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Unitary operators in the orthogonal complement of a type I von Neumann subalgebra in a type II_1 factor ☆

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A B S T R A C T

It is well-known that the equality

$$L_G \ominus L_H = \overline{\operatorname{span}\{L_g\colon\ g \in G - H\}}^{\mathrm{SOT}}$$

holds for G an i.c.c. group and H a subgroup in G, where L_G and L_H are the corresponding group von Neumann algebras and $L_G \ominus L_H$ is the set $\{x \in L_G\colon E_{L_H}(x) = 0\}$ with E_{L_H} the conditional expectation defined from L_G onto L_H. Inspired by this, it is natural to ask whether the equality

$$N \ominus A = \overline{\operatorname{span}\{u\colon\ u \text{ is unitary in } N \ominus A\}}^{\mathrm{SOT}}$$

holds for N a type II_1 factor and A a von Neumann subalgebra of N. In this paper, we give an affirmative answer to this question for the case A a type I von Neumann algebra.

1. Introduction

Throughout this paper, all Hilbert spaces discussed are *complex and separable*. Let (N, τ) be a finite von Neumann algebra with a faithful normal normalized trace τ and A be a von Neumann subalgebra of N. Then the trace τ induces an inner product $\langle\cdot,\cdot\rangle$ on N which is defined by $\langle x, y\rangle = \tau(y^*x)$, $\forall x, y \in N$. Denote by $L^2(N)$ (resp. $L^2(A)$) the completion of N (resp. A) with respect to the inner product, then $L^2(A)$ is a subspace of $L^2(N)$. Let e_A denote the projection from $L^2(N)$ onto $L^2(A)$. The trace-preserving conditional expectation E_A of N onto A is defined to be the restriction $e_A|_N$. By [2], E_A has the following properties:

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(1) $e_A|_N = E_A$ is a norm reducing map from N onto A with $E_A(1) = 1$;
(2) the equality $E_A(axb) = aE_A(x)b$ holds for all $x \in N$ and $a, b \in A$;
(3) $\tau(xE_A(y)) = \tau(E_A(x)E_A(y)) = \tau(E_A(x)y)$ for all $x, y \in N$;
(4) $e_A x e_A = E_A(x)e_A = e_A E_A(x)$ for all $x \in N$.

Let G be a (countable) discrete i.c.c. group and denote by $l^2(G)$ the Hilbert space of square-summable sequences. Given every g in G, the operator L_g is defined by $(L_g x)(h) = x(g^{-1}h)$, for every x in $l^2(G)$ and h in G. This is a unitary operator. Let L_G be the von Neumann algebra generated by $\{L_g\colon\ g \in G\}$. It is well-known that L_G is a type II_1 factor. For a subgroup H in G, define

$$L_G \ominus L_H \triangleq \bigl\{x \in L_G\colon\ E_{L_H}(x) = 0\bigr\}.$$

Thus we obtain that

$$L_G \ominus L_H = \overline{\operatorname{span}\{L_g\colon\ g \in G - H\}}^{\mathrm{SOT}}.$$

Inspired by this, it is natural to ask whether the equality

$$N \ominus A = \overline{\operatorname{span}\{u\colon\ u \text{ is unitary in } N \ominus A\}}^{\mathrm{SOT}}$$

holds for N a type II_1 factor and A a von Neumann subalgebra of N. In this paper, we give an affirmative answer to this question for the case A a type I von Neumann algebra in Theorem 2.6.

In [1], A. Ioana, J. Peterson and S. Popa proved a series of rigidity results for amalgamated free product II_1 factors, which can be viewed as von Neumann algebra versions of the "subgroup theorems" and "isomorphism theorems" for amalgamated free product groups in Bass–Serre theory. They introduced the concept "bounded homogeneous orthonormal basis" of M over B, where (M, τ) is a separable finite von Neumann algebra and $B \subset M$ is a von Neumann subalgebra. In the current paper, our result indicates that it is possible to choose unitary elements to form a bounded homogeneous orthonormal basis with respect to a type I von Neumann subalgebra of M.

