



A maximum principle for fully coupled forward–backward stochastic control systems with terminal state constraints[☆]



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ABSTRACT

We study a stochastic optimal control problem where the controlled system is described by a fully coupled forward–backward stochastic differential equation (FBSDE), while the forward state is constrained in a convex set at the terminal time. By introducing an equivalent backward control problem, we use terminal variation approach to obtain a stochastic maximum principle. Applications to the utility optimization problem in the financial market and state constrained stochastic linear quadratic control models are investigated.

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1. Introduction

The maximum principle for stochastic control systems is studied by Kushner [17], Bismut [4], Bensoussan [1], Haussmann [9], Hu [10], Peng [20], Øksendal and Sulem [16] (see [27] for the complete bibliography). In [20], Peng considered one kind of forward–backward stochastic control system with the control domain being convex and obtained the maximum principle. Since the paper of Peng [20], a number of developments in this direction were reported in Xu [25], Shi and Wu [23]. By assuming the diffusion coefficient in the forward equation does not contain the control variable, Xu [25] derived the maximum principle with non-convex control domain. Keeping the same assumption, Shi and Wu [23] obtained the maximum principle for fully coupled forward–backward stochastic control systems. In [26], Yong studied an optimal control problem for general coupled forward–backward stochastic differential equations (FBSDEs) with mixed initial–terminal conditions, where the control domain is not assumed to be convex. The maximum principle of Pontryagin's type for the optimal control is derived by means of spike variation techniques. For the control problems of other kinds of fully coupled FBSDEs, there are some optimal control problems considering the maximum principles, such as, the FBSDEs with delay [11], the FBSDEs with random jumps [22], and the partially-observed optimal control problem [24], etc. Note that the terminal state constraint is not considered in the above literatures.

In this paper, we study the following stochastic control system with terminal state constraint:

$$\begin{cases} dx(t) = b(t, x(t), y(t), z(t), u(t))dt + \sigma(t, x(t), y(t), z(t), u(t))dW(t), \\ x(0) = a, \\ dy(t) = -\bar{g}(t, x(t), y(t), z(t), u(t))dt + z(t)dW(t), \\ y(T) = h(x(T)), \quad t \in [0, T], \end{cases} \quad (1)$$

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where the terminal state $x(T)$ of the forward equation is subject to a convex constraint. More precisely, for a given convex set $K \subseteq \mathbb{R}^n$, we require $x(T) \in K$, a.s.. The above control system is a fully coupled forward–backward stochastic differential equation (FBSDE). It is well known that fully coupled FBSDE is an important tool in many fields, namely in finance, partial differential equation theory and especially in stochastic control theory (refer to [5,6,19]).

There are two difficulties in studying the control system (1). The first one is that our terminal state constraint is a sample-wise constraint, i.e., the terminal state $x(T)$ being in K with probability 1. As explained in [14], the classical theory is generally incapable of solving stochastic control with sample-wise state constraints. The second one is that the forward equation and the backward equation in (1) are fully coupled. Except for this, we also allow the diffusion coefficient in the forward equation to contain the control variable.

The main idea of this paper to overcome the above difficulties is due to some recently developed methods, i.e., the dual method, which is used to study a continuous-time mean–variance portfolio selection model [3] and terminal perturbation method in solving optimization problems with state constraints [8,13–15,2]. We first give a backward formulation of the FBSDE controlled system (1) in which the terminal state $X(T)$ of the forward equation is regarded as the control variable. There are two advantages of doing this: (i) it turns (1) into a purely backward system; (ii) since $X(T)$ now is the control variable, the state constraint becomes a control constraint, whereas a control constraint is much easier to deal with than a state constraint in control theory. However, the price of doing so is that the original initial condition of the forward state now becomes an additional constraint. Then we apply Ekeland's variational principle to deal with the additional constraint and obtain a stochastic maximum principle which characterizes the optimal terminal state.

Two applications of the established stochastic maximum principle are given. We first study an optimal portfolio selection problem under recursive utility. Then a stochastic linear quadratic control problem of fully coupled FBSDEs is investigated.

This paper is organized as follows. We list some preliminaries for fully coupled FBSDEs in Section 2. In Section 3, we re-formulate the control problem as an equivalent backward system. Using Ekeland's variational principle, we derive a stochastic maximum principle which characterizes the optimal terminal state. In Section 4, we study its applications to the optimal portfolio selection problem and stochastic linear quadratic control problem.

2. Preliminaries

Let $W(\cdot)$ be a standard d -dimensional Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is its natural filtration generated by $W(\cdot)$ and augmented by all the P -null sets. For any given Euclidean space H , we denote $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ by the scalar product and Euclidean norm of H , respectively. Now we introduce the following spaces:

$$\begin{aligned} L^2(\Omega, \mathcal{F}_t; H) &:= \{ \xi : \Omega \rightarrow H \mid \xi \text{ is } \mathcal{F}_t\text{-measurable, and } E|\xi|^2 < \infty \}, \\ \mathcal{M}^2(0, T; H) &:= \left\{ \varphi : [0, T] \times \Omega \rightarrow H \mid (\varphi_t)_{0 \leq t \leq T} \text{ is } \mathbb{F}\text{-progressively measurable process, and } \|\varphi\|^2 \right. \\ &\quad \left. = E \int_0^T |\varphi_t|^2 dt < +\infty \right\}. \end{aligned}$$

Consider the following fully coupled FBSDE

$$\begin{cases} dx(t) = \tilde{b}(t, x(t), y(t), z(t))dt + \tilde{\sigma}(t, x(t), y(t), z(t))dW(t), \\ x(t) = \eta, \\ dy(t) = -\tilde{g}(t, x(t), y(t), z(t))dt + z(t)dW(t), \\ y(T) = \Phi(x(T)), \quad t \in [0, T], \end{cases} \quad (2)$$

where $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, $T > 0$,

$$\begin{aligned} \tilde{b} : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} &\longrightarrow \mathbb{R}^n, & \tilde{\sigma} : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} &\longrightarrow \mathbb{R}^{n \times d}, \\ \tilde{g} : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} &\longrightarrow \mathbb{R}^m, & \Phi : \Omega \times \mathbb{R}^n &\longrightarrow \mathbb{R}^m \end{aligned}$$

are \mathbb{F} -progressively measurable processes. There are several approaches to solve this fully coupled FBSDE (see [18,12,21]).

