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Remarks on global regularity of 2D generalized MHD equations

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ABSTRACT

In this paper, we investigate the global regularity of 2D generalized MHD equations, in which the dissipation term and magnetic diffusion term are $\nu(-\Delta)^{\alpha}u$ and $\eta(-\Delta)^{\beta}b$ respectively. Let $(u_0, b_0) \in H^s$ with $s \geqslant 2$, it is showed that the smooth solution $(u(x,t), b(x,t))$ is globally regular for the case $0 \leqslant \alpha \leqslant \frac{1}{2}$, $\alpha + \beta > \frac{3}{2}$.

1. Introduction

In this paper, we consider the following 2D generalized magnetohydrodynamic (GMHD) equations

$$\begin{cases} u_t + \nu\Lambda^{2\alpha}u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + \eta\Lambda^{2\beta}b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{1.1}$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\nu \geqslant 0$ and $\eta \geqslant 0$ are real parameters, and u is the velocity of the flow, b is the magnetic field, p is the scalar pressure, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform by

$$\widehat{\Lambda f}(\xi) = |\xi|\widehat{f(\xi)}.$$

If $\alpha = \beta = 1$, (1.1) is the viscous MHD equations, and the global well-posedness of classical solution is well-known [7]. If $\nu = \eta = 0$, (1.1) is the inviscid magnetohydrodynamic equations.

We know that the 2D Euler equation is globally well-posed for smooth initial data. But for the 2D inviscid MHD equations, the global well-posedness of classical solution is still a big open problem. So the GMHD equations have attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements [9–12,15,17]. People pay attention to

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how the parameters ν, η, α, β influence the global regularity of the GMHD equations. It is well-known that the d-dimensional GMHD equations (1.1) with $\nu > 0$ and $\eta > 0$ have a unique global classical solution for every initial data $(u_0, b_0) \in H^s$ with $s \geqslant \max\{2\alpha, 2\beta\}$ if $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$ and $\beta \geqslant \frac{1}{2} + \frac{d}{4}$ [10]. An improved result by Wu [12] was established by reducing the requirement for α and β and the dissipation in (1.1) by a logarithmic factor. It is showed that the system is globally regular as long as the following conditions $\alpha \geqslant \frac{1}{2} + \frac{d}{4}$, $\beta > 0$, $\alpha + \beta \geqslant 1 + \frac{d}{2}$ are satisfied. As a special consequence, smooth solutions of the 2D GMHD equations with $\alpha \geqslant 1$, $\beta > 0$, $\alpha + \beta \geqslant 2$ are global.

However, for the 2D incompressible MHD equations with partial dissipation, the global regularity of the classical solutions is still a difficult problem. In 2011, Cao and Wu [2] showed an interesting result which considered the 2D MHD equations of the form

$$\begin{cases} u_t + u \cdot \nabla u = -p + \nu_1 u_{xx} + \nu_2 u_{yy} + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta_1 b_{xx} + \eta_2 b_{yy} + b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{1.2}$$

they stated that the classical solutions of Eqs. (1.2) with either $\nu_1 = 0$, $\nu_2 = \nu > 0$, $\eta_1 = \eta > 0$ and $\eta_2 = 0$ or $\nu_1 = \nu > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 = \eta > 0$ are globally existed for all time. If $\nu_1 = \nu_2 = 0$ and $\eta_1 = \eta_2 > 0$, the MHD equations (1.2) have a global H^1 weak solution [2,6]. But the existence of global classical solution is an open problem. When $\eta_1 = \eta_2 = 0$ and $\nu_1 = \nu_2 > 0$, it is also unknown for the existence of global classical solutions.

Recently, Tran, Yu and Zhai [9] obtained the global regularity of 2D GMHD equations (1.2) for the following three cases: (1) $\alpha \geqslant \frac{1}{2}$, $\beta \geqslant 1$; (2) $0 \leqslant \alpha < \frac{1}{2}$, $2\alpha + \beta > 2$; (3) $\alpha \geqslant 2$, $\beta = 0$. Combining them with the result of [12], we know that if $\alpha + \beta \geqslant 2$, (1.1) with $\nu > 0$ and $\eta > 0$ possesses a global smooth solution. Note that in this case, the end point $\alpha = 0$ ($\nu = 0$) and $\beta = 2$ is not included and it cannot ensure the global regularity for the system (1.1).

