J. Math. Anal. Appl. 417 (2014) 411–438

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Schrödinger–Poisson system with singular potential

Yongsheng Jiang, Huan-Song Zhou *

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, PR China

A R T I C L E I N F O

Article history:
Received 26 November 2013
Available online 18 March 2014
Submitted by V. Radulescu

Keywords:
Elliptic equations
Variational methods
Schrödinger–Poisson equation
Singular potential
Nonnegative Palais–Smale sequence

A B S T R A C T

Consider the following Schrödinger–Poisson system

$$(\mathrm{SP})\quad \begin{cases} -\Delta u + V_\lambda(x)u + \phi(x)u = |u|^{p-1}u, & x=(y,z)\in\mathbb{R}^2\times\mathbb{R}, \\ -\Delta\phi = u^2, \quad \lim\limits_{|x|\to+\infty}\phi(x)=0, \end{cases}$$

where $V_\lambda = \lambda + \frac{1}{|y|^\alpha}$ with $\lambda \geqslant 0$, $y=(x_1,x_2)\in\mathbb{R}^2$ and $|y|=\sqrt{x_1^2+x_2^2}$. When $\alpha\in[0,8)$ and $\max\{2,\frac{2+\alpha}{2}\}<p<5$, the existence and a priori estimate of positive solutions of problem (SP) are established in suitable weighted Sobolev space. Moreover, the asymptotic behavior of the solutions as $\lambda\to0$ is also discussed.

1. Introduction

In this paper, we study the following type of Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = |u|^{p-1}u, \\ -\Delta\phi = u^2, \quad \lim\limits_{|x|\to+\infty}\phi(x)=0, \quad x=(x_1,x_2,z)\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p\in(2,5)$, and the potential function V is of the form

(V) $V_\lambda(x)=\lambda+\frac{1}{|y|^\alpha}$, $\lambda\geqslant 0$, $\alpha\in[0,8)$, and $|y|=\sqrt{x_1^2+x_2^2}$.

Problem (1.1) arises in the study of the coupled Schrödinger–Poisson system:

* Corresponding author. Fax: +86 27 87199291.
E-mail addresses: jiangys@znufe.edu.cn (Y. Jiang), hszhou@wipm.ac.cn (H.-S. Zhou).

http://dx.doi.org/10.1016/j.jmaa.2014.03.034
0022-247X/

$$\begin{cases} i\psi_t - \Delta\psi + \phi(x)\psi = f(|\psi|)\psi, \\ -\Delta\phi = |\psi|^2, \qquad \lim\limits_{|x|\to+\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $f(|\psi|)\psi = |\psi|^{p-1}\psi + \omega_0\psi$, $\omega_0 > 0$, $2 < p < 5$ and $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$. Motivated by [7], we look for a solution of (1.2) with the form:

$$\psi(x,t) = u(x)e^{i(\eta(x)+\omega t)}, \quad u(x) \geqslant 0,\ \eta(x) \in \mathbb{R}/2\pi\mathbb{Z},\ \omega \geqslant \omega_0.$$

Then, u satisfies

$$\begin{cases} -\Delta u + \left(\omega - \omega_0 + \left|\nabla\eta(x)\right|^2\right)u + \phi(x)u = |u|^{p-1}u, \\ u\Delta\eta(x) + 2\nabla u\nabla\eta = 0, \\ -\Delta\phi = u^2, \qquad \lim\limits_{|x|\to+\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3. \end{cases}$$

Furthermore, similar to [6,8] we let $u(x) = u(y,z) = u(|y|,z)$ and

$$\eta(x) = \begin{cases} \arctan(x_2/x_1), & \text{if } x_1 > 0, \\ \arctan(x_2/x_1) + \pi, & \text{if } x_1 < 0, \\ \pi/2, & \text{if } x_1 = 0 \text{ and } x_2 > 0, \\ -\pi/2, & \text{if } x_1 = 0 \text{ and } x_2 < 0, \end{cases}$$

it is easy to see that $\eta(x) \in C^2(\mathbb{R}^3 \setminus T_-)$, where $T_- := \{(x_1,x_2,z) \in \mathbb{R}^3\colon\ x_1 = 0,\ x_2 \leqslant 0\}$. By a simple calculation we know that

$$\Delta\eta(x) = 0, \qquad \nabla\eta(x)\cdot\nabla u(x) = 0, \qquad \left|\nabla\eta(x)\right| = \frac{1}{|y|^2}, \quad \text{for } x \in \mathbb{R}^3 \setminus T_-.$$

These show that $u(|y|,z)$ is actually a nonnegative solution of (1.1) with $\alpha = 2$ and $\lambda = \omega - \omega_0$. Furthermore, $\psi(x)$ solves (1.2) with angular momentum:

$$M(\psi) = \operatorname{Re}\int\limits_{\mathbb{R}^3} i\bar{\psi}x \wedge \nabla\psi\,dx = -\int\limits_{\mathbb{R}^3} u^2 x \wedge \nabla v(x)\,dx = -\left(0,0,|u|_{L^2}^2\right).$$

Problem (1.1) has been studied under various conditions on the potential $V(x)$ and the power p. For example, if $V(x) =$ constant, that is $\alpha = 0$ in (V), the non-existence of nontrivial solutions of (1.1) for $p \notin (1,5)$ was proved in [12] by a Pohozaev type identity, a radially symmetric positive solution was obtained in [10,13] for $p \in [3,5)$, etc. It is known that a nontrivial weak solution of (1.1) can be obtained by searching for a nonzero critical point of the variational functional associated to problem (1.1). Usually, the weak limit of a bounded Palais–Smale, (PS) in short, sequence of the functional is actually a weak solution of (1.1), but it may be a trivial solution unless the functional satisfies (PS) condition, that is, any (PS) sequence has a strongly convergent subsequence. However, if there is no some further conditions on V_λ, such as (1.3) below, it seems hard to verify the (PS) condition, even difficult to have the boundedness of a (PS) sequence. In this paper, instead of trying to prove the (PS) condition, we adopt a trick used in [16], which is essentially a version of the concentration-compactness principle due to [22], to show directly that the weak limit of a (PS) sequence is indeed a nontrivial solution. But, this trick seems not working for the (PS) sequence simply obtained by the Mountain Pass Theorem because of the nonlocal term $\phi(x)u$ in (1.1). To overcome this difficult, based on the Deformation Lemma (cf. [24, Lemma 2.3]) we try to construct a special (PS) sequence which is nonnegative and such that $\phi(x)$ is bounded in $D^{1,2}(\mathbb{R}^3)$. Our approach also provides a simple way of getting a nonnegative (PS) sequence, see Lemma 2.6, which may be useful in certain situations. Note that in [5,9,11,16] the authors studied the single stationary Schrödinger equation, that is, taking $\phi(x)=0$ in the first equation of (1.1), in this case it is not necessary to seek a nonnegative (PS) sequence, see e.g. [5,16]. In this paper, we are concerned with the Schrödinger–Poisson system (1.1) under the condition (V) with $\alpha>0$. We point out that our results cover the case of $\alpha=0$ (i.e. constant potential). Moreover, we give also a priori estimate for solutions of (1.1), see Lemma 4.4, and get also a classical solution (except for $|y|=0$) to (1.1) with $\lambda=0$, $\alpha\in(0,8)$ and $\max\{2,\frac{2+\alpha}{2}\}<p<5$.

For problem (1.1) with $\alpha=0$ in (V), existence and nonexistence results were established by Ruiz in [21], he proved that (1.1) has always a positive radial solution if $p\in(2,5)$ and does not admit any nontrivial solution if $p\leqslant 2$. A ground state for (1.1) with $p\in(2,5)$ was proved in [3]. The existence of non-radially symmetric solutions was shown in [14] and multiple solutions for (1.1) were obtained in [2,10].

If the potential V is not a constant, problem (1.1) has been studied in [3] for $p\in(3,5)$ and [26] for $p\in(2,3]$. For more general nonlinearities, we refer the reader to the papers [1,4,20,23,25], etc. To ensure that the variational functional associated to problem (1.1) satisfies the (PS) condition, the following conditions are assumed in [3,26]

$$V(x)\leqslant V_\infty=\liminf_{|x|\to\infty}V(x), \tag{1.3}$$

$$2V(x)+\big(\nabla V(x),x\big)\geqslant 0 \quad \text{a.e. } x\in\mathbb{R}^3. \tag{1.4}$$

It is clear that our potential V does not satisfy the above conditions. So, we cannot use the same methods as that of [3,26] to deal with problem (1.1). Without condition (1.4), it seems difficult even showing that a (PS) sequence is bounded, specially for $p\in(2,3)$. Motivated by [6], here we try to find a bounded and nonnegative (PS) sequence directly from the well-known Deformation Lemma [24, Lemma 2.3].

Before stating our main results, we introduce some notations, definitions and recall some properties of the solution of the second equation (Poisson equation) in (1.1). For $\alpha\geqslant 0$ and $x=(y,z)\in\mathbb{R}^2\times\mathbb{R}$, define

$$E=\left\{u\in D^{1,2}(\mathbb{R}^3)\colon\ u(x)=u\big(|y|,z\big)\ \text{and}\ \int_{\mathbb{R}^3}\frac{u^2}{|y|^\alpha}\,dx<\infty\right\}, \tag{1.5}$$

and $D^{1,2}(\mathbb{R}^3)=\{u\in L^6(\mathbb{R}^3)\colon\ |\nabla u|\in L^2(\mathbb{R}^3)\}$. For $\lambda>0$, we denote

$$H=\left\{u\in E\colon\ \lambda\int_{\mathbb{R}^3}u^2\,dx<\infty\right\}.$$

Clearly $H\subset E$, $H\subset H^1(\mathbb{R}^3)$ and H is a Hilbert space, its scalar product and norm are given by

$$\langle u,v\rangle_H=\int_{\mathbb{R}^3}\nabla u\nabla v+V_\lambda(x)uv\,dx \quad\text{and}\quad \|u\|_H^2=\langle u,u\rangle_H, \tag{1.6}$$

respectively, where $V_\lambda(x)=\lambda+\frac{1}{|y|^\alpha}$.

Throughout this paper, we denote the standard norms of $H^1(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ $(1\leqslant p\leqslant+\infty)$ by $\|\cdot\|$ and $|\cdot|_p$, respectively. We observe that (1.6) implies that $\|\cdot\|_H$ is an equivalent norm of $\|\cdot\|$ if $\alpha=0$.

