Endemic threshold results for an age-structured SIS epidemic model with periodic parameters

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ABSTRACT

The main contribution of this paper is to obtain a threshold value for the existence and uniqueness of a nontrivial endemic periodic solution of an age-structured SIS epidemic model with periodic parameters. Under the assumption of the weak ergodicity of a non-autonomous Lotka–McKendrick system, we formulate a normalized system for an infected population as an initial boundary value problem of a partial differential equation. The existence problem for endemic periodic solutions is reduced to a fixed point problem of a nonlinear integral operator acting on a Banach space of locally integrable periodic L^1-valued functions. We prove that the spectral radius of the Fréchet derivative of the integral operator at zero plays the role of a threshold for the existence and uniqueness of a nontrivial fixed point of the operator corresponding to a nontrivial periodic solution of the original differential equation in a weak sense. If the Malthusian parameter of the host population is equal to zero, our threshold value is equal to the well-known epidemiological threshold value, the basic reproduction number R_0. However, if it is not the case, then two threshold values are different from each other and we have to pay attention on their actual biological implications.

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1. Introduction

The seasonality of infectious diseases is one of most important research interests in mathematical epidemiology, since the transmission parameters and host population behavior usually depend on season. Therefore many authors have examined differential equations systems with periodic parameters in order to model the seasonal spread of infectious diseases [12,13,28,2,1,37,30].

One of the most important concepts in this field is the basic reproduction number R_0. It is epidemiologically defined as the expected number of secondary cases produced by a typical infectious individual during its entire infectious period in a completely susceptible host population, and usually, it is mathematically obtained as the spectral radius of a linear operator, which is called the next generation operator [9,8,36,10].

Intuitively speaking, we can expect that the disease can invade into a completely susceptible host population if $R_0 > 1$, while it cannot if $R_0 < 1$ in a local sense. This is a principle of invasion threshold based on R_0. Moreover, usually we can also expect that there exists at least one endemic steady state if $R_0 > 1$, although it may not be a necessary condition for existence of endemic steady states. In fact, endemic steady states can exists under the subcritical condition $R_0 < 1$ (for instance, see [26,17]). This endemic threshold result has been widely established among autonomous epidemic systems for age-structured populations [18,26,17,19,20].

On the other hand, the general definition of the basic reproduction number in periodic environments was first successfully established by Bacaër and Guernaoui in 2006 [2], and then it has been extended to the case of more general time-heterogeneous environments by Thieme [34] and Inaba [21]. As is shown in Inaba [22], the invasion threshold principle for

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periodic structured epidemic models has been well-established based on the definition of Bacaër and Guernaoui. However, the endemic threshold result for periodic epidemic systems based on R_0 of the definition of Bacaër and Guernaoui has not yet been fully examined for structured population models, although for non-structured population models, Nakata and Kuniya [30] have shown existence of periodic endemic solution when $R_0 > 1$, where R_0 is given by the definition of Bacaër and Guernaoui.

Introduction of age-structure into epidemic models is a crucial point so that we deal with realistic situations. For example, any disease prevention policy depends on age structure of the host population. Due to their relatively complex form, the analysis is difficult and there are not a few open problems on the relation between R_0 and mathematical properties such as the existence, uniqueness, local and global stability of steady states. However, we strongly conjecture that the endemic threshold result for structured population in periodic environments will be properly formulated by R_0 defined by Bacaër and Guernaoui.

The main purpose of this paper is to obtain threshold results for an age-structured epidemic model with periodic parameters and study the relation between R_0 and the solution behavior of the model.

The model we study in this paper is an age-structured SIS epidemic model, in which total population is divided into two epidemiological classes, susceptibles and infectives, and individuals recovered from infection do not obtain immunity and directly go back to the susceptible class. The age-structured SIS epidemic models with time-independent parameters have been studied in [4,16,5,15,11]. A generalization of such models to a periodic system was given in [28], in which the periodic age-space-structured SIS epidemic model with reaction–diffusion terms is considered, so it has quite a general form. They showed the global asymptotic stability of a nontrivial endemic periodic solution, however, they relied on the assumption of the existence of such a periodic solution (for the case where there is no recruitment from other environments). Thus, the existence of a threshold value like R_0 for the existence of a nontrivial endemic periodic solution has not been investigated for any age-structured SIS periodic epidemic models, and this is the point we focus on in this paper.

Under the assumption of the weak ergodicity of a nonautonomous Lotka–McKendrick population system, the age-structured SIS periodic epidemic model we shall consider can be normalized to a single equation of fraction of infected population. Integrating it along the characteristic lines, we obtain an expression of the fraction of the infected by the force of infection. Substituting the expression into the definition of the force of infection, we obtain an integral equation, which can be a fixed point equation in a Banach space of locally integrable, time-periodic L^1-valued functions. We show that the spectral radius of the Fréchet derivative of the fixed point operator at zero is the desired threshold value, that is, if it is less than one, the trivial disease-free steady state of the normalized system is globally asymptotically stable, while if it is greater than one, there exists a unique nontrivial endemic periodic solution.

In case the Malthusian parameter of the host population is equal to zero, it is shown that the threshold value is equal to the basic reproduction number R_0 obtained by following to the definition in [2,34,22]. However, if it is not equal to zero, they can be different and therefore we have to pay attention to possible cases such as the relatively decreasing but absolutely increasing (or, vice versa) infected population.

The organization of this paper is as follows: In Section 2, we formulate the main model of this paper and normalize it to a single equation as mentioned above. In Section 3, we show the well-posedness of the time evolution problem. In Section 4, we obtain the threshold value as mentioned above, and prove the existence of a nontrivial endemic periodic solution of the system in case the threshold value is greater than one. In Section 5, we prove the uniqueness of such a nontrivial solution in case the threshold value is greater than one. In Section 6, we prove the global asymptotic stability of the trivial disease-free steady state of the system in case the threshold value is less than one. In Section 7, we investigate the relation between the threshold value and the basic reproduction number R_0. Finally, in Section 8, numerical illustration is given.

2. The basic model

Let $S(t, a)$ and $I(t, a)$ be the age-densities at time t and age $a \in [0, \omega]$ of susceptible and infective population respectively, where $\omega < \infty$ denotes the maximum attainable age. Let $P(t, a)$ be the age-density of host population at time t and thus we have $P(t, a) = S(t, a) + I(t, a)$. Let $N(t)$ be the total size of the population at time t and thus we have

$$N(t) = \int_0^\omega P(t, a) \, da.$$

Let $\mu(t, a)$ be the age-specific mortality rate at time t, $\gamma(t, a)$ be the age-specific recovery rate, $k(t, a, \sigma)$ be the transmission coefficient between susceptible individuals aged a and infective individuals aged σ, $f(t, a)$ be the age-specific fertility rate and $\lambda(t, a)$ be the force of infection to susceptible individuals aged a, at time t, respectively. We make the following technical assumption on vital parameters:

Assumption 1. The basic vital parameters f, γ, μ and k are periodic in time t with period $T > 0$, and we assume that $f(t, \cdot), \gamma(t, \cdot) \in L_+^\infty(0, \omega)$, $k(t, \cdot, \cdot) \in L_+^\infty([0, \omega] \times [0, \omega])$ and $\mu(t, a)$ is locally integrable along the characteristic line $t - a = \text{const.}$ and

$$\int_0^\omega \mu(t + \sigma, \sigma) d\sigma = \infty,$$

for all $t \in \mathbb{R}$, which implies that the cohort survival rate is zero at age ω.

Then the age-structured SIS epidemic model with time-periodic parameters is formulated as follows:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) S(t, a) = -\lambda(t, a) S(t, a) - \mu(t, a) S(t, a) + \gamma(t, a) I(t, a),$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) I(t, a) = \lambda(t, a) S(t, a) - (\mu(t, a) + \gamma(t, a)) I(t, a),$$

$$\lambda(t, a) = \frac{1}{N(t)} \int_0^\omega k(t, a, \sigma) I(t, \sigma) \, d\sigma, \tag{2.1}$$

$$S(t, 0) = \int_0^\omega f(t, a) P(t, a) \, da,$$

$$I(t, 0) = 0,$$

$$S(0, a) = S_0(a), \qquad I(0, a) = I_0(a),$$

where the force of infection is given by the standard type of incidence and the initial data are given by nonnegative integrable functions.

Since we neglect an additional death rate for infected, the host population dynamics is described by the Lotka–McKendrick system with periodic coefficients:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) P(t, a) = -\mu(t, a) P(t, a),$$

$$P(t, 0) = \int_0^\omega f(t, a) P(t, a) \, da, \tag{2.2}$$

$$P(0, a) = P_0(a) := S_0(a) + I_0(a).$$

To simplify the basic model, we introduce the age distribution of ratio for each epidemiological class as follows:

$$s(t, a) := \frac{S(t, a)}{P(t, a)}, \qquad i(t, a) := \frac{I(t, a)}{P(t, a)}. \tag{2.3}$$

Now we can rewrite model (2.1) as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s(t, a) = -\lambda(t, a) s(t, a) + \gamma(t, a) i(t, a),$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(t, a) = \lambda(t, a) s(t, a) - \gamma(t, a) i(t, a),$$

$$\lambda(t, a) = \int_0^\omega k(t, a, \sigma) \frac{P(t, \sigma)}{N(t)} i(t, \sigma) \, d\sigma, \tag{2.4}$$

$$s(t, 0) = 1, \qquad i(t, 0) = 0,$$

$$s(0, a) = s_0(a), \qquad i(0, a) = i_0(a).$$

Here we assume that the Lotka–McKendrick system (2.2) has a positive persistent solution as

$$P^*(t, a) = e^{r(t-a)} b(t-a) e^{-\int_0^a \mu(t-a+\sigma, \sigma) \, d\sigma},$$

where the Malthusian parameter r and a positive periodic function $b(t)$ satisfy the characteristic relation:

$$b(t) = \int_0^\omega f(t, a) e^{-ra - \int_0^a \mu(t-a+\sigma, \sigma) d\sigma} b(t-a) \, da.$$

Let $\theta(t, a)$ be the periodic age profile:

$$\theta(t, a) := \frac{P^*(t, a)}{\int_0^\omega P^*(t, x) dx}.$$

In this paper we assume that the age profile of the host population has already attained a periodic stable age profile θ, that is,

$$P(t, a) = N(t) \theta(t, a),$$

holds for all $t \geq 0$. We adopt a technical, but biologically reasonable, assumption as follows:

We remark that if we assume that the non-autonomous system (2.2) is *weakly ergodic*, the age profile of the host population converges to a periodic age profile θ in a L^1-sense, that is,

$$\lim_{t \to \infty} \left\| \frac{P(t, \cdot)}{N(t)} - \theta(t, \cdot) \right\|_{L^1} = 0,$$

where we use the notation θ neglecting the phase difference.

