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A note on Cantor boundary behavior☆

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ABSTRACT

For an analytic function f on the open unit disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, the Cantor boundary behavior (CBB) is used to describe the curve $f(\partial\mathbb{D})$ that forms infinitely many fractal-look loops everywhere. The class of analytic functions with the CBB was formulated and investigated in Dong et al. [6]. In this note, our main objective is to give further discuss of the criteria of CBB in Dong et al. [6]. We show that the two major criteria, the accumulation of the zeros of $f'(z)$ near the boundary and the fast mean growth rate of $f'(z)$ near the boundary, do not imply each other. Also we make an improvement of another criterion, which allows us to have more examples of CBB.

1. Introduction

We use $\mathbb{D}$ to denote the open unit disk with center $z = 0$, and let $\partial\mathbb{D}$ be its boundary. Let $A(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. For $f \in A(\mathbb{D})$, consider the decomposition $\mathbb{C}_\infty \setminus f(\partial\mathbb{D}) = \bigcup_{j\geq 0} \mathcal{W}_j$, where the $\mathcal{W}_j$'s are simply connected components. We say that $f \in A(\mathbb{D})$ has the Cantor boundary behavior (CBB) on $\mathbb{D}$ if

$$f^{-1}(\partial f(\mathbb{D})), \qquad f^{-1}(\partial \mathcal{W}_j) \cap \partial\mathbb{D}$$

are Cantor-type sets in $\partial\mathbb{D}$ (i.e., uncountable nowhere dense closed sets of $\partial\mathbb{D}$). The definition implies that for any open arc I on $\partial\mathbb{D}$, $f(I)$ contains infinitely many loops.

The concept of the Cantor boundary behavior (CBB) for analytic functions was first introduced in [4] and studied in detail in [6,3,5], it is used to describe some fractal behavior of analytic functions on the unit disk. The original idea comes from Strichartz's *Cantor set conjecture* (see [8]), which was proposed for the Cauchy transform $F(z) = \int_K d\mathcal{H}^\alpha(w)/(z-w)$ of the Hausdorff measure on the Sierpinski gasket K, it was observed that the curve $F(\partial\Delta_0)$, where Δ_0 is the unbounded component of $\mathbb{C} \setminus K$, is a fractal curve filled with loops within loops (due to the similarity). The conjecture is proved by Dong and Lau in [5].

By using some delicate analytic topology arguments, Lau and two of the authors established two criteria for the CBB. The first criterion concerns the distribution of the zeros of $f'(z)$ (see Theorem 5.3 of [6]). The second criterion for the CBB (see [6, Theorem 5.6]) makes use of the well known integral mean spectrum $\beta(\lambda)$ of normalized univalent functions on $\mathbb{D}$:

$$\beta(\lambda) = \sup_{f\in\mathcal{S}} \left(\limsup_{r\to 1^-} \frac{\log\left(\int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta\right)}{-\log(1-r)} \right)$$

where $\mathcal{S}$ denote the class of univalent functions f on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. The upper estimate of $\beta(\lambda)$ was given

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by Pommerenke ([9], [10, p. 178]):

$$\beta(\lambda) \le \lambda - \frac{1}{2} + \left(4\lambda^2 - \lambda + \frac{1}{4}\right)^{1/2} < 3\lambda^2 + 7\lambda^3, \quad \lambda > 0. \tag{1.1}$$

The exact statement of the two criteria is the following theorem.

Theorem A. *Let $f \in A(\mathbb{D})$. Then f has the Cantor boundary behavior* (CBB) *in $\mathbb{D}$, if either one of the following conditions holds:*

(i) *The set of limit points of $\mathcal{Z} = \{z \in \mathbb{D} : f'(z) = 0\}$ equals $\partial\mathbb{D}$; or*
(ii) *For every interval $E \subseteq [0, 2\pi]$ with Lebesgue measure $|E| > 0$, there exist $\lambda > 0$ and $\eta > \beta(\lambda)$, and $C > 0$, $0 < r_0 < 1$ (all depend on E) such that*

$$\int_E |f'(re^{i\theta})|^\lambda d\theta \ge \frac{C}{(1-r)^\eta}, \quad r_0 < r < 1. \tag{1.2}$$

In [6], we used the infinite Blaschke product to construct the following example which satisfies (i). Recall that for $p > 0$, the H^p-space on $\mathbb{D}$ is defined to be the class of analytic functions on $\mathbb{D}$ so that $\|f\|_p = \sup_{r<1}\left(\frac{1}{2\pi}\int_{\partial\mathbb{D}} |f(re^{i\theta})|^p d\theta\right)^{1/p} < \infty$.

