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A family of non-stationary subdivision schemes reproducing exponential polynomials

Byeongseon Jeong[a], Yeon Ju Lee[a], Jungho Yoon[b,*]

[a] Institute of Mathematical Sciences, Ewha W. University, Seoul, 120-750, South Korea

[b] Department of Mathematics, Ewha W. University, Seoul, 120-750, South Korea

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ABSTRACT

Exponential B-splines are the most well-known non-stationary subdivision schemes. A crucial limitation of these schemes is that they can reproduce at most two exponential polynomials (Jena et al., 2003) [26]. Although interpolatory schemes can improve the reproducing property of exponential polynomials, they are usually less smooth than the (exponential) B-splines of corresponding orders. In this regard, this paper proposes a new family of non-stationary subdivision schemes which extends the exponential B-splines to allow reproduction of more exponential polynomials. These schemes can represent exactly circular shapes, spirals or parts of conics which are important analytical shapes in geometric modeling. This paper also discusses the Hölder regularities of the proposed schemes. Lastly, some numerical examples are presented to illustrate the performance of the new schemes.

1. Introduction

Subdivision schemes are powerful tools for generating smooth curves and surfaces. These schemes have attracted much attention in the recent decades for they allow designs of efficient, local and hierarchical modeling algorithms in a wide range of applications related to computer-aided geometric design and computer graphics. A subdivision begins with a set of initial control points and then recursively produces denser control points by a linear combination of points at a lower refinement level. The control points at increasing refinement level typically converge to a smooth function called the *limit function*. If the refinement rule is the same at all levels and positions of each iteration, the scheme is called stationary. A general treatment of stationary schemes can be found in selected Refs. [5,16,18]. Among the most familiar examples of such schemes are the subdivisions of *B*-splines [4,7] and the interpolatory schemes [15]. The polynomial reproduction property of a subdivision scheme was investigated recently in [6,9,17,22].

This paper is mainly concerned with non-stationary subdivision schemes. The non-stationary refinement process is usually formalized as follows. Given a set of initial control points $\mathbf{f}^0 := \{f_n^0 \in \mathbb{R} : n \in \mathbb{Z}\}$, the new control points $\mathbf{f}^{k+1}$ are defined recursively by the rule depending on k:

$$f_n^{k+1} = \sum_{m\in\mathbb{Z}} a_{n-2m}^{[k]} f_m^k. \tag{1}$$

The sequence $\mathbf{a}^{[k]} := \{a_n^{[k]} \in \mathbb{R} : n \in \mathbb{Z}\}$ is called the subdivision *mask* at level k. It is assumed that only a finite number of coefficients $a_n^{[k]}$ are non-zero, and hence, the sum in (1) can be computed efficiently. To simplify the presentation of a

* Corresponding author.
E-mail address: yoon@ewha.ac.kr (J. Yoon).

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subdivision scheme and its analysis, it is convenient to assign to each mask $\mathbf{a}^{[k]}$ the z-transform

$$a^{[k]}(z) = \sum_{n\in\mathbb{Z}} a_n^{[k]} z^n,$$

which is called the *symbol* of the scheme. When only a finite number of coefficients $a_n^{[k]}$ are non-zero, the z-transform $a^{[k]}(z)$ becomes a Laurent polynomial.

A subdivision scheme is said to be C^0 (or convergent) if for an initial data $\mathbf{f}^0 = \{f_n^0 : n \in \mathbb{Z}\}$ in $\ell^\infty(\mathbb{Z})$, there exists a function $f \in C^0(\mathbb{R})$ such that for any compact set K in $\mathbb{R}$,

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}\cap 2^k K} |f_n^k - f(2^{-k}n)| = 0 \tag{2}$$

and f is not identically 0 for some initial data $\mathbf{f}^0$. In particular, the so-called *basic limit function* is obtained from the Dirac delta sequence $\mathbf{f}^0 = \{\delta_{n,0} : n \in \mathbb{Z}\}$. A non-stationary subdivision is said to *reproduce* an exponential polynomial ϑ if the scheme $\{S_{a^{[k]}}\}$ is convergent (i.e., C^0) and

$$\vartheta = \lim_{k\to\infty} S_{a^{[k]}} \cdots S_{a^{[0]}} \mathbf{f}^0$$

for the initial data $f_n^0 = \vartheta(n)$ with $n \in \mathbb{Z}$.

One of the most important properties for convergent stationary subdivision schemes is the ability of reproducing polynomials. Such schemes generate (or approximate) curves or surfaces very accurately with the corresponding approximation order but have a limitation in representing circular shapes, spirals or parts of conics, which are important analytical shapes in geometric modeling. This limitation can be overcome by employing non-stationary schemes which can reproduce or generate such families of trigonometric (or exponential) polynomials and sections (see, for example, [1–3,10, 11,13,14,19–21,23–25,27–29]). A necessary and sufficient condition for a symbol of a non-stationary scheme to reproduce exponential polynomials has been studied [12,26]. We note in passing that the condition in [12] involves more general parameterizations while the study [26] concerns with uniform dyadic parameterizations of subdivision.

The most familiar examples of such schemes are subdivisions of exponential B-splines [8,14,19,28]. However, these schemes may not reproduce any exponential polynomials without a suitable normalization factor. Even with a normalization factor, the exponential B-splines can reproduce at most two linearly independent exponential polynomials [26]. Recently, Conti et al. and Romani studied non-stationary subdivision schemes reproducing exponential polynomials [8,10,28]. In particular, for the purpose of reproducing a certain set of exponential polynomials, they modified the exponential B-spline to an interpolatory scheme at the expense of the mask length [8,28]. However, although interpolatory schemes provide high approximation orders, they are usually less smooth than the (exponential) B-splines of the corresponding orders. For instance, the 4-point (non-stationary) interpolatory scheme generates C^1 curves, while the fourth-order (exponential) B-spline is C^2. Moreover, in spite of being a very desirable property in curve (and surface) designs, interpolation may raise twisting artifacts to the parametric curves if the initial control points are highly irregular. From this point of view, there is a need for new non-stationary subdivision schemes that extend the exponential B-splines while overcoming their drawbacks at the same time.

This study proposes a new class of non-stationary subdivision schemes that can reproduce more exponential functions than the exponential B-splines do. This gives us flexibility in designs to accommodate the various design circumstances along with higher precision. To be more precise, we first introduce a new family of non-stationary schemes that reproduce up to four exponential polynomials. These schemes will be termed as 'exponential quasi-splines' to reflect the fact that each scheme in this family can be explained as the convolution of an exponential B-spline and a suitable distribution [14]. We see that most of the known schemes (both stationary and non-stationary) turn out to be special cases of this family. This paper discusses the Hölder regularities of the new schemes and provides some numerical results that illustrate the performance of new schemes. Furthermore, we provide a generalized version of the exponential quasi-spline for the purpose of extending the precision set.

The rest of the paper is organized as follows: Section 2 is devoted to provide a new family of non-stationary subdivision schemes. Their Hölder regularities are discussed in Section 3. Section 4 deals with the exponential polynomial reproducing property of the proposed scheme. In Section 5, we further generalize the new schemes to enhance the reproducing capability of exponential polynomials. The performance of the new schemes is demonstrated with some numerical examples in Section 6.