By Lemma 2.1 of [21], we know that $-\Delta\phi=u^2$ has a unique solution in $D^{1,2}(\mathbb{R}^3)$ with the form of

$$\phi(x):=\phi_u(x)=\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}\,dy, \quad \text{for any } u\in L^{\frac{12}{5}}\big(\mathbb{R}^3\big), \tag{1.7}$$

and there is a constant $C > 0$, independent of ϕ, such that

$$\big|\nabla\phi_u(x)\big|_2 \leqslant C|u|_{12/5}^2, \qquad \int_{\mathbb{R}^3} \phi_u(x)u^2\,dx \leqslant C|u|_{12/5}^4. \tag{1.8}$$

For $\lambda > 0$ and $u \in H$, we can define the variational functional associated to problem (1.1) as follows:

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V_\lambda(x)u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx. \tag{1.9}$$

Since (1.8), I_λ is well defined on H and $I_\lambda \in C^1(H,\mathbb{R})$ with

$$\big(I'_\lambda(u), v\big) = \int_{\mathbb{R}^3} \nabla u\nabla v + V_\lambda(x)uv\,dx + \int_{\mathbb{R}^3} \phi_u(x)uv\,dx - \int_{\mathbb{R}^3} |u|^{p-1}uv\,dx \tag{1.10}$$

for all $v \in H$ with $\lambda > 0$ and $p \in (1,5)$. Furthermore, it is known that a nontrivial weak solution of (1.1) corresponds to a nonzero critical point of the functional I in H if $\lambda > 0$.

In this paper, we want to establish existence results for problem (1.1) for both $\lambda > 0$ and $\lambda = 0$.

However, if $\lambda = 0$, then $H = E$. In this case, (1.7) (1.8) are not always true for $u \in E$. Therefore, the integrals $\int_{\mathbb{R}^3} |u|^p\,dx$, $\int_{\mathbb{R}^3} \phi_u(x)u^2\,dx$ and $\int_{\mathbb{R}^3} \phi_u(x)uv\,dx$ may not be well defined for $u, v \in E$.

To this end, we set

$$T = \big\{x \in \mathbb{R}^3\colon\ |y| = 0\big\} \quad \text{where } |y| = \sqrt{x_1^2 + x_2^2}. \tag{1.11}$$

Then, by using the results for $\lambda > 0$ and an approximation procedure ($\lambda \to 0$), see Section 4, we can get a solution $u \in E$ of (1.1) with $\lambda = 0$ in following sense

$$\int_{\mathbb{R}^3} \nabla u\nabla\varphi + \frac{1}{|y|^\alpha}u\varphi\,dx + \int_{\mathbb{R}^3} \phi_u(x)u\varphi\,dx = \int_{\mathbb{R}^3} |u|^{p-1}u\varphi\,dx, \quad \text{for } \varphi \in C_0^\infty\big(\mathbb{R}^3 \setminus T\big). \tag{1.12}$$

Note that $\int_{\mathbb{R}^3} \frac{1}{|y|^\alpha}u\varphi\,dx$ may be not integrable for $u \in E$ and $\varphi \in C_0^\infty(\mathbb{R}^3)$, this is why we take $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$ in (1.12) instead of $\varphi \in C_0^\infty(\mathbb{R}^3)$. So, it is reasonable for us to define a weak solution for (1.1) as follows.

Definition 1.1. $u \in E \setminus \{0\}$ is said to be a weak solution of (1.1) with $\lambda \geqslant 0$ if $\phi_u \in D^{1,2}(\mathbb{R}^3)$ and u satisfies

$$\int_{\mathbb{R}^3} \left[\nabla u\nabla\varphi + \left(\frac{1}{|y|^\alpha} + \lambda\right)u\varphi\right]dx + \int_{\mathbb{R}^3} \phi_u(x)u\varphi\,dx = \int_{\mathbb{R}^3} |u|^{p-1}u\varphi\,dx, \tag{1.13}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$.

We mention that the above definition also enables us to get a classical solution. In fact, if $u \in E$ and $\phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfy (1.13), by using our Lemmas 4.2 and 4.3, as well as Theorems 8.10 and 9.19 in [15], we can prove that $u \in C^2(\mathbb{R}^3 \setminus T)$, that is, u is a classical solution of (1.1), see Theorem 3.1 in Section 3.

For the single Schrödinger equation

$$-\Delta u + \frac{u}{|y|^\alpha} = f(u), \quad x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N,\ N \geqslant 3 \tag{1.14}$$

with $|y| = \sqrt{\Sigma_{k=1}^{N+1-i} x_k^2}$, $i < N$, the authors of paper [5] proved that (1.14) has a nontrivial solution in $H^1(\mathbb{R}^N)$ if $\alpha = 2$, $N > i \geqslant 2$ and $f(t)$ is supposed to have some kinds of double powers behavior which ensure that $F(u) = \int_0^u f(s)\,ds$ is well defined in $L^1(\mathbb{R}^N)$ for $u \in D^{1,2}(\mathbb{R}^N)$. In [5], the authors used a variational method to seek first a nontrivial solution of (1.14) in $D^{1,2}(\mathbb{R}^N)$, then proved this solution is also in $L^2(\mathbb{R}^N)$. Formally, (1.14) is nothing but the first equation of problem (1.1) with $\lambda = 0$, $N = 3$ and $\phi(x) \equiv 0$. However, even for $f(u) = |u|^{p-1}u$ with $p \in (2,5)$, $F(u)$ is not well defined in $D^{1,2}(\mathbb{R}^N)$, then the method and results of [5] do not work for our problem. So, when $\lambda = 0$ it seems difficult to have a good working space which can be directly used to solve (1.1). In this paper, we prove first that (1.1) has always a solution u_λ in $H^1(\mathbb{R}^3)$ for each $\lambda > 0$, then show that $\{u_\lambda\}$ (as a sequence of λ) is bounded in E. As mentioned above we can finally use an approximation process to get a weak solution of (1.1) for $\lambda = 0$ in the sense of (1.12).

The main results of this paper can be stated now as follows:

Theorem 1.1. *Let $\alpha \in [0,8)$, $\max\{2, \frac{2+\alpha}{2}\} < p < 5$ and let condition (V) be satisfied. Then, problem (1.1) has at least a positive solution $u_\lambda \in H \cap C^2(\mathbb{R}^3 \setminus T)$ for every $\lambda > 0$. Furthermore, if $\lambda \in (0,1]$, there exists $C > 0$ which is independent of $\lambda \in (0,1]$ such that the solution u_λ satisfies*

$$|\nabla u_\lambda|_2^2 + \int\limits_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2\,dx < C.$$

Theorem 1.2. *For $\lambda = 0$, let $\alpha \in [0,8)$ and $\max\{2, \frac{2+\alpha}{2}\} < p < 5$. Then, problem (1.1) has at least a positive solution $u \in E \cap C^2(\mathbb{R}^3 \setminus T)$ in the sense of (1.12).*

2. Bounded nonnegative (PS) sequence

In this section, $\lambda > 0$ is always assumed. Our aim is to know how the functional I_λ defined in (1.9) has always a bounded nonnegative (PS) sequence at some level $c > 0$ in H. As mentioned in the introduction, the authors in [6] developed an approach to get a bounded (PS) sequence for the single equation (1.14) with certain nonlinearities. By improving some techniques used in [6], we are able to obtain a bounded nonnegative (PS) sequence for (1.1), the nonnegativity of the (PS) sequence helps us to estimate the related term caused by the nonlocal term $\phi(x)u$, which leads to a nonzero weak limit of the (PS) sequence. Let us recall first a deformation lemma from [24].

Lemma 2.1. *(See [24, Lemma 2.3].) Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon, \delta > 0$ such that for any $u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$: $\varphi'(u) \geqslant 8\varepsilon/\delta$. Then there exists $\eta \in C([0,1] \times X, X)$ such that:*

(i) *$\eta(t,u) = u$, if $t = 0$ or $u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$.*
(ii) *$\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$, where $\varphi^{c\pm\varepsilon} = \{u \in X\colon\ \varphi(u) \leqslant c \pm \varepsilon\}$.*
(iii) *$\eta(t,\cdot)$ is a homeomorphism of X, for any $t \in [0,1]$.*
(iv) *$\varphi(\eta(\cdot,u))$ is non increasing, for any $u \in X$.*

Now, we give some lemmas, by which Lemma 2.1 can be used to get a desirable (PS) sequence.

Lemma 2.2. *Let $p \in (1,5)$ and $\lambda > 0$ in (1.6). If $u_1, u_2 \in H$ and $\|u_1\|_H \leqslant M$, $\|u_2\|_H \leqslant M$ for some $M > 0$, then there exists a positive constant $C := C(M,p,\lambda)$ such that,*

$$\big\|I'(u_1) - I'(u_2)\big\|_{H'} \leqslant C\big(\|u_1 - u_2\|_H + \|u_1 - u_2\|_H^3\big). \tag{2.1}$$

Proof. By (1.10) and (1.6),

$$\left\langle I'(u_1) - I'(u_2), \psi\right\rangle_H = \langle u_1 - u_2, \psi\rangle_H + \int_{\mathbb{R}^3} (\phi_{u_1} u_1 - \phi_{u_2} u_2)\psi\, dx - \int_{\mathbb{R}^3} \left(|u_1|^{p-1}u_1 - |u_1|^{p-1}u_1\right)\psi\, dx,$$

hence (2.1) is proved if we have that

$$\left| \int_{\mathbb{R}^3} \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right)\psi\, dx \right| \leqslant C\|u_1 - u_2\|_H \|\psi\|_H, \tag{2.2}$$

$$\left| \int_{\mathbb{R}^3} (\phi_{u_1} u_1 - \phi_{u_2} u_2)\psi\, dx \right| \leqslant C\left(\|u_1 - u_2\|_H + \|u_1 - u_2\|_H^3\right)\|\psi\|_H. \tag{2.3}$$

Indeed, using Taylor's formula and Hölder inequality as well as Minkovski inequality, we see that there is a function θ with $0 < \theta < 1$ such that

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right)\psi\, dx \right| &\leqslant p|u_1 - u_2|_{p+1}|\psi|_{p+1}\left|\theta u_1 + (1-\theta)u_2\right|_{p+1}^{p+1} \\ &\leqslant p\left(|u_1|_{p+1} + |u_2|_{p+1}\right)^{p+1}|u_1 - u_2|_{p+1}|\psi|_{p+1} \\ &\leqslant p(2M)^{p+1}|u_1 - u_2|_{p+1}|\psi|_{p+1}, \end{aligned}$$

hence (2.2) is obtained. To prove (2.3), we let $\upsilon = u_2 - u_1$, it follows from (1.7) that

$$\int_{\mathbb{R}^3} (\phi_{u_2} u_2 - \phi_{u_1} u_1)\psi\, dx = J_1 + J_2 + J_3 + J_4 + J_5, \tag{2.4}$$

where

$$J_1 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\upsilon^2(y)\upsilon(x)\psi(x)}{|x-y|}\, dx\, dy, \qquad J_2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\upsilon^2(y)u_1(x)\psi(x)}{|x-y|}\, dx\, dy,$$

$$J_3 = \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_1^2(y)\upsilon(x)\psi(x)}{|x-y|}\, dx\, dy, \qquad J_4 = 2\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_1(y)u_1(x)\upsilon(y)\psi(x)}{|x-y|}\, dx\, dy,$$

$$J_5 = 2\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_1(y)\upsilon(y)\upsilon(x)\psi(x)}{|x-y|}\, dx\, dy.$$

Now, we estimate J_1 to J_5 by using the following Hardy–Littlewood–Sobolev inequality [19, Theorem 4.3]

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)|x-y|^{-d}h(y)\, dx\, dy \right| \leqslant C(N,d,p)|f|_p|h|_r,$$

where $p, r > 1$ and $0 < d < N$ with $\frac{1}{p} + \frac{d}{N} + \frac{1}{r} = 2$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, the sharp constant $C(N,d,p)$, independent of f and h.