Example 1. Let $\theta_{k,m} = m/k$, $m = 1, 2, \ldots, k-1$, $k = 2, 3, \ldots$, and let $z_{k,m} = (1 - k^{-s})e^{i2\pi\theta_{k,m}}$. Since $\sum_{k=2}^\infty \sum_{m=1}^{k-1}(1 - |z_{k,m}|) = \sum_{k=2}^\infty (k-1)k^{-s} < \infty$ if $s > 2$, the Blaschke product

$$p_s(z) = \prod_{k=2}^\infty \prod_{m=1}^{k-1} \frac{|z_{k,m}|}{z_{k,m}} \frac{z_{k,m} - z}{1 - \bar{z}_{k,m} z}$$

converges uniformly for $|z| \le r < 1$ and $|p_s(z)| \le 1$ for $z \in \mathbb{D}$. For $s > 2$, we define a subclass $\mathcal{F}_s$ of analytic functions in $\mathbb{D}$:

$$\mathcal{F}_s = \left\{ f(z) = \int_0^z g(\xi) p_s(\xi) d\xi \;:\; g \in H^1(\mathbb{D}) \right\}.$$

Then for any $f \in \mathcal{F}_s$, f has the Cantor boundary behavior.

Beside the class of examples, it was shown that the complex Weierstrass functions

$$W_{q,\beta}(z) = \sum_{n=0}^\infty q^{-n\beta} z^{q^n}$$

where $0 < \beta < 1$ and $q > 1$ an integer, satisfy (ii) by Theorem 6.7 in [6], and satisfy (i) by Corollary 6.5 in [6] (see Fig. 1). We remark that if q is large and β is small, then $W_{q,\beta}(\partial\mathbb{D})$ can be a space filling curve [1,12,11]. Also, Strichartz's *Cantor set conjecture* for the Cauchy transform F on the Sierpinski gasket was answered positively in [5] by (ii).

For the two main criteria (i) and (ii) of CBB, it is clear that the (i) does not imply (ii) from the example of the Blaschke product. However it is not so clear whether (ii) will imply (i). In this paper, we first construct a function $f \in A(\mathbb{D})$ satisfies (ii) but not (i).

Theorem 1.1. *Let $f_{\rho,q}(z) = \int_0^z \exp\left(\rho \sum_{n=1}^\infty q^n w^{q^{q^n}}\right) dw$, $z \in \mathbb{D}$. Then $f_{\rho,q}(z)$ satisfies* (ii) *in Theorem A if $q \ge 4$; there is a constant $c > 0$ such that $f_{\rho,q}(z)$ is a Lipschitz function of order $1 - c\rho$ on $\overline{\mathbb{D}}$ if $\rho \in (0, c^{-1})$ and $q \ge 2$ an integer. Hence $f_{\rho,q}(z)$ has the Cantor boundary behavior* (CBB) *on $\mathbb{D}$ for $\rho \in (0, c^{-1})$ and $q \ge 4$.*

Remark 1. To sum up the above discussion and Theorem 1.1, we conclude that the criterions (i) and (ii) of the Cantor boundary behavior (CBB) for analytic function on $\mathbb{D}$ are independent of each other, i.e., there is a function $f \in A(\mathbb{D})$ satisfies the criterion (i) but not (ii), and there is a function $g \in A(\mathbb{D})$ satisfies the criterion (ii) and $g'(z) \ne 0$ (i.e., $g(z)$ does not satisfy (i)). However, the Weierstrass functions $W_{q,\beta}$ satisfy both criteria (i) and (ii).

We also want to construct a function f which has the CBB, and satisfies the criterion (i) on subset E of $\partial\mathbb{D}$ and not on its complement $\partial\mathbb{D} \setminus E$. To the end, we need to illustrate another sufficient condition of CBB using the pre-Schwartz derivatives, i.e., we extend the criterion (i) slightly, and make use of it to provide a new class having CBB.

