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Ergodic theorems for hybrid sequences in a Hilbert space with applications

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ABSTRACT

In this paper, we introduce the notion of generalized hybrid sequences, extending the notion of nonexpansive sequences introduced and studied in our previous work Djafari Rouhani (1981, 1990, 1990, 1997, 2002, 2004, 2002) [2–8], and prove ergodic and convergence theorems for such sequences in a Hilbert space H. Subsequently, we apply our results to prove new fixed point theorems for generalized hybrid mappings, first introduced in Kocourek et al. (2010) [14], Takahashi and Takeuchi (2011) [20], defined on arbitrary nonempty subsets of H.

1. Introduction

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. We denote weak convergence in H by $\rightharpoonup$ and strong convergence by $\to$. Let D be a nonempty subset of H. A self-mapping T of D is said to be nonexpansive if $|Tx - Ty| \le |x-y|$, $\forall x, y \in D$. Nonexpansive mappings have been studied extensively in all aspects. We refer the reader to the comprehensive books [10,11,18]. The first nonlinear ergodic theorem for nonexpansive mappings defined on convex subsets of H was proved by Baillon [1]. A sequence $\{x_n\}$ in H is called a nonexpansive sequence if $|x_{i+1} - x_{j+1}| \le |x_i - x_j|$, $\forall i, j \ge 0$. The study of nonexpansive sequences arose in connection with the study of iterates of nonexpansive mappings to show that it is possible to draw conclusions about asymptotic behavior in situations where the domains of such mappings may not be convex. The asymptotic behavior of such a sequence follows solely from the inherent properties of the sequence itself and the underlying space. The first ergodic theorem for such sequences in H was proved in [2], and then extended to almost nonexpansive sequences and curves which contain almost-orbits of solutions to quasi-autonomous dissipative evolution systems, as well as to more general mappings of nonexpansive type. We refer the reader to [2–8] and the references therein. Kohsaka and Takahashi [16] and Takahashi [19] introduced some new types of nonlinear mappings; they called them respectively nonspreading and hybrid. See also [13,15] and the references therein for related material. Recently, Kocourek, Takahashi and Yao [14] introduced a wide class of nonlinear mappings which they called generalized hybrid mappings, which contains the class of nonexpansive, nonspreading, as well as hybrid mappings, and they proved an ergodic theorem for such mappings, generalizing Baillon's ergodic theorem [1]. A self-mapping T of D is said to be generalized hybrid if there exist real numbers α, β such that

$$\alpha|Tx - Ty|^2 + (1-\alpha)|x - Ty|^2 \le \beta|Tx - y|^2 + (1-\beta)|x-y|^2, \quad \forall x, y \in D.$$

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Very recently, Takahashi and Takeuchi [20] proved a nonlinear ergodic theorem without convexity for generalized hybrid mappings in H, extending the results in [13–16,19].

In this paper, motivated by our previous work on nonexpansive sequences [2–8], and the new concept of generalized hybrid mappings introduced by Kocourek, Takahashi and Yao [14], we introduce the notion of generalized hybrid sequences, and we first prove an ergodic theorem for such sequences in H in Section 3. Our proof uses some modifications of the techniques used in our previous work [2–8] and is different and simpler, with stronger results than the one given by Takahashi and Takeuchi [20]. Then by a modification of our method, we are able to establish in Section 4, a new weak convergence theorem for such sequences in H.

The notion of absolute fixed points for a nonexpansive mapping was first introduced in [5], and the existence of such points in Hilbert space was established there. Using our methods described above, we are going to show in Section 5, the existence of absolute fixed points for generalized hybrid mappings in H, extending the results in [13–16,19,20]. Motivated by Goebel and Schöneberg [12], we proved in [6] some fixed point theorems for mappings of asymptotically nonexpansive type defined on nonconvex domains, where the Kirszbraun and Valentine's extension theorem (see [17]) used in [12] is not available anymore. In this paper, motivated by [6] and using our methods described above, we finally prove in Section 6, some new fixed point theorems for generalized hybrid mappings defined on nonconvex domains in H, extending the results in [13–16,19,20], and in Section 7, we present some examples of potential applications of our results.