Take $N = 3$ and $d = 1$, then take p, r suitably in the above inequality, we see that

$$|J_1| \leqslant C\|v\|_H^3\|\psi\|_H, \qquad |J_2| \leqslant C\|v\|_H^2\|u_1\|_H\|\psi\|_H, \qquad |J_3| \leqslant C\|u_1\|_H^2\|v\|_H\|\psi\|_H,$$
$$|J_4| \leqslant C\|u_1\|_H^2\|v\|_H\|\psi\|_H, \qquad |J_5| \leqslant C\|u_1\|_H\|v\|_H^2\|\psi\|_H.$$

These estimates and (2.4) imply that (2.3) holds. Thus, Lemma 2.2 is proved. □

Before giving our next lemma, we recall some basic properties of $\phi_u(x)$ given by (1.7). Let

$$u_t := u_t(x) = t^2u(tx) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^3,$$

then $u(x) = (u_t)_{\frac{1}{t}}(x) = (u_{\frac{1}{t}})_t(x)$ and

$$|\nabla u_t|_2^2 = t^3|\nabla u|_2^2, \qquad |u_t|_p^p = t^{2p-3}|u|_p^p \quad \text{for } 1 \leqslant p < \infty, \tag{2.5}$$

$$\int_{\mathbb{R}^3} \phi_{u_t}u_t^2\,dx = t^3\int_{\mathbb{R}^3} \phi_u u^2\,dx, \qquad \int_{\mathbb{R}^3} \frac{u_t^2}{|y|^\alpha}\,dx = t^{1+\alpha}\int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha}\,dx. \tag{2.6}$$

Lemma 2.3. *If $\alpha \in [0, 8)$ and $\max\{2, \frac{\alpha+2}{2}\} < p < 5$, then there exist $\rho > 0$, $\delta > 0$, $e \in H$ with $e \geqslant 0$ and $\|e\|_H > \rho$ such that:*

(i) *$I(u) \geqslant \delta$, for all $u \in H$ with $\|u\|_H = \rho$.*
(ii) *$I(e) < I(0) = 0$.*

Proof. (i) Since $H \hookrightarrow L^p(\mathbb{R}^3)$ for $2 \leqslant p < 6$, this conclusion is a straightforward consequence of the definition of I.

(ii) For $t > 0$ and $u \in H \setminus \{0\}$, by (2.5), (2.6) and the definition of I, we see that

$$I(u_t) = \frac{t^3}{2}|\nabla u|_2^2 + \frac{\lambda t}{2}|u|_2^2 + \frac{t^{1+\alpha}}{2}\int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha}\,dx + \frac{t^3}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2\,dx - \frac{t^{2p-1}}{p+1}|u|_{p+1}^{p+1}. \tag{2.7}$$

Since $\alpha \in [0, 8)$, $p > \max\{2, \frac{\alpha+2}{2}\}$, we see $I(u_t) \to -\infty$ as $t \to +\infty$. Hence, for each $u \in H \setminus \{0\}$, there is a $t_* > 0$ large enough such that (ii) holds with $e = u_{t_*}$. Moreover, we may assume that $e \geqslant 0$, otherwise, just replace e by $|e|$. □

For each $\lambda > 0$ and e given by Lemma 2.3, define

$$c := c_\lambda = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I_\lambda(u), \tag{2.8}$$

where $\Gamma := \{\gamma \in C([0, 1]; H)\colon\ \gamma(0) = 0,\ \gamma(1) = e\}$. Clearly, $c > 0$ by Lemma 2.3. Let $\{t_n\} \subset (0, +\infty)$ be a sequence such that $t_n \to 1$ as $n \to +\infty$, then by (2.5) it is easy to show that

$$e_{t_n} := t_n^2 e(t_n x) \to e \quad \text{in } H,\ \text{as } n \to +\infty. \tag{2.9}$$

Since $I \in C^1(H)$, it follows from Lemma 2.3 (ii) that there is $\varepsilon > 0$ small enough such that $I(u) < 0$ for all $u \in B_\varepsilon(e)$. Again using (2.9), there exists $t_0 \in (0, 1)$ such that

$$e_t := t^2e(tx) \in B_\varepsilon(e) \quad \text{for all } t \in (t_0, 1). \tag{2.10}$$

For this $t_0 \in (0, 1)$, similar to [6] we have

Lemma 2.4. *Let t_0 be given by (2.10). Then for all $t \in (t_0, 1)$, we have*

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u_t)$$

where c and Γ are defined in (2.8), $u_t = t^2 u(tx)$.

Proof. The proof is the same as that of Lemma 11 in [6]. □

By Lemma 2.4, we know that for any $s \in (t_0, 1)$ there exists $\gamma_s \in \Gamma$ such that

$$\max_{u \in \gamma_s([0,1])} I(u_s) \leqslant c + \left(1 - s^3\right). \tag{2.11}$$

For $s \in (t_0, 1)$, we define the set

$$U_s := \left\{u \in \gamma_s\left([0,1]\right):\ I(u) \geqslant c - \left(1 - s^3\right)\right\}, \tag{2.12}$$

then, (2.8) and the definition of U_s imply that $U_s \neq \emptyset$ for $s \in (t_0, 1)$.

Lemma 2.5. *If $\alpha \in [0, 8)$ and $\max\{2, \frac{\alpha+2}{2}\} < p < 5$, then for t_0 given by (2.10) there exist $t^* \in (t_0, 1)$ and $M = \frac{2(c+2)(2p-1)}{(p-2){t^*}^3} + \frac{4(c+2)(2p-1)}{(2p-2-\alpha){t^*}^{1+\alpha}}$ such that*

$$\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2\, dx < M \quad \text{for all } u \in U_s \text{ with } s \in \left(t^*, 1\right).$$

Proof. Let $u \in U_s$ and note that $u(x) = (u_s)_{\frac{1}{s}}(x)$, it follows from (2.5), (2.6) and the definition (1.9) that

$$\begin{aligned} I(u_s) - I(u) = {} & \frac{1}{2}\left(1 - \frac{1}{s^3}\right)|\nabla u_s|_2^2 + \frac{\lambda}{2}\left(1 - \frac{1}{s}\right)|u_s|_2^2 + \frac{1}{2}\left(1 - \frac{1}{s^{1+\alpha}}\right)\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\, dx \\ & + \frac{1}{4}\left(1 - \frac{1}{s^3}\right)\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\, dx - \frac{1}{p+1}\left(1 - \frac{1}{s^{2p-1}}\right)|u_s|_{p+1}^{p+1}. \end{aligned} \tag{2.13}$$

For $u \in U_s$, (2.11) and (2.12) imply that

$$I(u_s) - I(u) \leqslant 2\left(1 - s^3\right), \quad \text{for } s \in (t_0, 1). \tag{2.14}$$

By calculation, this and (2.13) show that, for any $u \in U_s$,

$$\begin{aligned} & \frac{\lambda}{2}\frac{s^2 - s^3}{s^3 - 1}|u_s|_2^2 + \frac{1}{2}\frac{s^2 - s^{3+\alpha}}{s^{3+\alpha} - s^\alpha}\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\, dx + \frac{1}{p+1}\frac{s^{2p+2} - s^3}{s^{2p+2} - s^{2p-1}}|u_s|_{p+1}^{p+1} \\ & - \frac{1}{2}|\nabla u_s|_2^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\, dx \leqslant 2s^3. \end{aligned} \tag{2.15}$$

To simplify (2.15), we need to use the following facts:

$$\frac{s^2 - s^3}{s^3 - 1} = \frac{-s^2}{s^2 + s + 1} \geqslant -1 \quad \text{for } s \geqslant 0,$$

$$g(s) \triangleq \frac{s^2 - s^{3+\alpha}}{s^{3+\alpha} - s^\alpha} = \frac{s^{2-\alpha} - s^3}{s^3 - 1} \xrightarrow{s \to 1^-} -\frac{1+\alpha}{3}, \quad \text{and} \quad g(s) \equiv g(1) = -1 \quad \text{if } \alpha = 2.$$

$p > \frac{\alpha+2}{2}$ implies that $\frac{2p-1}{1+\alpha} > 1$ and $\varepsilon_0 := \frac{2p+\alpha}{2(1+\alpha)} \in (1, \frac{2p-1}{1+\alpha})$. Hence, there is $\delta_1 > 0$ small enough such that $1 - \delta_1 > t_0$ and

$$g(s) \geqslant -\frac{\varepsilon_0(1+\alpha)}{3} = \frac{2p+\alpha}{6} \quad \text{for all } s \in (1-\delta_1, 1).$$

Let

$$h(s) = \frac{s^{2p+2} - s^3}{s^{2p+2} - s^{2p-1}} = \frac{s^3 - s^{4-2p}}{s^3 - 1} \quad \text{for } s \in (0,1),$$

then

$$\lim_{s\to 1^-} h(s) = \frac{2p-1}{3} \quad \text{and} \quad h'(s) = \frac{(2p-1)s^{6-2p} - 3s^2 - (2p-4)s^{3-2p}}{(s^3-1)^2},$$

$$h'(s) \xrightarrow{s\to 1^-} -\frac{(2p-1)(p-2)}{3} < 0 \quad \text{if } p > 2.$$