Theorem 1.2. *Let g be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose there exists a dense set E of $\partial\mathbb{D}$ such that for any $e^{i\theta} \in E$, there exists a sequence $z_n \to e^{i\theta}$ $(n \to \infty)$ such that*

$$g'(z_n) = 0 \quad \text{for all } n \tag{1.3}$$

or

$$\limsup_{n\to\infty}(1 - |z_n|^2)\left|\frac{g''(z_n)}{g'(z_n)}\right| > 24. \tag{1.4}$$

Then $g(z)$ has the Cantor boundary behavior (CBB) *on $\mathbb{D}$.*

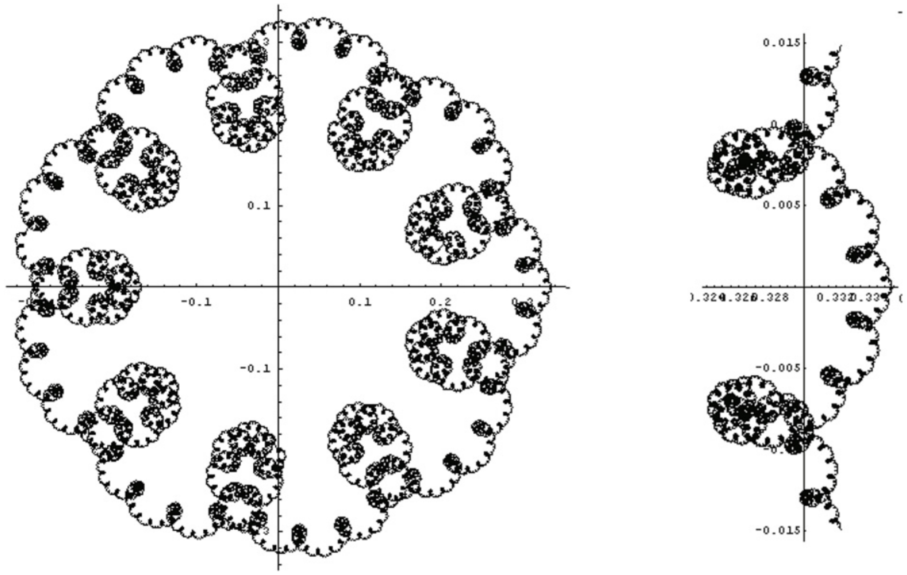

Fig. 1. The graph of $W_{q,\beta}(\partial\mathbb{D})$ with $q = 10$ and $\beta = 0.6$, the right is a magnification of the graph at a neighborhood of 1.

We remark that the above theorem is a modification of Theorem 7.2 of [6] by replacing $g'(z) \neq 0$ for all z with the addition part of $g'(z_n) = 0$ in (1.3), which allows us to construct more examples.

Theorem 1.3. *Let $a_n \nearrow +\infty$ as $n \to \infty$, and the positive integer sequence $\{b_n\}$ satisfy $b_n - b_{n-1} \geq ca_n (n \geq 1)$ for some constant $c > 0$. Let $s \geq 0$ an integer, $B(w) = e^{i\theta} z^s \prod_{j=1}^{N} \frac{|z_j|}{z_j} \frac{z_j - w}{1 - \overline{z}_j w}$ and*

$$g_{\rho,q}(z) = \int_0^z B(w) \exp\left(\rho \sum_{n=1}^{\infty} a_n w^{q^{b_n}}\right) dw$$

where N is finite or infinity, $z_j \in \mathbb{D}$ and $z_j \neq 0$ for all j, and $\sum_{j=1}^{N}(1 - |z_j|) < \infty$. Then $g_{\rho,q}(z)$ has the Cantor boundary behavior (CBB) *on $\mathbb{D}$ for small $\rho > 0$.*

Remark 2. Let $N = \infty$ and $\mathcal{Z}$ be the zero set of the infinite Blaschke product $B(z)$, it is easy to see that the set of the limit point of $\mathcal{Z}$ has to lie on $\partial\mathbb{D}$, hence there is at least a point ξ on $\partial\mathbb{D}$ to be a limit point of $\mathcal{Z}$. For any subset E of $\partial\mathbb{D}$, by using the method of Example 1, we can construct a $B(z)$ such that the set of the limit points of $\mathcal{Z}$ equals exactly $\overline{E}$, hence $B(z)$ has an extension analytically in $\partial\mathbb{D} \setminus \overline{E}$ (see [2]).