Given an $m \times n$ full-rank matrix G , we define:

$$\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A(t, \lambda) = \begin{pmatrix} -G^T \tilde{g} \\ G \tilde{b} \\ G \tilde{\sigma} \end{pmatrix} (t, \lambda),$$

where G^T is the transposed matrix of G .

We assume that

- (H₁) (i) $A(t, \lambda)$ is uniformly Lipschitz with respect to λ , and for any $\lambda, A(\cdot, \lambda) \in \mathcal{M}^2(0, T; R^{n+m+m \times d})$;
 (ii) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in R^n$, and for any $x \in R^n$, $\Phi(x) \in L^2(\Omega, F_T; R^m)$.
 (H₂) (i) $\langle A(t, \lambda) - A(t, \bar{\lambda}), \lambda - \bar{\lambda} \rangle \leq -\beta_1 |G\hat{x}|^2 - \beta_2 (|G^T \hat{y}|^2 + |G^T \hat{z}|^2)$,
 (ii) $\langle \Phi(x) - \Phi(\bar{x}), G(x - \bar{x}) \rangle \geq \mu_1 |G\hat{x}|^2$, $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$, $\hat{z} = z - \bar{z}$,
 where β_1, β_2, μ_1 are nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$. Moreover, we have $\beta_1 > 0$, $\mu_1 > 0$ (resp., $\beta_2 > 0$), when $m > n$ (resp., $m < n$).

Lemma 2.1. Under the assumptions (H₁), (H₂), for any $\eta \in L^2(\Omega, \mathcal{F}_T; R^n)$, FBSDE (2) has a unique adapted solution $(x(t), y(t), z(t))_{t \in [0, T]} \in \mathcal{M}^2(0, T; R^{n+m+m \times d})$.

For the proof, the readers may refer to Peng and Wu [21].

3. Stochastic optimization problem

3.1. Problem formulation

Now we consider the fully coupled FBSDE controlled system (1) with the cost function:

$$\bar{J}(u(\cdot)) := E \left[\int_0^T \bar{l}(t, x(t), y(t), z(t), u(t)) dt + \phi(x(T)) + \gamma(y(0)) \right],$$

where

$$\begin{aligned} b : [0, T] \times R^n \times R^m \times R^{m \times d} \times R^{n \times d} &\rightarrow R^n, & \sigma : [0, T] \times R^n \times R^m \times R^{m \times d} \times R^{n \times d} &\rightarrow R^{n \times d}, \\ \bar{g} : [0, T] \times R^n \times R^m \times R^{m \times d} \times R^{n \times d} &\rightarrow R^m, & \bar{l} : [0, T] \times R^n \times R^m \times R^{m \times d} \times R^{n \times d} &\rightarrow R, \end{aligned}$$

and $h : R^n \rightarrow R^m$, $\phi : R^n \rightarrow R$, $\gamma : R^m \rightarrow R$.

Let $\mathcal{U}_{ad} := \{u(\cdot) \mid u(\cdot) \in \mathcal{M}^2(0, T; R^{n \times d})\}$. Under the assumptions (H₁) and (H₂), for any $u(\cdot) \in \mathcal{U}_{ad}$, from Lemma 2.1, we know (1) has a unique adapted solution $(x^u(\cdot), y^u(\cdot), z^u(\cdot)) \in \mathcal{M}^2(0, T; R^{n+m+m \times d})$, and the cost function $\bar{J}(u(\cdot))$ is well-defined.

Our control problem is

Problem A. Minimize $\bar{J}(u(\cdot))$, subject to $u(\cdot) \in \mathcal{U}_{ad}$ and the terminal state constraint $x(T) \in K$, where $K \subseteq R^n$ is convex.

In order to derive a stochastic maximum principle, we need the following additional assumption.

- (H₃) (i) $b, \sigma, \bar{g}, h, \bar{l}, \phi$ and γ are continuous in their arguments and continuously differentiable in (x, y, z, u) ;
 (ii) the derivatives of b, σ, \bar{g} and h in (x, y, z, u) are bounded;
 (iii) the derivatives of \bar{l} in (x, y, z, u) are bounded by $C(1 + |x| + |y| + |z| + |u|)$, and the derivatives of ϕ and γ in x are bounded by $C(1 + |x|)$.

3.2. Backward formulation

In this subsection, we give an equivalent backward formulation of Problem A. We need the following additional assumption:

- (H₄) there exists $\alpha > 0$, such that $|\sigma(t, x, y, z, u_1) - \sigma(t, x, y, z, u_2)| \geq \alpha |u_1 - u_2|$ for all $(x, y, z) \in R^n \times R^m \times R^{m \times d}$, $t \in [0, T]$ and $u_1, u_2 \in R^{n \times d}$.

Under (H₃) and (H₄), we know the mapping $u \rightarrow \sigma(t, x, y, z, u)$ is a bijection from $R^{n \times d}$ onto itself for any (t, x, y, z) . Therefore, let $q \equiv \sigma(t, x, y, z, u)$, there exists the inverse function $\hat{\sigma}$, such that $u = \hat{\sigma}(t, x, y, z, q)$. In this way, we rewrite (1) as

$$\begin{cases} dx(t) = -f(t, x(t), y(t), z(t), q(t))dt + q(t)dW(t), \\ x(0) = a, \\ dy(t) = -g(t, x(t), y(t), z(t), q(t))dt + z(t)dW(t), \\ y(T) = h(x(T)), \quad t \in [0, T], \end{cases}$$

where

$$f(t, x, y, z, q) = -b(t, x, y, z, \hat{\sigma}(t, x, y, z, q)), \quad g(t, x, y, z, q) = \bar{g}(t, x, y, z, \hat{\sigma}(t, x, y, z, q)).$$

Since $u \rightarrow \sigma(t, x, y, z, u)$ is a bijection, we may regard $q(\cdot)$ as the control variable. Due to the existence and uniqueness theorem of BSDE, selecting $q(\cdot)$ is equivalent to selecting the terminal state $x(T)$. Then we obtain the following purely backward control system:

$$\begin{cases} dx(t) = -f(t, x(t), y(t), z(t), q(t))dt + q(t)dW(t), \\ x(T) = \xi, \\ dy(t) = -g(t, x(t), y(t), z(t), q(t))dt + z(t)dW(t), \\ y(T) = h(\xi), \quad t \in [0, T], \end{cases} \quad (3)$$

where ξ is the control variable to be chosen from

$$U = \{\xi \mid E|\xi|^2 < \infty, \xi \in K, a.s.\}.$$

The equivalent cost function is

$$J(\xi) := E \left[\int_0^T l(t, x(t), y(t), z(t), q(t))dt + \phi(\xi) + \gamma(y(0)) \right],$$

where $l(t, x, y, z, q) = \bar{l}(t, x, y, z, \hat{\sigma}(t, x, y, z, q))$.