2. Proofs

In this paper, the matrix representations of operators will be used frequently. We briefly recall the relationship between conditional expectations with respect to matrix representations of operators. Let $e_1, \ldots, e_n \in N$ be mutually equivalent orthogonal projections such that $\sum_{i=1}^n e_i = 1$, where 1 is the identity of N. Then for every $x \in N$, we can express x in the form

$$x = \begin{array}{cc} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} & \begin{array}{c} \operatorname{ran} e_1 \\ \vdots \\ \operatorname{ran} e_n \end{array} \end{array}$$

and there exists a $*$-isomorphism φ from N onto $\mathbb{M}_n(N_{e_1})$, where N_{e_1} is the restriction of $e_1 N e_1$ on $\operatorname{ran} e_1$ and denote by $\mathbb{M}_n(N_{e_1})$ the set n-by-n matrices with entries in N_{e_1}. On the other hand, let τ be a faithful normal normalized trace on N, and the trace τ_n on $\mathbb{M}_n(N)$ is defined by $\tau_n(x) = \frac{1}{n}(\sum_{i=1}^n \tau(x_{ii}))$, where x in $\mathbb{M}_n(N)$ is of the form

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

and x_{ij} is in N for $i, j = 1, \dots, n$. We observe that τ_n is a faithful normal normalized trace. For a von Neumann subalgebra A in N, there exist conditional expectations E_A from N onto A and $E_{\mathbb{M}_n(A)}$ from $\mathbb{M}_n(N)$ onto $\mathbb{M}_n(A)$. Given fixed i_0 and j_0, let $x_{i_0j_0}$ denote again the operator in $\mathbb{M}_n(N)$ with all entries 0 but the (i_0, j_0) entry $x_{i_0j_0}$. By the fact that $E_{\mathbb{M}_n(A)}$ is $\mathbb{M}_n(A)$-modular, the equality

$$E_{\mathbb{M}_n(A)}(x_{i_0j_0}) = E_{\mathbb{M}_n(A)}(e_{i_0}x_{i_0j_0}e_{j_0}) = e_{i_0}E_{\mathbb{M}_n(A)}(x_{i_0j_0})e_{j_0}$$

ensures that all but the (i_0, j_0) entry of $E_{\mathbb{M}_n(A)}(x_{i_0j_0})$ are 0, where e_i is the diagonal projection with all entries 0 except the (i, i) one being the identity of N. Therefore $x_{i_0j_0}$ is in $\mathbb{M}_n(N) \ominus \mathbb{M}_n(A)$ if and only if the (i_0, j_0) entry of $x_{i_0j_0}$ is in $N \ominus A$.

In what follows, N will always denote a von Neumann algebra. Every subalgebra of N we consider here is self-adjoint, weakly closed and contains the unit 1 of N. For a subset $\mathscr{S} \subseteq N$, denote by $\mathscr{U}(\mathscr{S})$ the unitary operators in $\mathscr{S}$.

Lemma 2.1. *Let N be a von Neumann algebra and*

$$M_n = \{x\colon\ x \in \mathbb{M}_n(N),\ x_{ii} = 0,\ i = 1, \dots, n\},$$

then $M_n = \operatorname{span}\{u\colon\ u \in \mathscr{U}(M_n)\}$.

Proof. For each x in M_n, we can write x in the form

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}_{n\times n},$$

where x_{ij} is in N and $x_{ii} = 0$, for $i, j = 1, \dots, n$. Without loss of generality, we may assume $x_{i_0j_0} \neq 0$, $i_0 < j_0$ and all other entries 0. Thus we can write x in the form of block matrix

$$x = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where X_{12} is a k-by-k matrix for some $k \leqslant n - 1$ with $x_{i_0j_0}$ on the main diagonal of X_{12}.

Note that $x_{i_0j_0} = \sum_{i=1}^4 \lambda_i u_i$ for some $u_i \in \mathscr{U}(N)$ and $\lambda_i \in \mathbb{C}$. For the sake of simplicity, we can write X_{12} in the form

$$x_{i_0j_0} \oplus 0^{(k-1)} = \sum_{i=1}^4 \frac{\lambda_i}{2}\bigl(u_i \oplus v^{(k-1)} + u_i \oplus (-v)^{(k-1)}\bigr),$$

where v is unitary in N. Write $\widetilde{u}_i = u_i \oplus v^{(k-1)}$ and $\widehat{u}_i = u_i \oplus (-v)^{(k-1)}$, then x can be written in the form

$$x = \sum_{i=1}^4 \frac{\lambda_i}{2}\left(\begin{pmatrix} 0 & \widetilde{u}_i \\ I_{X_{21}} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \widehat{u}_i \\ -I_{X_{21}} & 0 \end{pmatrix}\right),$$

where $I_{X_{21}}$ is the identity of $\mathbb{M}_{n-k}(N)$.