The stable population model with periodic parameters was first studied by A. J. Coale [6,7] based on the renewal integral equation. The reader may refer to [35,23,32,31,22] for the proof of the above statements. Note that our analysis covers a special case that only epidemic parameters are periodic and all demographic parameters are time independent, hence the Lotka–McKendrick system (2.2) becomes an autonomous system and there exists a disease-free steady state (see Section 8).

Since it follows from (2.3) that $s(t, a) \equiv 1 - i(t, a)$, system (2.4) can be reduced to a single equation for $i(t, a)$ as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i(t, a) = \lambda(t, a)(1 - i(t, a)) - \gamma(t, a)i(t, a),$$

$$\lambda(t, a) = \int_0^\omega \beta(t, a, \sigma) i(t, \sigma) \, d\sigma, \tag{2.5}$$

$$i(t, 0) = 0, \qquad i(0, a) = i_0(a),$$

where

$$\beta(t, a, \sigma) := k(t, a, \sigma)\theta(t, \sigma)$$

is a given time-periodic transmission kernel.

It is obvious that system (2.5) always has the trivial disease-free steady state $i \equiv 0$. The main interest of this paper is to investigate the existence and uniqueness of a nontrivial endemic periodic solution of system (2.5).

3. Abstract formulation

Although the mathematical well-posedness of the initial-boundary value problem (2.5) is well known [4,15], here we sketch a method to show it for the readers' convenience. Note that we can reformulate system (2.5) as an abstract Cauchy problem in Banach space $E := L^1(0, \omega)$.

Let us define a linear operator $A : D(A) \subset E \to E$ as

$$\begin{cases} (A\varphi)(a) := -\dfrac{d}{da}\varphi(a), \\ D(A) = \left\{ \varphi \in E : \varphi \in W^{1,1}(0, \omega), \ \varphi(0) = 0 \right\}, \end{cases} \tag{3.1}$$

where $W^{1,1}(0, \omega)$ denotes the set of absolutely continuous functions on $(0, \omega)$. Let C be a closed convex set defined by

$$C := \left\{ \varphi \in E_+ : \ 0 \le \varphi(a) \le 1 \text{ a.e.} \right\}, \tag{3.2}$$

where E_+ denotes the positive cone of E. Let us define family $\{F(t, \cdot)\}_{t \ge 0} : C \subset E \to E$ of nonlinear bounded operators as

$$F(t, \varphi)(a) := \lambda[t, a| \varphi](1 - \varphi(a)) - \gamma(t, a)\varphi(a), \tag{3.3}$$

where

$$\lambda[t, a|\varphi] := \int_0^\omega \beta(t, a, \sigma)\varphi(\sigma) \, d\sigma.$$

Then system (2.5) is reformulated as a semilinear nonautonomous Cauchy problem

$$\frac{d}{dt}i(t) = Ai(t) + F(t, i(t)), \qquad i(0) = i_0 \tag{3.4}$$

in E. It is easy to see that operator A is the infinitesimal generator of a C_0-semigroup $\left\{ e^{tA} \right\}_{t \ge 0}$ defined by

$$\left(e^{tA}\varphi \right)(a) = \begin{cases} 0, & t > a, \\ \varphi(a - t), & t < a \end{cases} \tag{3.5}$$

on E. Here note that although (3.5) is not defined for $t = a$, it does not matter since $\varphi(0) = 0$ always holds for $\varphi \in D(A)$.

From (3.5) we immediately have $e^{tA}(C) \subset C$.

Using the fact that γ and β are bounded above and the same kind of argument as in [4], it is easy to show that the following holds:

Proposition 3.1. *$F(t, \cdot) : C \to E$ is Lipschitz continuous for any fixed $t \in \mathbb{R}_+$. There exists a constant $\alpha \in (0, 1)$ such that if $\phi \in C$, then $\phi + \alpha F(t, \phi) \in C$.*

Using α appeared in Proposition 3.1, we can rewrite problem (3.4) as

$$\frac{d}{dt} i(t) = \left(A - \frac{1}{\alpha} \right) i(t) + \frac{1}{\alpha} \left(i(t) + \alpha F(t, i(t)) \right), \qquad i(0) = i_0. \tag{3.6}$$

Now we are in position to look for a mild solution $i \in C$ of (3.6), which is given by the solution of integral equation

$$i(t) = e^{-\frac{1}{\alpha} t} e^{tA} i_0 + \frac{1}{\alpha} \int_0^t e^{-\frac{1}{\alpha}(t-s)} e^{(t-s)A} \{ i(s) + \alpha F(s, i(s)) \} \, ds. \tag{3.7}$$

Eq. (3.7) provides the following scheme for the standard iterative procedure for obtaining mild solution $i \in C$:

$$i^0(t) = i_0,$$

$$i^{n+1}(t) = e^{-\frac{1}{\alpha} t} e^{tA} i_0 + \frac{1}{\alpha} \int_0^t e^{-\frac{1}{\alpha}(t-s)} e^{(t-s)A} \left\{ i^n(s) + \alpha F\left(s, i^n(s)\right) \right\} \, ds, \quad n = 0, 1, 2, \ldots.$$

Since it is easy to see that C is invariant with respect to the iteration process, that is, $i^{n+1} \in C$ if $i^n \in C$, according to the argument in [4,15], we can prove the following theorem [15]:

Theorem 3.1. *Let $i_0 \in C$. Then, abstract Cauchy problem (3.4) has a unique mild solution $i(t) = U(t, 0) i_0$ in C, where $U(t, s), t \geq s \geq 0$ defines an evolutionary systems with the following property:*

$$U(s, s) = I, \qquad U(t, \sigma) U(\sigma, s) = U(t, s), \quad U(t, s)(C) \subset C,$$

$$U(t, s) u \leq U(t, s) v, \quad \text{if } u \leq v.$$

4. Existence of an endemic periodic solution

In this section, we investigate the existence of an endemic T-periodic solution of system (2.5). Let X_T be the set of locally integrable T-periodic E-valued functions with norm

$$\|\varphi\|_{X_T} := \int_0^T \|\varphi(t)\|_E \, dt = \int_0^T \int_0^\omega |\varphi(t, a)| \, da \, dt,$$

and $X_{T,+}$ be its positive cone. Let Ω_T be the state space given by

$$\Omega_T := \left\{ \varphi \in X_{T,+} : \ 0 \leq \varphi(t, a) \leq 1 \ \text{a.e.} \right\}. \tag{4.1}$$

If an endemic T-periodic solution $i^* \in \Omega_T \setminus \{0\}$ of system (2.5) exists, it satisfies

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i^*(t, a) = \lambda^*(t, a) \left(1 - i^*(t, a) \right) - \gamma(t, a) i^*(t, a),$$

$$\lambda^*(t, a) = \int_0^\omega \beta(t, a, \sigma) i^*(t, \sigma) \, d\sigma, \tag{4.2}$$

$$i^*(t, 0) = 0.$$

Integrating the first equation of (4.2) along the characteristic lines, we have

$$i^*(t, a) = \int_0^a \lambda^*(t - a + \sigma, \sigma) e^{-\int_\sigma^a [\lambda^*(t-a+\rho, \rho) + \gamma(t-a+\rho, \rho)] \, d\rho} \, d\sigma. \tag{4.3}$$

Note that if a time-periodic E-valued function $\lambda^*(t, \cdot)$ is given, $i^*(t, a)$ calculated from (4.3) is differentiable along the characteristic line $t - a = const.$, that is, it follows that

$$Di^*(t, a) = \lambda^*(t, a) \left(1 - i^*(t, a) \right) - \gamma(t, a) i^*(t, a), \tag{4.4}$$

where operator D is a directional derivative defined by

$$(Df)(t, a) := \lim_{h \to 0} \frac{f(t + h, a + h) - f(t, a)}{h}.$$

In the following, we look for a time-periodic solution of (4.2) in a weak sense such that the differential operator $\partial_t + \partial_a$ is interpreted as the directional derivative D. If we assume differentiability of parameter β, the weak solution becomes a classical solution.