Therefore, [Problem A](#) is equivalent to the following optimization problem.

Problem B. Minimize $J(\xi)$, subject to $\xi \in U$ and $x^\xi(0) = a$.

Here $x^\xi(0)$ is the solution of (3) at time 0 under ξ . Although the initial condition $x^\xi(0) = a$ now becomes a constraint, as shown in the introduction, [Problem B](#) is much easier to deal with than [Problem A](#). Hereafter we focus on solving [Problem B](#) and try to characterize the optimal control $\xi^* \in U$.

Note that, according to the definitions of f, g, l and the assumption (H_3) , we know f, g, l also satisfy the similar conditions in (H_3) .

3.3. Variational equation

For $\xi^1, \xi^2 \in U$, define a metric in U by

$$d(\xi^1, \xi^2) := (E|\xi^1 - \xi^2|^2)^{\frac{1}{2}}.$$

Obviously, $(U, d(\cdot, \cdot))$ is a complete metric space.

Let ξ^* be an optimal control and $(x^*(\cdot), y^*(\cdot), q^*(\cdot), z^*(\cdot))$ be the state processes of (3) associated with ξ^* . For any $\xi \in U$, using the convexity of U , we know, for each $0 \leq p \leq 1$,

$$\xi^p := \xi^* + p(\xi - \xi^*) \in U.$$

Denote $(x_p(\cdot), y_p(\cdot), q_p(\cdot), z_p(\cdot))$ by the solution of (3) associated with $\xi = \xi^p$.

Consider the following BSDEs:

$$\begin{cases} d\delta x(t) = -[f_x^*(t)\delta x(t) + f_y^*(t)\delta y(t) + f_z^*(t)\delta z(t) + f_q^*(t)\delta q(t)]dt + \delta q(t)dW(t), \\ \delta x(T) = \xi - \xi^*, \\ d\delta y(t) = -[g_x^*(t)\delta x(t) + g_y^*(t)\delta y(t) + g_z^*(t)\delta z(t) + g_q^*(t)\delta q(t)]dt + \delta z(t)dW(t), \\ \delta y(T) = h_x(\xi^*)(\xi - \xi^*), \quad t \in [0, T], \end{cases} \quad (4)$$

where $a_k^*(t) = a_k(t, x^*(t), y^*(t), z^*(t), q^*(t))$, $a = f, g, k = x, y, z, q$, respectively. This equation is called the variational equation.

Remark 3.1. Note that the equations in (4) are two fully coupled linear BSDEs. First we explain the existence and uniqueness of solution of (4). To do it, we rewrite Eq. (4) into

$$\begin{cases} dY(t) = -[F(t)^T Y(t) + G(t)^T Z(t)]dt + Z(t)dW(t), \\ Y(T) = Y_T, \quad t \in [0, T], \end{cases} \quad (5)$$

where

$$\begin{aligned} Y(t) &= \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}, \quad Z(t) = \begin{pmatrix} \delta q(t) \\ \delta z(t) \end{pmatrix}, \quad Y_T = \begin{pmatrix} \xi - \xi^* \\ h_x(\xi^*)(\xi - \xi^*) \end{pmatrix}, \\ F(t) &= \begin{pmatrix} f_x^*(t)^T, g_x^*(t)^T \\ f_y^*(t)^T, g_y^*(t)^T \end{pmatrix}, \quad G(t) = \begin{pmatrix} f_q^*(t)^T, g_q^*(t)^T \\ f_z^*(t)^T, g_z^*(t)^T \end{pmatrix}. \end{aligned}$$

The elements of $F(\cdot), G(\cdot)$ are bounded, according to the existence and uniqueness result of solution of BSDE (see [20]), there exists a unique adapted $(Y(\cdot), Z(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R}^{n+m} \times \mathbb{R}^{(n+m) \times d})$ satisfying Eq. (5). Therefore, (4) has the unique adapted solution $(\delta x(\cdot), \delta q(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R}^n \times \mathbb{R}^{n \times d})$, $(\delta y(\cdot), \delta z(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R}^m \times \mathbb{R}^{m \times d})$.

Set

$$\tilde{x}_p(t) = p^{-1}[x_p(t) - x^*(t)] - \delta x(t),$$

$$\tilde{q}_p(t) = p^{-1}[q_p(t) - q^*(t)] - \delta q(t),$$

$$\tilde{y}_p(t) = p^{-1}[y_p(t) - y^*(t)] - \delta y(t),$$

$$\tilde{z}_p(t) = p^{-1}[z_p(t) - z^*(t)] - \delta z(t),$$

and denote

$$a_p(t) = a(t, x_p(t), y_p(t), z_p(t), q_p(t)), \quad a^*(t) = a(t, x^*(t), y^*(t), z^*(t), q^*(t)),$$

where $a = f, g$ respectively. We have the following results:

Lemma 3.1. Assuming (H_1) – (H_4) , we have

$$\lim_{p \rightarrow 0} \sup_{0 \leq t \leq T} E|\tilde{x}_p(t)|^2 = 0,$$

$$\lim_{p \rightarrow 0} E \left[\int_0^T |\tilde{q}_p(t)|^2 dt \right] = 0,$$

$$\lim_{p \rightarrow 0} \sup_{0 \leq t \leq T} E|\tilde{y}_p(t)|^2 = 0,$$

$$\lim_{p \rightarrow 0} E \left[\int_0^T |\tilde{z}_p(t)|^2 dt \right] = 0.$$

Proof.