Motivated by Tran, Yu and Zhai [9], we carried on a thorough investigation on whether the smooth solutions are global in the case $\alpha = 0$ and $\beta = 2$ for 2D GMHD equations. In fact, the system (1.1) has a global classical solution for this case. What is more, we find that when $\alpha = 0$, the condition $\beta = \alpha + \beta \geqslant 2$ can be reduced to $\beta > \frac{3}{2}$. When $0 < \alpha \leqslant \frac{1}{2}$, we also conclude that the system is globally regular provided that α and β satisfy the relation $\alpha + \beta > \frac{3}{2}$.

The topic of the global regularity of 2D MHD equations with partial dissipation has attracted much attention of many excellent scholars. After submission of this paper the authors were informed of the preprint version of [3,13,14] where the results are closely related to the topic. Moreover, Ref. [3] gives a more precise result which is, in the case $\alpha = 0$, the system (1.1) is globally regular as long as $\beta > 1$.

To this end, we state our regularity criteria as follows.

Theorem 1.1. *Consider the GMHD equations (1.1) in 2D case. Assume $(u_0, b_0) \in H^s$ with $s \geqslant 2$. Then the system is globally regular for α and β satisfying $0 \leqslant \alpha \leqslant \frac{1}{2}$, $\alpha + \beta > \frac{3}{2}$.*

Remark 1.1. In the special case $\alpha = \frac{1}{2}$, $\beta = 1$, Ref. [9] showed that Eq. (1.1) is globally regular. However, the global regularity of (1.1) with $0 \leqslant \alpha \leqslant \frac{1}{2}$, $\alpha + \beta = \frac{3}{2}$ is still a difficult problem.

Remark 1.2. To simplify the presentation, we will set $\nu = \eta = 1$. It is a standard exercise to adjust various constants to accommodate other values of ν, η, as long as both are positive.

2. Proof of the main result

In this section, we shall prove Theorem 1.1. The key idea here is to apply the standard L^2-energy estimates to carry out the H^1, H^2 and higher estimates.

2.1. L^2- and H^1-energy estimates

We consider the 2D GMHD equations (1.1) with $\alpha \geqslant 0$ and $\beta \geqslant 1$. It is easy to get the standard L^2-energy estimate. Multiplying the first two equations of (1.1) by u and b, respectively, integrating and adding the resulting equations together it follows that

$$\|u\|_2^2 + \|b\|_2^2 + 2\int_0^t \left\|\Lambda^\alpha u\right\|_2^2 \mathrm{d}s + 2\int_0^t \left\|\Lambda^\beta b\right\|_2^2 \mathrm{d}s = \|u_0\|_2^2 + \|b_0\|_2^2, \tag{2.1}$$

where we have used the incompressibility condition $\nabla \cdot u = \nabla \cdot b = 0$.

As $\beta \geqslant 1$, we can easily get

$$b \in L^2\big(0,T; H^\beta\big(\mathbb{R}^2\big)\big) \quad \Rightarrow \quad \nabla b \in L^2\big(0,T; L^2\big(\mathbb{R}^2\big)\big).$$

Let $\omega = \nabla \times u = -\partial_2 u_1 + \partial_1 u_2$ be the vorticity and $j = \nabla \times b = -\partial_2 b_1 + \partial_1 b_2$ be the current density. Applying $\nabla\times$ to the first two equations of (1.1) we obtain the governing equations:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = b \cdot \nabla j - \Lambda^{2\alpha}\omega, \\ j_t + u \cdot \nabla j = b \cdot \nabla \omega + T(\nabla u, \nabla b) - \Lambda^{2\beta} j. \end{cases} \tag{2.2}$$

Here

$$T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2(\partial_1 b_2 + \partial_2 b_1).$$

Multiplying the two equations of (2.2) by ω and j, respectively, integrating and applying the incompressibility condition we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \big(\omega^2 + j^2\big)\,\mathrm{d}x + \int_{\mathbb{R}^2} \big(\Lambda^\alpha \omega\big)^2 \mathrm{d}x + \int_{\mathbb{R}^2} \big(\Lambda^\beta j\big)^2 \mathrm{d}x = \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \,\mathrm{d}x. \tag{2.3}$$