2. Exponential quasi-splines

The main purpose of this section is to present a new family of non-stationary subdivision schemes reproducing exponential polynomials of the form $\varphi(x) = x^\alpha e^{\lambda x}$, where α is a nonnegative integer and λ belongs to $\mathbb{R}$ or $i\mathbb{R}$. To this end, we first briefly review the subdivisions of the exponential B-splines. Let

$$\Lambda := \{\lambda_n \in \mathbb{R} \text{ or } i\mathbb{R} : n = 1, \ldots, N\}$$

be a set of N numbers, where the values λ_n are not necessarily distinct. An exponential B-spline associated with the set Λ can be defined in terms of the following symbol:

$$\beta_N^{[k]}(z) = 2z^{-\lceil N/2\rceil} \prod_{n=1}^{N} \frac{1 + e^{\lambda_n 2^{-k-1}} z}{2}, \tag{3}$$

where $\lceil x\rceil$ indicates the smallest integer bigger than or equal to x. Since $\#\Lambda = N$, the corresponding exponential B-spline is termed as an Nth order scheme. The new family of non-stationary subdivision schemes suggested in this study is defined in terms of the following symbol:

$$a^{[k]}(z) = \beta_N^{[k]}(z) q^{[k]}(z) \tag{4}$$

with the Laurent polynomial

$$q^{[k]}(z) = \nu_k + \omega_k(z^{-1} + z). \tag{5}$$

As will be discussed in Section 3, the parameters ν_k and ω_k play a crucial role in determining the smoothness and the exponential polynomial reproducing property of the proposed scheme. For instance, a (normalized) exponential B-spline of any order (i.e., $\omega_k = 0$) reproduces at most two exponential polynomials that depend on the choice of the factor ν_k [26]. However, by choosing a suitable ω_k, the scheme reproduces (up to) four exponential polynomials. In fact, the proposed schemes can be generalized to extend the precision set. The specific form will be given in Section 5.

This new family unifies most of the well-known schemes (both stationary and non-stationary), such as the (classical and exponential) B-splines and the Deslauriers–Dubuc's four-point interpolatory schemes. In particular, when $\omega_k = 0$ for any $k \in \mathbb{Z}_+$, the suggested scheme becomes a normalized exponential B-spline with the symbol

$$b^{[k]}(z) = \nu_k \beta_N^{[k]}(z).$$

In fact, the scheme associated with $a^{[k]}$ in (4) can be explained as the convolution of an exponential B-spline and a distribution associated with the symbol $q^{[k]}(z)$ (see [14] for the details). In this regard, the proposed scheme is termed as an 'exponential quasi-spline' of order N. Before we further advance our discussion, let us observe some particular cases of the exponential quasi-splines.

2.1. *Stationary quasi-spline*

Let $\lambda_n = 0$ for any $\lambda_n \in \Lambda$. Then the proposed scheme becomes stationary and reproduces a certain set of algebraic polynomials. Accordingly, it is reasonable to write $\omega_k = \omega$ and set $\nu_k = 1 - 2\omega$ so that each sum of the even and the odd mask becomes 1. Then the corresponding symbol is of the form

$$a(z) = 2z^{-\lceil N/2\rceil} \left(\frac{1+z}{2}\right)^N q(z) \tag{6}$$

with

$$q(z) = 1 - 2\omega + \omega(z^{-1} + z). \tag{7}$$

The parameter ω plays an important role in determining the regularity and the polynomial reproducing property of the proposed scheme. For a fixed N, the Hölder regularities vary depending on ω (see Section 3). Moreover, this scheme reproduces cubic algebraic polynomials for the case that $\omega = -N/8$ [17,22]; see also Example 2.1. In the following example, we see that this family extends the subdivision schemes introduced by Hormann and Sabin [22].

Example 2.1 (*Hormann and Sabin's Schemes*). A family of stationary subdivision schemes reproducing cubic polynomials are proposed in [22], where the symbol of each scheme is of the form

$$a(z) = 2\left(\frac{1+z}{2}\right)^N (-N + (8 + 2N)z - Nz^2)/8.$$

This is identified as a special case of the stationary quasi-spline in (6) with $\omega = -N/8$. Theorem 3.1 tells that the smoothness of the quasi-spline can be improved when ω is bigger than $-N/8$ and that the maximal smoothness is obtained when $\omega = 1/4$.

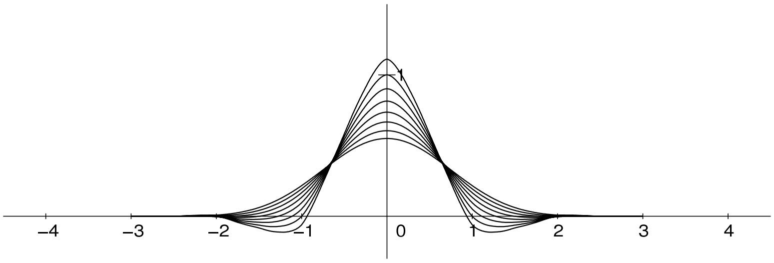

Fig. 1. Basic limit functions of the cubic quasi-spline. Here, from top with respect to the origin, $\omega = -5/8, -1/2, -3/8, -1/4, -1/8, 0, 1/8, 1/4$.

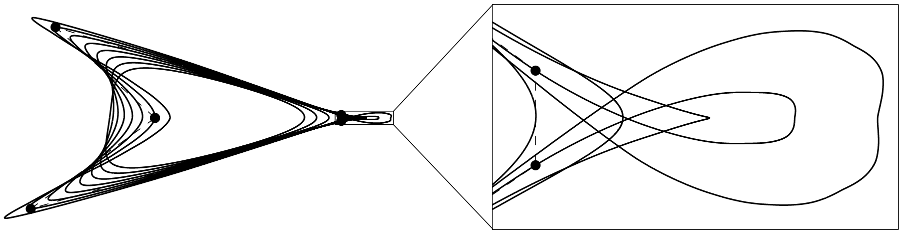

Fig. 2. Limit curves generated by the cubic quasi-spline. Here, from the outermost to innermost, $\omega = -5/8, -1/2, -3/8, -1/4, -1/8, 0, 1/8, 1/4$.

2.2. Fourth order exponential quasi-splines

Among the proposed schemes, the fourth order (i.e., $N = 4$) scheme may be of most interest for its practical usages. Put $\Lambda = \{\pm\lambda_1, \pm\lambda_2\}$ and

$$\gamma_{n,k} := e^{-\lambda_n 2^{-k-1}} + e^{\lambda_n 2^{-k-1}}, \quad n = 1, 2.$$

The mask of the corresponding scheme is supported in $\mathbb{Z} \cap [-3, 3]$ except for the case that $\omega_k = 0$. Its general form is given by

$$\begin{aligned} a_0^{[k]} &= ((2 + \gamma_{1,k}\gamma_{2,k})\nu_k + 2(\gamma_{1,k} + \gamma_{2,k})\omega_k)/8, \\ a_{-2}^{[k]} &= a_2^{[k]} = (\nu_k + (\gamma_{1,k} + \gamma_{2,k})\omega_k)/8, \\ a_{-1}^{[k]} &= a_1^{[k]} = ((\gamma_{1,k} + \gamma_{2,k})\nu_k + (3 + \gamma_{1,k}\gamma_{2,k})\omega_k)/8, \\ a_{-3}^{[k]} &= a_3^{[k]} = \omega_k/8. \end{aligned} \tag{8}$$

Some numerical results are provided in Section 6 to illustrate the performance of this fourth order method.