$$\begin{cases} d\tilde{x}_p(t) = -p^{-1}[f_p(t) - f^*(t) - pf_x^*(t)\delta x(t) - pf_y^*(t)\delta y(t) - pf_z^*(t)\delta z(t) - pf_q^*(t)\delta q(t)]dt + \tilde{q}_p(t)dW(t) \\ \quad = -[f_x^p(t)\tilde{x}_p(t) + f_y^p(t)\tilde{y}_p(t) + f_z^p(t)\tilde{z}_p(t) + f_q^p(t)\tilde{q}_p(t) + A^p(t)]dt + \tilde{q}_p(t)dW(t), \\ \tilde{x}_p(T) = 0, \\ d\tilde{y}_p(t) = -p^{-1}[g_p(t) - g^*(t) - pg_x^*(t)\delta x(t) - pg_y^*(t)\delta y(t) - pg_z^*(t)\delta z(t) - pg_q^*(t)\delta q(t)]dt + \tilde{z}_p(t)dW(t) \\ \quad = -[g_x^p(t)\tilde{x}_p(t) + g_y^p(t)\tilde{y}_p(t) + g_z^p(t)\tilde{z}_p(t) + g_q^p(t)\tilde{q}_p(t) + B^p(t)]dt + \tilde{z}_p(t)dW(t), \\ \tilde{y}_p(T) = p^{-1}[h(\xi^p) - h(\xi^*)] - h_x(\xi^*)(\xi - \xi^*), \end{cases}$$

where

$$f_k^p(t) = \int_0^1 f_k(t, A(\lambda, t), B(\lambda, t), C(\lambda, t), D(\lambda, t))d\lambda,$$

$$g_k^p(t) = \int_0^1 g_k(t, A(\lambda, t), B(\lambda, t), C(\lambda, t), D(\lambda, t))d\lambda,$$

$$A^p(t) = [f_x^p(t) - f_x^*(t)]\delta x(t) + [f_y^p(t) - f_y^*(t)]\delta y(t) + [f_z^p(t) - f_z^*(t)]\delta z(t) + [f_q^p(t) - f_q^*(t)]\delta q(t),$$

$$B^p(t) = [g_x^p(t) - g_x^*(t)]\delta x(t) + [g_y^p(t) - g_y^*(t)]\delta y(t) + [g_z^p(t) - g_z^*(t)]\delta z(t) + [g_q^p(t) - g_q^*(t)]\delta q(t),$$

$$k = x, y, z, q, \text{ respectively,}$$

and

$$A(\lambda, t) = x^*(t) + \lambda p(\delta x(t) + \tilde{x}_p(t)),$$

$$B(\lambda, t) = y^*(t) + \lambda p(\delta y(t) + \tilde{y}_p(t)),$$

$$C(\lambda, t) = z^*(t) + \lambda p(\delta z(t) + \tilde{z}_p(t)),$$

$$D(\lambda, t) = q^*(t) + \lambda p(\delta q(t) + \tilde{q}_p(t)).$$

Using Itô's formula to $|\tilde{x}_p(t)|^2 + |\tilde{y}_p(t)|^2$, we get

$$\begin{aligned} & E|\tilde{x}_p(t)|^2 + E|\tilde{y}_p(t)|^2 + E \int_t^T |\tilde{q}_p(s)|^2 ds + E \int_t^T |\tilde{z}_p(s)|^2 ds \\ &= E|\tilde{y}_p(T)|^2 + 2E \int_t^T \tilde{x}_p(s)[f_x^p(s)\tilde{x}_p(s) + f_y^p(s)\tilde{y}_p(s) + f_z^p(s)\tilde{z}_p(s) + f_q^p(s)\tilde{q}_p(s) + A^p(s)]ds \\ & \quad + 2E \int_t^T \tilde{y}_p(s)[g_x^p(s)\tilde{x}_p(s) + g_y^p(s)\tilde{y}_p(s) + g_z^p(s)\tilde{z}_p(s) + g_q^p(s)\tilde{q}_p(s) + B^p(s)]ds \\ & \leq CE \int_t^T [|\tilde{x}_p(s)|^2 + |\tilde{y}_p(s)|^2]ds + C_1E \int_t^T [|\tilde{z}_p(s)|^2 + |\tilde{q}_p(s)|^2]ds + E|\tilde{y}_p(T)|^2 + C_2E \int_t^T [|A^p(s)|^2 + |B^p(s)|^2]ds, \end{aligned}$$

where $C_1 < 1$. Applying Gronwall's inequality, we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} E|\tilde{x}_p(t)|^2 + \sup_{0 \leq t \leq T} E|\tilde{y}_p(t)|^2 + E \int_0^T |\tilde{q}_p(s)|^2 ds + E \int_0^T |\tilde{z}_p(s)|^2 ds \\ & \leq C'E|\tilde{y}_p(T)|^2 + C'E \int_0^T [|A^p(s)|^2 + |B^p(s)|^2] ds. \end{aligned} \quad (6)$$

It is easy to check that

$$\lim_{p \rightarrow 0} E|\tilde{y}_p(T)|^2 = \lim_{p \rightarrow 0} E|p^{-1}[h(\xi^p) - h(\xi^*)] - h_x(\xi^*)(\xi - \xi^*)|^2 = 0.$$

From Lebesgue's dominated convergence theorem, we have

$$\lim_{p \rightarrow 0} E \int_0^T |A^p(s)|^2 dt = \lim_{p \rightarrow 0} E \int_0^T |B^p(s)|^2 dt = 0.$$

Let $p \rightarrow 0$ in (6), we get the desired results. \square

3.4. Variational inequality

By applying Ekeland's variational principle, we deal with the initial constraint $x^\xi(0) = a$ and derive the variational inequality. Now recall Ekeland's variational principle [7].

Lemma 3.2 (Ekeland's Variational Principle). *Let $(V, d(\cdot, \cdot))$ be a complete metric space and $F(\cdot) : V \rightarrow R$ be a proper lower semi-continuous function bounded from below. Then, for every $\varepsilon > 0$, every $u \in V$ such that $F(u) \leq \inf_{v \in V} F(v) + \varepsilon$, there exists $u_\varepsilon \in V$ such that*

- (i) $F(u_\varepsilon) \leq F(u)$,
- (ii) $d(u, u_\varepsilon) \leq \varepsilon$,
- (iii) $F(v) + \sqrt{\varepsilon}d(v, u_\varepsilon) \geq F(u_\varepsilon), \quad \forall v \in V$.

We first consider the case where $l(t, x, y, z, q) = 0$ and then present the results for the general case.

Given the optimal $\xi^* \in U$, introduce a mapping $F_\varepsilon(\cdot) : U \rightarrow R$ by

$$F_\varepsilon(\xi) = \{|x^\xi(0) - a|^2 + (\max(0, \gamma(y^\xi(0)) - \gamma(y^*(0)) + \phi(\xi) - \phi(\xi^*) + \epsilon))^2\}^{\frac{1}{2}},$$

where a is the given initial state constraint and ϵ is an arbitrary positive constant.