Substituting (4.3) into the second equation of (4.2), we have

$$
\begin{aligned}
\lambda^* (t, a) &= \int_0^\omega \beta (t, a, \sigma) \int_0^\sigma \lambda^* (t - \sigma + \rho, \rho) \, \mathrm{e}^{- \int_\rho^\sigma [\lambda^*(t - \sigma + \eta, \eta) + \gamma(t - \sigma + \eta, \eta)] \, \mathrm{d}\eta} \, \mathrm{d}\rho \, \mathrm{d}\sigma \\
&= \int_0^\omega \int_0^\sigma \beta (t, a, \sigma) \lambda^* (t - \tau, \sigma - \tau) \, \mathrm{e}^{- \int_{\sigma - \tau}^\sigma [\lambda^*(t - \sigma + \eta, \eta) + \gamma(t - \sigma + \eta, \eta)] \, \mathrm{d}\eta} \, \mathrm{d}\tau \, \mathrm{d}\sigma \\
&= \int_0^\omega \int_0^\sigma \beta (t, a, \sigma) \lambda^* (t - \tau, \sigma - \tau) \, \mathrm{e}^{- \int_0^\tau [\lambda^*(t - \zeta, \sigma - \zeta) + \gamma(t - \zeta, \sigma - \zeta)] \, \mathrm{d}\zeta} \, \mathrm{d}\tau \, \mathrm{d}\sigma \\
&= \int_0^\omega \int_\tau^\omega \beta (t, a, \sigma) \, \mathrm{e}^{- \int_0^\tau [\gamma(t - \zeta, \sigma - \zeta) + \lambda^*(t - \zeta, \sigma - \zeta)] \, \mathrm{d}\zeta} \lambda^* (t - \tau, \sigma - \tau) \, \mathrm{d}\sigma \, \mathrm{d}\tau.
\end{aligned}
$$

Define a nonlinear positive operator

$$
\Phi (\varphi) (t, a) := \int_0^\omega \int_\tau^\omega \beta (t, a, \sigma) \, \mathrm{e}^{- \int_0^\tau [\gamma(t - \zeta, \sigma - \zeta) + \varphi(t - \zeta, \sigma - \zeta)] \, \mathrm{d}\zeta} \varphi (t - \tau, \sigma - \tau) \, \mathrm{d}\sigma \, \mathrm{d}\tau \tag{4.5}
$$

on X_T. Then if Φ has a nontrivial fixed point $\lambda^* = \Phi(\lambda^*)$ in $X_{T,+} \setminus \{0\}$, there exists an endemic T-periodic solution i^* in $\Omega_T \setminus \{0\}$ in a weak sense. In fact, if there exists such a fixed point $\lambda^* = \Phi(\lambda^*) \in X_{T,+} \setminus \{0\}$, then from (4.3) we easily see that $i^* \geq 0$, $i^* \neq 0$ and

$$
\begin{aligned}
i^* (t, a) &= \int_0^a \left(\lambda^* (t - a + \sigma, \sigma) + \gamma (t - a + \sigma, \sigma) \right) \mathrm{e}^{- \int_\sigma^a \lambda^*(t - a + \rho, \rho) + \gamma(t - a + \rho, \rho) \, \mathrm{d}\rho} \, \mathrm{d}\sigma \\
&\quad - \int_0^a \gamma (t - a + \sigma, \sigma) \, \mathrm{e}^{- \int_\sigma^a [\lambda^*(t - a + \rho, \rho) + \gamma(t - a + \rho, \rho)] \, \mathrm{d}\rho} \, \mathrm{d}\sigma \\
&= 1 - \mathrm{e}^{- \int_0^a [\lambda^*(t - a + \rho, \rho) + \gamma(t - a + \rho, \rho)] \, \mathrm{d}\rho} - \int_0^a \gamma (t - a + \sigma, \sigma) \, \mathrm{e}^{- \int_\sigma^a [\lambda^*(t - a + \rho, \rho) + \gamma(t - a + \rho, \rho)] \, \mathrm{d}\rho} \, \mathrm{d}\sigma \leq 1.
\end{aligned}
$$

Hence $i^* \in \Omega_T \setminus \{0\}$ and it satisfies (4.2) in the weak sense. Therefore, in what follows, we investigate the existence of a nontrivial fixed point λ^* of operator Φ in $X_{T,+} \setminus \{0\}$.

Let us define a positive bounded linear operator

$$
(K\varphi) (t, a) := \int_0^\omega \int_\tau^\omega \beta (t, a, \sigma) \, \mathrm{e}^{- \int_0^\tau \gamma(t - \zeta, \sigma - \zeta) \, \mathrm{d}\zeta} \varphi (t - \tau, \sigma - \tau) \, \mathrm{d}\sigma \, \mathrm{d}\tau, \quad \varphi \in X_T, \tag{4.6}
$$

which is the Fréchet derivative of operator Φ at $\varphi = 0$ and it is a majorant of Φ, that is $\Phi \leq K$. Without loss of generality, we can assume that β is uniformly bounded above, so Φ and K define maps from the positive cone of X_T into itself.

Let $\rho(K)$ be the spectral radius of operator K. Our main purpose here is to show the following proposition.

Proposition 4.1. *Suppose that $\rho(K) > 1$. Then operator Φ has at least one nontrivial fixed point $\lambda^* = \Phi(\lambda^*) \in X_{T,+} \setminus \{0\}$.*

For the proof of this proposition, we prepare two lemmas. The first one is as follows.

Lemma 4.1. *The operator Φ is monotone nondecreasing on X_T and $\Phi(\varphi), \varphi \in X_{T,+}$ is uniformly bounded.*

Proof. From (4.5), we have

$$
\begin{aligned}
\Phi (\varphi) (t, a) &= \int_0^\omega \beta (t, a, \sigma) \int_0^\sigma \{ \varphi (t - \tau, \sigma - \tau) + \gamma (t - \tau, \sigma - \tau) \} \, \mathrm{e}^{- \int_0^\tau \varphi(t - \zeta, \sigma - \zeta) + \gamma(t - \zeta, \sigma - \zeta) \, \mathrm{d}\zeta} \, \mathrm{d}\tau \mathrm{d}\sigma \\
&\quad - \int_0^\omega \beta (t, a, \sigma) \int_0^\sigma \gamma (t - \tau, \sigma - \tau) \, \mathrm{e}^{- \int_0^\tau \varphi(t - \zeta, \sigma - \zeta) + \gamma(t - \zeta, \sigma - \zeta) \, \mathrm{d}\zeta} \, \mathrm{d}\tau \, \mathrm{d}\sigma \\
&= \int_0^\omega \beta (t, a, \sigma) \left(1 - \mathrm{e}^{- \int_0^\sigma \varphi(t - \zeta, \sigma - \zeta) + \gamma(t - \zeta, \sigma - \zeta) \, \mathrm{d}\zeta} \right) \mathrm{d}\sigma \\
&\quad - \int_0^\omega \beta (t, a, \sigma) \int_0^\sigma \gamma (t - \tau, \sigma - \tau) \, \mathrm{e}^{- \int_0^\tau \varphi(t - \zeta, \sigma - \zeta) + \gamma(t - \zeta, \sigma - \zeta) \, \mathrm{d}\zeta} \, \mathrm{d}\tau \, \mathrm{d}\sigma. \tag{4.7}
\end{aligned}
$$

Then Φ is monotone nondecreasing. Moreover, from (4.7), we have

$$
\Phi (\varphi) (t, a) \leq \int_0^\omega \beta (t, a, \sigma) \, \mathrm{d}\sigma \leq k^+,
$$

where $k^+ := \sup k < +\infty$, which is well defined by Assumption 1. $\quad \square$

Now we make the following technical assumption, which is needed to ensure compactness of the fixed point operator and its derivative:

Assumption 2. For the transmission coefficient k, it holds that

$$\lim_{h \to 0} \int_0^T \int_0^\omega |k(t+h, a+h, \sigma) - k(t, a, \sigma)| \, da \, dt = 0 \quad \text{uniformly for } \sigma \in [0, \omega]. \tag{4.8}$$

Lemma 4.2. *Under the Assumption 2, if $\rho(K) > 0$, it is a positive eigenvalue of K associated with a positive eigenvector $v_0 \in X_{T,+} \setminus \{0\}$.*

Proof. First we show that K is regarded as a compact operator on $L^1([0, T] \times [0, \omega])$. Observe that K is a linear map from X_T into itself leaving the cone invariant, and we have

$$
\begin{aligned}
(K\varphi)(t, a) &= \int_0^\omega \int_\tau^\omega \beta(t, a, \sigma) \, e^{-\int_0^\tau \gamma(t-\zeta, \sigma-\zeta) \, d\zeta} \varphi(t-\tau, \sigma-\tau) \, d\sigma \, d\tau \\
&= \int_0^\omega \int_0^{\omega-\tau} \beta(t, a, \tau+x) \, e^{-\int_0^\tau \gamma(t-\zeta, \tau+x-\zeta) \, d\zeta} \varphi(t-\tau, x) \, dx \, d\tau \\
&= \int_{t-\omega}^t \int_0^{\omega-t+s} \beta(t, a, t-s+x) \, e^{-\int_0^{t-s} \gamma(t-\zeta, t-s+x-\zeta) \, d\zeta} \varphi(s, x) \, dx \, ds. \tag{4.9}
\end{aligned}
$$

If we extend the domain of parameter as $\beta(t, a, \sigma) = 0$ for $(a, \sigma) \notin [0, \omega] \times [0, \omega]$, we can rewrite (4.9) as

$$(K\varphi)(t, a) = \int_{-\infty}^t \int_0^\omega \beta(t, a, t-s+x) \, e^{-\int_0^{t-s} \gamma(t-\zeta, t-s+x-\zeta) \, d\zeta} \varphi(s, x) \, dx \, ds.$$