Example 2.2 (*Stationary Quasi-Spline*)**.** In the stationary case, as observed in Section 2.1, $\gamma_{1,k} = \gamma_{2,k} = 2$ since $\lambda_j = 0$. Also, $\omega_k = \omega$ and $\nu_k = 1 - 2\omega$. Then the mask of the cubic quasi-spline is of the form

$$\begin{aligned} &a_0 = (3 - 2\omega)/4, \qquad a_{-2} = a_2 = (1 + 2\omega)/8, \\ &a_{-1} = a_1 = (4 - \omega)/8, \qquad a_{-3} = a_3 = \omega/8. \end{aligned}$$

Fig. 1 displays its basic limit functions for $\omega = -5/8, -1/2, -3/8, -1/4, -1/8, 0, 1/8$ and $1/4$, respectively. In particular, when $\omega = 0$, $1/4$ or $-1/2$, the scheme becomes the cubic, the quintic *B*-splines, and Deslauriers–Dubuc's 4-point interpolatory scheme, respectively. Fig. 2 presents other results on the limit curves with the same choices for ω as above. If the given control points are highly irregular, the limit curve of the interpolatory scheme (i.e., $\omega = -1/2$) results in unpleasant artifacts. However, by choosing ω bigger than $-1/2$, we can obtain visually better curves without twisting artifacts.

Example 2.3. Non-stationary 4-point schemes reproducing conic sections are developed in [1,25]. These schemes can be obtained with suitable choices for ν_k and ω_k in (8). The exact values of ν_k, ω_k, and the corresponding subdivision mask are given in Theorem 4.4. Also, the well-known (exponential) *B*-splines of order 4 and 6 as well as the (non-stationary) 4-point interpolatory schemes are special cases of the fourth order exponential quasi-spline.

3. Smoothness analysis

For a given $\gamma = n + s$ with $n \in \mathbb{N}$ and $s \in (0, 1]$, the Hölder space C^γ is defined as the set of n-times continuously differentiable functions f whose n-th derivative $f^{(n)}$ satisfies the Lipschitz condition

$$\sup_{x,h\in\mathbb{R},h\neq 0} \frac{|f^{(n)}(x+h) - f^{(n)}(x)|}{|h|^s} \leq C$$

with a constant $C > 0$. In this section, we first discuss the Hölder regularity of stationary quasi-splines by using the method based on the joint spectral radius [18]. Based on this result, we then show that the exponential quasi-spline has the same integer smoothness as its stationary counterpart. To do this, we employ the concept of asymptotical equivalence among subdivision schemes [19,21]; see also (14).

The concept of the joint spectral radius is well-known in the literature. However, in order for this paper to be self-contained, we briefly introduce its definition. Consider the Laurent polynomial

$$a(z) = 2\left(\frac{1+z}{2}\right)^N b(z),$$

where N is the maximal number of the smoothing factor in this symbol. Let $b_n, n = 0, 1, \ldots, n_0$, be the coefficients of $2b(z)$, and let B_0 and B_1 be $n_0 \times n_0$ matrices that have components of the form

$$(B_0)_{i,j} = b_{i-2j+n_0}, \qquad (B_1)_{i,j} = b_{i-2j+1+n_0}, \tag{9}$$

for $i, j = 1, 2, \ldots, n_0$. The joint spectral radius ρ of B_0 and B_1 is given by

$$\rho := \rho(B_0, B_1) = \limsup_{n\to\infty}(\max\{\|B_{\epsilon_n}\cdots B_{\epsilon_1}\|_\infty^{1/n} : \epsilon_i \in \{0, 1\},\ i = 1, \ldots, n\}). \tag{10}$$

Then the associated stationary subdivision scheme has the Hölder regularity $N - \log_2 \rho$. From the definition of ρ in (10), we have

$$\max\{\rho(B_0), \rho(B_1)\} \leq \rho \leq \max\{\|B_0\|_\infty, \|B_1\|_\infty\}. \tag{11}$$

Based on this, we now compute the Hölder regularities of the stationary quasi-splines.

Theorem 3.1 (*Stationary Quasi-Spline*)**.** *Let $S_{N,\omega}$ be the quasi-spline scheme of order N associated with the symbol in (6). Then the scheme $S_{N,\omega}$ is $C^{N-\log_2\rho}$ with*

$$\begin{aligned}
&\rho = 2 - 4\omega, \quad \text{if } 2^{-1} - 2^{N-2} \leq \omega \leq 1/4,\\
&2 - 4\omega \leq \rho \leq 4\omega, \quad \text{if } 1/4 < \omega \leq 1/3,\\
&2\omega \leq \rho \leq 4\omega, \quad \text{if } 1/3 < \omega \leq 1,\\
&4\omega - 2 \leq \rho \leq 4\omega, \quad \text{if } 1 < \omega \leq 2^{N-2}.
\end{aligned} \tag{12}$$

Proof. The two matrices B_0 and B_1 in (9) corresponding to the Laurent polynomial $q(z)$ in (7) are of the form:

$$B_0 = \begin{bmatrix} 2(1-2\omega) & 0 \\ 2\omega & 2\omega \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 2\omega & 2\omega \\ 0 & 2(1-2\omega) \end{bmatrix}. \tag{13}$$

A direct calculation yields the spectral radii of B_0 and B_1 as follows:

$$\rho(B_0) = \rho(B_1) = 2\max\{|1-2\omega|, |\omega|\}.$$

It is also straightforward that

$$\|B_0\|_\infty = \|B_1\|_\infty = 2\max\{|1-2\omega|, 2|\omega|\}.$$

This in connection with (11) derives the result (12). □

Example 3.2 (*Smoothness of Cubic Quasi-Splines*)**.** Fig. 3 presents the Hölder regularities of the cubic quasi-spline with the parameter ω. The maximal smoothness is obtained when $\omega = 1/4$ which corresponds to the quintic B-spline.

We now estimate the Hölder regularity of exponential quasi-splines based on the concept of asymptotical equivalence between two schemes. A non-stationary scheme with the mask $\{\mathbf{a}^{[k]}\}$ is said to be asymptotically equivalent to a stationary scheme with the mask $\mathbf{a}$ [19] if

$$\sum_{k\in\mathbb{Z}_+} \|\mathbf{a}^{[k]} - \mathbf{a}\|_\infty < \infty. \tag{14}$$

Then, if the stationary scheme is C^0, then the corresponding non-stationary scheme is also C^0. The following theorem further details the Hölder regularity of a non-stationary scheme [19,21].