This shows that there is $\delta_2 > 0$ small enough and $1 - \delta_2 > t_0$ such that

$$h'(s) < 0 \quad \text{and} \quad h(s) \geqslant \lim_{s\to 1^-} h(s) = \frac{2p-1}{3} \quad \text{for all } s \in (1-\delta_2, 1) \text{ and } p > 2.$$

For $p = 2$, $h(s) \equiv \frac{2p-1}{3} = 1$, so we see that

$$h(s) \geqslant \frac{2p-1}{3} \quad \text{for all } s \in (1-\delta_2, 1) \text{ and } p \geqslant 2.$$

So, for $s \in (t^*, 1)$ with $t^* = 1 - \min\{\delta_1, \delta_2\}$, it follows from (2.15) that

$$-\frac{\lambda}{2}|u_s|_2^2 - \frac{2p+\alpha}{12}\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\,dx + \frac{1}{p+1}\frac{2p-1}{3}|u_s|_{p+1}^{p+1} - \frac{1}{2}|\nabla u_s|_2^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\,dx \leqslant 2s^3.$$

That is,

$$\begin{aligned} -\frac{1}{p+1}|u_s|_{p+1}^{p+1} \geqslant{}& -\frac{3}{2p-1}\left(\frac{\lambda}{2}|u_s|_2^2 + \frac{1}{2}|\nabla u_s|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\,dx\right) \\ & - \frac{2p+\alpha}{4(2p-1)}\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\,dx - \frac{6}{2p-1}s^3. \end{aligned} \tag{2.16}$$

For $u \in U_s$, by (2.11) it gives that

$$\left(\frac{\lambda}{2}|u_s|_2^2 + \frac{1}{2}|\nabla u_s|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\,dx\right) + \frac{1}{2}\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\,dx - \frac{1}{p+1}|u_s|_{p+1}^{p+1} \leqslant c + \left(1 - s^3\right). \tag{2.17}$$

Hence, it follows from (2.16) and (2.17) that

$$\begin{aligned} & \frac{2p-2-\alpha}{4(2p-1)}\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\,dx + \frac{2p-4}{2p-1}\left(\frac{\lambda}{2}|u_s|_2^2 + \frac{1}{2}|\nabla u_s|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\,dx\right) \\ & \leqslant c + 1 - \frac{2p-7}{2p-1}s^3 \leqslant c + 1 + \left|\frac{2p-7}{2p-1}\right| \\ & \leqslant c + 2 \quad \text{if } p > 2 \text{ and } s < 1. \end{aligned}$$

This implies that, if $5 > p > \max\{2, \frac{\alpha+2}{2}\}$ and $s \in (t^*, 1)$

$$\frac{\lambda}{2}|u_s|_2^2 + \frac{1}{2}|\nabla u_s|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_s} u_s^2\,dx \leqslant \frac{(c+2)(2p-1)}{2(p-2)}, \tag{2.18}$$

and

$$\frac{1}{4}\int_{\mathbb{R}^3} \frac{u_s^2}{|y|^\alpha}\,dx \leqslant \frac{(c+2)(2p-1)}{2p-2-\alpha} \quad \text{for } \alpha \in [0,8).$$

So, using (2.5) and (2.6), it follows from (2.18) that

$$\frac{\lambda}{2}s|u|_2^2 + \frac{1}{2}s^3|\nabla u|_2^2 + \frac{1}{4}s^3\int_{\mathbb{R}^3} \phi_u u^2\,dx \leqslant \frac{(c+2)(2p-1)}{2(p-2)}.$$

Since $s \in (t^*, 1)$, $s \geqslant s^3 \geqslant {t^*}^3$ and $s^{1+\alpha} \geqslant {t^*}^{1+\alpha}$ for $\alpha \in [0,8)$, those and $p > \max\{2, \frac{\alpha+2}{2}\}$ imply that

$$\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2\,dx \leqslant \frac{2(c+2)(2p-1)}{(p-2){t^*}^3} + \frac{4(c+2)(2p-1)}{(2p-2-\alpha){t^*}^{1+\alpha}}, \tag{2.19}$$

and Lemma 2.5 is proved by taking $M = \frac{2(c+2)(2p-1)}{(p-2){t^*}^3} + \frac{4(c+2)(2p-1)}{(2p-2-\alpha){t^*}^{1+\alpha}}$. $\square$

Note that M given by the above lemma depends on λ, since c depends on λ by the definition of I. The following lemma is used to get a bounded (PS) sequence. In this lemma, the constant M can be chosen independent of λ if $\lambda \in (0,1]$.

Lemma 2.6. *Let $\alpha \in [0,8)$, $\max\{2, \frac{\alpha+2}{2}\} < p < 5$ and c be given by (2.8). Then there exists a bounded nonnegative sequence $\{u_n\} \subset H$ such that*

$$I(u_n) \to c > 0, \qquad I'(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{2.20}$$

Moreover, if $\lambda \in (0,1]$ there exists $M > 0$ which is independent of $\lambda \in (0,1]$ such that

$$\|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \leqslant M.$$

Proof. For $t \in (t^*, 1)$ with t^* given in Lemma 2.5, let

$$W_t = \{|u|:\ u \in U_t\}, \quad U_t \text{ defined in (2.12)}, \tag{2.21}$$

and then for $u \in W_t$, by (2.19), (2.7) and (2.11) we have that

$$\begin{aligned} I(u) - I(u_t) &= \frac{1}{2}(1-t^3)|\nabla u|_2^2 + \frac{\lambda}{2}(1-t)|u|_2^2 + \frac{1-t^{1+\alpha}}{2}\int_{\mathbb{R}^3} \frac{u^2}{|y|^\alpha}\,dx + \frac{1-t^3}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{1-t^{2p-1}}{p+1}|u|_{p+1}^{p+1} \\ &\leqslant (1-t^3)\left(\frac{\lambda}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx\right) + \frac{1-t^{2p-1}}{t^{2p-1}}\left(\frac{1}{2}\int_{\mathbb{R}^3} \frac{u_t^2}{|y|^\alpha}\,dx - \frac{1}{p+1}|u_t|_{p+1}^{p+1}\right) \end{aligned}$$

$$\leqslant \left(1-t^3\right)\left(\frac{\lambda}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\frac{1}{4}\int\limits_{\mathbb{R}^3}\phi_u u^2\,dx\right)+\frac{1-t^{2p-1}}{t^{2p-1}}I(u_t)$$
$$\leqslant \left(1-t^3\right)\frac{(c+2)(2p-1)}{(2p-4){t^*}^3}+\left(1-t^{2p-1}\right)\frac{c+1}{{t^*}^{2p-1}}$$
$$\to 0 \quad \text{as } t\to 1^-.$$

On the other hand, similar to (2.14) we know that

$$I(u_t)-I(u)\leqslant 2\left(1-t^3\right)\to 0 \quad \text{as } t\to 1^-.$$

Hence,

$$\limsup_{t\to 1^-}\sup_{u\in W_t}\left|I(u_t)-I(u)\right|=0. \tag{2.22}$$

For $M>0$ given by Lemma 2.5, we define

$$S=\left\{|u|:\ u\in H \text{ and } \|u\|_H^2+\int\limits_{\mathbb{R}^3}\phi_u u^2\,dx<M\right\},$$
$$S_\delta=\left\{u:\ u\in H \text{ and } dist(u,S)<\delta\right\},\quad \delta\in(0,1). \tag{2.23}$$

Clearly, $\|v\|_H\leqslant\sqrt{M}+1$ for all $v\in S_\delta$. Then, by Lemma 2.2, there is a constant $K:=K(M)$ such that

$$\left\|I'(u)-I'(v)\right\|_{H'}\leqslant K\|u-v\|_H \quad \text{for all } u,v\in S_\delta, \tag{2.24}$$

and since $I\in C^1(H,\mathbb{R})$, there exists $C_S>0$ such that

$$\left|I(u)-I(v)\right|\leqslant C_S\|u-v\|_H \quad \text{for all } u,v\in S_\delta. \tag{2.25}$$

For any $m\in\mathbb{N}$ and M given by Lemma 2.5, let

$$\Lambda_m=\left\{|u|:\ u\in H,\ \|u\|_H^2+\int\limits_{\mathbb{R}^3}\phi_u u^2\,dx<M+\frac{1}{m} \text{ and } \left|I(u)-c\right|\leqslant\frac{C_S+1}{\sqrt{m}}\right\}. \tag{2.26}$$

We claim that $\Lambda_m\neq\emptyset$.

Indeed, for any $m\geqslant 1$, since (2.22) we can find $t_m\in(t^*,1)$ such that

$$1-{t_m}^3<\frac{1}{32m}\quad \text{and}\quad I(u)\leqslant I(u_{t_m})+\frac{1}{32m}\quad \text{for all } u\in W_{t_m}.$$

Then it follows from (2.11) and (2.12) that

$$c-\frac{1}{32m}\leqslant I(u)\leqslant c+\frac{1}{16m}\quad \text{for all } u\in W_{t_m}. \tag{2.27}$$

By the definition of W_{t_m}, Lemma 2.5 implies that

$$\|u\|_H^2+\int\limits_{\mathbb{R}^3}\phi_u u^2\,dx\leqslant M\quad \text{for all } u\in W_{t_m}.$$

This and (2.27) show that $W_{t_m}\subset\Lambda_m$, that is $\Lambda_m\neq\emptyset$.