Remark 3.2. It is easy to check that the mappings $|x^\xi(0) - a|^2$ and $\gamma(y^\xi(0))$, $\phi(\xi)$, both from U to R , are continuous on U . Therefore F_ε is a continuous function defined on U .

Theorem 3.1. *We suppose (H_1) – (H_4) . Let ξ^* be an optimal control to Problem B. Then there exist $h_1 \in R^n$ and $h_0 \in R$, with $h_0 \geq 0$ and $|h_0| + |h_1| \neq 0$, such that for any $\xi \in K$, the following variational inequality holds*

$$\langle h_1, \delta x(0) \rangle + h_0 \langle \gamma_y(y^*(0)), \delta y(0) \rangle + h_0 \langle \phi_x(\xi^*), \xi - \xi^* \rangle \geq 0, \quad (7)$$

where $(\delta x(0), \delta y(0))$ is the solution of (4) at time 0.

Proof. It is easy to check the following properties hold:

$$\begin{aligned} & F_\varepsilon(\xi^*) = \epsilon; \\ & F_\varepsilon(\xi) > 0, \quad \forall \xi \in U; \\ & F_\varepsilon(\xi^*) \leq \inf_{\xi \in U} F_\varepsilon(\xi) + \epsilon. \end{aligned}$$

Thus, from Lemma 3.2 (Ekeland's variational principle), there exists $\xi^\epsilon \in U$, such that

- (i) $F_\varepsilon(\xi^\epsilon) \leq F_\varepsilon(\xi^*)$;
- (ii) $d(\xi^*, \xi^\epsilon) \leq \sqrt{\epsilon}$;
- (iii) $F_\varepsilon(\xi) + \sqrt{\epsilon}d(\xi, \xi^\epsilon) \geq F_\varepsilon(\xi^\epsilon), \quad \forall \xi \in U$.

For any $\xi \in U$, $0 \leq p \leq 1$ we know $\xi_p^\epsilon = \xi^\epsilon + p(\xi - \xi^\epsilon) \in U$. Denote $(x_p^\epsilon(\cdot), y_p^\epsilon(\cdot), q_p^\epsilon(\cdot), z_p^\epsilon(\cdot))$ (resp. $x^\epsilon(\cdot), y^\epsilon(\cdot), q^\epsilon(\cdot), z^\epsilon(\cdot)$) by the solution of (3) with $\xi = \xi_p^\epsilon$ (resp. ξ^ϵ , and let $(\delta x^\epsilon(\cdot), \delta y^\epsilon(\cdot), \delta q^\epsilon(\cdot), \delta z^\epsilon(\cdot))$ be the solution of (4), in which $\xi^* = \xi^\epsilon$. Therefore, we have:

$$F_\varepsilon(\xi_p^\epsilon) - F_\varepsilon(\xi^\epsilon) + \sqrt{\epsilon}d(\xi_p^\epsilon, \xi^\epsilon) \geq 0. \quad (8)$$

Similarly to Lemma 3.1, we have

$$\lim_{p \rightarrow 0} \sup_{0 \leq t \leq T} E [p^{-1}(x_p^\epsilon(t) - x^\epsilon(t)) - \delta x^\epsilon(t)] = 0,$$

$$\lim_{p \rightarrow 0} \sup_{0 \leq t \leq T} E [p^{-1}(y_p^\epsilon(t) - y^\epsilon(t)) - \delta y^\epsilon(t)] = 0.$$

Thus,

$$x_p^\epsilon(0) - x^\epsilon(0) = p\delta x^\epsilon(0) + o(p),$$

$$y_p^\epsilon(0) - y^\epsilon(0) = p\delta y^\epsilon(0) + o(p)$$

which lead to the following expansions:

$$|x_p^\epsilon(0) - a|^2 - |x^\epsilon(0) - a|^2 = 2p\langle x^\epsilon(0) - a, \delta x^\epsilon(0) \rangle + o(p),$$

$$|\gamma(y_p^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi_p^\epsilon) - \phi(\xi^*) + \epsilon|^2 - |\gamma(y^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi^\epsilon) - \phi(\xi^*) + \epsilon|^2$$

$$= 2p[\gamma(y^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi^\epsilon) - \phi(\xi^*) + \epsilon][\langle \gamma_y(y^\epsilon(0)), \delta y^\epsilon(0) \rangle + \langle \phi_x(\xi^\epsilon), \xi - \xi^\epsilon \rangle] + o(p).$$

For the given $\epsilon > 0$, we consider the following two cases:

Case 1: There exists $p_0 > 0$ such that for all $p \in (0, p_0)$,

$$\gamma(y_p^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi_p^\epsilon) - \phi(\xi^*) + \epsilon \geq 0.$$

In this case

$$\lim_{p \rightarrow 0} \frac{F_\epsilon(\xi_p^\epsilon) - F_\epsilon(\xi^\epsilon)}{p} = \lim_{p \rightarrow 0} \frac{1}{F_\epsilon(\xi_p^\epsilon) + F_\epsilon(\xi^\epsilon)} \frac{F_\epsilon^2(\xi_p^\epsilon) - F_\epsilon^2(\xi^\epsilon)}{p}$$

$$= \frac{1}{F_\epsilon(\xi^\epsilon)} \{ \langle x^\epsilon(0) - a, \delta x^\epsilon(0) \rangle + [\langle \gamma_y(y^\epsilon(0)), \delta y^\epsilon(0) \rangle + \langle \phi_x(\xi^\epsilon), \xi - \xi^\epsilon \rangle] \cdot [\gamma(y^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi^\epsilon) - \phi(\xi^*) + \epsilon] \}.$$

Set

$$h_0^\epsilon = \frac{1}{F_\epsilon(\xi^\epsilon)} [\gamma(y^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi^\epsilon) - \phi(\xi^*) + \epsilon] \geq 0,$$

$$h_1^\epsilon = \frac{1}{F_\epsilon(\xi^\epsilon)} [x^\epsilon(0) - a].$$

From (8), we get

$$h_0^\epsilon \langle \gamma_y(y^\epsilon(0)), \delta y^\epsilon(0) \rangle + h_0^\epsilon \langle \phi_x(\xi^\epsilon), \xi - \xi^\epsilon \rangle + \langle h_1^\epsilon, \delta x^\epsilon(0) \rangle \geq -\sqrt{\epsilon} [E|\xi - \xi^\epsilon|^2]^{\frac{1}{2}}. \quad (9)$$

Case 2: There exists a positive sequence $\{p_n\}$ satisfying $p_n \rightarrow 0$, such that

$$\gamma(y_{p_n}^\epsilon(0)) - \gamma(y^*(0)) + \phi(\xi_{p_n}^\epsilon) - \phi(\xi^*) + \epsilon \leq 0.$$

In this case, by the definition of F_ϵ , for enough large n , $F_\epsilon(\xi_{p_n}^\epsilon) = \{|x_{p_n}^\epsilon(0) - a|^2\}^{\frac{1}{2}}$. Since $F_\epsilon(\cdot)$ is continuous, we have $F_\epsilon(\xi^\epsilon) = \{|x^\epsilon(0) - a|^2\}^{\frac{1}{2}}$.