According to the Biot–Savart law, we have the representations

$$\frac{\partial u}{\partial x_k} = R_k(R \times \omega); \quad k = 1, 2,$$

and

$$\frac{\partial b}{\partial x_k} = R_k(R \times j); \quad k = 1, 2,$$

where $R = (R_1, R_2)$, $R_k = \partial x_k(-\Delta)^{-\frac{1}{2}}$ denotes Riesz transformation. For details about the Riesz transformation please refer to [8]. By the boundedness of Riesz operator R in L^p space $(1 < p < \infty)$, we arrive at

$$\|\nabla u\|_{L^2} \leqslant C\|\omega\|_{L^2} \quad \text{and} \quad \|\nabla b\|_{L^4} \leqslant C\|j\|_{L^4}.$$

Using Hölder and Young's inequalities one has

$$\begin{aligned} \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \,\mathrm{d}x &\leqslant C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}\|j\|_{L^4} \leqslant C\|\omega\|_{L^2}\|j\|_{L^2}^{2-\frac{1}{\beta}}\left\|\Lambda^\beta j\right\|_{L^2}^{\frac{1}{\beta}} \\ &\leqslant C(\varepsilon)\|\omega\|_{L^2}^{\frac{2\beta}{2\beta-1}}\|j\|_{L^2}^2 + \varepsilon\left\|\Lambda^\beta j\right\|_{L^2}^2 \leqslant C(\varepsilon)\big(\|\omega\|_{L^2}^2 + 1\big)\|j\|_{L^2}^2 + \varepsilon\left\|\Lambda^\beta j\right\|_{L^2}^2, \end{aligned}$$

where we have used the following Gagliardo–Nirenberg inequality

$$\|j\|_{L^4} \leqslant C\|j\|_{L^2}^{1-\frac{1}{2\beta}} \big\|\Lambda^\beta j\big\|_{L^2}^{\frac{1}{2\beta}}.$$

Inserting the above estimate into (2.3), and taking ε small enough so that $\varepsilon < 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\big) + \big\|\Lambda^\alpha \omega\big\|_{L^2}^2 + \big\|\Lambda^\beta j\big\|_{L^2}^2 \leqslant C(\varepsilon)\big(\|\omega\|_{L^2}^2 + 1\big)\|j\|_{L^2}^2.$$

Gronwall's inequality [5, Appendix B.j] and L^2-energy estimate imply that

$$\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \int_0^t \big\|\Lambda^\alpha \omega\big\|_{L^2}^2\,\mathrm{d}s + \int_0^t \big\|\Lambda^\beta j\big\|_{L^2}^2\,\mathrm{d}s \leqslant \big(\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2\big)\exp\Bigg[\int_0^t \|j\|_{L^2}^2\,\mathrm{d}s\Bigg] < \infty.$$

2.2. Higher estimates for $\alpha = 0$

In this case we have $\beta > \frac{3}{2}$, and the GMHD equations now read

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = b\cdot\nabla u - \Lambda^{2\beta} b, \\ \nabla\cdot u = \nabla\cdot b = 0. \end{cases} \tag{2.4}$$

First of all, we estimate b_t. Taking the inner product of the second equation of (2.4) with b_t and using Hölder and Young's inequalities we obtain

$$\begin{aligned} \|b_t\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\Lambda^\beta b\big\|_{L^2}^2 &\leqslant \int_{\mathbb{R}^2} |u\cdot\nabla b\cdot b_t|\,\mathrm{d}x + \int_{\mathbb{R}^2} |b\cdot\nabla u\cdot b_t|\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\|b_t\|_{L^2}^2 + \frac{1}{2}\big(\|u\|_{L^4}^2\|\nabla b\|_{L^4}^2 + \|\nabla u\|_{L^2}^2\|b\|_{L^\infty}^2\big). \end{aligned}$$

Application of the following Gagliardo–Nirenberg inequalities

$$\begin{aligned} \|f\|_{L^4} &\leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}, \\ \|f\|_{L^\infty} &\leqslant C\|f\|_{L^2}^{1-\frac{1}{\beta}}\big\|\Lambda^\beta f\big\|_{L^2}^{\frac{1}{\beta}}, \end{aligned}$$

yields that

$$\begin{aligned} \|b_t\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big\|\Lambda^\beta b\big\|_{L^2}^2 &\leqslant C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla b\|_{L^2}\|\nabla j\|_{L^2} + C\|\nabla u\|_{L^2}^2\|b\|_{L^2}^{1-\frac{1}{\beta}}\big\|\Lambda^\beta b\big\|_{L^2}^{\frac{1}{\beta}} \\ &\leqslant C\|\nabla j\|_{L^2} + C\big\|\Lambda^\beta b\big\|_{L^2}^{\frac{1}{\beta}}. \end{aligned}$$