Next, we claim that there are infinitely many elements in $\{\Lambda_m\}_{m=1}^{+\infty}$, which we still simply denote by Λ_m $(m = 1, 2, \ldots)$, such that for each $m \geqslant 1$, there is $u_m \in \Lambda_m$ with

$$\|I'(u_m)\|_{H'} < \frac{1+K}{\sqrt{m}}, \quad K \text{ is given by (2.24).} \tag{2.28}$$

Then, to prove Lemma 2.6 we need only to show the above claim. By contradiction, if the claim is false, then there must be a number $\bar{m} \in \mathbb{N}$ with $\bar{m} > \max\{\frac{1}{8c}, 4\}$ such that

$$\|I'(u)\|_{H'} \geqslant \frac{1+K}{\sqrt{m}}, \quad \text{for all } m > \bar{m} \text{ and } u \in \Lambda_m. \tag{2.29}$$

By the above discussion we know that $W_{t_m} \subset \Lambda_m$. For any $u \in W_{t_m}$, the definition of W_{t_m} and Lemma 2.5 show that $\|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u u^2\,dx \leqslant M$ and $W_{t_m} \subset S$. Hence,

$$W_{t_m} \subset S \cap \left\{u \in H\colon \left|I(u) - c\right| < \frac{1}{8m}\right\} \subset S \cap \left\{u \in H\colon \left|I(u) - c\right| < \frac{C_S + 1}{\sqrt{m}}\right\} \subset \Lambda_m,$$

where (2.27) is used. Then

$$S \cap \left\{u \in H\colon \left|I(u) - c\right| < \frac{C_S + 1}{\sqrt{m}}\right\} \neq \emptyset.$$

Let $\varepsilon = \frac{1}{16m}$, $\delta = \frac{1}{2\sqrt{m}}$, then $\frac{8\varepsilon}{\delta} = \frac{1}{\sqrt{m}} < \frac{1}{2} < 1$, since $\bar{m} > \max\{\frac{1}{8c}, 4\}$. So,

$$S_{2\delta} = S_{\frac{1}{\sqrt{m}}} = \left\{u\colon\ u \in H \text{ and } \mathit{dist}(u, S) < \frac{1}{\sqrt{m}}\right\}.$$

By the definitions of S and Λ_m, we see that

$$S \cap \left\{u \in H\colon \left|I(u) - c\right| < \frac{C_S + 1}{\sqrt{m}}\right\} \subset \Lambda_m.$$

Hence, for any $u \in S \cap \{u \in H\colon |I(u) - c| < \frac{C_S+1}{\sqrt{m}}\} \subset \Lambda_m$, we have

$$\|I'(u)\|_{H'} \geqslant \frac{1+K}{\sqrt{m}}, \quad \text{for all } m > \bar{m}. \tag{2.30}$$

For any $v \in S_{\frac{1}{\sqrt{m}}} \cap \{u \in H\colon |I(u) - c| < \frac{1}{8m}\}$, there exists $u_0 \in S$ such that

$$\|u_0 - v\|_H < \frac{1}{\sqrt{m}}. \tag{2.31}$$

This and (2.25) show that

$$\begin{aligned} \left|I(u_0) - c\right| &\leqslant \left|I(v) - I(u_0)\right| + \left|I(v) - c\right| \\ &\leqslant \left|I(v) - c\right| + \frac{C_S}{\sqrt{m}} \leqslant \frac{1}{8m} + \frac{C_S}{\sqrt{m}} \leqslant \frac{C_S + 1}{\sqrt{m}}. \end{aligned}$$

Hence, $u_0 \in S \cap \{u \in H\colon |I(u) - c| < \frac{C_S+1}{\sqrt{m}}\}$. Then, it follows from (2.24), (2.30) and (2.31) that, for $v \in S_{\frac{1}{\sqrt{m}}} \cap \{u \in H\colon |I(u) - c| < \frac{1}{8m}\}$,

$$\begin{aligned}\big\|I'(v)\big\|_{H'} &= \big\|I'(v) - I'(u_0) + I'(u_0)\big\|_{H'} \\ &\geqslant \big\|I'(u_0)\big\|_{H'} - \big\|I'(v) - I'(u_0)\big\|_{H'} \\ &\geqslant \frac{1+K}{\sqrt{m}} - K\|u_0 - v\|_H \\ &\geqslant \frac{1+K}{\sqrt{m}} - K\frac{1}{\sqrt{m}} = \frac{1}{\sqrt{m}}.\end{aligned}$$

Applying Lemma 2.1 with $X = H$, $\varphi = I$, we know that there is a homeomorphism $\eta(t,\cdot) : [0,1] \times H \to H$ such that

$$\eta(t,u) = u, \quad \text{if } t = 0 \text{ or } u \notin S_{\frac{1}{\sqrt{m}}} \cap \left\{u \in H\colon \big|I(u) - c\big| \leqslant \frac{1}{8m}\right\}; \tag{2.32}$$

$$I\big(\eta(1,u)\big) \leqslant c - \frac{1}{16m}, \quad \text{for } u \in S \cap \left\{u \in H\colon \big|I(u) - c\big| \leqslant \frac{1}{8m}\right\}; \tag{2.33}$$

$$I\big(\eta(t,u)\big) \leqslant I(u), \quad \text{for any } u \in H. \tag{2.34}$$

Let $\xi(u) := \eta(1,u)$ and $\bar\gamma(t) = \xi(|\gamma_{t_m}(t)|) \in C([0,1],H)$. By $m > \bar m > \max\{\frac{1}{8c}, 4\}$, $c > \frac{1}{8m}$, then $\{0,e\} \nsubseteq S_{\frac{1}{\sqrt{m}}} \cap \{u \in H\colon |I(u) - c| < \frac{1}{8m}\}$, since $I(e) < 0$ and $|I(e) - c| = c + |I(e)| > c$ where e is given by Lemma 2.3. With this observation and (2.32) we see that $\bar\gamma(0) = \xi(|\gamma_{t_m}(0)|) = \xi(0) = \eta(1,0) = 0$, $\bar\gamma(1) = \xi(|\gamma_{t_m}(1)|) = \xi(e) = \eta(1,e) = e$. Hence, $\bar\gamma \in \Gamma$, with Γ defined in (2.8). For each $m \geqslant \bar m$, let $u_m \in \bar\gamma([0,1])$ be such that

$$I\big(\xi\big(|u_m|\big)\big) = \max_{u \in \gamma_{t_m}[0,1]} I\big(\xi\big(|u|\big)\big) = \max_{v \in \bar\gamma[0,1]} I(v) \geqslant c. \tag{2.35}$$

Since $u_m \in \gamma_{t_m}[0,1]$, $|u_m| \in |\gamma_{t_m}[0,1]| = \{|u|\colon u \in \gamma_{t_m}[0,1]\}$. We are ready to get a contradiction in both of the following two cases.

Case A. If $|u_m| \in |\gamma_{t_m}[0,1]| \setminus U_{t_m}$, then (2.34) and the definition of U_{t_m} imply that

$$I\big(\xi\big(|u_m|\big)\big) = I\big(\eta\big(1,|u_m|\big)\big) \leqslant I(u_m) \leqslant c - \big(1 - t_m^3\big) < c,$$

which contradicts (2.35).

Case B. If $|u_m| \in U_{t_m}$, then by (2.21) $|u_m| \in W_{t_m}$ and (2.27) implies that $|I(|u_m|) - c| \leqslant \frac{1}{16m}$. Moreover, $\|u_m\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_m} {u_m}^2\,dx \leqslant M$ by Lemma 2.5. Hence $|u_m| \in S \cap \{u \in H\colon |I(u) - c| \leqslant \frac{1}{16m}\}$, and it follows from (2.33) that

$$I\big(\xi\big(|u_m|\big)\big) = I\big(\eta\big(1,|u_m|\big)\big) \leqslant c - \frac{1}{16m} < c,$$

this is a contradiction to (2.35). □

3. Existence for $\lambda > 0$: Proof of Theorem 1.1

Motivated by [5], we prove Theorem 1.1 by a result due to Solimini [22], which is a version of so called concentration-compactness principle. To state this result, we introduce first the operator $T_{s,\xi}$ and its basic properties. Let $s > 0$, $N \geqslant 3$ and $\xi \in \mathbb{R}^N$ be fixed, for any $u \in L^q(\mathbb{R}^N)$, $q \in (1,+\infty)$, we define

$$T_{s,\xi}u(x) \triangleq T(s,\xi)u(x) := s^{-\frac{N-2}{2}} u\big(s^{-1}x + \xi\big), \quad \forall x \in \mathbb{R}^N. \tag{3.1}$$

Clearly, $T(s,\xi)u \in L^q(\mathbb{R}^N)$ if $u \in L^q(\mathbb{R}^N)$ and $T(s,\xi)$ is also well defined on Hilbert space $D^{1,2}(\mathbb{R}^N)$ with scalar product

$$\langle u,v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx, \quad \text{for } u,v \in D^{1,2}(\mathbb{R}^N), \tag{3.2}$$

since $T(s,\xi)u \in D^{1,2}(\mathbb{R}^N)$ if $u \in D^{1,2}(\mathbb{R}^N)$. It is not difficult to see that the linear operators

$$u \in L^{2^*}(\mathbb{R}^N) \rightarrowtail T(s,\xi)u \in L^{2^*}(\mathbb{R}^N) \quad \text{and} \quad u \in D^{1,2}(\mathbb{R}^N) \rightarrowtail T(s,\xi)u \in D^{1,2}(\mathbb{R}^N)$$

are isometric, where $2^* = \frac{2N}{N-2}$. Moreover, we have that

$$T_{s,\xi}^{-1} = T(s^{-1}, -s\xi), \qquad T_{s,\xi}T_{\mu,\eta} = T(s\mu, \xi/\mu + \eta), \tag{3.3}$$

$$|\nabla T_{s,\xi}u|_2^2 = |\nabla u|_2^2, \qquad |T_{s,\xi}u|_q^q = s^{N-\frac{q(N-2)}{2}}|u|_q^q. \tag{3.4}$$

For $N \geqslant 3$, $k \in [2,N)$ and $x \in \mathbb{R}^N$, in this section we denote by

$$x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad \text{i.e. } y \in \mathbb{R}^k,\ z \in \mathbb{R}^{N-k},$$

$\tilde{y} = (y,0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\tilde{z} = (0,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Similarly, $x_n = (y_n,z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\tilde{y}_n = (y_n,0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

Lemma 3.1. *(See [5, Proposition 22].) Let $\{\eta_n\} \subset \mathbb{R}^N$ be such that $\lim_{n\to\infty}|\eta_n| = \infty$ and fix $R > 0$. Then for any $m \in \mathbb{N} \setminus \{0,1\}$ there exists $N_m \in \mathbb{N}$ such that for any $n > N_m$ one can find a sequence of unit orthogonal matrices $\{g_i\}_{i=1}^m \in O(N)$ satisfying the condition*

$$B_R(g_i\eta_n) \cap B_R(g_j\eta_n) = \emptyset, \quad \textit{for } i \neq j.$$

Lemma 3.2. *(See [5, Proposition 11].) Let $q \in (1,\infty)$ and $\{s_n\} \subset (0,\infty)$, $\{\xi_n\} \subset \mathbb{R}^N$ be such that $s_n \xrightarrow{n} s \neq 0$, $\xi_n \xrightarrow{n} \xi$. Then*

$$T_{s_n,\xi_n}u_n \overset{n}{\rightharpoonup} T_{s,\xi}u \quad \textit{weakly in } L^q(\mathbb{R}^N),$$

if $u_n \overset{n}{\rightharpoonup} u$ weakly in $L^q(\mathbb{R}^N)$.