By a similar method, every x in M_n can be written as a linear combination of finitely many unitary operators. Thus we finish the proof. $\square$

Lemma 2.2. *If N is a von Neumann algebra, $A \subseteq N$ is a von Neumann subalgebra and $N \ominus A = \overline{\mathrm{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}}^{\mathrm{SOT}}$, then*

$$\mathbb{M}_n(N \ominus A) = \overline{\mathrm{span}\bigl\{u\colon\ u \in \mathscr{U}\bigl(\mathbb{M}_n(N \ominus A)\bigr)\bigr\}}^{\mathrm{SOT}}.$$

Proof. For each $x \in N \ominus A$, there exists a sequence $\{x_n\}_{n\in\Lambda}$ in $N \ominus A$, such that $x_n \xrightarrow{\mathrm{SOT}} x$, $x_n = \sum_{i=1}^{k_n} \lambda_{n_i} u_{n_i}$, $u_{n_i} \in \mathscr{U}(N \ominus A)$, $\lambda_{n_i} \in \mathbb{C}$. Without loss of generality and for the sake of simplicity, we may assume

$$\widetilde{x} = \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \widetilde{x}_n = \begin{pmatrix} x_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that x can be moved to the (i,j) entry by multiplying u_i on the left and u_j on the right, where u_i (resp. u_j) is the elementary matrix obtained by swapping row 1 (resp. column 1) and row i (resp. column j) of the identity matrix for $1 \leqslant i, j \leqslant n$.

Then the result $\widetilde{x} \in \overline{\mathrm{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}}^{\mathrm{SOT}}$ follows from the two relations

$$\widetilde{x}_n \xrightarrow{\mathrm{SOT}} \widetilde{x},$$
$$\widetilde{x}_n = \sum_{i=1}^{k_n} \frac{\lambda_{n_i}}{2} \left(\begin{pmatrix} u_{n_i} & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix} + \begin{pmatrix} u_{n_i} & & & \\ & -v & & \\ & & \ddots & \\ & & & -v \end{pmatrix} \right),$$

where v is a unitary operator in $\mathscr{U}(N \ominus A)$. $\square$

Lemma 2.3. *If N is a type II_1 factor with a faithful normal normalized trace τ and $A \subseteq N$ is a diffuse abelian von Neumann subalgebra, then*

$$N \ominus A = \mathrm{span}\bigl\{u\colon\ u \in \mathscr{U}(N \ominus A)\bigr\}.$$

Proof. Since N is a type II_1 factor, there exist four equivalent mutually orthogonal projections $\{e_i\}_{1\leqslant i\leqslant 4} \subseteq A$, such that $\sum_{i=1}^4 e_i = 1$. Denote by M the reduced von Neumann algebra $e_1 N e_1$. Then there exists a $*$-isomorphism φ from N onto $\mathbb{M}_4(M)$ so that $\varphi(A) = \bigoplus_{i=1}^4 A_i$, where A_i is a diffuse abelian von Neumann subalgebra in M. For the sake of simplicity, we assume $N = \mathbb{M}_4(M)$, $A = \bigoplus_{i=1}^4 A_i$.

Denote $M_4 = \{(x_{ij})_{1\leqslant i,j\leqslant 4} \in N\colon\ x_{ii} = 0 \text{ for } 1 \leqslant i \leqslant 4\}$, then

$$M_4 = \mathrm{span}\bigl\{u\colon\ u \in \mathscr{U}(M_4)\bigr\}$$

following from Lemma 2.1. Thus we only need to prove

$$\bigoplus_{i=1}^4 M \ominus A_i \subseteq \mathrm{span}\bigl\{u\colon\ u \in \mathscr{U}(N \ominus A)\bigr\}, \tag{2.1}$$

since $M_4 \subseteq N \ominus A$.

Consider the matrix

$$\widetilde{x} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $x_1 = x_1^* \in M \ominus A_1$, $x_2 = x_2^* \in M \ominus A_2$, $\|x_1\| < 1$, $\|x_2\| < 1$. Since for each $y \in N$, we have $y = \lambda_1 y_1 + \lambda_2 y_2$, where $y_i = y_i^*$, $\lambda_i \in \mathbb{C}$, $\|y_i\| < 1$, for $1 \leqslant i \leqslant 2$. Let

$$\widetilde{u}_1 = \begin{pmatrix} x_1 & 0 & \sqrt{1-x_1^2} & 0 \\ 0 & x_2 & 0 & \sqrt{1-x_2^2} \\ 0 & -\sqrt{1-x_2^2} & 0 & x_2 \\ -\sqrt{1-x_1^2} & 0 & x_1 & 0 \end{pmatrix},$$

$$\widetilde{u}_2 = \begin{pmatrix} x_1 & 0 & \sqrt{1-x_1^2} & 0 \\ 0 & x_2 & 0 & \sqrt{1-x_2^2} \\ 0 & \sqrt{1-x_2^2} & 0 & -x_2 \\ \sqrt{1-x_1^2} & 0 & -x_1 & 0 \end{pmatrix},$$

$$\widetilde{u}_3 = \begin{pmatrix} 0 & 0 & \sqrt{1-x_1^2} & 0 \\ 0 & 0 & 0 & \sqrt{1-x_2^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then we obtain that

$$\widetilde{x} = \frac{1}{2}\widetilde{u}_1 + \frac{1}{2}\widetilde{u}_2 - \widetilde{u}_3. \tag{2.2}$$