By the results of the L^2-energy estimate and H^1 estimate, we deduce that

$$\big\|\Lambda^\beta b\big\|_{L^2}^2 + \int_0^t \|b_t\|_{L^2}^2\,\mathrm{d}s \leqslant \big\|\Lambda^\beta b_0\big\|_{L^2}^2 + C\int_0^t \big\|\Lambda^\beta b\big\|_{L^2}^{\frac{1}{\beta}}\,\mathrm{d}s + C\int_0^t \|\nabla j\|_{L^2}\,\mathrm{d}s < \infty. \tag{2.5}$$

Now we go back to the equation $b_t + u\cdot\nabla b = b\cdot\nabla u - \Lambda^{2\beta} b$, and using the similar way with the estimate of b_t we get

$$\big\|\Lambda^{2\beta}b\big\|_{L^2}^2 \leqslant \|b_t\|_{L^2}^2 + \|u\cdot\nabla b\|_{L^2}^2 + \|b\cdot\nabla u\|_{L^2}^2 \leqslant \|b_t\|_{L^2}^2 + C\|\nabla j\|_{L^2} + C\big\|\Lambda^\beta b\big\|_{L^2}^{\frac{1}{\beta}}.$$

Recall that $j = \nabla\times b$, one can deduce, thanks to (2.5), that

$$\begin{aligned}\int_0^t \|\nabla j\|_{\dot H^{2\beta-2}}^2 \,\mathrm{d}s &\leqslant \int_0^t \big\|\Lambda^{2\beta}b\big\|_{L^2}^2\,\mathrm{d}s \\ &\leqslant \int_0^t \|b_t\|_{L^2}^2\,\mathrm{d}s + C\int_0^t \|\nabla j\|_{L^2}\,\mathrm{d}s + C\int_0^t \big\|\Lambda^\beta b\big\|_{L^2}^{\frac{1}{\beta}}\,\mathrm{d}s < \infty. \end{aligned} \tag{2.6}$$

Since $\beta > \frac{3}{2}$, by Sobolev embedding theorem, it is easily to see

$$\nabla j \in L^2\big(0,T;H^{2\beta-2}\big(\mathbb{R}^2\big)\big) \hookrightarrow L^2\big(0,T;L^\infty\big(\mathbb{R}^2\big)\big).$$

Secondly, we estimate ω. From the first equation of (2.4), we have the vorticity equation $\omega_t + u\cdot\nabla\omega = b\cdot\nabla j$. Multiplying both sides of it by $p|\omega|^{p-2}\omega$ and integrating both sides over $\mathbb{R}^2$, it follows, by Hölder inequality, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p}^p + p\int_{\mathbb{R}^2} u\cdot\nabla\omega\cdot|\omega|^{p-2}\omega\,\mathrm{d}x \leqslant p\|b\cdot\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1} \leqslant p\|b\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1}.$$

Note that $p\int_{\mathbb{R}^2} u\cdot\nabla\omega\cdot|\omega|^{p-2}\omega\,\mathrm{d}x = 0$. Now let $p\to\infty$, we infer that

$$\|\omega\|_{L^\infty} \leqslant \|\omega_0\|_{L^\infty} + \int_0^t \|b\|_{L^\infty}\|\nabla j\|_{L^\infty}\,\mathrm{d}s < \infty.$$

This leads to

$$\omega \in L^\infty\big(0,T;L^\infty\big(\mathbb{R}^2\big)\big).$$

Lastly, according to the classical BKM-type blow-up criterion [1] which is the MHD system stays regular beyond T provided that $\int_0^T (\|\omega\|_{L^\infty} + \|j\|_{L^\infty})\,\mathrm{d}t < \infty$, the proof of the case $\alpha = 0$ is thus completed.