2. Estimate of integral means

It is known that the growth rate of the integral mean of $|f'(z)|$ plays an important role in the theory of univalent functions. In [6], the authors used this to establish a criterion for the function f having the Cantor boundary behavior: the fast mean growth rate of $|f'(z)|^\lambda$ near boundary as in (1.2), i.e., it is faster than that (the integral mean spectrum $\beta(\lambda)$) of the univalent function. To prove Theorem 1.1, we need to estimate the mean growth rate of

$$f_{\rho,q}(z) = \int_0^z \exp\left(\rho \sum_{n=1}^{\infty} q^n w^{q^{q^n}}\right) dw, \quad z \in \mathbb{D} \tag{2.1}$$

where $\rho > 0$ is a small constant, and $q \geq 2$ an integer.

Proposition 2.1. *Let $f_{\rho,q}(z)$ be defined by (2.1) with $q \geq 4$, then for any interval $[a, b] \subset [0, 2\pi]$ with $b - a \neq 0$, there exist $\lambda_0, C > 0$ and $0 < r_0 < 1$ (depending only on the interval $[a, b]$) such that*

$$\int_a^b |f'_{\rho,q}(re^{i\theta})|^\lambda d\theta \geq \frac{C}{(1-r)^{\rho\kappa\lambda}}, \quad 0 < \lambda < \lambda_0,\ r_0 < r < 1 \tag{2.2}$$

where $\kappa > 0$ is an absolute constant.

Proof. Let $u(re^{i\theta}) = \sum_{n=1}^{\infty} q^n r^{q^{q^n}} \cos(q^{q^n}\theta)$, it is easy to see

$$|f'_{\rho,q}(re^{i\theta})|^{\lambda} = \left|\exp\left\{\rho \sum_{n=1}^{\infty} q^n (re^{i\theta})^{q^{q^n}}\right\}\right|^{\lambda} = \exp\{\rho\lambda u(re^{i\theta})\}.$$

Let integer $N_0 > 2$ such that $(b-a)q^{q^{N_0}} \geq 4\pi$. It follows that there exist constants c, d with $a \leq c < d \leq b$ and $s \in \mathbb{Z}$ such that

$$[cq^{q^{N_0}}, dq^{q^{N_0}}] = \left[2s\pi - \frac{\pi}{2},\ 2s\pi + \frac{3\pi}{2}\right].$$

For integer $M > N_0$, we have $(d-c)q^{q^M} \geq (d-c)q^{q^{N_0+1}} = 2\pi q^{q^{N_0}(q-1)} \geq 16\pi$, and hence there exist c_M, d_M with $c \leq c_M < d_M \leq d$, $d_M - c_M \geq \frac{1}{2}(d-c)$ and $k, l \in \mathbb{Z}, l \geq k+3$ such that

$$[c_M q^{q^M}, d_M q^{q^M}] = \left[2k\pi - \frac{\pi}{2},\ 2l\pi + \frac{3\pi}{2}\right].$$

Let $c_M = c_0 < c_1 < \cdots < c_{l-k} < c_{l-k+1} = d_M$ such that $(c_{i+1} - c_i)q^{q^M} = 2\pi$ for $0 \leq i \leq l-k$. For each interval $[c_i, c_{i+1}]$, there exist $[c'_i, d'_i] \subset [c_i, c_{i+1}]$ with $d'_i - c'_i = \frac{1}{4}(c_{i+1} - c_i)$ such that

$$\cos(q^{q^M}\theta) \geq \frac{\sqrt{2}}{2}, \quad \theta \in [c'_i, d'_i]. \tag{2.3}$$

Let $r \in [1 - q^{-q^M}, 1 - q^{-q^{M+1}}]$. Since $(1 - q^{-q^M})^{q^{q^M}}$ is increasing and $(1 - q^{-q^M})^{q^{q^M}} \to 1/e$ $(M \to \infty)$, then there exists $N_1 > N_0$ such that

$$1/3 < (1 - q^{-q^M})^{q^{q^M}} < r^{q^{q^M}}, \qquad r^{q^{q^{M+1}}} < (1 - q^{-q^{M+1}})^{q^{q^{M+1}}} < 1/2 \tag{2.4}$$