Now,

$$\lim_{n \rightarrow \infty} \frac{F_\epsilon(\xi_{p_n}^\epsilon) - F_\epsilon(\xi^\epsilon)}{p} = \lim_{n \rightarrow \infty} \frac{1}{F_\epsilon(\xi_{p_n}^\epsilon) + F_\epsilon(\xi^\epsilon)} \frac{F_\epsilon^2(\xi_{p_n}^\epsilon) - F_\epsilon^2(\xi^\epsilon)}{p_n} = \frac{\langle x^\epsilon(0) - a, \delta x^\epsilon(0) \rangle}{F_\epsilon(\xi^\epsilon)}.$$

Similar to Case 1, it follows from (8),

$$\langle h_1^\epsilon, \delta x^\epsilon(0) \rangle \geq -\sqrt{\epsilon} [E|\xi - \xi^\epsilon|^2]^{\frac{1}{2}}$$

where $h_0^\epsilon = 0$, $h_1^\epsilon = \frac{1}{F_\epsilon(\xi^\epsilon)} [x^\epsilon(0) - a]$. All in all, we have

$$\begin{cases} (9) \text{ holds,} \\ h_0^\epsilon \geq 0, \\ |h_0^\epsilon|^2 + |h_1^\epsilon|^2 = 1. \end{cases}$$

Hence, there exist a convergent subsequence of $(h_1^\epsilon, h_0^\epsilon)$ whose limit is denote by (h_1, h_0) . Due to $d(\xi^\epsilon, \xi^*) < \epsilon$, we have $\xi^\epsilon \rightarrow \xi^*$ in U , as $\epsilon \rightarrow 0$. So, from the regularity of the solutions of BSDEs, we have $\delta x^\epsilon(0) \rightarrow \delta x(0)$, $\delta y^\epsilon(0) \rightarrow \delta y(0)$, as $\epsilon \rightarrow 0$. Let $\epsilon \rightarrow 0$ in (9), we get (7). \square

When $l(t, x, y, z, q) \neq 0$, using similar analysis, we can get the following variational inequality:

Theorem 3.2. Suppose (H_1) – (H_4) . Let ξ^* be an optimal control to Problem B. Then there exist $h_1 \in R^n$ and $h_0 \in R$, with $h_0 \geq 0$ and $|h_0| + |h_1| \neq 0$, such that for any $\xi \in K$, the following variational inequality holds

$$\begin{aligned} & \langle h_1, \delta x(0) \rangle + h_0 \langle \gamma_y(y^*(0)), \delta y(0) \rangle + h_0 \langle \phi_x(\xi^*), \xi - \xi^* \rangle + h_0 \int_0^T \langle l_x^*(t), \delta x(t) \rangle dt \\ & + h_0 \int_0^T \langle l_y^*(t), \delta y(t) \rangle dt + h_0 \int_0^T \langle l_z^*(t), \delta z(t) \rangle dt + h_0 \int_0^T \langle l_q^*(t), \delta q(t) \rangle dt \geq 0, \end{aligned} \quad (10)$$

where $l_k^*(t) = l_k(t, x^*(t), y^*(t), z^*(t), q^*(t))$, $k = x, y, z, q$, respectively, and $(\delta x(\cdot), \delta y(\cdot), \delta q(\cdot), \delta z(\cdot))$ is the solution of (4).

3.5. Maximum principle

In this subsection we derive the stochastic maximum principle. To this end, we introduce the following adjoint equation:

$$\begin{cases} dm(t) = [f_x^*(t)^T m(t) + g_x^*(t)^T n(t) + h_0 l_x^*(t)] dt + [f_q^*(t)^T m(t) + g_q^*(t)^T n(t) + h_0 l_q^*(t)] dW(t), \\ m(0) = h_1, \\ dn(t) = [f_y^*(t)^T m(t) + g_y^*(t)^T n(t) + h_0 l_y^*(t)] dt + [f_z^*(t)^T m(t) + g_z^*(t)^T n(t) + h_0 l_z^*(t)] dW(t), \\ n(0) = h_0 \gamma_y(y^*(0)), \quad t \in [0, T], \end{cases} \quad (11)$$

where $f_k^*(\cdot), g_k^*(\cdot), k = x, y, z, q$ are the same as the notations in (4), $l_k^*(\cdot), k = x, y, z, q$ are the notations in Theorem 3.2. The solution $(m(\cdot), n(\cdot))$ of (11) is called the adjoint process.

Remark 3.3. We have seen that the variational equation (4) are two fully coupled BSDEs. Note that the adjoint equations (11) are two fully coupled SDEs. Similar to Remark 3.1, let

$$Q(t) = \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}, \quad Q_0 = \begin{pmatrix} h_1 \\ h_0 \gamma_y(y^*(0)) \end{pmatrix}, \quad B(t) = \begin{pmatrix} h_0 l_x^*(t) \\ h_0 l_y^*(t) \end{pmatrix}, \quad D(t) = \begin{pmatrix} h_0 l_q^*(t) \\ h_0 l_z^*(t) \end{pmatrix}.$$

We can rewrite (11) into the following form:

$$\begin{cases} dQ(t) = (F(t)Q(t) + B(t))dt + (G(t)Q(t) + D(t))dW(t), \\ Q(0) = Q_0, \quad t \in [0, T]. \end{cases} \quad (12)$$

From the boundedness of the elements of $F(\cdot), G(\cdot)$, we know there is a unique process $Q(\cdot)$ solving (12).