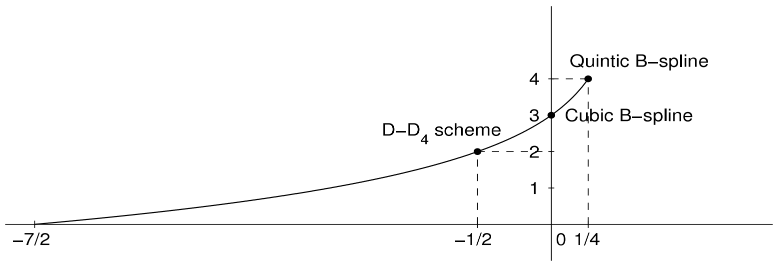

Fig. 3. Plot of Hölder regularity $4 - \log_2 \rho$ of the cubic quasi-spline with for $-7/2 \le \omega \le 1/4$. Here, D–D$_4$ indicates Deslauriers–Dubuc's four-point interpolatory scheme.

Theorem 3.3 (*[19,21]*)**.** *Let $\{S_{a^{[k]}}\}$ be a non-stationary subdivision scheme with the mask $\{\mathbf{a}^{[k]}\}$ and let S_a be a stationary scheme with the mask* $\mathbf{a}$. *Assume that* $\mathrm{supp}(\mathbf{a}) = \mathrm{supp}(\mathbf{a}^{[k]})$ *for $k \in \mathbb{Z}_+$ and S_a is C^μ for some $\mu \in (0, 1)$. If*

$$\|\mathbf{a}^{[k]} - \mathbf{a}\|_\infty < c2^{-k}, \quad k \in \mathbb{Z}_+,$$

then $\{S_{a^{[k]}}\}$ is C^s for some $s \in (0, 1)$.

Furthermore, we cite the following result from [21].

Lemma 3.4 (*[21]*)**.** *Let $\{S_{a^{[k]}}\}$ be a non-stationary subdivision scheme associated with the symbol of the form $a^{[k]}(z) = \frac{1}{2}(1 + r_k z)b^{[k]}(z)$. Suppose*

$$|1 - r_k| \le c2^{-k}, \quad k \in \mathbb{Z}_+,$$

and the scheme corresponding to $\{S_{b^{[k]}}\}$ is $C^{\ell+\mu}$ with $\ell \in \mathbb{Z}_+$ and $\mu \in (0, 1)$. Then the scheme $\{S_{a^{[k]}}\}$ is $C^{\ell+1+s}$ for some $s \in (0, 1)$.

In the following theorem, we derive a simple condition on the parameters ν_k and ω_k such that the exponential quasi-spline scheme has the same integer smoothness as its stationary counterpart.

Theorem 3.5. *Let $S_{N,\omega}$ be the quasi-spline of order N with the symbol in (6), and assume that $S_{N,\omega}$ is $C^{\ell+\mu}$ for some $\mu \in (0, 1)$. Let $\{S_{a^{[k]}}\}$ be the exponential quasi-spline of order N associated with the symbol $a^{[k]}$ in (4). Suppose that the parameters ν_k and ω_k in $a^{[k]}$ satisfy the following conditions:*

$$|\nu_k - (1 - 2\omega)|, \qquad |\omega_k - \omega| \le c2^{-k}, \quad k \in \mathbb{Z}_+. \tag{15}$$

Then $\{S_{a^{[k]}}\}$ is $C^{\ell+s}$ with $s \in (0, 1)$.

Proof. Let $b(z) = a(z)2^\ell(1 + z)^{-\ell}$. Since the quasi-spline $S_{N,\omega}$ is C^ℓ, the subdivision associated with $b(z)$ is C^μ with $\mu \in (0, 1)$. Its non-stationary counterpart of $b(z)$ can be given as follows. Let Λ_ℓ be an arbitrary subset of Λ so that $\#\Lambda_\ell = \ell$. Set

$$b^{[k]}(z) = a^{[k]}(z)\left(\prod_{\lambda_n \in \Lambda_\ell} \frac{1 + e^{\lambda_n 2^{-k-1}} z}{2}\right)^{-1}$$

and let $\{S_{b^{[k]}}\}$ be the corresponding scheme. Then, seeing that

$$|e^{\lambda_n 2^{-k-1}} - 1| < c2^{-k}, \quad \forall \lambda_n \in \Lambda,$$

for some constant $c > 0$ and using the condition (15), we can claim from the expression of $a^{[k]}(z)$ in (4) that the scheme $\{S_{b^{[k]}}\}$ is asymptotically equivalent to S_b and that $\{S_{b^{[k]}}\}$ is C^{s_1} for some $s_1 \in (0, 1)$ by Theorem 3.3. Moreover, by applying Lemma 3.4 repeatedly, we can conclude that $S_{a^{[k]}}$ is $C^{\ell+s}$ for $s \in (0, 1)$. The proof is done. $\square$

4. Exponential polynomial reproducing property

The goal of this section is to show that the exponential quasi-spline reproduces up to four exponential polynomials with suitable parameters ν_k and ω_k. A general version of the proposed scheme reproducing more exponential polynomials will be discussed later in Section 5. We cite the following result which serves as a basic tool for our proof.

Theorem 4.1 (*[26, Theorem 2.3]*)**.** *A non-stationary subdivision scheme with the symbol $a^{[k]}(z)$ reproduces a set of exponential polynomials $\{x^\ell e^{\lambda x} : \ell = 0, \ldots, \mu - 1\}$ if and only if for some Laurent polynomial $b^{[k]}(z)$,*

$$2 - a^{[k]}(z^{1+\tau})z^\tau = b^{[k]}(z)(1 - e^{\lambda 2^{-k-1-\tau}} z)^\mu, \tag{16}$$

where $\tau = 0$ if N is even, and $\tau = 1$ if N is odd.

Remark 4.2. In [12], Conti and Romani studied a necessary and sufficient condition for a symbol of a non-stationary scheme to reproduce exponential polynomials. This condition is indeed equivalent to (16) but involves more general parameterizations while Theorem 4.1 concerns with the typical (called *primal* and *dual* [9,17]) parameterizations of subdivision.

Now, recall that the symbol of exponential quasi-spline is of the form

$$a^{[k]}(z) = \beta_N^{[k]}(z)q^{[k]}(z) \tag{17}$$

with $q^{[k]}(z) = \nu_k + \omega_k(z^{-1} + z)$. In order for a subdivision scheme to be symmetric, the set $\Lambda = \{\lambda_j : j = 1, \ldots, N\}$ is forced to be of the form

$$\Lambda = \begin{cases} \{\lambda_n, -\lambda_n : n = 1, \ldots, N/2\}, & \text{if } N \text{ is even,} \\ \{\lambda_0, \lambda_n, -\lambda_n : \lambda_0 = 0, n = 1, \ldots, (N-1)/2\}, & \text{if } N \text{ is odd.} \end{cases} \tag{18}$$

For simplicity in notation, we use the abbreviation

$$\beta_{j,k} := \beta_N^{[k]}(e^{-\lambda_j 2^{-k-1}})e^{-\lambda_j 2^{-k-2}\tau}, \tag{19}$$

where $\beta_N^{[k]}(z)$ is the symbol of the exponential *B*-spline in (3). We first verify that the exponential quasi-spline reproduces at least two linearly independent exponential polynomials under a suitable choice of ν_k.