Notice that $\widetilde{u}_1$, $\widetilde{u}_2$ are unitary operators in $N \ominus A$ and $\widetilde{u}_3$ belongs to M_4, then (2.2) and Lemma 2.1 allow us to conclude that

$$\widetilde{x} \in \operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}.$$

Similarly, we can also show that

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}$$

is a linear combination of finitely many unitary operators in $N \ominus A$, where $x_3 \in M \ominus A_3$, $x_4 \in M \ominus A_4$, thus we finish the proof of (2.1). $\square$

Lemma 2.4. *If N is a type* II_1 *factor with a faithful normal normalized trace τ and $A \subseteq N$ is an atomic abelian von Neumann subalgebra, then*

$$N \ominus A = \overline{\operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}}^{\mathrm{SOT}}.$$

Proof. We now consider four cases respectively.

(i) $A = \mathbb{C}1$.

Since N is a type II_1 factor, there exist two equivalent mutually orthogonal projections p and q in N such that $p + q = 1$. Denote by M the reduced von Neumann algebra pNp with a faithful normal normalized trace τ_M. Then there exists a $*$-isomorphism φ from N onto $\mathbb{M}_2(M)$ so that $\varphi(A) = \mathbb{C}p^{(2)}$. If we write $\widetilde{N} = \mathbb{M}_2(M)$, $\widetilde{A} = \varphi(A)$, then we obtain

$$\widetilde{N} \ominus \widetilde{A} = \bigl\{\widetilde{x} \in \widetilde{N}\colon\ \tau_M(x_{11} + x_{22}) = 0,\ \widetilde{x} = (x_{ij})_{1 \leqslant i,j \leqslant 2}\bigr\}.$$

Note that

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + x_{22} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -x_{22} & 0 \\ 0 & x_{22} \end{pmatrix} + \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}. \tag{2.3}$$

For each $x_{22} \in M$, $x_{22} = \sum_{i=1}^{4} \lambda_i u_i$, where $u_1, \dots, u_4$ are unitary operators in M so that

$$\begin{pmatrix} -x_{22} & 0 \\ 0 & x_{22} \end{pmatrix} = \sum_{i=1}^{4} \lambda_i \begin{pmatrix} -u_i & 0 \\ 0 & u_i \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} -u_i & 0 \\ 0 & u_i \end{smallmatrix}\right) \in \widetilde{N} \ominus \widetilde{A}$, we have

$$\begin{pmatrix} -x_{22} & 0 \\ 0 & x_{22} \end{pmatrix} \in \operatorname{span}\bigl\{u\colon\ u \in \mathscr{U}(\widetilde{N} \ominus \widetilde{A})\bigr\}. \tag{2.4}$$

Denote

$$M_2 = \left\{\begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}:\ x_{12}, x_{21} \in M\right\}.$$

Since $M_2 \subseteq \widetilde{N} \ominus \widetilde{A}$, by Lemma 2.1 we obtain

$$M_2 \subseteq \operatorname{span}\bigl\{u\colon\ u \in \mathscr{U}(\widetilde{N} \ominus \widetilde{A})\bigr\}. \tag{2.5}$$

For $x \in M$ and $\tau_M(x) = 0$, we may assume $x = x^*$, $\|x\| < 1$, since

$$x = \frac{x + x^*}{2} + i\frac{x - x^*}{2i} \quad \text{and} \quad \tau_M\left(\frac{x + x^*}{2}\right) = \tau_M\left(\frac{x - x^*}{2i}\right) = 0.$$

Let

$$\widetilde{x} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

$$\widetilde{u}_1 = \begin{pmatrix} x & \sqrt{1 - x^2} \\ -\sqrt{1 - x^2} & x \end{pmatrix}, \qquad \widetilde{u}_2 = \begin{pmatrix} x & \sqrt{1 - x^2} \\ \sqrt{1 - x^2} & -x \end{pmatrix}, \qquad \widetilde{u}_3 = \begin{pmatrix} 0 & \sqrt{1 - x^2} \\ 0 & 0 \end{pmatrix},$$

then we have

$$\widetilde{x} = \frac{1}{2}\widetilde{u}_1 + \frac{1}{2}\widetilde{u}_2 - \widetilde{u}_3. \tag{2.6}$$

Observe that $\widetilde{u}_1$, $\widetilde{u}_2$ are both unitary in $\widetilde{N} \ominus \widetilde{A}$ and $\widetilde{u}_3 \in M_2$.