2. Preliminaries

Here we recall and introduce some notations and definitions we shall use in the sequel.

Definition 2.1. Let $(x_n)_{n\geq 0}$ be a sequence in H.

(a) $(x_n)_{n\geq 0}$ is said to be a generalized hybrid sequence in H, if there exist real numbers α, β such that

$$\alpha|x_{i+1}-x_{j+1}|^2+(1-\alpha)|x_i-x_{j+1}|^2\leq\beta|x_{i+1}-x_j|^2+(1-\beta)|x_i-x_j|^2,\quad \forall i,j\geq 0.$$

(b) $(x_n)_{n\geq 0}$ is said to be asymptotically regular (abbreviated a.r.) (resp. weakly asymptotically regular (abbreviated w.a.r.)) if $x_{n+1}-x_n\to 0$, (resp. $x_{n+1}-x_n\rightharpoonup 0$), as $n\to+\infty$.

Definition 2.2. Given a bounded sequence $(x_n)_{n\geq 0}$ in H, the asymptotic center c of $(x_n)_{n\geq 0}$ is defined as follows (cf. [9]). For every $u\in H$, let $\phi(u)=\limsup_{n\to+\infty}|x_n-u|^2$. Then ϕ is a continuous, strictly convex function on H, satisfying $\phi(u)\to+\infty$, as $|u|\to+\infty$. Thus ϕ achieves its minimum on H at a unique point c, called the asymptotic center of the sequence $(x_n)_{n\geq 0}$. It is known that $c\in clco\{(x_n)_{n\geq 0}\}$, where *clco* U denotes the closed convex hull of a subset U of H.

Definition 2.3. Let D be a nonempty subset of H, and let T be a generalized hybrid self-mapping of D. A point $p\in H$ is said to be an absolute fixed point for T if there exists a generalized hybrid extension S of T from $D\cup\{p\}$ to $D\cup\{p\}$ such that $Sp=p$, and if p is a fixed point for every generalized hybrid extension of T to the union of D and a subset of H containing p. We denote by $F(T)$ (resp. $AF(T)$) the set of fixed (resp. absolute fixed) points of T in H.

Notations 2.4. (a) Given a sequence $(x_n)_{n\geq 0}$ in H, we use the following notations introduced in [2–4]: $F:=\{q\in H;\ \lim_{n\to+\infty}|x_n-q|$ exists$\}$ and $F_1:=\{q\in H;$ the sequence $|x_n-q|$ is nonincreasing$\}$. It is clear that $F_1\subset F$, and it was shown in [2–4] that F_1 and F are closed convex (possibly empty) subsets of H. For a self-map T of a nonempty subset D of H, Takahashi and Takeuchi [20] introduced the set $A(T):=\{x\in H;\ |Ty-x|\leq|y-x|,\ \forall y\in D\}$, and called it the set of attractive points of the map T. It is clear that $A(T)\subset F_1$, and that the two sets coincide when D consists of the orbit under T of some $x\in H$.

(b) We denote $s_n:=\frac{1}{n}\sum_{i=0}^{n-1}x_i$.

(c) If K is a nonempty closed and convex subset of H, we denote by P_K the metric projection map of H onto K. We recall (see e.g. [18]) that for any $x\in H$, we have $y=P_Kx$ if and only if $y\in K$ and $(x-y,z-y)\leq 0$, $\forall z\in K$.

Definition 2.5. We say that a nonempty subset D of H is Chebyshev with respect to its convex closure, if for any $y\in$ clco D, there is a unique $x\in D$ such that $|y-x|=\inf\{|y-z|;\ z\in D\}$.