Lemma 3.3. *Let $\{s_n\} \subset (0,\infty)$, $\{\xi_n\} \subset \mathbb{R}^N$ be such that $s_n \xrightarrow{n} s_0 \neq 0$, $\xi_n \xrightarrow{n} \xi$. If $v_n \overset{n}{\rightharpoonup} v$ weakly in $D^{1,2}(\mathbb{R}^N)$, then*

$$T_{s_n,\xi_n}v_n \overset{n}{\rightharpoonup} T_{s_0,\xi}v \quad \textit{weakly in } D^{1,2}(\mathbb{R}^N).$$

Proof. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, by (3.2) we get that

$$\langle T_{s_n,0}^{-1}v_n, \varphi\rangle = \langle v_n, T_{s_n,0}\varphi\rangle = \langle v_n, T_{s_0,0}\varphi\rangle + \langle v_n, T_{s_n,0}\varphi - T_{s_0,0}\varphi\rangle. \tag{3.5}$$

Since

$$\lim_{n\to\infty}\left|\nabla(T_{s_n,0}\varphi - T_{s_0,0}\varphi)\right|_2^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla T_{s_n,0}\varphi|^2\,dx + \int_{\mathbb{R}^N}|\nabla T_{s_0,0}\varphi|^2\,dx$$
$$- 2\lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla T_{s_n,0}\varphi\nabla T_{s_0,0}\varphi\,dx = 0,$$

we have

$$\langle v_n, T_{s_n,0}\varphi - T_{s_0,0}\varphi\rangle \leqslant |\nabla v_n|_2 \big|\nabla(T_{s_n,0}\varphi - T_{s_0,0}\varphi)\big|_2 \xrightarrow{n} 0. \tag{3.6}$$

By $T_{s_0,0}\varphi \in C_0^\infty(\mathbb{R}^N)$ and $v_n \overset{n}{\rightharpoonup} v$ weakly in $D^{1,2}(\mathbb{R}^N)$, we have

$$\langle v_n, T_{s_0,0}\varphi\rangle \xrightarrow{n} \langle v, T_{s_0,0}\varphi\rangle = \big\langle T_{s_0,0}^{-1}v, \varphi\big\rangle. \tag{3.7}$$

It follows from (3.5) to (3.7) that

$$\big\langle T_{s_n,0}^{-1}v_n, \varphi\big\rangle \xrightarrow{n} \big\langle T_{s_0,0}^{-1}v, \varphi\big\rangle, \quad \text{for any } \varphi \in C_0^\infty\big(\mathbb{R}^N\big). \tag{3.8}$$

On the other hand, for any $\psi \in D^{1,2}(\mathbb{R}^N)$ and any $\epsilon > 0$, there exists $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $|\nabla(\psi-\varphi)|_2 < \epsilon$ and

$$\big\langle T_{s_n,0}^{-1}v_n, \psi - \varphi\big\rangle \leqslant \big|\nabla\big(T_{s_n,0}^{-1}v_n\big)\big|_2\big|\nabla(\psi-\varphi)\big|_2 = |\nabla v_n|_2\big|\nabla(\psi-\varphi)\big|_2,$$

this and (3.8) imply that

$$\big\langle T_{s_n,0}^{-1}v_n, \varphi\big\rangle \xrightarrow{n} \big\langle T_{s_0,0}^{-1}v, \varphi\big\rangle, \quad \text{for any } \varphi \in D^{1,2}\big(\mathbb{R}^N\big). \qquad \square$$

Lemma 3.4. *(See [22, Theorem 1].) If $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ is bounded, then, up to a subsequence, either $u_n \xrightarrow{n} 0$ in $L^{2^*}(\mathbb{R}^N)$ or there exist $\{s_n\} \subset (0,\infty)$ and $\{\xi_n\} \subset \mathbb{R}^N$ such that*

$$T_{s_n,\xi_n}u_n \overset{n}{\rightharpoonup} u \neq 0 \quad \textit{weakly in } L^{2^*}\big(\mathbb{R}^N\big).$$

Let

$$D_s^{1,2}\big(\mathbb{R}^N\big) \triangleq \big\{u \in D^{1,2}\big(\mathbb{R}^N\big)\colon\ u(x) = u(y,z) = u\big(|y|, z\big)\big\},$$

we see that $D_s^{1,2}(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N)$ is a closed set, hence $D_s^{1,2}(\mathbb{R}^N)$ is a Hilbert space with scalar product as (3.2). Based on Lemmas 3.1 to 3.4, we have the following lemma which ensures us to get a nontrivial solution for (1.1) without proving the (PS) condition.

Lemma 3.5. *If $\{u_n\} \subset D_s^{1,2}(\mathbb{R}^N)$ is bounded and there exist $\{s_n\} \subset (0,+\infty)$ and $\{x_n\} \subset \mathbb{R}^N$ with $x_n = (y_n, z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ such that*

$$T(s_n, x_n)u_n \overset{n}{\rightharpoonup} u \neq 0 \quad \textit{weakly in } L^{2^*}\big(\mathbb{R}^N\big). \tag{3.9}$$

Then

$$v_n = T(s_n, 0)w_n \overset{n}{\rightharpoonup} v \not\equiv 0 \quad \textit{weakly in } D_s^{1,2}\big(\mathbb{R}^N\big),$$

where $w_n = T(1, \tilde{z}_n)u_n$ and $\tilde{z}_n = (0, z_n)$. Moreover, if $\{u_n\}$ is also bounded in $L^q(\mathbb{R}^N)$ for some $1 < q < 2^*$, then, there exists a constant $l > 0$ such that $s_n > l$ for all n.

Proof. The proof of this lemma is almost the same as that of Lemma 23 in [5]. But for the sake of completeness, we give its proof.

Since $\{u_n\}$ is bounded in $D_s^{1,2}(\mathbb{R}^N)$, by the definition of $T_{s,\xi}$ we see that $\{v_n\}$ is also bounded in $D_s^{1,2}(\mathbb{R}^N)$. Then there is $v \in D_s^{1,2}(\mathbb{R}^N)$ such that

$$v_n = T(s_n, 0)w_n \overset{n}{\rightharpoonup} v \quad \text{weakly in } D_s^{1,2}\big(\mathbb{R}^N\big).$$

We claim that $v \not\equiv 0$. Otherwise, if $v \equiv 0$, then it leads to a contradiction in the following two cases. For $x_n = (y_n, z_n)$, we recall that

$$\tilde{y}_n = (y_n, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \qquad \tilde{z}_n = (0, z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}.$$

Case A. If $\{s_n\tilde{y}_n\} \subset \mathbb{R}^N$ is bounded. Then, there is $\tilde{y}_0 = (y_0, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ such that $s_n\tilde{y}_n \xrightarrow{n} \tilde{y}_0$ and from (3.3) we have

$$T_{1,-s_n\tilde{y}_n} T_{s_n,\tilde{y}_n} w_n = T_{1,-s_n\tilde{y}_n} T_{s_n,x_n} u_n \overset{n}{\rightharpoonup} T_{1,-\tilde{y}_0} u \neq 0 \quad \text{in } L^{2^*}\big(\mathbb{R}^N\big),$$

where we used the assumption (3.9) and Lemma 3.2. On the other hand, since $v \equiv 0$, it follows from (3.3) that

$$T_{1,-s_n\tilde{y}_n} T_{s_n,\tilde{y}_n} w_n = T_{s_n,0} w_n = v_n \overset{n}{\rightharpoonup} 0 \quad \text{in } D^{1,2}\big(\mathbb{R}^N\big),$$

then we get a contradiction.

Case B. If $|s_n\tilde{y}_n| \to +\infty$. We claim that there is also a contradiction. Indeed, since $u \not\equiv 0$, there exist $\Omega \subset \mathbb{R}^N$, $|\Omega| \neq 0$ and $\kappa > 0$ such that $u > \kappa$ or $u < -\kappa$ a.e. in Ω. So we can choose $R > 0$ such that $|B_R \cap \Omega| > 0$ and

$$\left| \int\limits_{\mathbb{R}^N} T_{s_n,\tilde{y}_n} w_n \chi_{B_R \cap \Omega}\, dx \right| \xrightarrow{n} \left| \int\limits_{\mathbb{R}^N} u \chi_{B_R \cap \Omega}\, dx \right| \geqslant \kappa |B_R \cap \Omega| > 0.$$

But,

$$T_{s_n,\tilde{y}_n} w_n = T_{s_n,\tilde{y}_n} T_{s_n^{-1},0} v_n = T_{1,s_n\tilde{y}_n} v_n.$$

Then,

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^N} T_{s_n,\tilde{y}_n} w_n \chi_{B_R \cap \Omega}\, dx \right| &\leqslant \int\limits_{B_R} |T_{s_n,\tilde{y}_n} w_n|\, dx = \int\limits_{B_R(s_n\tilde{y}_n)} |v_n|\, dx \\ &\leqslant C_R \Bigg\{ \int\limits_{B_R(s_n\tilde{y}_n)} |v_n|^{2^*}\, dx \Bigg\}^{\frac{1}{2^*}}. \end{aligned}$$

This implies

$$\inf_n \int\limits_{B_R(s_n\tilde{y}_n)} |v_n|^{2^*}\, dx > \epsilon > 0.$$

Since $|s_n\tilde{y}_n| \to +\infty$, by Lemma 3.1 we know that, for any $m \in \mathbb{N}$, there exist $n_m \in \mathbb{N}$ and $\{g_i\}_{i=1}^m \subset O(N)$ such that

$$\int_{\mathbb{R}^N} |u_n|^{2^*}\,dx = \int_{\mathbb{R}^N} |v_n|^{2^*}\,dx \geqslant \sum_{i=1}^{m} \int_{B_R(g_i(s_n\tilde{y}_n))} |v_n|^{2^*}\,dx$$
$$= m \int_{B_R(s_n\tilde{y}_n)} |v_n|^{2^*}\,dx > m\epsilon \quad \text{for } n > n_m,$$

where we used (3.4) and $v(y,z) = v(|y|,z)$. Let $m \to \infty$, then $|u_n|_{2^*} \xrightarrow{n} +\infty$, which contradicts that $\{u_n\} \subset L^{2^*}$ is bounded.