Note that

$$\int_{-\infty}^t = \int_0^t + \sum_{n=0}^\infty \int_{-(n+1)T}^{-nT},$$

and

$$
\begin{aligned}
&\int_{-(n+1)T}^{-nT} \int_0^\omega \beta(t, a, t-s+x) \, e^{-\int_0^{t-s} \gamma(t-\zeta, t-s+x-\zeta) \, d\zeta} \varphi(s, x) \, dx \, ds \\
&= \int_0^T \int_0^\omega \beta(t, a, t-s+(n+1)T+x) \, e^{-\int_0^{t-s+(n+1)T} \gamma(t-\zeta, t-s+(n+1)T+x-\zeta) \, d\zeta} \varphi(s-(n+1)T, x) \, dx \, ds \\
&= \int_0^T \int_0^\omega \Psi(t, a, t-s+(n+1)T, x) \, \varphi(s, x) \, dx \, ds,
\end{aligned}
$$

where Ψ is defined by

$$\Psi(t, a, z, x) := \beta(t, a, z+x) \, e^{-\int_0^z \gamma(t-\zeta, z+x-\zeta) \, d\zeta}. \tag{4.10}$$

According to [1], we define

$$\hat{\Psi}(t, a, s, x) := \begin{cases} \displaystyle\sum_{n=0}^\infty \Psi(t, a, t-s+nT, x) & \text{for } t > s, \\ \displaystyle\sum_{n=1}^\infty \Psi(t, a, t-s+nT, x) & \text{for } t < s. \end{cases} \tag{4.11}$$

Then $\hat{\Psi}$ is well defined, because the right hand side is a finite sum due to the fact that $\Psi(t, a, z) = 0$ for $z > \omega$.

Then it follows from (4.9) that

$$(K\varphi)(t, a) = \int_0^T \int_0^\omega \hat{\Psi}(t, a, s, x) \, \varphi(s, x) \, dx \, ds.$$

Hence, we can regard K as an operator on $L^1([0, T] \times [0, \omega])$. From (4.8) and the well-known compactness criteria in L^1 (see, for instance, [38, p. 275]), we see that K is compact. Since K is positive, linear and compact, it follows from the Krein–Rutman theorem [27] that if $\rho(K) > 0$, it is a positive eigenvalue of K associated with a positive eigenvector $\tilde{v}_0 \in L^1_+([0, T] \times [0, \omega]) \setminus \{0\}$. That is

$$(K\tilde{v}_0)(t, a) = \rho(K) \tilde{v}_0(t, a).$$

Hence, it is easy to see that there exists a periodic eigenvector v_0 in $X_{T,+} \setminus \{0\}$ of K, which is associated with eigenvalue $\rho(K)$ and is the periodization of $\tilde{v}_0$. $\quad\square$

Using the above two lemmas, we prove Proposition 4.1.

Proof of Proposition 4.1. It follows from Lemma 4.2 that for vector $v_0 \in X_{T,+} \setminus \{0\}$ we have

$$(Kv_0)(t,a) = \rho(K) v_0(t,a) = \int_0^T \int_0^\omega \hat{\Psi}(t,a,s,x) v_0(s,x) \, dx \, ds. \tag{4.12}$$

Hence, we have

$$\rho(K) v_0(t,a) \leq \hat{\Psi}^+ \|v_0\|_{X_T}, \tag{4.13}$$

where $\hat{\Psi}^+ := \sup \hat{\Psi}(t,a,s,x) < \infty$. Let

$$\lambda_0 := \frac{\rho(K) \log \rho(K)}{\hat{\Psi}^+ \omega} \frac{v_0}{\|v_0\|_{X_T}} \in X_{T,+} \setminus \{0\}, \tag{4.14}$$

whose positivity follows from $\rho(K) > 1$. Then, from (4.12)–(4.14), we have

$$
\begin{aligned}
\Phi(\lambda_0)(t,a) &= \int_0^\omega \int_\tau^\omega \beta(t,a,\sigma) \, e^{-\int_0^\tau \gamma(t-\zeta,\sigma-\zeta)+\lambda_0(t-\zeta,\sigma-\zeta)\,d\zeta} \lambda_0(t-\tau,\sigma-\tau) \, d\sigma \, d\tau \\
&\geq \int_0^\omega \int_\tau^\omega e^{-\int_0^\omega \lambda_0(t-\zeta,\sigma-\zeta)\,d\zeta} \beta(t,a,\sigma) \, e^{-\int_0^\tau \gamma(t-\zeta,\sigma-\zeta)\,d\zeta} \lambda_0(t-\tau,\sigma-\tau) \, d\sigma \, d\tau \\
&= \int_0^\omega \int_\tau^\omega e^{-\frac{\log \rho(K)}{\hat{\Psi}^+ \omega \|v_0\|_{X_T}} \int_0^\omega \rho(K) v_0(t-\zeta,\sigma-\zeta)\,d\zeta} \beta(t,a,\sigma) \, e^{-\int_0^\tau \gamma(t-\zeta,\sigma-\zeta)\,d\zeta} \lambda_0(t-\tau,\sigma-\tau) \, d\sigma \, d\tau \\
&\geq e^{-\log \rho(K)} (K\lambda_0)(t,a) \\
&= \lambda_0(t,a).
\end{aligned}
$$

Hence, from the monotonicity of operator Φ proved in Lemma 4.1, we can define a monotone sequence as

$$\lambda_n = \Phi(\lambda_{n-1}), \quad \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots.$$

From Lemma 4.1 we see that λ_n is bounded above. Therefore, it follows from B. Levi's theorem that there exists $\lambda^* \in X_{T,+} \setminus \{0\}$ such that $\lim_{n \to \infty} \lambda_n = \lambda^*$ and $\lambda^* = \Phi(\lambda^*)$. $\quad\square$

From Proposition 4.1 and the arguments stated in the above, we immediately have the following proposition:

Proposition 4.2. *Suppose that $\rho(K) > 1$. Then, system (2.5) has at least one endemic T-periodic weak solution.*

5. Uniqueness of endemic periodic solution

Next we investigate the uniqueness of endemic T-periodic solution. For our purpose, we add the following assumptions:

Assumption 3. There exists a positive number $\epsilon > 0$ such that $k(t,a,\sigma) \geq \epsilon$ for all $(t,a,\sigma) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$.

Biologically speaking, we assume that transmission can occur between every susceptible and infected. In order to prove the uniqueness result, we prepare a following lemma:

Lemma 5.1. *If an endemic T-periodic solution $i^* \in \Omega_T \setminus \{0\}$ satisfying (4.2) exists, there exist numbers $0 < \alpha_1 = \alpha_1(i^*) < \alpha_2 = \alpha_2(i^*)$ such that $\alpha_1 \leq \lambda^* \leq \alpha_2$ for the force of infection λ^* corresponding to i^*.*

Proof. Let i^* be a periodic solution of (4.2). Observe that

$$\epsilon V(t) \leq \lambda^*(t,a) \leq k^+ V(t)$$

where

$$V(t) := \int_0^\omega \theta(t,a) i^*(t,a) \, da.$$

Then $V(t)$ is the prevalence (proportion of infected) at time t, and it is clear $0 \leq V \leq 1$. To complete our proof, it is sufficient to show $\inf_{t \in \mathbb{R}} V(t) > 0$. Without loss of generality, we can assume that $V(t)$ is a positive, continuous periodic function, since

$$V(t) = \frac{1}{N(t)} \int_0^\omega I^*(t,a) \, da,$$

where $I^*(t,a) = P^*(t,a) i^*(t,a)$ is a periodic solution of the original system. Then its global minimum is nonnegative. If there exists a time t_0 such that $V(t_0) = 0$, then $i^*(t_0,a) = 0$ for almost all $a \in [0,\omega]$, which implies that $i^*(t,a) = 0$ for all $t \geq t_0$ by the uniqueness of solution. This is a contradiction, so the global minimum of $V(t)$ is positive. $\quad\square$

Lemma 5.2. *Let λ^* be the force of infection corresponding to a periodic endemic classical solution i^*. For a number $\kappa \in (0, 1)$, there exists a positive number $\eta(i^*) > 0$ such that*

$$\Phi\left(\kappa\lambda^*\right)(t, a) \geq \kappa\Phi\left(\lambda^*\right)(t, a) + \eta. \tag{5.1}$$

Proof. From Eq. (4.5), we have

$$\Phi\left(\kappa\lambda^*\right)(t, a) - \kappa\Phi\left(\lambda^*\right)(t, a) = \int_0^\omega \int_\tau^\omega \beta(t, a, \sigma)\, e^{-\int_0^\tau \gamma(t-\zeta, \sigma-\zeta)\, d\zeta} \kappa\lambda^*(t-\tau, \sigma-\tau)\, e^{-\int_0^\tau \lambda^*(t-\zeta, \sigma-\zeta)\, d\zeta}$$

$$\times \left(e^{(1-\kappa)\int_0^\tau \lambda^*(t-\zeta, \sigma-\zeta)\, d\zeta} - 1\right)\, d\sigma\, d\tau$$

$$\geq \epsilon\kappa\alpha_1\left(i^*\right) \int_0^\omega d\sigma\theta(t, \sigma) \int_0^\sigma e^{-(\gamma^+ + \alpha_2)\tau}\left(e^{(1-\kappa)\alpha_1\tau} - 1\right)\, d\tau,$$

where $\gamma^+ := \sup \gamma < \infty$. Then if we define

$$\eta := \epsilon\kappa\alpha_1(i^*) \int_0^\omega d\sigma\theta(t, \sigma) \int_0^\sigma e^{-(\gamma^+ + \alpha_2)\tau}\left(e^{(1-\kappa)\alpha_1\tau} - 1\right)\, d\tau,$$

then $\eta = \eta(i^*)$ is positive. This completes our proof. $\square$

Proposition 5.1. *The basic system (2.1) has at most one endemic T-periodic classical solution.*