3. Ergodic theorem

In this section, we prove an ergodic theorem for generalized hybrid sequences in H, extending with a simpler proof, the ergodic theorem of Takahashi and Takeuchi [20, Theorem 3.1].

Theorem 3.1. *Let $\{(x_n)_{n\geq 0}\}$ be a generalized hybrid sequence in H. Then the following are equivalent:*

(i) $F_1\neq\emptyset$.
(ii) $F\neq\emptyset$.

(iii) $\{x_n\}$ *is bounded in* H.
(iv) s_n *converges weakly to some* $p \in H$, *as* $n \to +\infty$.

Moreover, in this case p *is the asymptotic center of the sequence* $\{x_n\}$ *in* H, *and* $p = \lim_{n\to+\infty} P_{F_1} x_n$.

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Let us show that (iii) $\Rightarrow$ (iv). Since $\{x_n\}$ is bounded in H, $\{s_n\}$ has a weakly convergent subsequence, say $s_{n_j} \rightharpoonup_{j\to\infty} p$. By the polarization identity, we have $2(x_l - p, x_m - p) = |x_l - p|^2 + |x_m - p|^2 - |x_l - x_m|^2$. Writing this identity in the following four ways with the following indices and multiplied by the corresponding coefficients: $l = i+1$, $m = k+1$, multiplied by α; $l = i$, $m = k+1$, multiplied by $(1-\alpha)$; $l = i+1$, $m = k$, multiplied by $-\beta$; and $l = i$, $m = k$, multiplied by $-(1-\beta)$, and then adding up the four identities obtained, after some manipulations, we get the following inequality:

$$\begin{aligned} &2\alpha(x_{i+1} - p, x_{k+1} - p) - 2\beta(x_{i+1} - p, x_k - p) + 2(1-\alpha)(x_i - p, x_{k+1} - p) - 2(1-\beta)(x_i - p, x_k - p) \\ &\quad \geq (\alpha - \beta)(|x_{i+1} - p|^2 - |x_i - p|^2) + |x_{k+1} - p|^2 - |x_k - p|^2; \end{aligned}$$

$\{x_n\}$ being bounded, we have $\frac{1}{n}\sum_{i=0}^{n-1} x_{i+1} - s_n = \frac{x_n - x_0}{n} \to_{n\to\infty} 0$. Therefore, summing up the above inequality from $i = 0$ to $i = n-1$, dividing by n, replacing n by n_j, and letting $j \to +\infty$, we get $0 \geq |x_{k+1} - p|^2 - |x_k - p|^2$, which implies that $p \in F_1$. Now if $s_{m_l} \rightharpoonup_{l\to\infty} q$, then by the above proof, we also have $q \in F_1$. This implies that $\lim_{n\to\infty}(x_n, p - q) = \frac{1}{2}\lim_{n\to\infty}(|x_n - q|^2 - |x_n - p|^2) + \frac{1}{2}(|p|^2 - |q|^2)$ exists.

Hence $(p, p-q) = (q, p-q)$, which implies that $p = q$. This shows that $\{s_n\}$ converges weakly to some $p \in H$, as $n \to +\infty$, and proves our assertion. Finally, (iv) $\Rightarrow$ (i) is already shown in the above proof since $p \in F_1$. Now the proof is completed by using [4, Theorems 3.3 and 3.4] which show respectively that p is the asymptotic center of the sequence $\{x_n\}$ in H, and that $p = \lim_{n\to+\infty} P_{F_1} x_n$. ■

Remark 3.1. Example 3.5 in [4] shows that the sequence $P_F x_n$ may not converge in H.

Remark 3.2. The above proof shows that if $\alpha \geq \beta$, then (iii) can be replaced with the weaker condition $\liminf_{n\to\infty} |s_n| < +\infty$.

4. Weak convergence theorem

In this section, we prove a weak convergence theorem for generalized hybrid sequences in H, which is completely new, to the best of our knowledge.