Hence $\widetilde{N} \ominus \widetilde{A} = \operatorname{span}\{u\colon\ u \in \mathscr{U}(\widetilde{N} \ominus \widetilde{A})\}$ follows from (2.3), (2.4), (2.5) and (2.6).

(ii) $A = \mathbb{C}p + \mathbb{C}q$, where p and q are mutually orthogonal projections in N with sum 1.
Each $x \in N \ominus A$ can be written as

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with respect to the decomposition $1 = p+q$, where $x_{11} \in p(N\ominus A)p$, $x_{12} \in pNq$, $x_{21} \in qNp$, $x_{22} \in q(N\ominus A)q$. By (i), we obtain that

$$pNp \ominus \mathbb{C}p = \operatorname{span}\{u\colon\ u \in \mathscr{U}(pNp \ominus \mathbb{C}p)\},$$
$$qNq \ominus \mathbb{C}q = \operatorname{span}\{u\colon\ u \in \mathscr{U}(qNq \ominus \mathbb{C}q)\},$$

therefore

$$\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} \in \operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}.$$

We only need to prove

$$\begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix} \in \operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}. \tag{2.7}$$

If $\tau(p)$ is rational, then we assume

$$\frac{\tau(p)}{\tau(q)} = \frac{m}{n}, \quad \text{for some } m, n \in \mathbb{N}^+.$$

Let $\{p_i\}_{1\leqslant i\leqslant m}$ and $\{q_j\}_{1\leqslant j\leqslant n}$ be two families of mutually orthogonal projections in N such that $p_1 + p_2 + \cdots + p_m = p$, $q_1 + q_2 + \cdots + q_n = q$ and $\tau(p_i) = \tau(q_j)$, for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Denote $M = p_1Np_1$ with a faithful normal normalized trace τ_M, then there exists a $*$-isomorphism φ from N onto $\mathbb{M}_{m+n}(M)$ so that $\varphi(A) = \mathbb{C}p_1^{(m)} \oplus \mathbb{C}p_1^{(n)}$. Denote $\widetilde{N} = \varphi(N)$, $\widetilde{A} = \varphi(A)$,

$$N_0 = \left\{\begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}\colon\ x_{12} \in pNq,\ x_{21} \in qNp\right\},$$
$$M_{m+n} = \{\widetilde{x} \in \mathbb{M}_{m+n}(M)\colon\ x_{ii} = 0,\ \text{for } 1 \leqslant i \leqslant m+n\}.$$

Note that

$$\widetilde{N} \ominus \widetilde{A} = \left\{(x_{ij})_{1\leqslant i,j\leqslant m+n} \in \widetilde{N}\colon \sum_{i=1}^{m} \tau_M(x_{ii}) = 0,\ \sum_{i=m+1}^{m+n} \tau_M(x_{ii}) = 0\right\},$$

then $\varphi(N_0) \subseteq M_{m+n} \subseteq \widetilde{N} \ominus \widetilde{A}$. Then applying Lemma 2.1 to M_{m+n}, we obtain that

$$M_{m+n} = \operatorname{span}\{u\colon\ u \in \mathscr{U}(M_{m+n})\},$$

so that

$$\varphi(N_0) \subseteq \operatorname{span}\{u\colon\ u \in \mathscr{U}(\widetilde{N} \ominus \widetilde{A})\},$$

thus (2.7) holds.

If $\tau(p)$ is irrational, then let $\{p_n\}_{n\in\Lambda}$, $\{q_n\}_{n\in\Lambda}$ be two families of increasing subprojections of p and q respectively such that $\tau(p_n)\to\tau(p)$, $\tau(q_n)\to\tau(q)$ and for all $n\in\Lambda$, both $\tau(p_n)$ and $\tau(q_n)$ are rational. Thus for $n\in\Lambda$, $x\in N$, we have

$$p_n x q_n \xrightarrow{\text{SOT}} pxq.$$

Next we show that

$$p_n x q_n \in \operatorname{span}\{u:\ u\in\mathscr{U}(N\ominus A)\}. \tag{2.8}$$

For each $n\in\Lambda$, suppose

$$\frac{\tau(p_n)}{\tau(q_n)}=\frac{k_n}{l_n}\quad\text{with } k_n,l_n\in\mathbb{N}^+.$$