Proof. Suppose that there exist two endemic periodic classical solution. Let λ_1^* and λ_2^* be corresponding force of infections. From the above lemma, there exist positive numbers $\alpha_{jk} = \alpha_{jk}(i_j^*)$, $(j, k = 1, 2)$ such that

$$0 < \alpha_{j1} \leq \lambda_j^*(t, a) \leq \alpha_{j2}, \quad j = 1, 2. \tag{5.2}$$

From inequality (5.2) we have

$$\lambda_1^* \geq \alpha_{11} = \alpha_{11}\alpha_{22}^{-1}\alpha_{22} \geq \alpha_{11}\alpha_{22}^{-1}\lambda_2^*. \tag{5.3}$$

Let $\kappa := \sup\left\{\eta : \lambda_1^* \geq \eta\lambda_2^*\right\}$. Then $\kappa > 0$ follows from inequality (5.3). Suppose that $\kappa < 1$. Then it follows from the above lemma that there exist positive numbers η_1 and η_2 such that

$$\Phi(\kappa\lambda_j^*) \geq \kappa\Phi\left(\lambda_j^*\right)(t, a) + \eta_j, \quad j = 1, 2.$$

Therefore, from Lemma 4.1 and the fact that $\lambda_j^* = \Phi(\lambda_j^*)$, $(j = 1, 2)$, we have

$$\lambda_1^* = \Phi\left(\lambda_1^*\right) \geq \Phi\left(\kappa\lambda_2^*\right) \geq \kappa\Phi\left(\lambda_2^*\right) + \eta_2 = \kappa\lambda_2^* + \eta_2\alpha_{22}^{-1}\alpha_{22} \geq \kappa\lambda_2^* + \eta_2\alpha_{22}^{-1}\lambda_2^* = \left(\kappa + \eta_2\alpha_{22}^{-1}\right)\lambda_2^*,$$

which contradicts the definition of κ. Thus, we have $\kappa \geq 1$ and $\lambda_1^* \geq \kappa\lambda_2^* \geq \lambda_2^*$. Exchanging the role of λ_1^* and λ_2^*, we can prove $\lambda_2^* \geq \lambda_1^*$. Therefore, $\lambda_1^* = \lambda_2^*$ and hence $i_1^* = i_2^*$. $\square$

6. Global stability of the disease-free steady state

In this section, we investigate the case where $\rho(K) < 1$. First it is easy to see that under the subcritical condition $\rho(K) < 1$, system (2.5) does not have any endemic T-periodic solutions. In fact, if we assume that there exists an endemic T-periodic solution $i^* \in \Omega_T \setminus \{0\}$. Then we see that operator Φ has a nontrivial fixed point $\lambda^* = \Phi(\lambda^*) \in X_+ \setminus \{0\}$. Then we have $\lambda^* = \Phi(\lambda^*) \leq K\lambda^*$, which implies $\rho(K) \geq 1$.

As is shown in Theorem 5.6 of [28], we can expect that if system (2.5) does not have any endemic T-periodic solutions in $\Omega_T \setminus \{0\}$, then the disease-free steady state $i^* \equiv 0$ of system (2.5) is globally asymptotically stable in region Ω_T. In fact, we have the following proposition.

Proposition 6.1. *Suppose that $\rho(K) < 1$. Then, the disease-free state $i^* \equiv 0$ of system (2.5) is globally asymptotically stable.*

Proof. Instead of (2.5), let us consider a linear system as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)y(t, a) = -\gamma(t, a)y(t, a) + \int_0^\omega \beta(t, a, \sigma)y(t, \sigma)d\sigma,$$

$$y(t, 0) = 0, \qquad y(0, a) = i_0(a).$$

Then it is easy to see that $0 \leq i(t, a) \leq y(t, a)$. Integrating along the characteristic line, we obtain an expression:

$$y(t, a) = \begin{cases} \int_0^a e^{-\int_\sigma^a \gamma(t-a+z,z)dz} B(t-a+\sigma, \sigma)d\sigma, & t-a > 0, \\ e^{-\int_0^t \gamma(\sigma, a-t+\sigma)d\sigma} i_0(a-t) + \int_0^t e^{-\int_\sigma^t \gamma(z, a-t+z)dz} B(\sigma, a-t+\sigma)d\sigma, & a-t > 0 \end{cases}$$

where

$$B(t, a) := \int_0^\omega \beta(t, a, \sigma) y(t, \sigma)d\sigma,$$

is the density of newly infected in the linear phase of the normalized system. Inserting the above expression of y into the definition of B and changing the order of integrals, we have a renewal equation:

$$B(t, a) = G(t, a) + \int_0^t d\eta \int_\eta^\omega \beta(t, a, \sigma) e^{-\int_0^\eta \gamma(t-x, \sigma-x)dx} B(t-\eta, \sigma-\eta)d\sigma,$$

where

$$G(t, a) := \int_t^\omega \beta(t, a, \sigma) e^{-\int_0^t \gamma(z, \sigma-t+z)dz} i_0(\sigma - t)d\sigma, \quad 0 < t < \omega,$$

and $G = 0$ for $t > \omega$. Define a time-periodic operator Ψ acting on $L^1(0, \omega)$ as

$$(\Psi(t, \eta)f)(a) := \int_\eta^\omega \beta(t, a, \sigma) e^{-\int_0^\eta \gamma(t-x, \sigma-x)dx} f(\sigma - \eta)d\sigma,$$

we arrive at an abstract renewal equation for L^1-valued functions:

$$B(t) = G(t) + \int_0^t \Psi(t, \eta)B(t-\eta)d\eta,$$

where $B(t) = B(t, \cdot) \in L^1$, etc. Then we note that the operator K on X_T is expressed as

$$(K\varphi)(t, a) = \int_0^\omega (\Psi(t, \eta)\varphi(t-\eta, \cdot))(a)d\eta, \quad \varphi \in X_T,$$

which is a kind of next generation operator introduced by Bacaër and Guernaoui [2]. Let us introduce a Laplace transform of Ψ by

$$(\hat{K}(z)\varphi)(t, a) := \int_0^\omega (e^{-z\eta}\Psi(t, \eta)\varphi(t-\eta, \cdot))(a)d\eta.$$

Using the periodic renewal theorem [35,22,21], we can conclude that there exists a periodic (vector-valued) function $\varphi_0(t)$ such that $B(t) \sim e^{r_0 t}\varphi_0(t)$ as $t \to \infty$, where φ_0 is a positive eigenfunction of $\hat{K}(r_0)$ associated with eigenvalue $\rho(K(r_0)) = 1$. Therefore if $\rho(K) = \rho(K(0)) < 1$, the Malthusian parameter r_0 is negative, we have $\lim_{t\to\infty} B(t) = 0$. Therefore we have $\lim_{t\to\infty} y(t, a) = \lim_{t\to\infty} i(t, a) = 0$, which shows the global asymptotic stability of the disease-free state. □

7. The basic reproduction number R_0

Here let us introduce the *basic reproduction number* R_0 for the periodic SIS epidemic system (2.1) to discuss the relation between R_0 and the threshold parameter $\rho(K)$. Originally, the basic reproduction number is defined as the average number of secondary cases produced by a typical infected individual, introduced into a completely susceptible host population, during its entire period of infectiousness. In this definition, the host population is assumed to be in a demographic steady state [9,8].

Recently, the definition of the basic reproduction number has been extended so that it can be applied to the case of time periodic environments [2,1,22,21]. However, even in this extended definition for periodic environments, the Malthusian parameter of the host susceptible population is assumed to be zero.

If the Malthusian parameter of host susceptible population is not zero, we have to distinguish two cases of the growth of infected population, absolute growth and relative growth. Based on the general theory of R_0 in a time-heterogeneous environment [21], we can calculate the basic reproduction number for a small group of infected invading into a growing susceptible population. However, the ratio of infected to the total population can go to zero if the growth rate of infected is less than the Malthusian parameter of the total population (relative eradication). Therefore we can define another threshold parameter which determines whether the ratio of infected can increase or not.

First let us introduce the basic reproduction number R_0 as a threshold value for absolute growth of infected population. Let $y(t, a)$ be a perturbation from the disease-free persistent solution $(P^*(t, a), 0)$ of (2.1). Then the linearized system around

the persistent solution is written as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(t, a) = v(t, a) - (\mu(t, a) + \gamma(t, a)) y(t, a),$$

$$v(t, a) = \int_0^\omega \theta(t, a) k(t, a, \sigma) y(t, \sigma) \, d\sigma, \tag{7.1}$$

where $v(t, a)$ denotes the density of newly infected individuals in the linear invasion phase.