Theorem 3.3. Suppose (H_1) – (H_4) hold. If ξ^* is optimal to Problem B with $(x^*(\cdot), y^*(\cdot), q^*(\cdot), z^*(\cdot))$ being the corresponding state of (3), then there exist $h_1 \in R^n$ and $h_0 \in R$ with $h_0 \geq 0$ and $|h_0| + |h_1| \neq 0$ such that $\forall v \in K$

$$\langle m(T) + h_x(\xi^*)^T n(T) + h_0 \phi_x(\xi^*), v - \xi^* \rangle \geq 0, \quad a.s. \quad (13)$$

where $(m(\cdot), n(\cdot))$ is the solution of the adjoint equation (11).

Proof. For any $\xi \in U$, let $(Y(\cdot), Z(\cdot))$ be the solution to (5). Applying Itô's formula to $\langle Y(t), Q(t) \rangle$ yields

$$\begin{aligned} d\langle Y(t), Q(t) \rangle &= [\langle Y(t), B(t) \rangle + \langle Z(t), D(t) \rangle]dt + \{\cdot \cdot \cdot\}dW(t) \\ &= [h_0 l_x^*(t)^T \delta x(t) + h_0 l_y^*(t)^T \delta y(t) + h_0 l_z^*(t)^T \delta z(t) + h_0 l_q^*(t)^T \delta q(t)]dt + \{\cdot \cdot \cdot\}dW(t). \end{aligned}$$

Integrating from 0 to T , taking expectation and using the variational inequality (10), we obtain

$$\begin{aligned} & E[\langle m(T) + h_x(\xi^*)^T n(T) + h_0 \phi_x(\xi^*), \xi - \xi^* \rangle] \\ &= \langle h_1, \delta x(0) \rangle + h_0 \langle \gamma_y(y^*(0)), \delta y(0) \rangle + h_0 \langle \phi_x(\xi^*), \xi - \xi^* \rangle + h_0 \int_0^T \langle l_x^*(t), \delta x(t) \rangle dt \\ &+ h_0 \int_0^T \langle l_y^*(t), \delta y(t) \rangle dt + h_0 \int_0^T \langle l_z^*(t), \delta z(t) \rangle dt + h_0 \int_0^T \langle l_q^*(t), \delta q(t) \rangle dt \geq 0. \end{aligned}$$

From the arbitrariness of $\xi \in U$, we have $\forall v \in K$,

$$\langle m(T) + h_x(\xi^*)^T n(T) + h_0 \phi_x(\xi^*), v - \xi^* \rangle \geq 0, \quad a.s. \quad \square$$

Denote ∂K by the boundary of K . Set $\Omega_0 := \{\omega \in \Omega \mid \xi^*(\omega) \in \partial K\}$.

According to the above theorem, we have:

Corollary 3.1. Under the assumptions of Theorem 3.3, for each $v \in K$,

$$\begin{aligned} \langle m(T) + h_x(\xi^*)^T n(T) + h_0 \phi_x(\xi^*), v - \xi^* \rangle &\geq 0, \quad \text{a.s. on } \Omega_0 \\ m(T) + h_x(\xi^*)^T n(T) + h_0 \phi_x(\xi^*) &= 0, \quad \text{a.s. on } \Omega_0^c. \end{aligned} \quad (14)$$

Remark 3.4. When $K = \mathbb{R}^n$, the terminal state constraint $X(T) \in K$ will disappear. Due to the diffusion coefficient σ depends on the control variable u , the control problem cannot be degenerated into the one in [23]. Note that we derive a local maximum principle and the one in [23] is global, since they applied the spike variational method.

4. Applications

4.1. An optimal portfolio selection problem

There are $d + 1$ investment instruments in the market. One of the instruments is a bank account (free risk); the others are stocks. The price processes are described by the following equations:

$$\begin{cases} dP_0(t) = P_0(t)r(t)dt, \\ P_0(0) = p_0, \\ dP_i(t) = P_i(t) \left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t) \right], \\ P_i(0) = p_i > 0, \quad i = 1, \dots, d, t \in [0, T]. \end{cases}$$

We assume:

- (H₅) (i) The interest rate $r(\cdot)$ is a nonnegative, predictable and uniformly bounded scalar-valued process.
(ii) The stock-appreciation rate $b(\cdot) = (b_1(\cdot), \dots, b_d(\cdot))^T$ is a predictable and uniformly bounded vector-valued process.
(iii) The matrix process $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i, j \leq d}$ is the stock-volatility which is a predictable and bounded process. $\sigma(\cdot)$ is assumed to be invertible and $\sigma^{-1}(\cdot)$ is assumed to be bounded uniformly in $(t, \omega) \in [0, T] \times \Omega$.

An investor whose initial wealth is $x > 0$, decides to invest in the i 'th stock ($i = 1, 2, \dots, d$) with the amount π_i . Denote $X(\cdot)$ and $Y(\cdot)$ by the wealth process and the recursive utility of the investor, respectively. Here, we suppose that the consumption function c depends on the wealth process $X(\cdot)$ and the utility $Y(\cdot)$. Let $B(t) := (b_1(t) - r(t), \dots, b_d(t) - r(t))$, $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_d(\cdot))'$ be the portfolio process and $\theta(t) = \sigma(t)^{-1}B(t)'$ be the risk premium process. By the conventional calculation, we have:

$$\begin{cases} dX(t) = [r(t)X(t) + \pi(t)'\sigma(t)\theta(t)]dt - c(X(t), Y(t))dt + \pi(t)'\sigma(t)dW(t), \\ X(0) = x, \\ dY(t) = -g(t, Y(t), Z(t))dt + Z(t)dW(t), \\ Y(T) = u(X(T)), \quad t \in [0, T] \end{cases} \quad (15)$$

where c, g satisfies (H₃).