Let $\{p_{n_i}\}_{1\leqslant i\leqslant k_n}$ and $\{q_{n_j}\}_{1\leqslant j\leqslant l_n}$ be two families of mutually orthogonal equivalent projections in N such that $p_{n_1}+p_{n_2}+\cdots+p_{n_{k_n}}=p_n$, $q_{n_1}+q_{n_2}+\cdots+q_{n_{l_n}}=q_n$. Then there exists a $*$-isomorphism φ from N onto $\varphi(N)$ such that

$$\varphi\big((p_n+q_n)N(p_n+q_n)\big)=\mathbb{M}_{k_n+l_n}(p_{n_1}Np_{n_1})$$

and $\varphi|_{(1-p_n-q_n)N(1-p_n-q_n)}$ is the identity map. Denote

$$\begin{aligned}
N_{n_0}&=\left\{\begin{pmatrix}0&x\\ y&0\end{pmatrix}:\ x\in p_nNq_n,\ y\in q_nNp_n\right\},\\
N_1&=\big((p-p_n)N(p-p_n)\big)\ominus\big((p-p_n)A(p-p_n)\big),\\
N_2&=\big((q-q_n)N(q-q_n)\big)\ominus\big((q-q_n)A(q-q_n)\big),\\
M_{k_n+l_n}&=\big\{\{x_{ij}\}_{1\leqslant i,j\leqslant k_n+l_n}\in\mathbb{M}_{k_n+l_n}(p_{n_1}Np_{n_1}):\ x_{ii}=0,\ \text{for } 1\leqslant i\leqslant k_n+l_n\big\}.
\end{aligned}$$

By Lemma 2.1 and Case (i), we have that each operator in

$$M_{k_n+l_n}\oplus N_1\oplus N_2$$

can be written as a linear combination of finitely many unitary operators in this set. Note that

$$\varphi(N_{n_0})\subseteq M_{k_n+l_n}\quad\text{and}\quad M_{k_n+l_n}\oplus N_1\oplus N_2\subseteq\varphi(N)\ominus\varphi(A),$$

so that

$$\varphi(N_{n_0})\oplus N_1\oplus N_2\subseteq\operatorname{span}\{u:\ u\in\mathscr{U}\big(\varphi(N)\ominus\varphi(A)\big)\},$$

thus (2.8) holds.

(iii) $A=\sum_{i=1}^n\mathbb{C}p_i$, where $\{p_i\}_{1\leqslant i\leqslant n}$ is a family of mutually orthogonal projections in N with sum 1. Each $x\in N\ominus A$ can be written as

$$x=\begin{pmatrix}x_{11}&x_{12}&\cdots&x_{1n}\\ x_{21}&x_{22}&\cdots&x_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ x_{n1}&x_{n2}&\cdots&x_{nn}\end{pmatrix}$$

with respect to the decomposition $1=\sum_{1\leqslant i\leqslant n}p_i$, where $x_{ii}\in p_iNp_i\ominus\mathbb{C}p_i$, $x_{ij}\in p_iNp_j$, for $1\leqslant i\neq j\leqslant n$.

For $1 \leqslant i < j \leqslant n$, denote

$$P_{ij} = \bigoplus_{k \neq i,j;\, 1 \leqslant k \leqslant n} p_k N p_k \ominus \mathbb{C} p_k,$$
$$N_{ij} = (p_i + p_j)(N \ominus A)(p_i + p_j),$$

then

$$P_{ij} = \operatorname{span}\{u\colon\ u \in \mathscr{U}(P_{ij})\}, \qquad N_{ij} = \overline{\operatorname{span}\{u\colon\ u \in \mathscr{U}(N_{ij})\}}^{\mathrm{SOT}}$$

following from Case (i) and Case (ii), therefore

$$N_{ij} \oplus P_{ij} = \overline{\operatorname{span}\{u\colon\ u \in \mathscr{U}(N_{ij} \oplus P_{ij})\}}^{\mathrm{SOT}}.$$

(iv) $A = \sum_{i=1}^{\infty} \mathbb{C} p_i$, where $\{p_i\}_{i=1}^{\infty}$ is a family of mutually orthogonal projections in A with sum 1. Denote

$$q_i = \sum_{k=1}^{i} p_k,$$
$$N_i = q_i (N \ominus A) q_i,$$
$$P_i = \bigoplus_{i+1 \leqslant k < \infty} p_k N p_k \ominus \mathbb{C} p_k,$$

then the strong-operator closure of $\bigcup_{i=1}^{\infty} N_i$ is $N \ominus A$. By Case (i) and Case (iii), we have

$$N_i \oplus P_i \subseteq \overline{\operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}}^{\mathrm{SOT}}.$$