Now we can choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} v\varphi\,dx \neq 0$. Take $R > 0$ such that $\operatorname{supp}\varphi \subset B_R$. Since $u \in D^{1,2}(\mathbb{R}^N) \to T(s,\xi)u \in D^{1,2}(\mathbb{R}^N)$ is isometric, we know that $\{T_{\lambda_n,0}w_n\}$ is also bounded in $D^{1,2}(B_R)$, hence in $L^2(B_R)$, then $T_{s_n,0}w_n \rightharpoonup v$ in $L^2(B_R)$. Thus,

$$\int_{\mathbb{R}^N} T_{s_n,0}w_n\varphi\,dx = \int_{B_R} T_{s_n,0}w_n\varphi\,dx \to \int_{B_R} v\varphi\,dx = \int_{\mathbb{R}^N} v\varphi\,dx \neq 0.$$

On the other hand, since $1 < q < 2^*$ and $\frac{N}{q} - \frac{N-2}{2} > 0$,

$$\left|\int_{\mathbb{R}^N} T_{s_n,0}w_n\varphi\,dx\right| \leqslant |\varphi|_\infty |B_R|^{\frac{q-1}{q}} |T_{s_n,0}w_n|_{L^q(B_R)}$$
$$\leqslant s_n^{\frac{N}{q}-\frac{N-2}{2}} |\varphi|_\infty |B_R|^{\frac{q-1}{q}} \sup_n |u_n|_q.$$

Therefore, if $\lim_{n\to\infty} s_n = 0$, we obtain a contradiction. This implies that there exists $l > 0$ such that $\inf_n s_n > l$, since $s_n > 0$ for all n. $\square$

Lemma 3.6. *Let $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$ be a nonnegative function, and $K \subset \mathbb{R}^N$ be a closed set with zero measure. Then there exists $\varphi \in C_0^\infty(\mathbb{R}^N \setminus K)$ with $\varphi \geqslant 0$ such that $\int_{\mathbb{R}^N} \nabla u \nabla\varphi\,dx > 0$.*

Proof. Since $K \subset \mathbb{R}^N$ is closed and $u \not\equiv 0$, we can choose a ball $B \subset\subset \mathbb{R}^N \setminus K$, and a nonnegative function $f \in C_0^\infty(B) \subset C_0^\infty(\mathbb{R}^N \setminus K)$ such that $\int_{\mathbb{R}^N} uf\,dx > 0$. Otherwise, $u(x) = 0$ a.e. in $x \in \mathbb{R}^N \setminus K$, and it follows from $|K| = 0$ that $u(x) = 0$ a.e. in $x \in \mathbb{R}^N$, which contradicts $u \not\equiv 0$ in $D^{1,2}(\mathbb{R}^N)$. Then the problem

$$\begin{cases} -\Delta v = f, & x \in B, \\ v = 0, & x \in \partial B \end{cases}$$

has a nontrivial solution $\tilde{\varphi} \geqslant 0$ on B and $\tilde{\varphi} \in C_0^\infty(B)$. Setting

$$\varphi = \begin{cases} \tilde{\varphi}, & x \in B, \\ 0, & x \in \mathbb{R}^N \setminus B. \end{cases}$$

Hence,

$$\int_{\mathbb{R}^N} \nabla u \nabla\varphi\,dx = \int_{\mathbb{R}^N} uf\,dx > 0. \qquad \square$$

Based on Lemmas 3.5 and 3.6, we prove now the following theorem, which is important in proving our main Theorems 1.1 and 1.2.

Theorem 3.1. *Let $\{u_n\} \subset E$ be a nonnegative sequence such that*

$$\|u_n\|_E + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \leqslant C, \quad \text{and for any } \varphi \in C_0^\infty(\mathbb{R}^3 \setminus T), \text{ there holds}$$

$$\int_{\mathbb{R}^3} \left[\nabla u_n \nabla \varphi + \left(\frac{1}{|y|^\alpha} + \lambda_n\right) u_n \varphi\right] dx + \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n \varphi\,dx = \int_{\mathbb{R}^3} {u_n}^p \varphi\,dx + o(1), \tag{3.10}$$

where $\alpha \geqslant 0$, $p \in (2,5)$ and $\lambda_n \geqslant 0$ with $\lambda_n \xrightarrow{n} \lambda_0 < +\infty$. If $\{u_n\}$ does not converge to 0 in $L^6(\mathbb{R}^3)$, then there exist $\{\tilde{z}_n\} = \{(0, z_n)\} \subset \mathbb{R}^2 \times \mathbb{R}$ and a nonnegative function $w \in E \setminus \{0\}$ such that

$$w_n = T_{1,\tilde{z}_n} u_n \stackrel{n}{\rightharpoonup} w \quad \text{weakly in } E,$$

and, for any $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$,

$$\int_{\mathbb{R}^3} \left[\nabla w \nabla \varphi + \left(\frac{1}{|y|^\alpha} + \lambda_0\right) w \varphi\right] dx + \int_{\mathbb{R}^3} \phi_w(x) w \varphi\,dx = \int_{\mathbb{R}^3} w^p \varphi\,dx. \tag{3.11}$$

Moreover, $\|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2\,dx \leqslant C$ and $w \in C^2(\mathbb{R}^3 \setminus T)$.

Proof. If $\{u_n\} \subset E$ does not converge to 0 in $L^6(\mathbb{R}^3)$, by Lemma 3.4 with $N = 3$, there exist $\{s_n\} \subset (0, +\infty)$ and $\{x_n\} \subset \mathbb{R}^3$ with $x_n = (y_n, z_n) \in \mathbb{R}^2 \times \mathbb{R}$ such that

$$T_{s_n, x_n} u_n \stackrel{n}{\rightharpoonup} u \neq 0 \quad \text{weakly in } L^6(\mathbb{R}^3). \tag{3.12}$$

Let

$$\tilde{z}_n = (0, z_n) \in \mathbb{R}^2 \times \mathbb{R}^1, \qquad w_n = T_{1,\tilde{z}_n} u_n = T(1, \tilde{z}_n) u_n(x). \tag{3.13}$$

By (3.12) and Lemma 3.5 with $N = 3$, we have

$$v_n = T_{s_n,0} w_n \stackrel{n}{\rightharpoonup} v \not\equiv 0, \quad \text{weakly in } D_s^{1,2}(\mathbb{R}^3), \tag{3.14}$$

where v is nonnegative. And we claim that $s_n > l > 0$ for all $n \in \mathbb{N}$. Indeed, since $-\Delta \phi_{u_n} = u_n^2$, we easily conclude

$$\int_{\mathbb{R}^3} |u_n|^3\,dx = \int_{\mathbb{R}^3} \nabla \phi_{u_n} \nabla u_n\,dx \quad \text{and} \quad \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx = \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2\,dx.$$

By applying Hölder inequality, we deduce that

$$2\int_{\mathbb{R}^3} |u_n|^3\,dx \leqslant \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2\,dx = \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \leqslant C.$$

So, taking $N = 3$ and $q = 3$ in Lemma 3.5, we know that there is $l > 0$ such that $s_n > l$ for all $n \in \mathbb{N}$.

Step 1. There exists $L > l > 0$ such that $s_n < L$ for $n \in \mathbb{N}$ large.

Recalling the definition of T in (1.11), we have $|T| = 0$. Since $v \geqslant 0$, by Lemma 3.6, there exists a nonnegative function $\varphi_1 \in C_0^\infty(\mathbb{R}^3 \setminus T)$ such that

$$\int_{\mathbb{R}^3} \nabla v \nabla \varphi_1 \, dx > 0.$$

It follows from (3.13) and (3.14) that

$$\int_{\mathbb{R}^3} \nabla(T_{s_n,\tilde{z}_n} u_n) \nabla \varphi_1 \, dx \to \int_{\mathbb{R}^3} \nabla v \nabla \varphi_1 \, dx > 0. \tag{3.15}$$

Noting that $T^{-1}_{s_n,\tilde{z}_n} \varphi_1(x) = s_n^{\frac{1}{2}} \varphi_1(s_n x - s_n \tilde{z}_n)$, then $T^{-1}_{s_n,\tilde{z}_n} \varphi_1(x) \in C_0^\infty(\mathbb{R}^3 \setminus T)$. Taking $\varphi = T^{-1}_{s_n,\tilde{z}_n} \varphi_1(x)$ in (3.10), we see that

$$\begin{aligned} &\int_{\mathbb{R}^3} \phi_{u_n} u_n T^{-1}_{s_n,\tilde{z}_n} \varphi_1 \, dx + \int_{\mathbb{R}^3} \nabla u_n \nabla \big(T^{-1}_{s_n,\tilde{z}_n} \varphi_1\big) + \left(\lambda_n + \frac{1}{|y|^\alpha}\right) u_n T^{-1}_{s_n,\tilde{z}_n} \varphi_1 \, dx \\ &\quad = \int_{\mathbb{R}^3} u_n^p T^{-1}_{s_n,\tilde{z}_n} \varphi_1 \, dx + o(1). \end{aligned}$$

It follows from $u_n \geqslant 0$ and $\lambda_n \geqslant 0$ that

$$\int_{\mathbb{R}^3} \nabla u_n \nabla \big(T^{-1}_{s_n,\tilde{z}_n} \varphi_1\big) \, dx \leqslant \int_{\mathbb{R}^3} u_n^p T^{-1}_{s_n,\tilde{z}_n} \varphi_1 \, dx + o(1).$$

That is,

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla(T_{s_n,\tilde{z}_n} u_n) \nabla \varphi_1 \, dx &\leqslant s_n^{\frac{p-5}{2}} \int_{\mathbb{R}^3} (T_{s_n,\tilde{z}_n} u_n)^p \varphi_1 \, dx + o(1) \\ &\leqslant C s_n^{\frac{p-5}{2}} \int_{\operatorname{supp} \varphi_1} (T_{s_n,\tilde{z}_n} u_n)^p \, dx + o(1) \\ &\leqslant C s_n^{\frac{p-5}{2}} |T_{s_n,\tilde{z}_n} u_n|_6^p + o(1) \quad \text{for } 2 < p < 5 \\ &\leqslant C s_n^{\frac{p-5}{2}} |\nabla u_n|_2^p + o(1), \quad \text{by (3.1)}. \end{aligned}$$

Since $\{u_n\}$ is bounded in E, if $s_n \to \infty$, it follows that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} \nabla(T_{s_n,\tilde{z}_n} u_n) \nabla \varphi \, dx \leqslant 0,$$

which contradicts (3.15).