for $M \geq N_1$. This and (2.3) imply that for $\theta \in [c'_i, d'_i]$,

$$\begin{aligned} u(re^{i\theta}) &\geq \frac{\sqrt{2}}{6} q^M + q^{M+1} r^{q^{q^{M+1}}} \cos(q^{q^{M+1}}\theta) - \left(\sum_{n=1}^{M-1} + \sum_{n=M+2}^{\infty}\right) q^n r^{q^{q^n}} \\ &\geq \frac{\sqrt{2}}{6} q^M + q^{M+1} r^{q^{q^{M+1}}} \cos(q^{q^{M+1}}\theta) - \frac{q^M - q}{q-1} - I \end{aligned} \tag{2.5}$$

where $I = \sum_{n=M+2}^{\infty} q^n r^{q^{q^n}}$. It follows from (2.4) that

$$\begin{aligned} I &= q^{M+2} r^{q^{q^{M+2}}} \sum_{n=M+2}^{\infty} q^{(n-M-2)} r^{q^{q^n} - q^{q^{M+2}}} \\ &\leq \frac{1}{2} q^{M+2} r^{q^{q^{M+2}} - q^{q^{M+1}}} \sum_{n=M+2}^{\infty} q^{(n-M-2)} r^{q^{q^n} - q^{q^{M+2}}}. \end{aligned}$$

Let $\beta = q^{q^{M+2}}$, noting that $\beta^{q^{n}-1} - 1 \geq \beta^{qn} - 1 \geq (\beta^q - 1)n$ for $n \geq 2$, we have

$$q^{q^{M+2+n}} - q^{q^{M+2}} = q^{q^{M+2}}(\beta^{q^n - 1} - 1) \geq (q^{q^{M+3}} - q^{q^{M+2}})n.$$

Obviously, the above inequality also holds for $n = 0, 1$. Thus

$$I \leq \frac{1}{2} q^{M+2} r^{q^{q^{M+2}} - q^{q^{M+1}}} \sum_{n=0}^{\infty} q^n (r^{q^{q^{M+3}} - q^{q^{M+2}}})^n = \frac{1}{2} q^{M+2} \frac{r^{q^{q^{M+2}} - q^{q^{M+1}}}}{1 - qr^{q^{q^{M+3}} - q^{q^{M+2}}}}.$$

In view of $q^{q^{M+2}} - q^{q^{M+1}} \geq q^{q^{M+1}} q^{q^{M+1}}$ and $q^{q^{M+3}} - q^{q^{M+2}} \geq q^{q^{M+1}} q^{q^{M+1}}$, we have

$$I \leq \frac{1}{2} q^{M+2} \frac{r^{q^{q^{M+1}} q^{q^{M+1}}}}{1 - qr^{q^{q^{M+1}} q^{q^{M+1}}}} \leq \frac{1}{2} q^{M+2} \frac{\left(\frac{1}{2}\right)^{q^{q^{M+1}}}}{1 - q\left(\frac{1}{2}\right)^{q^{q^{M+1}}}} \leq \frac{q^{M+2}}{2(2^{q^{q^{M+1}}} - q)} < \frac{1}{q-1}$$

for large $M \geq N_2 \geq N_1$. By (2.5),

$$u(re^{i\theta}) \geq \left(\frac{\sqrt{2}}{6} - \frac{1}{q-1}\right) q^M + q^{M+1} r^{q^{q^{M+1}}} \cos(q^{q^{M+1}}\theta),$$

hence for $q \geq 7$,

$$\int_{c_i'}^{d_i'} \exp\{\rho\lambda u(re^{i\theta})\} d\theta \geq \exp\left\{\rho\lambda \frac{\sqrt{2}-1}{6} q^M\right\} \int_{c_i'}^{d_i'} \exp\{\rho\lambda q^{M+1} r^{q^{q^{M+1}}} \cos(q^{q^{M+1}}\theta)\} d\theta.$$

Since $(d_i' - c_i')q^{q^{M+1}} = \frac{1}{4}(c_{i+1} - c_i)q^{q^{M+1}} = \frac{\pi}{2} q^{q^M(q-1)} \geq \frac{\pi}{2} q^{q^M} \geq \frac{\pi}{2} 2^{2^M} \geq 128\pi$ for $M \geq N_0 > 2$, we can find $[c_i'', d_i''] \subset [c_i', d_i']$ and positive integers s, h with $d_i'' - c_i'' \geq (d_i' - c_i')/2$ and $h \geq s + 31$ satisfying

$$[c_i'' q^{q^{M+1}}, d_i'' q^{q^{M+1}}] = \left[2s\pi - \frac{\pi}{2}, 2h\pi + \frac{3\pi}{2}\right].$$