Theorem 4.1. *Let* $(x_n)_{n\geq 0}$ *be a generalized hybrid sequence in* H. *Assume that* $\{x_n\}$ *is weakly asymptotically regular. Then the following are equivalent:*

(i) $F_1 \neq \emptyset$.
(ii) $F \neq \emptyset$.
(iii) $\{x_n\}$ *is bounded in* H.
(iv) x_n *converges weakly to some* $p \in H$, *as* $n \to +\infty$.

Moreover, in this case p *is the asymptotic center of the sequence* $\{x_n\}$ *in* H, *and* $p = \lim_{n\to+\infty} P_{F_1} x_n$.

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Let us show that (iii) $\Rightarrow$ (iv). It follows from (iii) that $\{x_n\}$ has a weakly convergent subsequence, say $x_{n_j} \rightharpoonup_{j\to\infty} p$. Let $m \geq 1$ be a fixed integer, and let $M = sup_{n\geq 0}|x_n|$. As in Theorem 3.1, writing the polarization identity in the following four ways with the following indices and multiplied by the corresponding coefficients: $l = n_j + i + 1$, $m = k+1$, multiplied by α; $l = n_j + i$, $m = k+1$, multiplied by $(1-\alpha)$; $l = n_j + i + 1$, $m = k$, multiplied by $-\beta$; and $l = n_j + i$, $m = k$, multiplied by $-(1-\beta)$, and then adding up the four identities obtained, after some manipulations, we get the following inequality:

$$\begin{aligned} &2\alpha(x_{n_j+i+1} - p, x_{k+1} - p) - 2\beta(x_{n_j+i+1} - p, x_k - p) + 2(1-\alpha)(x_{n_j+i} - p, x_{k+1} - p) - 2(1-\beta)(x_{n_j+i} - p, x_k - p) \\ &\quad \geq (\alpha - \beta)(|x_{n_j+i+1} - p|^2 - |x_{n_j+i} - p|^2) + |x_{k+1} - p|^2 - |x_k - p|^2. \end{aligned}$$

Now summing up the above inequality from $i = 0$ to $i = m-1$, dividing by m, letting $j \to +\infty$ and using the asymptotic regularity of $\{x_n\}$, we get

$$\begin{aligned} 0 &\geq (\alpha - \beta)\limsup_{j\to+\infty} \frac{1}{m}(|x_{n_j+m} - p|^2 - |x_{n_j} - p|^2) + |x_{k+1} - p|^2 - |x_k - p|^2 \\ &\geq \frac{-2|\alpha - \beta|(M + |p|)^2}{m} + |x_{k+1} - p|^2 - |x_k - p|^2. \end{aligned}$$

Letting $m \to +\infty$, we get $|x_{k+1} - p|^2 - |x_k - p|^2 \leq 0$, which implies that $p \in F_1$. Now a similar argument as in Theorem 3.1 shows that if $x_{n_j} \rightharpoonup_{j\to\infty} p$ and $x_{m_l} \rightharpoonup_{l\to\infty} q$, then $p = q$. Therefore $\{x_n\}$ converges weakly to some $p \in H$, as $n \to +\infty$, and by Theorem 3.1, p is the asymptotic center of the sequence $\{x_n\}$ in H, as well as $p = \lim_{n\to+\infty} P_{F_1} x_n$. The proof is now complete. ■

Remark 4.1. The above proof shows that if $\alpha \geq \beta$, then (iii) can be replaced with the weaker condition $\liminf_{n\to\infty} |x_n| < +\infty$.

5. Absolute fixed points

In this section, we establish the existence of absolute fixed points for generalized hybrid mappings in H, extending our results for nonexpansive maps in [5], as well as the corresponding results in [13–16,19,20]. We start with the following proposition.