Integrating the first equation of the linearized system (7.1) along the characteristic lines, we have

$$y(t, a) = \int_0^a v(t - a + \sigma, \sigma) \, e^{-\int_\sigma^a [\mu(t-a+\eta, \eta) + \gamma(t-a+\eta, \eta)] \, d\eta} \, d\sigma. \tag{7.2}$$

Substituting (7.2) into the second equation of (7.1), we have

$$v(t, a) = \int_0^\omega \theta(t, a) k(t, a, \sigma) \int_0^\sigma v(t - \sigma + \rho, \rho) \, e^{-\int_\rho^\sigma [\mu(t-\sigma+\eta, \eta) + \gamma(t-\sigma+\eta, \eta)] \, d\eta} \, d\rho \, d\sigma$$

$$= \int_0^\omega \int_\tau^\omega \theta(t, a) k(t, a, \sigma) v(t - \tau, \sigma - \tau) \, e^{-\int_0^\tau [\mu(t-\zeta, \sigma-\zeta) + \gamma(t-\zeta, \sigma-\zeta)] \, d\zeta} \, d\sigma \, d\tau$$

$$= \int_0^\omega \int_\tau^\omega \theta(t, a) k(t, a, \sigma) \, e^{-\int_0^\tau \mu(t-\zeta, \sigma-\zeta) + \gamma(t-\zeta, \sigma-\zeta) \, d\zeta} v(t - \tau, \sigma - \tau) \, d\sigma \, d\tau. \tag{7.3}$$

Let us define linear operator $A(t, \tau)$ from $E = L^1(0, \omega)$ into itself as

$$(A(t, \tau) \varphi)(a) := \int_\tau^\omega \theta(t, a) k(t, a, \sigma) \, e^{-\int_0^\tau \mu(t-\zeta, \sigma-\zeta) + \gamma(t-\zeta, \sigma-\zeta) \, d\zeta} \varphi(\sigma - \tau) \, d\sigma.$$

Then (7.3) can be written as an abstract homogeneous renewal equation:

$$v(t, a) = \int_0^\omega (A(t, \tau) v(t - \tau))(a) \, d\tau.$$

From the periodic renewal theorem [35,22], $v(t, a)$ is asymptotically proportional to an exponential solution $e^{r_0 t} w(t)$. The Malthusian parameter r_0 is a real root of the characteristic equation $\rho(\hat{A}(z)) = 1$, where $\hat{A}(z), z \in \mathbb{C}$ is a linear operator on X_T defined by

$$(\hat{A}(z)\varphi)(t) := \int_0^\omega e^{-z\tau} A(t, \tau) \varphi(t - \tau) \, d\tau,$$

and $w \in X_T$ is a positive eigenfunction of $\hat{A}(r_0)$ associated with the positive eigenvalue unity. Therefore the sign relation $\text{sign}(r_0) = \text{sign}(\rho(\hat{A}(0)) - 1)$ holds.

Based on the above observation, R_0 in a periodic environment is defined by the spectral radius of the next generation operator, denoted by K_T, on the space of periodic vector-valued functions X_T [2,22]:

$$(K_T \varphi)(t) := (\hat{A}(0)\varphi)(t) = \int_0^\omega A(t, \tau) \varphi(t - \tau) \, d\tau, \quad \varphi \in X_T.$$

We know that K_T is a positive operator from $X_{T,+}$ into itself:

$$(K_T \varphi)(t, a) = \theta(t, a) \int_0^\omega \int_\tau^\omega k(t, a, \sigma) \, e^{-\int_0^\tau [\mu(t-\zeta, \sigma-\zeta) + \gamma(t-\zeta, \sigma-\zeta)] \, d\zeta} \varphi(t - \tau, \sigma - \tau) \, d\sigma \, d\tau. \tag{7.4}$$

As is shown in [21], $R_0 = \rho(K_T)$ gives the asymptotic per generation growth factor of infected population. Now we can establish a following relation between $R_0 = \rho(K_T)$ and $\rho(K)$:

Proposition 7.1. *Suppose that K_T and K are compact positive operators. Then it holds that $R_0 \geq \rho(K)$ if $r > 0$, $R_0 = \rho(K)$ if $r = 0$ and $R_0 \leq \rho(K)$ if $r < 0$.*

Proof. Define a formal multiplication operator $L : \psi \to q\psi$ on X_T, where $q \in X_T$ is given by

$$q(t, a) := e^{-rt} P^*(t, a) = e^{-ra} b(t - a) \, e^{-\int_0^a \mu(t-\zeta, a-\zeta) d\zeta}.$$

Then we have

$$((K_T L)\psi)(t, a) = \frac{P^*(t, a)}{N(t)} \int_0^\omega \int_\tau^\omega k(t, a, \sigma) \, e^{-\int_0^\tau [\mu(t-\zeta, \sigma-\zeta) + \gamma(t-\zeta, \sigma-\zeta)] \, d\zeta} q(t - \tau, \sigma - \tau) \psi(t - \tau, \sigma - \tau) \, d\sigma \, d\tau,$$

where we can observe that

$$q(t - \tau, \sigma - \tau) = e^{-r(t-\tau)} P^*(t, \sigma) \, e^{\int_0^\tau \mu(t-\zeta, \sigma-\zeta) d\zeta}.$$

Therefore we obtain a formal relation that

$$((K_T L)\psi)(t, a) = q(t, a) \int_0^\omega \int_\tau^\omega k(t, a, \sigma) \, e^{r\tau - \int_0^\tau \gamma(t-\zeta, \sigma-\zeta) \, d\zeta} \frac{P^*(t, \sigma)}{N(t)} \psi(t - \tau, \sigma - \tau) \, d\sigma \, d\tau.$$

Therefore we have $L^{-1} K_T L > K$ if $r > 0$, $L^{-1} K_T L = K$ if $r = 0$ and $L^{-1} K_T L < K$ if $r < 0$. Since $\rho(L^{-1} K_T L) = \rho(K_T) = R_0$, we arrive at the conclusion. $\square$

Note that if K and K_T are semi-nonsupporting compact operators, we can apply the comparison theorem by Marek [29] to obtain a more sharp sign relation as

$$\text{sign}(r) = \text{sign}(R_0 - \rho(K)).$$

If $r = 0$, that is, the Malthusian parameter of the host susceptible population is zero, the threshold value $\rho(K)$ coincides with the basic reproduction number R_0, so we do not need to distinguish absolute growth and relative growth (in the normalized system) of infected population, and we obtain an endemic threshold result that there exists a unique periodic endemic state if $R_0 > 1$, while the disease-free periodic state is globally stable if $R_0 < 1$. As is mentioned above, if $r > 0$, there is a possibility that $\rho(K_T) > 1 > \rho(K)$. In this case, the size of infected population increases, but the proportion of infected population to the total population decrease. On the other hand, if $r < 0$ and $\rho(K_T) < 1 < \rho(K)$, the proportion of infected population can increase, although the size of infected decreases.

8. Numerical examples

In this section, providing some numerical examples, we verify the validity of our results obtained in the previous sections. To simplify, we consider the case where parameters $f(t, a) \equiv f(a)$ and $\mu(t, a) \equiv \mu(a)$ are only age-dependent, $\gamma(t, a) \equiv \gamma$ is constant and $k(t, a) \equiv k(t)$ is only time-periodic.

8.1. Calculation of threshold values

In the case of time-independent and age-dependent vital rates $f(a)$ and $\mu(a)$, from the strong ergodicity theorem (see, e.g., [14,24]), we have

$$\theta(t, a) \equiv \frac{e^{-ra} \, e^{-\int_0^a \mu(\rho) \, d\rho}}{\int_0^\omega e^{-ra} \, e^{-\int_0^a \mu(\rho) \, d\rho} \, da} =: \theta(a),$$

where r denotes the Malthusian parameter as in the previous sections. Hence, operators K and K_T are given by

$$(K\varphi)(t, a) = k(t) \int_0^\omega \int_\tau^\omega \theta(\sigma) \, e^{-\gamma\tau} \varphi(t - \tau, \sigma - \tau) \, d\sigma \, d\tau \tag{8.1}$$

and

$$(K_T\varphi)(t, a) = k(t) \theta(a) \int_0^\omega \int_\tau^\omega e^{-\int_0^\tau \mu(\sigma-\zeta) \, d\zeta - \gamma\tau} \varphi(t - \tau, \sigma - \tau) \, d\sigma \, d\tau, \tag{8.2}$$

respectively, where $\varphi \in X_T$. Since the right-hand side of (8.1) is independent of a, the spectral radius $\rho(K)$ is obtained by solving the eigenvalue problem

$$\rho(K) v(t) = k(t) \int_0^\omega \phi_1(\tau) v(t - \tau) \, d\tau, \quad v \in X_T, \tag{8.3}$$

where

$$\phi_1(\tau) := \int_\tau^\omega \theta(\sigma) \, d\sigma \, e^{-\gamma\tau}. \tag{8.4}$$

Moreover, since

$$(K_T v\theta)(t, a) = k(t) \theta(a) \int_0^\omega \left(\int_\tau^\omega \theta(\sigma - \tau) \, e^{-\int_0^\tau \mu(\sigma-\zeta) \, d\zeta} \, d\sigma \right) e^{-\gamma\tau} v(t - \tau) \, d\tau,$$

we see that the basic reproduction number $R_0 = \rho(K_T)$ is obtained by solving the eigenvalue problem

$$R_0 v(t) = k(t) \int_0^\omega \phi_2(\tau) v(t - \tau) \, d\tau, \quad v \in X_T, \tag{8.5}$$

where

$$
\phi_2(\tau) := \int_\tau^\omega \theta(\sigma - \tau)\, e^{-\int_0^\tau \mu(\sigma - \zeta)\, d\zeta}\, d\sigma\, e^{-\gamma\tau}
$$

$$
= \int_\tau^\omega \frac{e^{-r(\sigma-\tau)}\, e^{-\int_0^{\sigma-\tau} \mu(\rho)\, d\rho}}{\int_0^\omega e^{-ra}\, e^{-\int_0^a \mu(\rho)\, d\rho}\, da}\, e^{-\int_{\sigma-\tau}^\sigma \mu(\rho)\, d\rho}\, d\sigma\, e^{-\gamma\tau}
$$

$$
= \int_\tau^\omega \theta(\sigma)\, d\sigma\, e^{(r-\gamma)\tau}. \tag{8.6}
$$