Let $E_i = \{\theta \in [c_i'', d_i''] : \cos(q^{q^{M+1}}\theta) > 0\}$. It follows that the Lebesgue measure of E_i is $|E_i| = (d_i'' - c_i'')/2 \geq (d_i' - c_i')/4 = (c_{i+1} - c_i)/16$. Hence

$$\int_{c_i'}^{d_i'} \exp\{\rho\lambda q^{M+1} r^{q^{q^{M+1}}} \cos(q^{q^{M+1}}\theta)\} d\theta \geq \int_{E_i} d\theta \geq |E_i| = \frac{1}{16}(c_{i+1} - c_i),$$

by summing, we have

$$\int_a^b |f_{\rho,q}'(re^{i\theta})|^\lambda d\theta \geq \sum_{i=0}^{l-k} \int_{c_i'}^{d_i'} \exp\{\rho\lambda u(re^{i\theta})\} d\theta \geq \frac{d-c}{32} \exp\left\{\rho\lambda \frac{\sqrt{2}-1}{6} q^M\right\}. \tag{2.6}$$

Noting that the inequality (2.6) holds for all $M \geq N_2$ and $1 - q^{-q^M} \leq r < 1 - q^{-q^{M+1}}$. Now we take $r_0 = 1 - q^{-q^{N_2+1}}$ and $C = (d-c)/32$. Then for any $r \in (r_0, 1)$ close to 1, we can find a $M > N_2$ such that $1 - q^{-q^M} \leq r < 1 - q^{-q^{M+1}}$, which implies $q^M \geq \frac{1}{q \log q} \log \frac{1}{1-r}$. Hence (2.6) gives that for $q \geq 7$,

$$\int_a^b |f_{\rho,q}'(re^{i\theta})|^\lambda d\theta \geq \tfrac{d-c}{32} \exp\left\{\rho\lambda \tfrac{\sqrt{2}-1}{6q\log q} \log \tfrac{1}{1-r}\right\} = \tfrac{C}{(1-r)^{\kappa\rho\lambda}},$$

where $\kappa = \frac{\sqrt{2}-1}{6q\log q} > 0$.

For the cases $q = 4, 5, 6$, we only need to take $\cos(q^{q^M}\theta) \geq 1 - \varepsilon$, $\theta \in [c_i', d_i']$ in (2.3) and take $\frac{1}{e} - \varepsilon < (1 - q^{-q^M})^{q^{q^M}} < r^{q^{q^M}}$ in (2.4), by using some small modifications, we complete the rest of the proof. $\square$

To prove Theorem 1.1, we need

Hardy–Littlewood Theorem (*[7, Theorem 5.1]*). *For an analytic function h in $\mathbb{D}$, it has a continuous extension to $\overline{\mathbb{D}}$ and has Lipschitz order $0 < \alpha < 1$ on $\partial\mathbb{D}$ if and only if*

$$h'(z) = O\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad r = |z| \to 1.$$

Proof of Theorem 1.1. From Lemma 7.3 of [6], there exists a constant $c > 0$ such that

$$\left|\sum_{n=1}^{\infty} q^n w^{q^{q^n}}\right| \leq c \log \frac{1}{1-|w|}.$$

It follows from Hardy–Littlewood Theorem that $f_{\rho,q}(z)$ is a Lipschitz function of order $1 - c\rho$ on $\overline{\mathbb{D}}$ if $\rho \in (0, c^{-1})$ and $q \geq 2$ an integer. By (1.1), we see that for λ sufficiently small, $\beta(\lambda) < 3\lambda^2 + 7\lambda^3 < \rho\kappa\lambda$, thus the $f_{\rho,q}(z)$ satisfies (ii) by Proposition 2.1 and taking $\eta = \rho\kappa\lambda$. $\square$

3. A new criterion for CBB and examples

In this section, we consider another criterion for CBB, it is an improvement of (i) slightly, the criterion allows us to have more examples having CBB. The following lemma is the crux to prove criteria (i) and (ii) of the CBB [6, Propositions 4.1 and 4.3].