Proposition 5.1. *Let D be a nonempty subset of H and let T be a generalized hybrid self-mapping of D. Assume that the sequence $x_n = T^n x$ is bounded for some $x \in D$ (i.e. T has a bounded orbit). Let c be the asymptotic center of the sequence $\{x_n\}$ in H. Then for every $y \in D$, the orbit $y_n = T^n y$ is bounded, and moreover the sequence $\{|y_n - c|\}$ is nonincreasing.*

Proof. We already know from Theorem 3.1 that $s_n = \frac{1}{n}\sum_{i=0}^{n-1} x_i \rightharpoonup_{n\to\infty} c$, and that $c \in F_1$. Let $k \geq 0$ be a fixed integer. We have

$$\begin{aligned}
|y_{k+1} - c|^2 &= \frac{\alpha}{n}\sum_{i=0}^{n-1} |y_{k+1} - x_{i+1} + x_{i+1} - c|^2 + \frac{(1-\alpha)}{n}\sum_{i=0}^{n-1} |y_{k+1} - x_i + x_i - c|^2 \\
&= \frac{1}{n}\sum_{i=0}^{n-1} (\alpha|y_{k+1} - x_{i+1}|^2 + (1-\alpha)|y_{k+1} - x_i|^2) + \frac{1}{n}\sum_{i=0}^{n-1} (\alpha|x_{i+1} - c|^2 + (1-\alpha)|x_i - c|^2) \\
&\quad + \frac{2\alpha}{n}\sum_{i=0}^{n-1} (y_{k+1} - x_{i+1}, x_{i+1} - c) + \frac{2(1-\alpha)}{n}\sum_{i=0}^{n-1} (y_{k+1} - x_i, x_i - c) \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1} (\beta|x_{i+1} - y_k|^2 + (1-\beta)|x_i - y_k|^2) + \frac{1}{n}\sum_{i=0}^{n-1} |x_i - c|^2 \\
&\quad + \frac{2\alpha}{n}\sum_{i=0}^{n-1} (y_{k+1} - x_{i+1}, x_{i+1} - c) + \frac{2(1-\alpha)}{n}\sum_{i=0}^{n-1} (y_{k+1} - x_i, x_i - c). \qquad (5.1)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} |x_{i+1} - y_k|^2 &= \frac{1}{n}\sum_{i=0}^{n-1} |x_i - y_k|^2 + \frac{(|x_n - y_k|^2 - |x_0 - y_k|^2)}{n} \\
&= \frac{1}{n}\sum_{i=0}^{n-1} |x_i - y_k|^2 + o(1)
\end{aligned}$$

where $o(1)$ is a function of n, tending to zero as $n \to +\infty$. Similarly, simple computations show that we have

$$\frac{1}{n}\sum_{i=0}^{n-1} (y_{k+1} - x_{i+1}, x_{i+1} - c) = \frac{1}{n}\sum_{i=0}^{n-1} (y_k - x_i, x_i - c) + o(1)$$

and

$$\frac{1}{n}\sum_{i=0}^{n-1} (y_{k+1} - x_i, x_i - c) = \frac{1}{n}\sum_{i=0}^{n-1} (y_k - x_i, x_i - c) + o(1).$$

Replacing the above estimates in inequality (5.1), we get

$$\begin{aligned}
|y_{k+1} - c|^2 &\leq \frac{1}{n}\sum_{i=0}^{n-1} |x_i - y_k|^2 + \frac{1}{n}\sum_{i=0}^{n-1} |x_i - c|^2 + \frac{2}{n}\sum_{i=0}^{n-1} (y_k - x_i, x_i - c) + o(1) \\
&= \frac{1}{n}\sum_{i=0}^{n-1} |(y_k - x_i) + (x_i - c)|^2 + o(1) = |y_k - c|^2 + o(1).
\end{aligned}$$

Letting $n \to +\infty$, we get $|y_{k+1} - c| \leq |y_k - c|$, $\forall k \geq 0$, as desired, which implies also that the sequence $\{y_n\}$ is bounded in H. The proof is now complete. ■