In what follows we consider the transmission coefficient with form $k(t) = p\{1 + q\cos(2\pi t/T)\}$, where $p > 0$, $0 < q < 1$ and $T > 0$. Thus, using the method of [3], we can compute $\rho(K)$ as the largest real root of

$$
\frac{\rho(K)}{\phi_{1,0}} - 1 = 2\mathrm{Re}\, \frac{q^2/4}{\frac{\rho(K)}{\phi_{1,1}} - 1 - \frac{q^2/4}{\frac{\rho(K)}{\phi_{1,2}} - 1 - \frac{q^2/4}{\cdots}}},
$$

where

$$
\phi_{1,n} := p \int_0^\omega \phi_1(\tau)\, e^{-i\frac{2\pi}{T} n\tau}\, d\tau, \quad n = 0, 1, 2, \ldots,
$$

and R_0 as the largest real root of

$$
\frac{R_0}{\phi_{2,0}} - 1 = 2\mathrm{Re}\, \frac{q^2/4}{\frac{R_0}{\phi_{2,1}} - 1 - \frac{q^2/4}{\frac{R_0}{\phi_{2,2}} - 1 - \frac{q^2/4}{\cdots}}},
$$

where

$$
\phi_{2,n} := p \int_0^\omega \phi_2(\tau)\, e^{-i\frac{2\pi}{T} n\tau}\, d\tau, \quad n = 0, 1, 2, \ldots.
$$

As in [3], both of them are easily obtained by using the backward iterative algorithm

$$
z_n := \frac{x_i}{\phi_{i,n}} - 1, \qquad z_{k-1} := \frac{x_i}{\phi_{i,k-1}} - 1 - \frac{q^2/4}{z_k}, \quad i = 1, 2, \ k = n, n-1, \ldots, 2,
$$

where $x_1 = \rho(K)$ and $x_2 = R_0$. From (8.4) and (8.6), we see that if $r = 0$, then $\phi_1(\tau) = \phi_2(\tau)$ and hence $\rho(K) = R_0$. This coincides with the statement of Proposition 7.1.

8.2. Case $r = 0$

In what follows, we fix $\mu(a) = (a - 30)^2 \times 10^{-4}$, $\gamma = 0.2$, $\omega = 100$ and vary $f(a)$ and $k(t)$. First we set

$$
f(a) = \begin{cases} \dfrac{1}{6.09923} \sin^2\left(\dfrac{a - 15}{30}\pi\right), & a \in [15, 45], \\ 0, & \text{otherwise} \end{cases}
$$

and $k(t) = 0.25(1 + 0.8\cos t)$. In this case, we have $r \simeq 0$ and $\phi_{1,0} \simeq \phi_{2,0} \simeq 1.07885$ and $R_0 \simeq \rho(K) \simeq 1.07921 > 1$. From Propositions 4.2 and 5.1, we can expect that system (2.5) has a unique periodic solution with period 2π. In fact, (a) of Fig. 1 exhibits a solution $j(t, a) = \int_0^{100} i(t, a)\, da$ converging to a periodic solution.
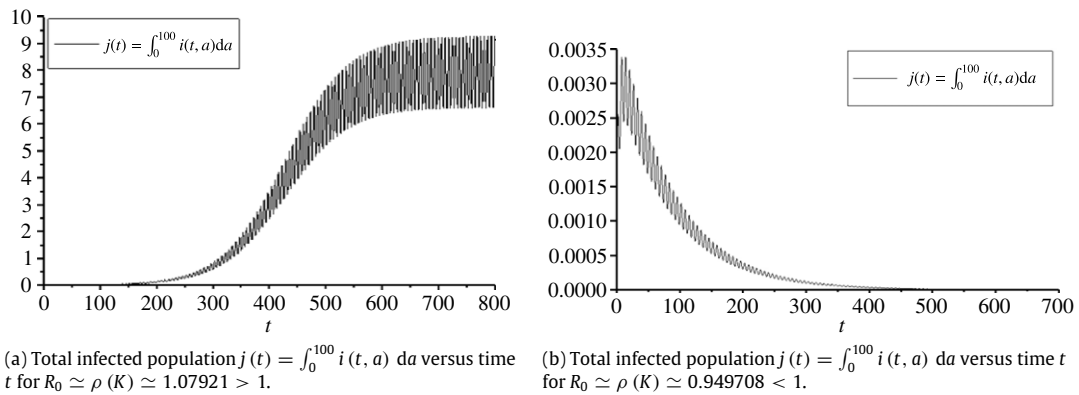
Next we change $k(t)$ to $0.22(1 + 0.8\cos t)$. In this case, we have $\phi_{1,0} \simeq \phi_{2,0} \simeq 0.949388$ and $R_0 \simeq \rho(K) \simeq 0.949708 < 1$. Hence, from Proposition 6.1 we can expect that the disease-free steady state of system (2.5) is globally asymptotically stable. In fact, (b) of Fig. 1 exhibits a solution $j(t, a) = \int_0^{100} i(t, a)\, da$ converging to the disease-free steady state 0.
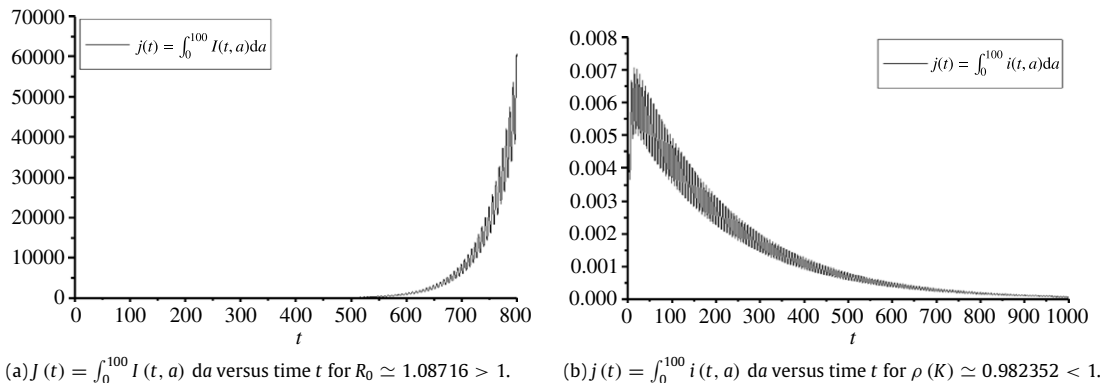
8.3. Case $r > 0$

Next we change $k(t)$ to $0.25(1 + 0.8\cos t)$ and $f(a)$ to

$$
f(a) = \begin{cases} \dfrac{1}{3} \sin^2\left(\dfrac{a - 15}{30}\pi\right), & a \in [15, 45], \\ 0, & \text{otherwise.} \end{cases}
$$

In this case, we have $r \simeq 0.0239891 > 0$. Hence, from Proposition 7.1, we can expect that $R_0 > \rho(K)$. In fact, in this case, we have $\phi_{1,0} \simeq 1.0865 > 0.981739 \simeq \phi_{2,0}$ and $R_0 \simeq 1.08716 > 1 > 0.982352 \simeq \rho(K)$. This inequality particularly implies the absolutely growing but relatively decaying infected population (see Fig. 2).

(a) Total infected population $j(t) = \int_0^{100} i(t,a)\,\mathrm{d}a$ versus time t for $R_0 \simeq \rho(K) \simeq 1.07921 > 1$.

(b) Total infected population $j(t) = \int_0^{100} i(t,a)\,\mathrm{d}a$ versus time t for $R_0 \simeq \rho(K) \simeq 0.949708 < 1$.

Fig. 1. Solution behavior of $j(t) = \int_0^{100} i(t,a)\,\mathrm{d}a$ for $R_0 \simeq \rho(K) \simeq 1.07921 > 1$ (left) and $R_0 \simeq \rho(K) \simeq 0.949708 < 1$ (right).

(a) $J(t) = \int_0^{100} I(t,a)\,\mathrm{d}a$ versus time t for $R_0 \simeq 1.08716 > 1$.

(b) $j(t) = \int_0^{100} i(t,a)\,\mathrm{d}a$ versus time t for $\rho(K) \simeq 0.982352 < 1$.

Fig. 2. Behavior of absolutely growing (a) but relatively decaying (b) infected population for $r \simeq 0.0239891 > 0$, $R_0 \simeq 1.08716 > 1$ and $\rho(K) \simeq 0.982352 < 1$.

8.4. Case $r < 0$

Finally we change $k(t)$ to $0.23(1 + 0.8\cos t)$ and $f(a)$ to

$$f(a) = \begin{cases} \dfrac{1}{10}\sin^2\left(\dfrac{a-15}{30}\pi\right), & a \in [15, 45], \\ 0, & \text{otherwise.} \end{cases}$$
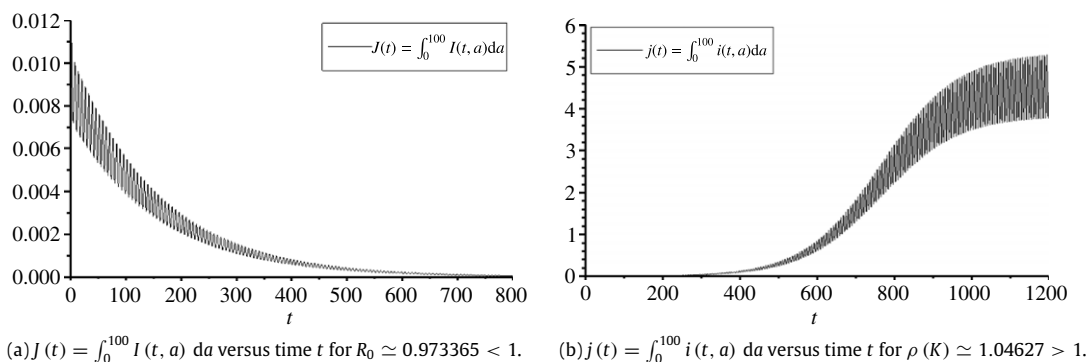
In this case, we have $r \simeq -0.016387 < 0$. Hence, from Proposition 7.1, we can expect that $R_0 < \rho(K)$. In fact, in this case, we have $\phi_{1,0} \simeq 0.973187 < 1.04609 \simeq \phi_{2,0}$ and $R_0 \simeq 0.973365 < 1 < 1.04627 \simeq \rho(K)$. This inequality particularly implies the absolutely decaying but relatively growing infected population (see Fig. 3).