Lemma 3.1 (*[6]*). *Let $f \in A(\mathbb{D})$ and suppose there is a non-degenerated arc $I \subset \partial\mathbb{D}$ such that*

$$f(I) \subset \partial f(\mathbb{D}) \quad \text{or} \quad f(I) \subset \partial \mathcal{W}_j \tag{3.1}$$

for a connected component $\mathcal{W}_j$ of the complement of $f(\partial\mathbb{D})$. Then there exists a non-degenerated sub-arc $J \subset I$ and a Jordan domain $D \subset \mathbb{D}$ such that $J \subset \partial D$ and f is univalent in D.

Lemma 3.2. *Let $f(z)$ be analytic in $\mathbb{D}$. Suppose that there exists a simply connected domain $D \subset \mathbb{D}$ such that $f(z)$ is univalent in D. Then*

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le 24, \quad z \in D.$$

The proof of this lemma can be find in the proof of Theorem 7.2 in [6].

Proof of Theorem 1.2. We only need to show that $g^{-1}(\partial g(\mathbb{D}))$ (or $g^{-1}(\partial \mathcal{W}_j)$) does not contain any sub-arc of $\partial\mathbb{D}$. Suppose otherwise, there exists an arc I such that $g(I) \subset \partial g(\mathbb{D})$ (or $g(I) \subset \partial \mathcal{W}_j$). By Lemma 3.1, there exists a non-degenerated sub-arc I' of I, and a Jordan domain $D \subset \mathbb{D}$ with $I' \subset \partial D$ such that $g(z)$ is univalent in D. This and Lemma 3.2 give

$$g'(z) \neq 0 \quad \text{and} \quad (1-|z|^2)\left|\frac{g''(z)}{g'(z)}\right| \le 24, \quad z \in D,$$

which contradict the assumptions (1.3)–(1.4). $\square$

By applying Lemma 6.6 (with $s = 1$) of [6], we have:

Lemma 3.3. *Let $f(z) = \sum_{n=1}^{\infty} z^{q^n}$ with integer $q \ge 2$. Then there exist $c > 0$ such that*

$$f(r) \le c\ \log\frac{1}{(1-r)}, \quad \frac{1}{2} < r < 1.$$

Lemma 3.4. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^{q^{b_n}}$ where $q \ge 2$ is an integer and $a_n \nearrow \infty$ as $n \to \infty$, and $\{b_n\}$ is a positive integer sequence with $c'(b_n - b_{n-1}) \ge a_n (n \ge 1)$ for some constant c'. Then there exists $c > 0$ such that*

$$f(r) \le c \log\frac{1}{(1-r)}, \quad \frac{1}{2} < r < 1.$$

Proof. For any integer $n \ge 1$ and $\frac{1}{2} < r < 1$, we have

$$a_n r^{q^{b_n}} \le c'(b_n - b_{n-1}) r^{q^{b_n}} < c'(r^{q^{b_{n-1}+1}} + \cdots + r^{q^{b_n - 1}} + r^{q^{b_n}}).$$

By Lemma 3.3, we conclude

$$\sum_{n=1}^{\infty} a_n r^{q^{b_n}} \le c' \sum_{n=1}^{\infty} r^{q^n} \le c \log\frac{1}{(1-r)}. \quad \square$$

Proof of Theorem 1.3. Let $c > 0$ be in Lemma 3.4. It follows from Hardy–Littlewood Theorem that $g_{\rho,q}(z)$ has Lipschitz order $1 - c\rho$ at the boundary for $\rho \in (0, c^{-1})$, hence $g_{\rho,q}(z) \in A(\mathbb{D})$.

Now we find a dense subset E of $\partial\mathbb{D}$ such that (1.3) or (1.4) in Theorem 1.2 is satisfied. For $k \ge 3$ and $m = 0, 1, 2, \ldots, q^{b_k} - 1$, let $\theta_{k,m} = 2\pi m q^{-b_k}$. Then $E := \{e^{i\theta_{k,m}}\}$ is dense on $\partial\mathbb{D}$. Let $\mathcal{Z} = \{z \in \mathbb{D} : g'_{\rho,q}(z) = 0\}$, it is easy to see that $\mathcal{Z}$ is the set of zero points of $B(z)$, i.e., $\mathcal{Z} = \{z_j : j = 1, 2, \ldots\}$ if $s = 0$, or $\mathcal{Z} = \{z_j : j = 1, 2, \ldots\} \bigcup \{z_0 = 0\}$ if $s \neq 0$.