Lemma 4.3. *Let $a^{[k]}(z)$ be the symbol of the exponential quasi-spline scheme as in (17). For a given $\lambda_j \in \Lambda$ and a parameter ω_k, choose the parameter ν_k to be*

$$\nu_k = 2\beta_{j,k}^{-1} - \omega_k\gamma_{j,k}. \tag{20}$$

Then the exponential quasi-spline reproduces at least two linearly independent exponential polynomials associated with $\pm\lambda_j$.

Proof. In view of Theorem 4.1, we need to show that $2 - a^{[k]}(z^{1+\tau})z^{\tau} = 0$ at $z = e^{\pm\lambda_j 2^{-k-1-\tau}}$. To do this, we first consider the case that $\tau = 0$ and $z = e^{-\lambda_j 2^{-k-1}}$. Then a direct calculation gives the result

$$a^{[k]}(e^{-\lambda_j 2^{-k-1}}) = \beta_{j,k} \cdot 2\beta_{j,k}^{-1} = 2$$

with $\beta_{j,k}$ in (19). We can prove similarly for the cases that $z = e^{\lambda_j 2^{-k-1}}$ and $\tau = 1$ (i.e., $z = e^{\pm\lambda_j 2^{-k-2}}$). $\square$

The following theorem shows that the proposed scheme can enhance the reproducing capability of exponential polynomials with a suitable choice of ω_k. Our proof is divided into two cases: (i) N is even, (ii) N is odd. The case that N is even is considered below.

Theorem 4.4. *Suppose $\lambda_j, \lambda_\ell \in \Lambda$ and $\#\Lambda = N$ is even. Let the parameter ω_k be given as*

$$\omega_k = \begin{cases} -\dfrac{2(\beta_{j,k} - \beta_{\ell,k})}{\beta_{j,k}\beta_{\ell,k}(\gamma_{j,k} - \gamma_{\ell,k})} & \text{if } \lambda_\ell \neq \lambda_j, \\ -\dfrac{2}{\beta_{j,k}}\displaystyle\sum_{n=1}^{N/2}\dfrac{1}{\gamma_{j,k} + \gamma_{n,k}} & \text{if } \lambda_\ell = \lambda_j. \end{cases} \tag{21}$$

If ν_k is chosen as in (20), then the corresponding exponential quasi-spline reproduces four linearly independent exponential polynomials associated with $\pm\lambda_j$ and $\pm\lambda_\ell$.

Proof. By Theorem 4.3, it suffices to show that the scheme $\{S_{a^{[k]}}\}$ reproduces the exponential polynomials associated with λ_ℓ and $-\lambda_\ell$. To this end, we first consider the case $\lambda_\ell \neq \lambda_j$. Then by Theorem 4.1, we need to show that $2 - a^{[k]}(z) = 0$ at $z = e^{-\lambda_\ell 2^{-k-1}}$. This is indeed proved by a direct calculation as follows:

$$a^{[k]}(e^{-\lambda_\ell 2^{-k-1}}) = \beta_{\ell,k} \cdot 2\beta_{\ell,k}^{-1} = 2.$$

Similarly, we can get the same result for $z = e^{\lambda_\ell 2^{-k-1}}$. Next, consider the case that $\lambda_\ell = \lambda_j$. We then need to show that $\frac{d}{dz}(2 - a^{[k]})(z) = 0$ at $z = e^{-\lambda_j 2^{-k-1}}$. This is verified by the calculation

$$\frac{d}{dz}(2 - a^{[k]})(e^{-\lambda_j 2^{-k-1}}) = -\frac{d}{dz}a^{[k]}(e^{-\lambda_j 2^{-k-1}}) = 0.$$

Similarly, we have the same result for $z = e^{\lambda_\ell 2^{-k-1}}$. Therefore, by applying Theorem 4.1, the proof is done. $\square$

The following theorem treats the case that N is odd.

Theorem 4.5. *Suppose $\lambda_j, \lambda_\ell \in \Lambda$ and $\#\Lambda = N$ is odd. Let the parameter ω_k be given as*

$$\omega_k = \begin{cases} -\dfrac{2(\beta_{j,k}-\beta_{\ell,k})}{\beta_{j,k}\beta_{\ell,k}(\gamma_{j,k}-\gamma_{\ell,k})} & \text{if } \lambda_\ell \neq \lambda_j, \\ -\dfrac{2}{\beta_{j,k}}\left(\dfrac{1}{2(2+\gamma_{j,k})} + \displaystyle\sum_{n=1}^{(N-1)/2}\dfrac{1}{\gamma_{j,k}+\gamma_{n,k}}\right) & \text{if } \lambda_\ell = \lambda_j. \end{cases} \tag{22}$$

If ν_k is chosen as in (20), then the corresponding exponential quasi-spline reproduces four linearly independent exponential polynomials associated with $\pm\lambda_j$ and $\pm\lambda_\ell$.

Proof. The proof is analogous to the proof of the previous theorem. □

Remark 4.6 (*The Four-Point Interpolatory Schemes*)**.** When $N = 4$, the exponential quasi-splines reproducing four exponential polynomials with the choice of ν_k in (20) and ω_k in (21) become the four-point interpolatory schemes. The corresponding mask is of the form

$$a_{2n}^{[k]} = \delta_{n,0}, \quad n \in \mathbb{Z},$$
$$a_{-1}^{[k]} = a_1^{[k]} = \frac{\gamma_{1,k}^2 + \gamma_{1,k}\gamma_{2,k} + \gamma_{2,k}^2 - 3}{\gamma_{1,k}\gamma_{2,k}(\gamma_{1,k}+\gamma_{2,k})}, \qquad a_{-3}^{[k]} = a_3^{[k]} = \frac{-1}{\gamma_{1,k}\gamma_{2,k}(\gamma_{1,k}+\gamma_{2,k})}.$$

5. Generalized exponential quasi-spline

As observed in Section 4, the (exponential) quasi-spline reproduces up to four exponential polynomials. In this section, we introduce its generalized version that reproduces more exponential polynomials. More specifically, for a given set $\Lambda^\circ = \{\pm\lambda_n : n = 1, \dots, M\} \subset \Lambda$, our aim is to construct a scheme reproducing $2M$ exponential polynomials associated with the set Λ°. It indeed can be achieved by employing the extended form of the symbol

$$a^{[k]}(z) = \beta_N^{[k]}(z) q_M^{[k]}(z) \tag{23}$$

with the Laurent polynomial

$$q_M^{[k]}(z) = \sum_{m=0}^{M-1} \omega_{m,k}(z^{-1}+z)^m, \tag{24}$$

where $2M \le N$. A natural question arising here is in the existence of the parameters $\omega_{m,k}$, $m = 0, \dots, M-1$, such that the corresponding scheme reproduces exponential polynomials associated with Λ°. In view of Theorem 4.1, we see that determining the parameters $\omega_{m,k}$, $m = 0, \dots, M-1$, is translated into solving the linear system

$$\left[\frac{d^\alpha(2 - a^{[k]}(z^{1+\tau})z^\tau)}{dz^\alpha}\right](e^{-\lambda_n 2^{-k-1-\tau}}) = 0, \quad \alpha = 0, \dots, \mu_n - 1, \tag{25}$$

where μ_n indicates the number of values of λ_n that are duplicated in Λ°. The existence of the solution of this linear system is treated below.

