35}$$

Moreover, since the constants κ and $\tilde{\kappa}$ are such that $\kappa\tilde{\kappa} < 2/3$, there exists a constant γ such that $\frac{\kappa\tilde{\kappa}}{2(1-\kappa\tilde{\kappa})} < \gamma < 1$. Setting $\alpha_0 := \gamma\|x_0 - \bar{x}\|^2$, the pseudo-Lipschitz continuity of H^{-1} around $(0, 0)$ together with relations (34) and (35) yield

$$\begin{aligned} d(\bar{x}, \Phi_0(\bar{x})) &\le e\big(H^{-1}(-f(\bar{x}) - z_0) \cap \mathbb{B}_a(0), H^{-1}(-f(x_0) - z_0 - \nabla f(x_0)(\bar{x} - x_0))\big) \\ &\le \kappa\|f(\bar{x}) - f(x_0) - \nabla f(x_0)(\bar{x} - x_0)\| \\ &\le \frac{\kappa\tilde{\kappa}}{2}\|\bar{x} - x_0\|^2 < \alpha_0(1 - \kappa\tilde{\kappa}). \end{aligned}$$

The first assumption in Theorem 2.5 is therefore satisfied. Now take arbitrary points u and v in $\mathbb{B}_{\alpha_0}(\bar{x}) \subset \mathbb{B}_r(\bar{x})$ and define $\Psi^0_{u,v} := e\big(\Phi_0(u) \cap \mathbb{B}_{\alpha_0}(\bar{x}), \Phi_0(v)\big)$ along with the points z^0_u and z^0_v as in the proof of Theorem 3.3. As in the proof of Theorem 3.3, Lemma 3.2 together with the choice of $\tilde{\kappa}$ and a ensure that both z^0_u and z^0_v are in $\mathbb{B}_b(0)$.

Then, the pseudo-Lipschitz continuity of the mapping H^{-1} around $(0, 0)$ implies

$$\Psi^0_{u,v} \le e(H^{-1}(z^0_u) \cap \mathbb{B}_a(0), H^{-1}(z^0_v)) \le \kappa\tilde{\kappa}\|u - v\|.$$

Applying Theorem 2.5 it follows that there exists a fixed point $x_1 \in \mathbb{B}_{\alpha_0}(\bar{x}) \subset \mathbb{B}_r(\bar{x})$ of Φ_0, *i.e.*, a point x_1 such that

$$\|x_1 - \bar{x}\| \le \gamma\|x_0 - \bar{x}\|^2,$$

and satisfying

$$f(x_0) + \nabla f(x_0)(x_1 - x_0) + F(x_0) + H(x_1 - x_0) \ni 0.$$

The induction step is now clear, we assume that there are $x_1, x_2, \ldots, x_n$ in the ball $\mathbb{B}_r(\bar{x})$, such that, for $i = 0, 1, \ldots, n-1$, relation (8) holds and $\|x_{i+1} - \bar{x}\| \le \gamma\|x_i - \bar{x}\|^2$. Repeating exactly the same arguments we used above for $n = 0$ we obtain that the mapping Φ_n (defined in the proof of Theorem 3.3) has a fixed point x_{n+1} lying in the ball of radius $\alpha_n := \gamma\|x_n - \bar{x}\|^2$ and centered at $\bar{x}$. Hence, such a point satisfies

$$f(x_n) + \nabla f(x_n)(x_{n+1} - x_n) + F(x_n) + H(x_{n+1} - x_n) \ni 0,$$

and

$$\|x_{n+1} - \bar{x}\| \le \gamma\|x_n - \bar{x}\|^2.$$

Consequently, the sequence x_n converges Q-quadratically to the solution $\bar{x}$ to (6). □

5. Concluding remarks

There is no doubt that, among the works we found in the literature, the Ref. [5] by Azé and Chou and [6] by Dias and Smirnov are the closest to our concerns. This is the reason why we will endeavor to give, in the present section, more specific information and comments about these works. This will allow us to emphasize the interest of our results.

A Newton type method by Azé and Chou.

In [5], given two Banach spaces X and Y and a closed set-valued mapping $G : X \rightrightarrows Y$, Azé and Chou proposed a Newton type method for solving the inclusion

$$G(x) \ni 0. \tag{36}$$

They consider a new notion of tangent cone based on the definition of the Clarke's tangent cone (also known as the circatangent cone) from which they derive a concept of derivative for set-valued maps: *the strict lower differentiability*. More precisely, according to Azé and Chou, a cone $C \subset X$ is said to be *equi-circatangent* to $A \subset X$ at a if

$$\lim_{(t,b)\to(0,a)} e(C \cap \mathbb{B}, t^{-1}(A - b)) = 0.$$

If $C(A, a)$ and $T_A(a)$ denote respectively the Clarke's tangent cone and the contingent cone, for any equi-circatangent cone C to A at a we have $C \subset C(A, a) \subset T_A(a)$. For more details regarding these concepts of tangent cones the reader could refer to the comprehensive monograph by Aubin and Frankowska [19].