We take a $\eta_0 = e^{i\theta_{k,m}} \in E$, if η_0 is a limit point of $\mathcal{Z}$, then there exists $z_{j_n} \in \mathbb{D}$ such that

$$g'_{\rho,q}(z_{j_n}) = 0 \quad \text{and} \quad z_{j_n} \to \eta_0. \tag{3.2}$$

If η_0 is not a limit point of $\mathcal{Z}$, then there exists $\delta > 0$ such that $\operatorname{dist}(\eta_0, \overline{\mathcal{Z}}) = 2\delta$. Let $r_l = 1 - q^{-b_{k+l}}$ $(l = 1, 2 \cdots)$ and $\xi_l = r_l \eta_0$, it is easy to see that there exist $l_0 > 0$ such that $|\xi_l - \eta_0| < \delta$ and $r_l^{q^{b_{k+l}}} \ge \frac{1}{3}$ for $l > l_0$. Hence $|\xi_l - z_j| \ge \delta$ for $l > l_0$ and $z_j \in \mathcal{Z}$ since $|\eta_0 - z_j| \ge \operatorname{dist}(\eta_0, \mathcal{Z}) = 2\delta$, which implies

$$(1-|\xi_l|^2)\left|\frac{B'(\xi_l)}{B(\xi_l)}\right| \le \frac{s}{\delta} + \sum_{j=1}^{N} \frac{(1-r_l^2)(1-|z_j|^2)}{|1-\overline{z_j}\xi_l||\xi_l - z_j|} \le \frac{s}{\delta} + \frac{4}{\delta}\sum_{j=1}^{N}(1-|z_j|).$$

Thus for $l > l_0$, we have

$$\begin{aligned}(1-|\xi_l|^2)\left|\frac{g''_{\rho,q}(\xi_l)}{g'_{\rho,q}(\xi_l)}\right| &= (1-r_l^2)\left|\frac{\rho\sum_{n=1}^{\infty} a_n q^{b_n}(r_l e^{i\theta_{k,m}})^{q^{b_n}}}{r_l e^{i\theta_{k,m}}} + \frac{B'(\xi_l)}{B(\xi_l)}\right| \\ &\ge (1-r_l^2)\rho\left(\sum_{n=k}^{\infty} a_n q^{b_n} r_l^{q^{b_n}} - \sum_{n=1}^{k-1} a_n q^{b_n}\right) - \frac{s}{\delta} - \frac{4}{\delta}\sum_{j=1}^{N}(1-|z_j|).\end{aligned}$$

With a similar proof of Lemma 3.4, we have

$$\sum_{n=1}^{k-1} a_n q^{b_n} \le c' \sum_{n=0}^{b_{k-1}} q^n \le c' q^{b_{k-1}} \sum_{n=0}^{\infty} q^{-n} \le c' \frac{q^{b_k}}{q-1}.$$

This and the fact $r_l^{q^{b_{k+l}}} \ge \frac{1}{3}$ give that for large l,

$$\begin{aligned}(1-|\xi_l|^2)\left|\frac{g''_{\rho,q}(\xi_l)}{g'_{\rho,q}(\xi_l)}\right| &\ge \rho q^{-b_{k+l}}\left(\frac{a_{k+l}q^{b_{k+l}}}{3} - c'\frac{q^{b_k}}{q-1}\right) - \frac{s}{\delta} - \frac{4}{\delta}\sum_{j=1}^{N}(1-|z_j|) \\ &\ge \rho\frac{a_{k+l}}{4} - \frac{s}{\delta} - \frac{4}{\delta}\sum_{j=1}^{N}(1-|z_j|).\end{aligned}$$

It follows from $\sum_{j=1}^{N}(1-|z_j|) < \infty$ and $a_n \to \infty$ that

$$(1-|\xi_l|^2)\left|\frac{g''_{\rho,q}(\xi_l)}{g'_{\rho,q}(\xi_l)}\right| \to \infty, \quad \xi_l \to e^{i\theta_{k,m}} \text{ as } l \to \infty. \tag{3.3}$$

(3.2) and (3.3) show that $g_{\rho,q}(z)$ satisfies the conditions of Theorem 1.2, hence has the Cantor boundary behavior on $\mathbb{D}$. □

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