2.3. Higher estimates for $0 < \alpha \leqslant \frac{1}{2}$, $\alpha + \beta > \frac{3}{2}$

In this case, we can easily get $\beta > 1$. Firstly, we estimate $\|\omega\|_{L^p}$. Multiplying both sides of the first equation of (2.2) by $p|\omega|^{p-2}\omega$ and integrating both sides over $\mathbb{R}^2$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p}^p + p\int_{\mathbb{R}^2} \Lambda^{2\alpha}\omega\cdot|\omega|^{p-2}\omega\,\mathrm{d}x \leqslant p\|b\cdot\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1}.$$

For the dissipation term, we know by the property of Riesz potential that $\int_{\mathbb{R}^2}\Lambda^{2\alpha}\omega\cdot|\omega|^{p-2}\omega\,\mathrm{d}x \geqslant 0$. For the details on it see [4]. Thus, we have

$$\|\omega\|_{L^p} \leqslant \|\omega_0\|_{L^p} + \int_0^t \|b\cdot\nabla j\|_{L^p}\,\mathrm{d}s. \tag{2.7}$$

By the Gagliardo–Nirenberg inequality, one has the following estimate

$$\|b\cdot\nabla j\|_{L^p}\leqslant C\|b\|_{L^\infty}\|\nabla j\|_{L^p}\leqslant C\|b\|_{L^2}^{\frac{\beta}{1+\beta}}\big\|\Lambda^\beta j\big\|_{L^2}^{\frac{1}{1+\beta}}\|j\|_{L^2}^{\frac{2\beta-3}{2\beta-1}+\frac{2}{(2\beta-1)p}}\big\|\Lambda^{2\beta-1}j\big\|_{L^2}^{\frac{2(p-1)}{(2\beta-1)p}}, \tag{2.8}$$

where p satisfies $p>\frac{1}{\alpha}$. So, inserting (2.8) into (2.7) and applying with L^2 and H^1 estimates and (2.6), it can be derived that

$$\begin{aligned}\|\omega\|_{L^p}&\leqslant\|\omega_0\|_{L^p}+C\int_0^t\|b\|_{L^2}^{\frac{\beta}{1+\beta}}\big\|\Lambda^\beta j\big\|_{L^2}^{\frac{1}{1+\beta}}\|j\|_{L^2}^{\frac{2\beta-3}{2\beta-1}+\frac{2}{(2\beta-1)p}}\big\|\Lambda^{2\beta-1}j\big\|_{L^2}^{\frac{2(p-1)}{(2\beta-1)p}}\,\mathrm{d}s\\&\leqslant\|\omega_0\|_{L^p}+C\int_0^t\big\|\Lambda^\beta j\big\|_{L^2}^{\frac{1}{1+\beta}}\big\|\Lambda^{2\beta-1}j\big\|_{L^2}^{\frac{2(p-1)}{(2\beta-1)p}}\,\mathrm{d}s\\&\leqslant\|\omega_0\|_{L^p}+C\int_0^t\Big(\big\|\Lambda^\beta j\big\|_{L^2}^2+\big\|\Lambda^{2\beta}b\big\|_{L^2}^{\frac{4(p-1)(1+\beta)}{(2\beta-1)(2\beta+1)p}}\Big)\,\mathrm{d}s<\infty.\end{aligned}$$

Note that as long as $p>\frac{1}{\alpha}$, we have $\frac{4(p-1)(1+\beta)}{(2\beta-1)(2\beta+1)p}\leqslant 2$.

Secondly, we derive the estimates of $\|\omega\|_{H^1}$ and $\|j\|_{H^1}$. We differentiate Eqs. (2.2) with respect to x_i over $\mathbb{R}^2$, then multiply the resulting equations by $\partial_{x_i}\omega$ and $\partial_{x_i}j$ for $i=1,2$, integrate with respect to x over $\mathbb{R}^2$ and sum them up. It follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla\omega\|_{L^2}^2+\|\nabla j\|_{L^2}^2\big)+\big\|\Lambda^\alpha\nabla\omega\big\|_{L^2}^2+\big\|\Lambda^\beta\nabla j\big\|_{L^2}^2\\&\quad\leqslant\int|\nabla u||\nabla\omega|^2\,\mathrm{d}x+\int|\nabla b||\nabla j||\nabla\omega|\,\mathrm{d}x+\int|\nabla u||\nabla j|^2\,\mathrm{d}x\\&\qquad+\int|\nabla b||\nabla\omega||\nabla j|\,\mathrm{d}x+\int\big|\nabla^2u\big||\nabla b||\nabla j|\,\mathrm{d}x+\int|\nabla u|\big|\nabla^2b\big||\nabla j|\,\mathrm{d}x\\&\quad=I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned}\tag{2.9}$$

It is easy to see that the estimates of I_4 and I_5 are the same as I_2 while I_6 is the same as I_3. Therefore, it suffices to estimate I_1, I_2, I_3.