Theorem 5.1. *The linear system in (25) has a unique solution for $\omega_{m,k}$ in (24) with $m = 0, \dots, M-1$.*

Proof. The Laurent polynomial $q_M^{[k]}$ in (24) can be rewritten as

$$q_M^{[k]}(z) = \widetilde{\omega}_{0,k} + \sum_{m=1}^{M-1} \widetilde{\omega}_{m,k} \prod_{\ell=1}^{m} (z^{-1} + z - \gamma_{\ell,k})$$

for some suitable $\widetilde{\omega}_{m,k}$ with $m = 0, \dots, M-1$. Then, we show inductively that the parameters $\widetilde{\omega}_{m,k}$ for $m = 0, \dots, M-1$ are uniquely determined by Eq. (25). Here, recalling that $a^{[k]}(z) = \beta_N^{[k]}(z) q_M^{[k]}(z)$, we consider only the case that N is even since the other case can be proved similarly. Observe first that

$$a^{[k]}(e^{-\lambda_1 2^{-k-1}}) = \beta_{1,k}\widetilde{\omega}_{0,k}.$$

Thus Eq. (25) with $n = 1$ and $\alpha = 0$ gives the solution $\widetilde{\omega}_{0,k} = 2\beta_{1,k}^{-1}$. Next, assume that $\widetilde{\omega}_{\ell,k}$ for $\ell = 0, \dots, m$ have been determined. To solve Eq. (25) for $\widetilde{\omega}_{m+1,k}$, let

$$\Lambda_\nu := \{\lambda_n : n = 1, \dots, \nu\}$$

and denote by $\mu_{\nu,\ell}$ the duplication of λ_ℓ in Λ_ν. It can easily be seen that Eq. (25) with $n = m+2$ and $\alpha = \mu_{m+1,m+2} - 1$ has only one unknown variable, namely, $\widetilde{\omega}_{m+1,k}$. Thus $\widetilde{\omega}_{m+1,k}$ is obtained uniquely. This completes the proof. □

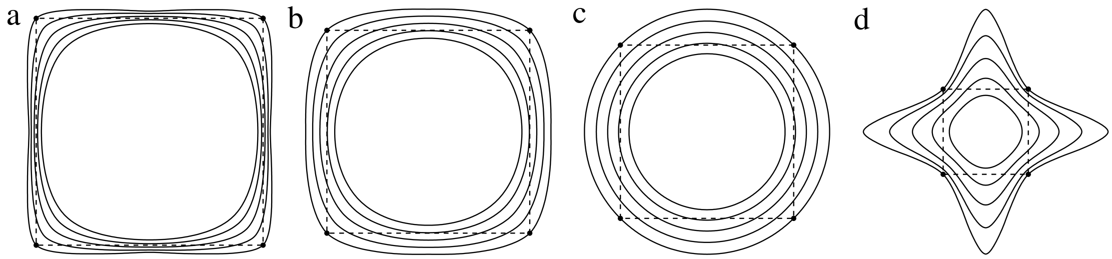

Fig. 4. Limit curves generated by using the mask (26) with various (λ_1, λ_2). Here, from the innermost to the outermost, $t = 0, 1/4, 1/2, 3/4, 1$, and $(\lambda_1, \lambda_2) =$ (a) $(0, 3)$, (b) $(0, 1)$, (c) $(0, i\pi/2)$, (d) $(0, 2.7i)$.

Theorem 5.2. *Let N be a positive even integer and put $M = N/2$. Assume that the exponential quasi-spline associated with the symbol in (23) reproduces $2M$ linearly independent exponential polynomials. Then the scheme is interpolatory.*

Proof. This theorem holds immediately from Theorem 2.5 and Remark 2.6 in [20]. $\square$

Remark 5.3 (*Approximation Order*)**.** It is discussed in [26] that a non-stationary scheme provides the approximation order N if it reproduces N linearly independent exponential polynomials. More precisely, assume that the initial data $\mathbf{f}^0$ is of the form $\mathbf{f}^0 := \{f_n^0 = f(2^{-\kappa}n) : n \in \mathbb{Z}\}$ for some $\kappa \in \mathbb{Z}_+$ with a smooth function f satisfying $\|f^{(\ell)}\|_{L_\infty(K)} < \infty$ for $\ell = 0, \ldots, N$. Then

$$\|f^\infty - f\|_{L_\infty(K)} \le c_f 2^{-\kappa N}$$

with a constant $c_f > 0$ depending on f but independent of κ, where K is a compact set in $\mathbb{R}$.

Remark 5.4. In [10], the authors constructed a family of non-stationary subdivision schemes based on the exponential *B*-splines. The associated symbols are in fact equivalent to the one in (23) with $M = 3$ so that the corresponding schemes reproduce up to six exponential polynomials. However, the study [10] is mainly concerned with Λ in (18) which satisfies the following condition. If $\pm\lambda_n$, $n = 1, \ldots, q$, are the distinct elements in Λ, then λ_n's need to be of the form

$$\lambda_n = n\lambda, \quad n = 1, \ldots, q,$$

where $\lambda \in \mathbb{R}$ or $i\mathbb{R}$.

6. Numerical example

In this section, we present some numerical results which illustrate the performance of the proposed schemes. To do this, we first employ the fourth-order exponential quasi-spline with the mask in (8). We especially employ ν_k in (20) and set

$$\omega_k = \frac{-8t}{\gamma_{1,k}\gamma_{2,k}(\gamma_{1,k} + \gamma_{2,k})},$$

which is just the mask in (21) multiplied by $t \in \mathbb{R}$. When $t = 0, 1$, the scheme results in the cubic exponential *B*-spline and the non-stationary four-point interpolatory scheme, respectively. Furthermore, we assume that $\lambda_1 = 0$ (i.e., $\gamma_{1,k} = 2$) to guarantee $\sum_{n\in\mathbb{Z}} a_{j-2n}^{[k]} = 1$ for $j = 1, 2$. Then the resulting mask is given by

$$\begin{aligned} a_0^{[k]} &= \frac{1 + \gamma_{2,k} + t}{2 + \gamma_{2,k}}, \qquad a_{-2}^{[k]} = a_2^{[k]} = \frac{1 - t}{2(2 + \gamma_{2,k})} \\ a_{-1}^{[k]} &= a_1^{[k]} = \frac{\gamma_{2,k}(2 + \gamma_{2,k}) + t}{2\gamma_{2,k}(2 + \gamma_{2,k})}, \qquad a_{-3}^{[k]} = a_3^{[k]} = \frac{-t}{2\gamma_{2,k}(2 + \gamma_{2,k})}. \end{aligned} \tag{26}$$

Example 6.1 (*Various Shapes and Tensions*)**.** Fig. 4 presents the limit curves generated with $t = 0, 1/4, 2/1, 3/4, 1$, and various values of λ_2. It demonstrates how different tension parameters ω_k affect the limit function. Other curves generated with real λ_n $(n = 1, 2)$ and $t = 1/4$ are given in Fig. 5. Here, we used $(\lambda_1, \lambda_2) = (0, 1), (0, 4), (0, 8)$ and $(0, 16)$.