Theorem 5.2. *With the same notations and assumptions as in Proposition 5.1, assume that S is a generalized hybrid extension of T to a subset K of H containing $D \cup \{c\}$. Then we have $Sc = c$.*

Proof. By the polarization identity, we have

$$\begin{aligned}
&2\alpha(x_{i+1} - Sc, Sc - c) + 2(1-\alpha)(x_i - Sc, Sc - c)\\
&\quad = \alpha|x_{i+1} - c|^2 + (1-\alpha)|x_i - c|^2 - \alpha|x_{i+1} - Sc|^2 - (1-\alpha)|x_i - Sc|^2 - |Sc - c|^2\\
&\quad \geq \alpha|x_{i+1} - c|^2 + (1-\alpha)|x_i - c|^2 - \beta|x_{i+1} - c|^2 - (1-\beta)|x_i - c|^2 - |Sc - c|^2\\
&\quad = (\alpha - \beta)(|x_{i+1} - c|^2 - |x_i - c|^2) - |Sc - c|^2.
\end{aligned}$$

Summing up the above inequality from $i = 0$ to $i = n-1$, dividing by n, and letting $n \to +\infty$, we get

$$-2|Sc - c|^2 = 2(c - Sc, Sc - c) \geq -|Sc - c|^2$$

which implies that $|Sc - c|^2 \leq 0$, and hence $Sc = c$. ■

Corollary 5.3. *Let C be a nonempty, closed and convex subset of H, and T be a generalized hybrid self-mapping of C with a bounded orbit. Then $F(T) \neq \emptyset$.*

Proof. Assume the orbit of $x \in C$ is bounded, and let c be the asymptotic center of the sequence $x_n = T^n x$ in H. Since C is closed and convex, we know that $c \in C$, so that Tc is well defined and belongs to C. Now it follows from Theorem 5.2 that $Tc = c$. ■

In our next lemma, we give necessary and sufficient conditions for a generalized hybrid self-mapping of D with a bounded orbit, to have a generalized hybrid extension to $D \cup \{c\}$, where c is the asymptotic center of the bounded orbit.

Lemma 5.4. *Let D be a nonempty subset of H, and T be a generalized hybrid self-mapping of D with corresponding constants α and β, and with a bounded orbit $x_n = T^n x$, with $x \in D$. Let c be the asymptotic center of the sequence $\{x_n\}$ in H. Then the map $S : D \cup \{c\} \to D \cup \{c\}$ defined as $Sz = Tz$, $\forall z \in D$, and $Sc = c$ is a generalized hybrid self-mapping of $D \cup \{c\}$, if and only if either $\alpha \geq \beta$, or $\alpha < \beta$ and the orbit of every $z \in D$ lies on the sphere centered at z, with radius $|z - c|$.*

Proof. S is a generalized hybrid self-mapping of $D \cup \{c\}$ if and only if the following inequality holds:

$$\alpha|Tz - c|^2 + (1-\alpha)|z - c|^2 \leq \beta|Tz - c|^2 + (1-\beta)|z - c|^2, \quad \forall z \in D.$$

This is equivalent to $(\alpha - \beta)(|z-c|^2 - |Tz-c|^2) \geq 0$, $\forall z \in D$. Since from Proposition 5.1, we know that $|z-c|^2 - |Tz-c|^2 \geq 0$, $\forall z \in D$, then the above inequality holds if and only if either $\alpha \geq \beta$, or $\alpha < \beta$ and $|Tz - c| = |z - c|$, $\forall z \in D$. The latter condition is equivalent to $|T^n z - c| = |z - c|$, $\forall n \geq 0$, $\forall z \in D$, i.e. the orbit of every $z \in D$ lies on the sphere centered at z, with radius $|z - c|$. The proof is now complete. ■

We are now ready to state our main result on the existence of absolute fixed points for generalized hybrid mappings in H.