A set-valued mapping $F : X \rightrightarrows Y$ is said to be strictly lower differentiable at $(a, b) \in \operatorname{gph} F$ if there exists a closed positive homogeneous mapping $DF(a, b) : X \rightrightarrows Y$ that is equi-circatangent to F at (a, b).

In their paper, Azé and Chou make the following assumptions:

(α) the mapping G in (36) is strictly lower differentiable at $(a, b) \in \operatorname{gph} G$;
(β) the mapping $DG(a, b)$ is surjective;
(γ) $\|DG(a, b)^{-1}\| < +\infty$.

The "norm" $\|\cdot\|$ in assumption (γ) is the one defined in [19] by

$$\|F\| = \sup_{u\in\operatorname{dom} F} \|u\|^{-1} d(0, F(u)), \quad \text{for any } F : X \rightrightarrows Y.$$

It turns out that the definition of this norm agrees with the one of the so-called *inner norm* (see, *e.g.*, [20]) given by

$$\|F\|^- = \sup_{\|x\|\le 1} \inf_{y\in F(x)} \|y\|,$$

then, thanks to [21, Theorem 1.2], we know that assumption (γ) implies that the mapping G is metrically regular at a for b the latter being equivalent to the pseudo-Lipschitz continuity (see Definition 2.3) of G^{-1} around (b, a).

The Newton-type method proposed in [5] consists in constructing sequences u_n in X converging to 0, δ_n in $\mathbb{R}$ converging to 0 and $(x_n, y_n) \subset \operatorname{gph} G$ converging to some element $(\bar{x}, 0) \in \operatorname{gph} G$ (*i.e.*, $\bar{x}$ is a solution to the inclusion (36)) such that for each integer n

(1) $y_n + DG(a, b)(u_n) \ni 0$,
(2) $\lim_{n\to\infty} d((x_n + u_n, 0),\ \operatorname{gph} G) = 0$,
(3) $d((x_{n+1}, y_{n+1}),\ (x_n + u_n, 0)) \le d((x_n + u_n, 0),\ \operatorname{gph} G) + \delta_n$.

In particular assertions (2) and (3) imply that, for each n, u_n is close to $x_{n+1} - x_n$, thus assertion (1) is not very far from the classical Newton iteration for solving nonlinear equations of the form $f(x) = 0$ which can be written as $f(x_n) + Df(a)(x_{n+1} - x_n) = 0$, where a is a point close to the initial guess x_0.

Under assumptions (α), (β) and (γ) Azé and Chou generate a Newton sequence x_n satisfying the above assertions (1)–(3) and strongly converging to a solution to the inclusion (36).

A Newton type method by Dias and Smirnov.

In [6], Dias and Smirnov consider the inclusion

$$F(x) \ni 0, \tag{37}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a locally Lipschitz continuous set-valued mapping with closed values. They study the convergence of a Newton-type method consisting in generating a sequence $(x_n, v_n) \in \operatorname{gph} F$ satisfying

$$x_{n+1} = x_n + t_n\bar{x}_n \quad \text{and} \quad v_{n+1} \in \pi(0, F(x_{n+1})), \tag{38}$$

where the positive number t_n is the step-length, $\pi(0, F(x_{n+1}))$ denotes the set of all the projections of 0 onto $F(x_{n+1})$ and $\bar{x}_n$ is a solution to the inclusion $-v_n \in DF(x_n, v_n)(\bar{x}_n)$.

Assuming, in addition, that the mapping F is metrically regular at some reference point (*i.e.*, F^{-1} is pseudo-Lipschitz continuous around this point) the authors prove the Q-quadratic convergence of their Newton-type method to a solution to (37).

A significant advance in the solving of variational inclusions was made in both [5,6] where the Newton-type methods proposed by the authors involve a linearization of the set-valued mapping. Here, we follow the same path but as we said in the introduction we chose to focus on the particular case of generalized equations of the form (6). Moreover, contrary to Dias and Smirnov we do not work in the finite dimensional framework and we neither need the metrically regularity nor the local Lipschitz continuity of the set-valued mapping F. Our assumptions and setting are closer to the ones used by Azé and Chou, nevertheless we do know the rate of convergence (Q-linear or Q-quadratic) of our iteration while the Newton-type algorithm they propose is strongly convergent to a solution to the problem without any information regarding the speed of the convergence. Finally, it would be interesting to be able to compare the two concepts of graphical derivatives used in [5,6] with the generalized derivative we consider here; unfortunately the lack of relations between these concepts has been pointed out in the finite dimensional setting by Pang in [9].

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