Our control problem is

$$\begin{aligned} &\text{minimize } Y(0) \\ &\text{subject to } \pi(\cdot) \in \mathcal{M}^2(0, T; \mathbb{R}^{1 \times d}), \quad X(T) \geq 0. \end{aligned}$$

Using the method in Section 3, let $\pi(t)'\sigma(t) = q(t)$, then we get the following equivalent control system

$$\begin{cases} dX(t) = -b(t, X(t), Y(t), q(t))dt + q(t)dW(t), \\ X(T) = \xi, \\ dY(t) = -g(t, Y(t), Z(t))dt + Z(t)dW(t), \\ Y(T) = u(\xi), \quad t \in [0, T], \end{cases} \quad (16)$$

where $b(t, x, y, q) = -r(t)x - \theta(t)q - c(x, y)$ and the equivalent control problem:

$$\begin{aligned} &\text{minimize } Y(0) \\ &\text{subject to } \xi \in L^2(\Omega, \mathcal{F}_T, P), \quad \xi \geq 0, \quad X(0) = x. \end{aligned} \quad (17)$$

Let ξ^* be an optimal terminal wealth and $X^*(\cdot), Y^*(\cdot)$ be the wealth process and the utility associated with ξ^* , respectively. According to Section 3.5 the adjoint equation is

$$\begin{cases} dm(t) = -[r(t) + c_x(X(t), Y(t))]m(t)dt - \theta(t)m(t)dW(t), \\ m(0) = h_1, \\ dn(t) = [g_y^*(t)n(t) - c_y(X(t), Y(t))m(t)]dt + g_z^*(t)n(t)dW(t), \\ n(0) = h_0, \end{cases}$$

where $g_j^*(t) = g_j(t, Y^*(t), Z^*(t))$, $j = y, z$, respectively. The solution is

$$\begin{aligned} m(t) &= h_1 \exp \left\{ \int_0^t \left[-r(s) - c_x(X(s), Y(s)) + \frac{1}{2}\theta(s)^2 \right] ds - \int_0^t \theta(s)dW(s) \right\}, \\ n(t) &= \left\{ h_0 - \int_0^t c_y(X(s), Y(s))m(s)F_s ds \right\} F_t^{-1}, \end{aligned}$$

where $F_t = \exp \{ -\int_0^t [g_y^*(s) + \frac{1}{2}g_z^*(s)^2] ds - \int_0^t g_z^*(s)dW(s) \}$.

Set $\Omega_0 = \{ \omega \in \Omega \mid \xi^*(\omega) = 0 \}$, by Theorem 3.3, we deduce that, there exist constants $h_1, h_0 \in \mathbb{R}$ with $h_0 \geq 0$ and $|h_1| + |h_0| \neq 0$, such that

$$m(T) + u_x(\xi^*)n(T) = 0, \quad \text{if } \xi^*(\omega) > 0, \text{ a.s.}$$

$$m(T) + u_x(\xi^*)n(T) \geq 0, \quad \text{if } \xi^*(\omega) = 0, \text{ a.s.}$$

Once $g(t, \cdot, \cdot)$, $c(\cdot, \cdot)$ and $u(\cdot)$ are given, we will derive the expression of the optimal control ξ^* . For example, we take $g(t, y, z) = \rho(t)z$, $c(x, y) = ax + by$, $u(x) = -\frac{1}{2}x^2$. It is easy to see that the optimal control can be represented as

$$\xi^* = \begin{cases} \left(\frac{m(T)}{(h_0 - \int_0^T bm(s)F_s ds) F_T^{-1}} \right)^+, & \text{if } h_0 > 0; \\ 0, & \text{if } h_0 = 0, \end{cases} \quad (18)$$

where

$$\begin{aligned} m(T) &= h_1 \exp \left\{ \int_0^T \left(-r(s) - a + \frac{1}{2}\theta(s)^2 \right) ds - \int_0^T \theta(s)dW(s) \right\}, \\ F_t &= \exp \left\{ -\int_0^t \frac{1}{2}\rho(s)^2 ds - \int_0^t \rho(s)dW(s) \right\}. \end{aligned}$$

4.2. Stochastic LQ control with terminal state constraints

Next we consider a linear quadratic (LQ, for short) control problem of fully coupled FBSEs.

Consider the following linear control system ($m = n = d = 1$):

$$\begin{cases} dx_t = (2x_t - 4y_t + z_t + u_t)dt + (-2x_t - y_t - 3z_t + u_t)dW(t), \\ x_0 = a, \\ dy_t = -(x_t + 2y_t - 2z_t + 3u_t)dt + z_t dW(t), \\ y_T = x_T^2, \quad t \in [0, T]. \end{cases} \quad (19)$$

The objective of our control problem is to minimize the following cost function:

$$J(u(\cdot)) = \frac{1}{2} E [x_T^2 + y_0^2]$$

subject to $u(\cdot) \in \mathcal{M}^2(0, T; \mathbb{R})$, $x(T) \in \mathbb{R}^+$, a.s..

Following the procedures in previous section, let $-2x_t - y_t - 3z_t + u_t = q_t$, and treat $q(\cdot)$ as a control, then the system can be rewritten as

$$\begin{cases} dx_t = (4x_t - 3y_t + 4z_t + q_t)dt + q_t dW(t), \\ x_T = \xi, \\ dy_t = -(7x_t + 5y_t + 7z_t + 3q_t)dt + z_t dW(t), \\ y_T = \xi^2. \end{cases} \quad (20)$$

Accordingly, the equivalent control problem is

$$\text{Minimize } J(u(\cdot)) = \frac{1}{2} E [\xi^2 + y_0^2],$$

$$\text{subject to } \xi \in \mathbb{R}^+, \quad x_0^\xi = a.$$

Then from Theorem 3.3, if ξ^* is optimal, there exist $h_1, h_0 \in \mathbb{R}$, with $h_0 \geq 0$ and $|h_0| + |h_1| \neq 0$ such that, for any $\xi \geq 0$,

$$(m(T) + 2\xi^*n(T) + \xi^*)(\xi - \xi^*) \geq 0, \quad a.s.$$

where $(m(\cdot), n(\cdot))$ is the solution of the following equation:

$$\begin{cases} dm(t) = (-4m(t) + 7n(t))dt + (-m(t) + 3n(t))dW(t), \\ m(0) = h_1, \\ dn(t) = (3m(t) + 5n(t))dt + (-4m(t) + 7n(t))dW(t), \\ n(0) = h_0y^*(0). \end{cases} \quad (21)$$

Denote $\Omega_0 := \{\omega \in \Omega \mid \xi^*(\omega) = 0\}$. From the arbitrariness of ξ , we get the following necessary condition of the optimal control ξ^* :

$$m(T) + 2\xi^*n(T) + \xi^* \geq 0, \quad a.s. \text{ on } \Omega_0.$$

$$m(T) + 2\xi^*n(T) + \xi^* = 0 \quad a.s. \text{ on } \Omega_0^c,$$

where $(m(\cdot), n(\cdot))$ is the solution of (21).

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