Thus we finish the proof. □

Lemma 2.5. *If N is a type II_1 factor and $A \subseteq N$ is an abelian von Neumann subalgebra, then $N \ominus A = \overline{\operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}}^{\mathrm{SOT}}$.*

Proof. Since A is an abelian von Neumann algebra, there exist two mutually orthogonal projections $p, q \in A$ with sum 1 such that pAp is a diffuse abelian von Neumann algebra with unit p and qAq is an atomic abelian von Neumann algebra with unit q. Each $x \in N \ominus A$ can be written as

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with respect to the decomposition $1 = p + q$, where $x_{11} \in p(N \ominus A)p$, $x_{12} \in pNq$, $x_{21} \in qNp$, $x_{22} \in q(N \ominus A)q$.

Denote

$$N_0 = \left\{ \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}\colon\ x_{12} \in pNq,\ x_{21} \in qNp \right\}.$$

By Lemma 2.3 and Lemma 2.4, we only need to prove

$$N_0 \subseteq \overline{\operatorname{span}\{u\colon\ u \in \mathscr{U}(N \ominus A)\}}^{\mathrm{SOT}}.$$

If $\tau(p)$ is rational, then idea of the proof is the same as that used in Case (ii) for $\tau(p)$ rational in Lemma 2.4.

If $\tau(p)$ is irrational, then let $\{p_n\}_{n\in\Lambda}$ and $\{q_n\}_{n\in\Lambda}$ be two families of increasing subprojections of p and q respectively such that $\tau(p_n)\to\tau(p)$, $\tau(q_n)\to\tau(q)$ and for all $n\in\Lambda$, both $\tau(p_n)$ and $\tau(q_n)$ are rational. Then for $n\in\Lambda$, $x\in N$, we have

$$p_n x q_n \xrightarrow{\text{SOT}} pxq.$$

Next we show that

$$p_n x q_n \in \operatorname{span}\{u\colon\ u\in\mathscr{U}(N\ominus A)\}. \tag{2.9}$$

For each $n\in\Lambda$, suppose

$$\frac{\tau(p_n)}{\tau(q_n)}=\frac{k_n}{l_n}\quad\text{with } k_n,l_n\in\mathbb{N}^+.$$

Let $\{p_{n_i}\}_{1\leqslant i\leqslant k_n}$ and $\{q_{n_j}\}_{1\leqslant j\leqslant l_n}$ be two families of mutually orthogonal equivalent projections in N such that $p_{n_1}+p_{n_2}+\cdots+p_{n_{k_n}}=p_n$, $q_{n_1}+q_{n_2}+\cdots+q_{n_{l_n}}=q_n$. Then there exists a $*$-isomorphism φ from N onto $\varphi(N)$ such that

$$\varphi\big((p_n+q_n)N(p_n+q_n)\big)=\mathbb{M}_{k_n+l_n}(p_{n_1}Np_{n_1})$$

and $\varphi|_{(1-p_n-q_n)N(1-p_n-q_n)}$ is the identity map. Denote

$$\begin{aligned} N_{n_0}&=\left\{\begin{pmatrix}0 & x\\ y & 0\end{pmatrix}\colon\ x\in p_nNq_n,\ y\in q_nNp_n\right\},\\ N_1&=\big((p-p_n)N(p-p_n)\big)\ominus\big((p-p_n)A(p-p_n)\big),\\ N_2&=\big((q-q_n)N(q-q_n)\big)\ominus\big((q-q_n)A(q-q_n)\big),\\ M_{k_n+l_n}&=\big\{\{x_{ij}\}_{1\leqslant i,j\leqslant k_n+l_n}\in\mathbb{M}_{k_n+l_n}(p_{n_1}Np_{n_1})\colon\ x_{ii}=0,\ \text{for } 1\leqslant i\leqslant k_n+l_n\big\}.\end{aligned}$$

By Lemma 2.1, we have

$$M_{k_n+l_n}=\operatorname{span}\{u\colon\ u\in\mathscr{U}(M_{k_n+l_n})\}.$$

By Lemma 2.3 and Lemma 2.4, there is a unitary operator $v\in N_1\oplus N_2$. Note that

$$\varphi(N_{n_0})\subseteq M_{k_n+l_n}\quad\text{and}\quad M_{k_n+l_n}\oplus v\subseteq\varphi(N)\ominus\varphi(A),$$

so that

$$\varphi(N_{n_0})\oplus v\subseteq\operatorname{span}\big\{u\colon\ u\in\mathscr{U}\big(\varphi(N)\ominus\varphi(A)\big)\big\},$$

thus (2.9) holds. $\square$

Theorem 2.6. *If N is a type* II_1 *factor and $A\subseteq N$ is a type* I *von Neumann subalgebra, then* $N\ominus A=\overline{\operatorname{span}\{u\colon\ u\in\mathscr{U}(N\ominus A)\}}^{\text{SOT}}$.