Hölder, Young and Gagliardo–Nirenberg inequalities together give

$$I_1\leqslant\|\nabla u\|_{L^p}\|\nabla\omega\|_{L^{2q}}^2\leqslant C\|\omega\|_{L^p}\|\nabla\omega\|_{L^2}^{\frac{2(\alpha p-1)}{\alpha p}}\big\|\Lambda^\alpha\nabla\omega\big\|_{L^2}^{\frac{2}{\alpha p}}\leqslant C(\varepsilon)\|\nabla\omega\|_{L^2}^2+\varepsilon\big\|\Lambda^\alpha\nabla\omega\big\|_{L^2}^2,$$

where p and q satisfy $\frac{1}{p}+\frac{1}{q}=1$ and $p>\frac{1}{\alpha}$.

Arguing similarly as the estimate of I_1, thanks to the L^2 and H^1 estimates, one has

$$\begin{aligned}I_2&\leqslant\|\nabla b\|_{L^\infty}\|\nabla j\|_{L^2}\|\nabla\omega\|_{L^2}\leqslant C\|\nabla b\|_{L^2}^{1-\frac{1}{\beta}}\big\|\Lambda^\beta\nabla b\big\|_{L^2}^{\frac{1}{\beta}}\big(\|\nabla\omega\|_{L^2}^2+\|\nabla j\|_{L^2}^2\big)\\&\leqslant C\big\|\Lambda^\beta\nabla b\big\|_{L^2}^{\frac{1}{\beta}}\big(\|\nabla\omega\|_{L^2}^2+\|\nabla j\|_{L^2}^2\big)\end{aligned}$$

where use has been made of the following Gagliardo–Nirenberg inequality

$$\|\nabla b\|_{L^\infty}\leqslant C\|\nabla b\|_{L^2}^{1-\frac{1}{\beta}}\big\|\Lambda^\beta\nabla b\big\|_{L^2}^{\frac{1}{\beta}}.$$

The estimate of I_3 can also be obtained by Hölder, Young and Sobolev embedding inequalities

$$I_3 \leqslant \|\nabla u\|_{L^2}\|\nabla j\|_{L^4}^2 \leqslant C\|\nabla u\|_{L^2}\|\nabla j\|_{L^2}^{\frac{2\beta-1}{\beta}}\|\Lambda^\beta \nabla j\|_{L^2}^{\frac{1}{\beta}} \leqslant C(\varepsilon)\|\nabla j\|_{L^2}^2 + \varepsilon\|\Lambda^\beta \nabla j\|_{L^2}^2.$$

Combining the above estimates into (2.9), and taking ε small enough we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right) + \|\Lambda^\alpha \nabla\omega\|_{L^2}^2 + \|\Lambda^\beta \nabla j\|_{L^2}^2 \leqslant C(\varepsilon)\|\Lambda^\beta \nabla b\|_{L^2}^{\frac{1}{\beta}}\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right).$$

Gronwall's inequality and H^1 estimate imply that

$$\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \nabla\omega\|_{L^2}^2\,\mathrm{d}s + \int_0^t \|\Lambda^\beta \nabla j\|_{L^2}^2\,\mathrm{d}s \leqslant C\left(\|\nabla\omega_0\|_{L^2}^2 + \|\nabla j_0\|_{L^2}^2\right).$$

Thus, we arrive at

$$\omega \in L^\infty\left(0,T;H^1\left(\mathbb{R}^2\right)\right) \cap L^2\left(0,T;H^{\alpha+1}\left(\mathbb{R}^2\right)\right),$$
$$j \in L^\infty\left(0,T;H^1\left(\mathbb{R}^2\right)\right) \cap L^2\left(0,T;H^{\beta+1}\left(\mathbb{R}^2\right)\right).$$

In the end, by the embedding relation $H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for $s > 1$, we can get $\omega \in L^2(0,T;L^\infty(\mathbb{R}^2)), j \in L^2(0,T;L^\infty(\mathbb{R}^2))$, and combining the BKM-type blow-up criterion [1], this completes the proof. Obviously, the fact $H^1(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}(\mathbb{R}^2)$ and the blow-up criterion [16] can also give the proof. □

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