Theorem 5.5. *Let D be a nonempty subset of H, and T be a generalized hybrid self-mapping of D with corresponding constants α and β, and with a bounded orbit. Then the asymptotic center of this orbit in H is an absolute fixed point of T if and only if either $\alpha \geq \beta$, or $\alpha < \beta$ and the orbit of every $x \in D$ lies on the sphere centered at x, with radius $|x - c|$.*

Proof. This is an immediate consequence of Theorem 5.2 and Lemma 5.4. ■

6. Fixed point theorems

In this section, we prove some fixed point theorems for generalized hybrid mappings defined on nonconvex domains in H, that are new to the best of our knowledge, and extend the corresponding results in [13–16,19,20].

Theorem 6.1. *Let T be a generalized hybrid self-mapping of a nonempty subset D of H. Then T has a fixed point in D if and only if T has a bounded orbit $\{T^n x\}$, for some $x \in D$, and for any $y \in \operatorname{clco}\{T^n x;\ n \geq 0\}$, there is a unique $p \in D$ such that $|y - p| = \inf\{|y - z|;\ z \in D\}$. In this case, every orbit of T is bounded.*

Proof. Necessity is obvious; let us prove the sufficiency. Assume that $\{T^n x\}$ is bounded for some $x \in D$, and let c be the asymptotic center of $\{T^n x\}$ in H. Since $c \in \operatorname{clco}\{T^n x;\ n \geq 0\}$, there exists a unique $p \in D$ such that $|c - p| \leq |c - z|$, $\forall z \in D$. From Proposition 5.1, we know that for every $y \in D$, the sequence $\{|T^n y - c|\}$ is nonincreasing. Hence $\{T^n y\}$ is also bounded for every $y \in D$. In particular, the sequence $|T^n p - c|$ is nonincreasing. Hence we have $|c - p| = \inf\{|c - z|;\ z \in D\} \leq |c - Tp| \leq |c - p|$. Then the unicity of p implies that $Tp = p$ and completes the proof. ■

Corollary 6.2. *Let D be a nonempty subset of H which is Chebyshev with respect to its convex closure, and let T be a generalized hybrid self-mapping of D. Then T has a fixed point in D, if and only if T has a bounded orbit.*

Proof. This is a direct consequence of Theorem 6.1. ■

7. Examples of applications

Example 7.1. The following is a modification of [13, Example 2]. Let $T : H \to H$ be defined as follows: $Tx = \begin{cases} 0 & \text{if } |x| \leq 2 \\ \frac{x}{|x|} & \text{if } |x| > 2. \end{cases}$ Then T is not continuous, hence not nonexpansive. On the other hand, a simple computation shows that $2|Tx - Ty|^2 \leq |Tx - y|^2 + |x - Ty|^2$, $\forall x, y \in H$. This means that T is nonspreading, or equivalently, a generalized hybrid mapping with $\alpha = 2$ and $\beta = 1$. Then every orbit of T is a generalized hybrid sequence in H. In fact, in this simple example, every orbit is eventually constant and converges to zero. This simple example shows how our results on sequences could prove to be useful in drawing conclusions about the asymptotic behavior of maps that can have a more complicated structure than their orbits.

Example 7.2. Let $\gamma > 0$ and D be a nonempty subset of H. A map $A : D \to H$ is called a γ inverse strongly monotone mapping of D into H if $(Ax - Ay, x - y) \geq \gamma |Ax - Ay|^2$, $\forall x, y \in D$. It is easy to see that every 1-inverse strongly monotone operator is nonspreading, hence generalized hybrid with $\alpha = 2$ and $\beta = 1$. However, such a mapping is also nonexpansive, hence generalized hybrid with $\alpha = 1$ and $\beta = 0$. Also if T is nonexpansive, then $A = I - T$ is $\frac{1}{2}$ inverse strongly monotone, where I is the identity operator on H; However, it is not a generalized hybrid mapping. See [21] for more details and applications to variational inequalities. We are now going to provide examples of inverse strongly monotone operators that are not nonexpansive, but are generalized hybrid, and then see how our results in the previous sections on generalized hybrid sequences could be applied to the study of the existence and convergence theorems for solutions to variational inequalities associated with such operators defined on nonconvex domains. If A is a γ inverse strongly monotone operator of D into H, then by a well known identity in Hilbert space, we have

$$2(Ax - Ay, x - y) = |Ax - y|^2 + |Ay - x|^2 - |x - Ax|^2 - |y - Ay|^2 \geq 2\gamma |Ax - Ay|^2, \quad \forall x, y \in D.$$