Proof. Since A is a type I von Neumann algebra, there exists a family of mutually orthogonal central projections $\{p_i\}_{i\in\Lambda} \subseteq A$ with sum 1 such that

$$p_i A p_i \cong \mathbb{M}_{k_i}(A_i),$$

where for each $i \in \Lambda$, A_i is an abelian von Neumann subalgebra and k_i is some positive integer. So we assume

$$A = \bigoplus_{i\in\Lambda} \mathbb{M}_{k_i}(A_i),$$
$$p_i N p_j = \mathbb{M}_{k_i\times k_j}(N_{ij}),$$

where $N_{ij} = e_{i_1} N e_{j_1}$, $\{e_{i_n}\}_{1\leqslant n\leqslant k_i}$ is a family of mutually orthogonal equivalent subprojections of p_i with sum p_i.

For each $i, j \in \Lambda$, $i < j$, denote

$$P_{ij} = \bigoplus_{k\in\Lambda,\, k\neq i,j} p_k(N \ominus A)p_k,$$
$$\widetilde{N}_{ij} = (p_i + p_j)(N \ominus A)(p_i + p_j),$$
$$\widehat{N}_{ij} = \left\{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}:\ x_1 \in N_{ii} \ominus A_i,\ x_2 \in N_{ij},\ x_3 \in N_{ji},\ x_4 \in N_{jj} \ominus A_j\right\},$$
$$\widetilde{S}_{ij} = \left\{\begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix}:\ X_2 \in p_i N p_j,\ X_3 \in p_j N p_i\right\},$$
$$\widehat{S}_{ij} = \left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}:\ X_1 \in p_i(N \ominus A)p_i,\ X_4 \in p_j(N \ominus A)p_j\right\}.$$

By Lemma 2.2 and Lemma 2.5, we have

$$p_k(N \ominus A)p_k = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(p_k(N \ominus A)p_k)\}}^{\mathrm{SOT}}, \tag{2.10}$$

so that

$$\widehat{S}_{ij} = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(\widehat{S}_{ij})\}}^{\mathrm{SOT}}.$$

Next we show

$$\widetilde{S}_{ij} \subseteq \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(\widetilde{N}_{ij})\}}^{\mathrm{SOT}}. \tag{2.11}$$

For each $x = \{x_{kl}\}_{1\leqslant k,l\leqslant k_i+k_j} \in \widetilde{S}_{ij}$, we have

$$\begin{pmatrix} 0 & x_{st} \\ x_{ts} & 0 \end{pmatrix} \in \widehat{N}_{ij},$$

for $1 \leqslant s \leqslant k_i$, $k_i + 1 \leqslant t \leqslant k_i + k_j$. By Lemma 2.5, we have

$$\widehat{N}_{ij} = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(\widehat{N}_{ij})\}}^{\mathrm{SOT}}$$

and

$$(N_{ii} \ominus A_i)^{(k_i-1)} = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}((N_{ii} \ominus A_i)^{(k_i-1)})\}}^{\mathrm{SOT}},$$

so that each operator in

$$\widehat{N}_{ij} \oplus (N_{ii} \ominus A_i)^{(k_i-1)} \oplus (N_{jj} \ominus A_j)^{(k_j-1)}$$

can be approximated in the strong-operator topology by a linear combination of finitely many unitary operators in this set. Then relation (2.11) holds since

$$\widehat{N}_{ij} \oplus (N_{ii} \ominus A_i)^{(k_i-1)} \oplus (N_{jj} \ominus A_j)^{(k_j-1)} \subseteq \widetilde{N}_{ij}.$$

Thus we have

$$\widetilde{N}_{ij} = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(\widetilde{N}_{ij})\}}^{\mathrm{SOT}}.$$

By (2.10), we have

$$P_{ij} = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(P_{ij})\}}^{\mathrm{SOT}},$$

so that

$$\widetilde{N}_{ij} \oplus P_{ij} = \overline{\operatorname{span}\{u:\ u \in \mathscr{U}(\widetilde{N}_{ij} \oplus P_{ij})\}}^{\mathrm{SOT}}.$$

Thus, we finish the proof of this theorem. □

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