Example 6.2 (*Reproduction of Surfaces*)**.** Polynomials, trigonometric functions and hyperbolic functions are frequently used in many applications. The proposed scheme can reproduce such functions exactly by employing a suitable set Λ. Fig. 6 presents reconstructions of the surfaces termed as a tear drop, a Möbius strip, a figure-8 Klein bottle and a toroidal knot, respectively, by using the mask (26) with $t = 1$. The specific parametric equations of these surfaces can be given as follows.

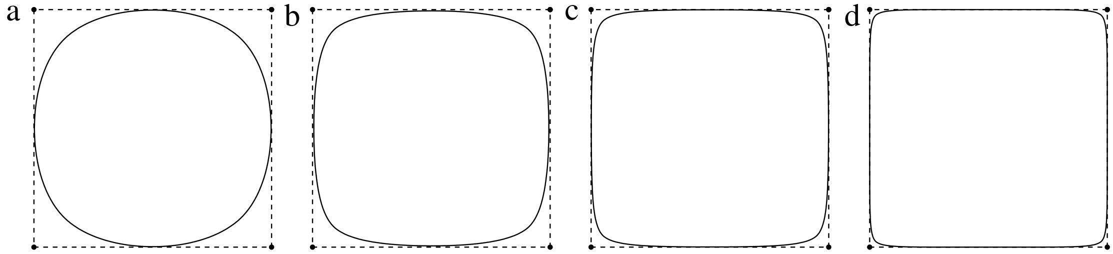

Fig. 5. Limit curves with $t = 1/2$ and $(\lambda_1, \lambda_2) =$ (a) (0, 1), (b) (0, 4), (c) (0, 8), (d) (0, 16).

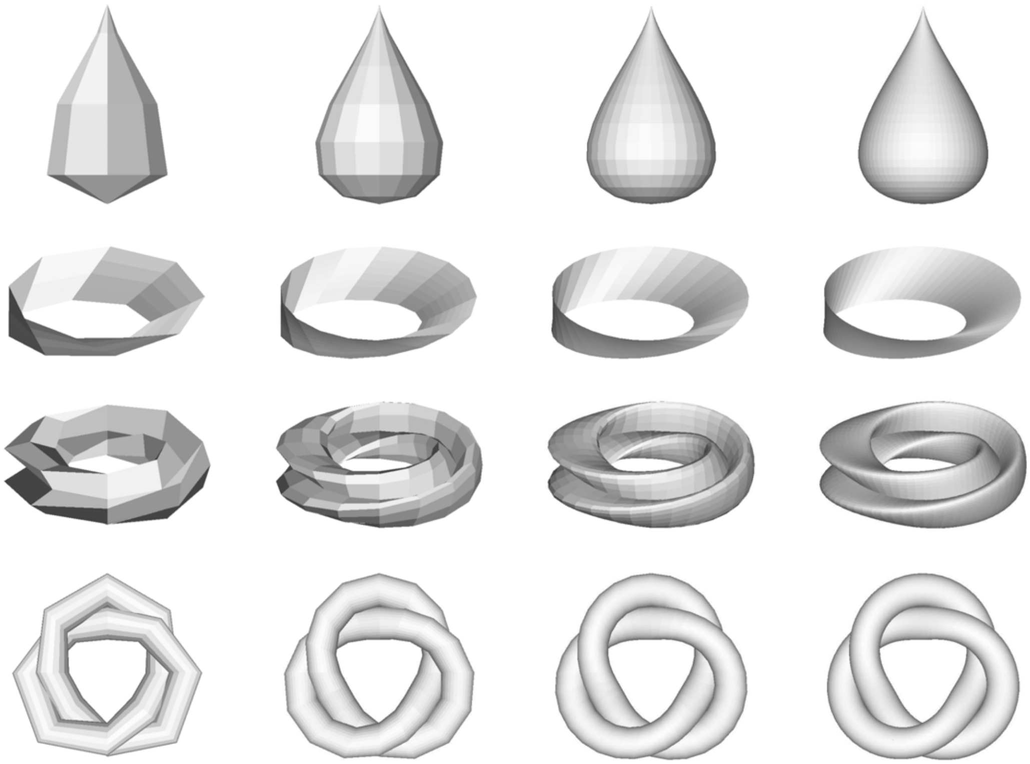

Fig. 6. From top to bottom, reproduction of surfaces termed as a tear drop, a Möbius strip, a figure-8 Klein bottle and a toroidal knot.

(Tear drop): for $0 \le u \le 4$ and $0 \le v \le 2$,

$$x(u, v) = \frac{1}{2} \cos \frac{\pi u}{2} \left(1 - \cos \frac{\pi v}{2}\right) \sin \frac{\pi v}{2},$$
$$y(u, v) = \frac{1}{2} \sin \frac{\pi u}{2} \left(1 - \cos \frac{\pi v}{2}\right) \sin \frac{\pi v}{2},$$
$$z(u, v) = \cos \frac{\pi v}{2}.$$

(Möbius strip): for $0 \le u \le 4$ and $-1 \le v \le 1$,

$$x(u, v) = 3 \cos \frac{\pi u}{2} + v \cos \frac{\pi u}{4},$$
$$y(u, v) = 3 \sin \frac{\pi u}{2} + v \cos \frac{\pi u}{4},$$
$$z(u, v) = v \sin \frac{\pi u}{4}.$$

(Figure-8 Klein bottle): for $0 \le u, v \le 4$,

$$x(u, v) = \left(3 + \cos \frac{\pi u}{4} \sin \frac{\pi v}{2} - \sin \frac{\pi u}{4} \sin \pi v\right) \cos \frac{\pi u}{2},$$

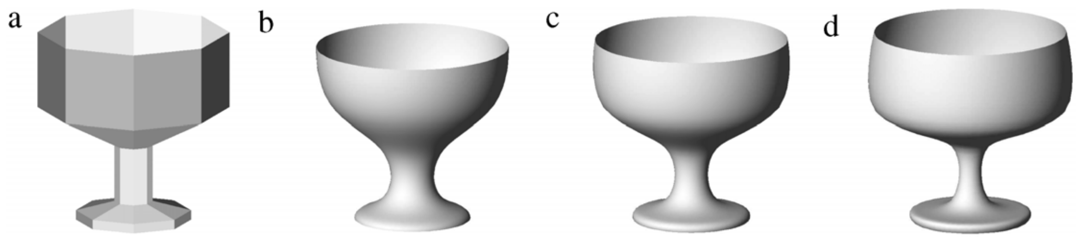

Fig. 7. Surfaces of revolutions generated by using the mask (26) with various vertical tensors. Here, (a) the initial control polyhedron, (b) $t = 0$, (c) $t = 1/2$, (d) $t = 1$.