9. Discussion

In this paper, we have formulated an age-structured SIS epidemic model with periodic parameters, and studied the existence of a threshold value which can determine the asymptotic behavior of the model. We have proven that for normalized system (2.5), the spectral radius $\rho(K)$ of the linear operator K plays the role of a threshold for the global asymptotic stability of the disease-free steady state and the existence of a unique endemic periodic solution, that is, if $\rho(K) < 1$, then the disease-free steady state of system (2.5) is globally asymptotically stable, while if $\rho(K) > 1$, then system (2.5) has a unique endemic T-periodic solution $i^* \neq 0$. This is the first endemic threshold result for age-structured periodic epidemic models based on R_0 defined by Bacaër and Guernaoui.

We strongly believe that not only for SIS epidemic models but also for SIR and SEIR age-structured periodic epidemic models, our method might be applied and so the endemic threshold principle will be established under most general conditions by using R_0 defined by Bacaër and Guernaoui.

On the other hand, the reader should note that Hethcote [12] studied the non-structured SIS model with periodic coefficients and showed that the disease-free steady state is globally asymptotically stable in the subcritical case, while there is a unique positive periodic solution in the supercritical case, which is globally asymptotically stable. However, this

(a)$J(t) = \int_0^{100} I(t, a)\, \mathrm{d}a$ versus time t for $R_0 \simeq 0.973365 < 1$.   (b)$j(t) = \int_0^{100} i(t, a)\, \mathrm{d}a$ versus time t for $\rho(K) \simeq 1.04627 > 1$.

Fig. 3. Behavior of absolutely decaying (a) but relatively growing (b) infected population for $r = -0.016387 < 0$, $R_0 \simeq 0.973365 < 1$ and $\rho(K) \simeq 1.04627 > 1$.

simple dichotomy is only partially true for periodic SIR models. The global stability may be lost when $R_0 > 1$ because there may exist subharmonic solutions [33]. In such a case, the main problem is to understand for which class of models the global stability holds.

For our age-dependent case, the global stability of the endemic periodic solution i^* for $\rho(K) > 1$ has been left as an open problem. The corresponding global stability result obtained in [28] is limited to the case of vertically transmitted diseases (the proportion ε of newborn offspring of infective parents who are themselves infective is positive). This implies that their stability result cannot be directly applied to our case. However, on the contrary, if our result can be extended to the case of vertically transmitted diseases, their result may be applied to show the global stability result. This is also a future task.

In Section 7, we have shown that if the Malthusian parameter r equals zero, then our threshold value $\rho(K)$ equals to the basic reproduction number R_0, while if it is not, then $\rho(K) \neq R_0$ and this causes possibilities of the relative growth in the sense of percentage of infected, but absolute decay (or, vice versa) of infected population. This fact suggests us that in the situation where we estimate the long-time behavior of infected population using a threshold value, we have to pay attention on the host population growth. For age-dependent, autonomous endemic models, the effect of the host population growth on R_0 was considered in [19,20,25], although R_0 introduced in [19,20] was not the basic reproduction number, but the threshold value $\rho(K)$ of the normalized system.

Acknowledgments

References

[1] N. Bacaër, Approximation of the basic reproduction number R_0 for vector-borne diseases with a periodic vector population, Bull. Math. Biol. 69 (2007) 1067–1091.
[2] N. Bacaër, S. Guernaoui, The epidemic threshold of vector-borne diseases with seasonality, J. Math. Biol. 53 (2006) 421–436.
[3] N. Bacaër, R. Ouifki, Growth rate and basic reproduction number for population models with a simple periodic factor, Math. Biosci. 210 (2007) 647–658.
[4] S.N. Busenberg, M. Iannelli, H.R. Thieme, Global behavior of an age-structured epidemic model, SIAM J. Math. Anal. 22 (1991) 1065–1080.
[5] S.N. Busenberg, M. Iannelli, H.R. Thieme, Dynamics of an age structured epidemic model, in: Dynamical Systems, World Scientific, Singapore, 1993, pp. 1–19.
[6] A.J. Coale, The use of Fourier analysis to express the relation between time variations in fertility and the time sequence of births in a closed human population, Demography 7 (1970) 93–120.
[7] A.J. Coale, The Growth and Structure of Human Populations: A Mathematical Investigation, Princeton UP, Princeton, 1972.
[8] O. Diekmann, J.A.P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation, John Wiley and Sons, Chichester, 2000.
[9] O. Diekmann, J.A.P. Heesterbeek, J.A.J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, J. Math. Biol. 28 (1990) 365–382.
[10] O. Diekmann, J.A.P. Heesterbeek, M.G. Roberts, The construction of next-generation matrices for compartmental epidemic models, J. Roy. Soc. Interface 7 (2010) 873–885.
[11] Z. Feng, W. Huang, C. Castillo-Chavez, Global behavior of a multi-group SIS epidemic model with age structure, J. Differential Equations 218 (2005) 292–324.
[12] H.W. Hethcote, Asymptotic behavior in a deterministic epidemic model, Bull. Math. Biol. 35 (1973) 607–614.
[13] H.W. Hethcote, S.A. Levin, Periodicity in epidemiological models, in: S.A. Levin, T.G. Hallam, L.J. Gross (Eds.), Applied Mathematical Ecology, Springer, Berlin, 1989, pp. 193–211.
[14] M. Iannelli, Mathematical Theory of Age-structured Population Dynamics, Giardini editori e stampatori, Pisa, 1995.
[15] M. Iannelli, M.Y. Kim, E.J. Park, Asymptotic behavior for an SIS epidemic model and its approximation, Nonlinear Anal. 35 (1999) 797–814.

[16] M. Iannelli, F.A. Milner, A. Pugliese, Analytical and numerical results for the age-structured SIS epidemic model with mixed inter-intracohort transmission, SIAM J. Math. Anal. 23 (1992) 662–688.
[17] H. Inaba, Endemic threshold results for age-duration-structured population model for HIV infection, Math. Biosci. 201 (2006) 15–47.
[18] H. Inaba, Threshold and stability results for an age-structured epidemic model, J. Math. Biol. 28 (1990) 411–434.
[19] H. Inaba, Mathematical analysis of an age-structured SIR epidemic model with vertical transmission, Discrete Contin. Dyn. Syst. Ser. B 6 (1) (2006) 69–96.
[20] H. Inaba, Age-structured homogeneous epidemic systems with application to the MSEIR epidemic model, J. Math. Biol. 54 (2007) 101–146.
[21] H. Inaba, On a new perspective of the basic reproduction number in heterogeneous environments, J. Math. Biol. 65 (2012) 309–348.
[22] H. Inaba, The Malthusian parameter and R_0 for heterogeneous populations in periodic environments, Math. Biosci. Eng. 9 (2012) 313–346.
[23] H. Inaba, Weak ergodicity of population evolution processes, Math. Biosci. 96 (1989) 195–219.
[24] H. Inaba, A semigroup approach to the strong ergodic theorem of the multistate stable population process, Math. Popul. Stud. 1 (1988) 49–77.
[25] H. Inaba, H. Nishiura, The basic reproduction number of an infectious disease in a stable population: the impact of population growth rate on the eradication threshold, Math. Model. Nat. Phenom. 3 (7) (2008) 194–228.
[26] H. Inaba, H. Sekine, A mathematical model for Chagas disease with infection-age-dependent infectivity, Math. Biosci. 190 (2004) 39–69.
[27] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Transl. 10 (1948) 199–325.
[28] M. Langlais, S.N. Busenberg, Global behaviour in age structured S.I.S. models with seasonal periodicities and vertical transmission, J. Math. Anal. Appl. 213 (1997) 511–533.
[29] I. Marek, Frobenius theory of positive operators: comparison theorems and applications, SIAM J. Appl. Math. 19 (1970) 607–628.
[30] Y. Nakata, T. Kuniya, Global dynamics of a class of SEIRS epidemic models in a periodic environment, J. Math. Anal. Appl. 363 (2010) 230–237.
[31] B. Perthame, Transport Equations in Biology, Birkhäuser Verlag, Basel, 2007.
[32] R. Rundnicki, M.C. Mackey, Asymptotic similarity and Malthusian growth in autonomous and nonautonomous populations, J. Math. Anal. Appl. 187 (1994) 548–566.
[33] H.L. Smith, Multiple stable subharmonics for a periodic epidemic model, J. Math. Biol. 17 (1983) 179–190.
[34] H.R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, SIAM J. Appl. Math. 70 (2009) 188–211.
[35] H.R. Thieme, Renewal theorems for linear periodic Volterra integral equations, J. Integral Equations 7 (1984) 253–277.
[36] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29–48.
[37] W. Wang, X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, J. Dynam. Differential Equations 20 (2008) 699–717.
[38] K. Yosida, Functional Analysis, Springer, Berlin, 1980.