Assume that A satisfies $(Ax, x) \leq \frac{|Ax|^2 - |x|^2}{2}$, $\forall x \in D$, which is in particular satisfied if $(Ax, x) \leq \frac{-|x|^2}{2}$, $\forall x \in D$. Then this implies that

$$\begin{aligned} 2\gamma |Ax - Ay|^2 - |x - Ay|^2 &\leq |Ax - y|^2 - |x - Ax|^2 - |y - Ay|^2 \\ &= |Ax - y|^2 - 2(|x|^2 + |y|^2) + (x, x + 2Ax) + (y, y + 2Ay) - |Ax|^2 - |Ay|^2 \\ &\leq |Ax - y|^2 - |x - y|^2, \quad \forall x, y \in D. \end{aligned}$$

Or equivalently

$$|Ax - Ay|^2 - \frac{1}{2\gamma}|x - Ay|^2 \leq \frac{1}{2\gamma}|Ax - y|^2 - \frac{1}{2\gamma}|x - y|^2, \quad \forall x, y \in D.$$

This implies that A is a generalized hybrid mapping with α and β determined by the following four constraints: $\alpha > 0$, $\frac{\alpha - 1}{\alpha} = \frac{1}{2\gamma}$, $\frac{\beta}{\alpha} = \frac{1}{2\gamma}$, $\frac{\beta - 1}{\alpha} \leq \frac{1}{2\gamma}$. From these constraints, we get $\alpha = \frac{2\gamma}{2\gamma - 1}$ with $\gamma > \frac{1}{2}$, and $\beta = \frac{\alpha}{2\gamma} = \frac{1}{2\gamma - 1}$ which satisfies the last constraint if $\gamma > \frac{1}{2}$. Therefore for A satisfying the above conditions, we get generalized hybrid mappings that are not nonexpansive for $\frac{1}{2} < \gamma < 1$. Then each of their orbits is a generalized hybrid sequence. For $\gamma = 1$, A is nonexpansive, and for $\gamma > 1$, it is even a contraction mapping. Now similar to the study done by Takahashi and Toyoda [21] for the convex case, by using our convergence and fixed point theorems in the previous sections (which use Proposition 5.1 and the results in Section 5), we are able to study the existence and convergence theorems for solutions to variational inequalities associated with generalized hybrid mappings exhibited above, on nonconvex domains, via their orbits that generate generalized hybrid sequences.

Example 7.3. As mentioned above, if T is nonexpansive, then $I - T$ is a $\frac{1}{2}$ inverse strongly monotone operator. Therefore, according to our study in Example 7.2, for $1 < r < 2$, the operator $A = r(I - T)$ is $\frac{r}{2}$ inverse strongly monotone, which is generalized hybrid (and not nonexpansive), if it satisfies the following additional condition: $r(x - Tx, x) \leq \frac{r^2|x - Tx|^2 - |x|^2}{2}$, $\forall x \in D$, or equivalently $\frac{r+1}{r}|x|^2 \leq |Tx|^2 + (r - 1)|x - Tx|^2$, $\forall x \in D$. For $r = 1$, we have $A = I - T$ which is not generalized hybrid; for $r = 2$, we have $\gamma = 1$, so that A is nonexpansive, and for $r > 2$, we have $\gamma > 1$, so that A is a contraction mapping. Then the results mentioned in Example 7.2 apply to this special case.

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