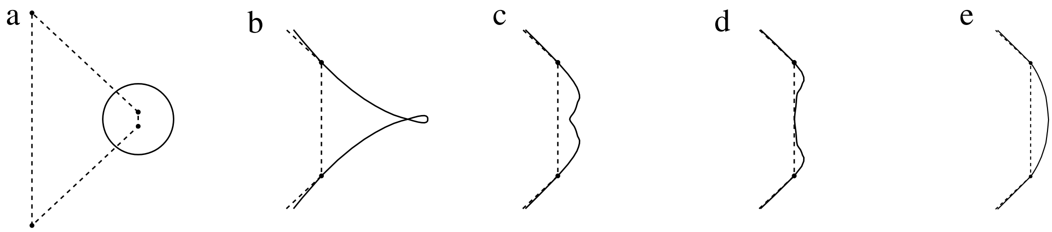

Fig. 8. Limit curves by the interpolatory scheme with the mask in (27) and the parameter ω_2 in (28): (a) initial control points, (b) $(t, \lambda_2) = (1, 0)$, (c) $(t, \lambda_2) = (1, 4)$, (d) $(t, \lambda_2) = (1, 8)$, (e) $(t, \lambda_2) = (0.25, 1.5)$.

$$y(u, v) = \left(3 + \cos\frac{\pi u}{4}\sin\frac{\pi v}{2} - \sin\frac{\pi u}{4}\sin\pi v\right)\sin\frac{\pi u}{2},$$
$$z(u, v) = \sin\frac{\pi u}{4}\sin\frac{\pi v}{2} + \cos\frac{\pi u}{4}\sin\pi v.$$

(Toroidal knot): for $0 \le u, v \le 4$,

$$x(u, v) = \left(4 - \cos\frac{\pi u}{2} + \sin\frac{3\pi v}{2}\right)\cos\pi v,$$
$$y(u, v) = \left(4 - \cos\frac{\pi u}{2} + \sin\frac{3\pi v}{2}\right)\sin\pi v,$$
$$z(u, v) = \sin\frac{\pi u}{2} - \cos\frac{3\pi v}{2}.$$

In order to reproduce such surfaces, we used tensor product schemes of two independently parameterized exponential quasi-splines. More specifically, seeing that the parametric equations of these surfaces are of the form $\sum_{\ell=1}^{L} f_\ell(u)g_\ell(v)$ with suitable trigonometric functions f_ℓ and g_ℓ, we first construct tensor product schemes reproducing their component functions $f_\ell(u)g_\ell(v)$ for $L = 1, \ldots, L$, respectively. Then, these tensor product schemes are combined together to reconstruct the given parametric surfaces. For this experiment, we adopt the initial control points $f_\ell(n2^{-k})g_\ell(n2^{-k})$ with $k = 1$ (tear drop), 0 (Möbius strip), 1 (figure-8 Klein bottle), 2 (toroidal knot).

Example 6.3 (*Surface of Revolution*)**.** The surface of revolution is commonly used in many geometric design applications. From the initial control polyhedron in Fig. 7(a), different types of wine glasses are obtained as presented in Fig. 7(b)–(d). For the horizontal components of the surfaces, the mask (26) with $\lambda_2 = \pi/2$ and $t = 1$ was used to reproduce a circle. The vertical components were generated by using the mask with $t = 0, 1/2, 1$, respectively.

Example 6.4. The interpolatory schemes often generate limit curves with unpleasant oscillatory artifacts if the given control points are highly irregular, as depicted in Fig. 8(b)–(d). However, by choosing proper parameters in the exponential quasi-spline, this drawback can be overcome. Consider the 4-point interpolatory scheme with the symbol

$$\begin{aligned} a_2^{[k]}(z) &= \beta_2^{[k]}(z)q_2^{[k]}(z) \\ &= \left(\frac{z^{-1} + z + \gamma_{1,k}}{2}\right)\left(\omega_{0,k} + \omega_{1,k}(z^{-1} + z) + \omega_{2,k}(z^{-1} + z)^2\right). \end{aligned}$$

In particular, by choosing $\lambda_1 = 0$, $\omega_{0,k} = 1$, and $\omega_{1,k} = -2\omega_{2,k}$, we get the interpolatory mask

$$\begin{aligned} &a_{2n}^{[k]} = \delta_{n,0}, \\ &a_{-1}^{[k]} = a_1^{[k]} = \frac{1}{2} - \frac{\omega_{2,k}}{2}, \qquad a_{-3}^{[k]} = a_3^{[k]} = \frac{\omega_{2,k}}{2}. \end{aligned} \tag{27}$$

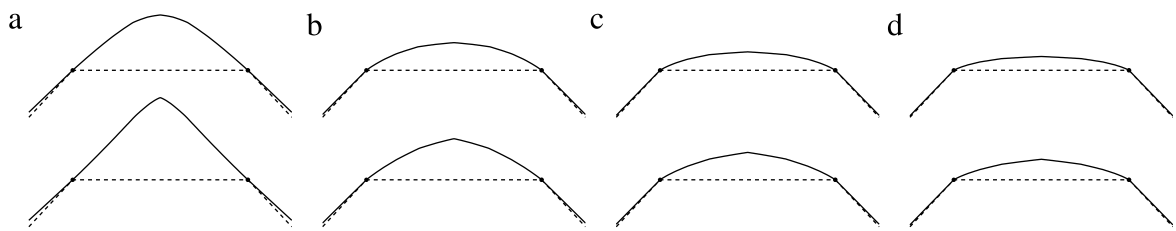

Fig. 9. Comparison between the exponential quasi-spline (top) and the four-point scheme (bottom) for the encircled region in Fig. 8: (a) $t = 1/2$, (b) $t = 1/4$, (c) $t = 1/6$, (d) $t = 1/8$.

Set

$$\omega_{2,k} = -\frac{t}{\gamma_{2,k}(2+\gamma_{2,k})}. \tag{28}$$

Note that when $t = 1$, we obtain the non-stationary 4-point interpolatory scheme reproducing $1, x, e^{\lambda_2 x}$ and $e^{-\lambda_2 x}$ if $\lambda_2 \neq 0$. In Fig. 8(e), we see that the suggested non-stationary scheme has an advantage in eliminating the artifact. For the results in Fig. 8(b)–(d), we used the parameter (28) with $t = 1$ and $\lambda_2 = 0, 4, 8$. For Fig. 8(e), we employed $t = 1/4$ and $\lambda_2 = 1.5$. Fig. 9 shows an interesting comparison of the known 4-point scheme by Dyn et al. [16], which also reduces the oscillatory behavior. Nonetheless, the advantage of the proposed scheme is more noticeable. In this experiment, we employed $\lambda_2 = 1.5$ for our non-stationary scheme.

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