Step 2. $\{w_n\}$ is a bounded sequence in E such that, for any $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$,

$$\int_{\mathbb{R}^3} \left[\nabla w_n \nabla \varphi + \left(\frac{1}{|y|^\alpha} + \lambda_n\right) w_n \varphi\right] dx + \int_{\mathbb{R}^3} \phi_{w_n}(x) w_n \varphi \, dx = \int_{\mathbb{R}^3} {w_n}^p \varphi \, dx + o(1). \tag{3.16}$$

By the properties of $T_{s,\xi}$ in (3.4), we have

$$\left|\nabla(T_{1,\tilde{z}_n}u_n)\right|_2 = |\nabla u_n|_2, \qquad \int\limits_{\mathbb{R}^3} \frac{|T_{1,\tilde{z}_n}u_n|^2}{|y|^\alpha}\,dx = \int\limits_{\mathbb{R}^3} \frac{|u_n|^2}{|y|^\alpha}\,dx,$$

hence, $\|w_n\|_E^2 = \|T_{1,\tilde{z}_n}u_n\|_E^2 = \|u_n\|_E^2$ and $\{w_n\}$ is bounded in E. By the properties of $T_{1,\tilde{z}_n}$ in (3.3) and ϕ_u in (1.7), it is easy to see that

$$\int\limits_{\mathbb{R}^3} \phi_{u_n}u_nT^{-1}_{1,\tilde{z}_n}\varphi\,dx = \int\limits_{\mathbb{R}^3} T_{1,\tilde{z}_n}(\phi_{u_n}u_n)\varphi\,dx = \int\limits_{\mathbb{R}^3} \phi_{w_n}w_n\varphi\,dx,$$

$$\int\limits_{\mathbb{R}^3}\left[\nabla u_n\nabla T^{-1}_{1,\tilde{z}_n}\varphi + \left(\frac{1}{|y|^\alpha}+\lambda_n\right)u_nT^{-1}_{1,\tilde{z}_n}\varphi\right]dx = \int\limits_{\mathbb{R}^3}\left[\nabla w_n\nabla\varphi + \left(\frac{1}{|y|^\alpha}+\lambda_n\right)w_n\varphi\right]dx,$$

and

$$\int\limits_{\mathbb{R}^3} u_n^pT^{-1}_{1,\tilde{z}_n}\varphi\,dx = \int\limits_{\mathbb{R}^3} w_n^p\varphi\,dx.$$

Hence, (3.10) implies that (3.16) holds.

Step 3. $w_n \overset{n}{\rightharpoonup} w \not\equiv 0$ in E and $w(x) \geqslant 0$ a.e. in $x \in \mathbb{R}^3$.

By Step 1, there exists $s_0 \in [l, L]$ such that, passing to a subsequence, $s_n \xrightarrow{n} s_0$. Then, it follows from (3.14) and Lemma 3.3 that

$$w_n = T^{-1}_{s_n,0}v_n \overset{n}{\rightharpoonup} T_{\frac{1}{s_0},0}v \not\equiv 0 \quad \text{weakly in } D^{1,2}_s\left(\mathbb{R}^3\right). \tag{3.17}$$

By Step 2, there exists $w \in E$ such that, passing to a subsequence, $w_n \overset{n}{\rightharpoonup} w$ weakly in E. Since $E \subset D^{1,2}_s(\mathbb{R}^3)$, we have $(D^{1,2}_s(\mathbb{R}^3))^* \subset E^*$. Hence, $w_n \overset{n}{\rightharpoonup} w$ weakly in $D^{1,2}_s(\mathbb{R}^3)$. So, it follows from (3.17) that $w = T_{\frac{1}{s_0},0}v \not\equiv 0$ and $w(x) \geqslant 0$ a.e. in $x \in \mathbb{R}^3$, since $v \geqslant 0$ in (3.14).

Step 4. $\phi_w \in D^{1,2}(\mathbb{R}^3)$ and (3.11) holds.

For each $n \in \mathbb{N}$, $|\nabla\phi_{w_n}|_2^2 = \int_{\mathbb{R}^3}\phi_{w_n}w_n^2\,dx = \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx$, hence, $\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx < C$ implies that $\{\phi_{w_n}\}$ is bounded in $D^{1,2}_s(\mathbb{R}^3)$. So, there exists $\phi \in D^{1,2}_s(\mathbb{R}^3)$ such that $\phi_{w_n} \overset{n}{\rightharpoonup} \phi$ weakly in $D^{1,2}_s(\mathbb{R}^3)$, that is

$$\int\limits_{\mathbb{R}^3}\nabla\phi_{w_n}\nabla\varphi\,dx \xrightarrow{n} \int\limits_{\mathbb{R}^3}\nabla\phi\nabla\varphi\,dx, \quad \text{for any } \varphi \in C_0^\infty\left(\mathbb{R}^3\right). \tag{3.18}$$

On the other hand, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int\limits_{\mathbb{R}^3}\nabla\phi_{w_n}\nabla\varphi\,dx = \int\limits_{\mathbb{R}^3} w_n^2\varphi\,dx \quad\text{and}\quad \int\limits_{\mathbb{R}^3} w_n^2\varphi\,dx \xrightarrow{n} \int\limits_{\mathbb{R}^3} w^2\varphi\,dx. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\int\limits_{\mathbb{R}^3}\nabla\phi\nabla\varphi\,dx = \int\limits_{\mathbb{R}^3} w^2\varphi\,dx \quad \text{for any } \varphi \in C_0^\infty\left(\mathbb{R}^3\right).$$

So, ϕ is a solution of $-\Delta\phi = w^2$ in the sense of distribution. Since $w \in E \subset L^6(\mathbb{R}^3)$, $\phi_w(x) = \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \in W^{2,3}(\mathbb{R}^3)$ by Theorem 9.9 in [15], hence ϕ_w satisfies $-\Delta\phi_w = w^2$ in the sense of distribution (Theorem 6.21 in [19]). By uniqueness, we have $\phi_w = \phi \in D_s^{1,2}(\mathbb{R}^3)$. It follows from (3.18) that

$$\phi_{w_n} \overset{n}{\rightharpoonup} \phi_w \quad \text{weakly in } D_s^{1,2}(\mathbb{R}^3).$$

Then (see (3.18) in Step 3 of the proof of Theorem 1.1 in [17] for the details), for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \phi_{w_n}(x) w_n \varphi \, dx \xrightarrow{n} \int_{\mathbb{R}^3} \phi_w(x) w \varphi \, dx.$$

For each bounded domain $\Omega \subset \mathbb{R}^3$ and $q \in (1,6)$, it follows from (3.18) and the compactness of Sobolev embedding that $w_n \xrightarrow{n} w$ strongly in $L^q(\Omega)$. Hence, for any $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus T)$,

$$\int_{\mathbb{R}^3} \left[\nabla w_n \nabla \varphi + \left(\frac{1}{|y|^\alpha} + \lambda_n\right) w_n \varphi\right] dx \xrightarrow{n} \int_{\mathbb{R}^3} \left[\nabla w \nabla \varphi + \left(\frac{1}{|y|^\alpha} + \lambda_0\right) w \varphi\right] dx$$

and

$$\int_{\mathbb{R}^3} w_n^p \varphi \, dx \xrightarrow{n} \int_{\mathbb{R}^3} w^p \varphi \, dx.$$

Hence, (3.16) implies that (3.11) holds.

Step 5. $\|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 \, dx < C$.

By Step 3, we have $w_n \overset{n}{\rightharpoonup} w$ weakly in E, and Step 4 implies that

$$\int_{\mathbb{R}^3} \phi_w w^2 \, dx = |\nabla \phi_w|_2^2 \quad \text{and} \quad \phi_{w_n} \overset{n}{\rightharpoonup} \phi_w \quad \text{weakly in } D^{1,2}(\mathbb{R}^3),$$

and the lower semi-continuity of norm implies that

$$\|w\|_E \leqslant \liminf_{n\to+\infty} \|w_n\|_E,$$

and

$$\int_{\mathbb{R}^3} \phi_w w^2 \, dx = |\nabla \phi_w|_2^2 \leqslant \liminf_{n\to+\infty} |\nabla \phi_{w_n}|_2^2 = \liminf_{n\to+\infty} \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 \, dx.$$

Hence, we know from (3.13) that

$$\begin{aligned} \|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 \, dx &\leqslant \liminf_{n\to+\infty} \left\{ \|w_n\|_E + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 \, dx \right\} \\ &= \liminf_{n\to+\infty} \left\{ \|u_n\|_E + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx \right\} \leqslant C. \end{aligned}$$

Step 6. $w(x) \in C^2(\mathbb{R}^3 \setminus T)$.

Since $\lambda_0 \geqslant 0$ and $w(x) \geqslant 0$ for a.e. $x \in \mathbb{R}^3$, it follows from (3.11) that, for any nonnegative function $v \in C^\infty(\mathbb{R}^3 \setminus T)$,

$$\int_{\mathbb{R}^3} \nabla w \nabla v \, dx \leqslant \int_{\mathbb{R}^3} w^p v \, dx. \tag{3.20}$$

Then, Lemma 4.2 in Section 4 implies that (3.20) holds also for any nonnegative function $v \in H^1(\mathbb{R}^3)$. Note that, for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^3)$ and any nonnegative piecewise smooth function h on $[0, +\infty)$, $h(w)\varphi \in H^1(\mathbb{R}^3)$. Take $v = h(w)\varphi$ in (3.20), then we see that (4.7) in Section 4 holds with $u = w$ and $N = 3$. Hence, by Lemma 4.3, we have $w \in L^\infty(\mathbb{R}^3)$. Let $\Omega \subset\subset \mathbb{R}^3 \setminus T$ be a bounded domain with smooth boundary, then $\frac{1}{|y|}$ is a smooth function in Ω and $w \in W^{1,2}(\Omega)$ is a weak solution of

$$-\Delta w(x) = f(x), \quad x \in \Omega, \tag{3.21}$$

where $f(x) = |w|^{p-1}w(x) - \phi_w(x)w(x) - (\lambda_0 + \frac{1}{|y|})w(x)$. Since $w, \phi_w \in W^{1,2}(\Omega)$ and $w \in L^\infty(\Omega)$, we have $f(x) \in W^{1,2}(\Omega)$. By using Theorem 8.10 in [15], we get $w \in W^{3,2}_{loc}(\Omega)$. Then, Sobolev imbedding theorem implies that $w \in C^{1/4}_{loc}(\Omega)$, hence $\phi_w(x) \in C^{2,1/4}_{loc}(\Omega)$ since $\phi_w(x)$ is a weak solution of $-\Delta\phi(x) = w^2(x)$ in $D^{1,2}(\Omega)$. It follows that $f(x) \in C^{1/4}_{loc}(\Omega)$. By applying Theorem 9.19 in [15] to (3.21), we have $w \in C^{2,1/4}_{loc}(\Omega)$. So $w \in C^2(\mathbb{R}^3 \setminus T)$. □

Proof of Theorem 1.1. Let $\{u_n\} \subset H$ be the bounded nonnegative (PS) sequence obtained by Lemma 2.6, then there exists $C > 0$, which is independent of λ if $\lambda \in (0, 1]$, such that

$$\|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 \, dx \leqslant C \quad \text{and} \quad u_n(x) \geqslant 0 \quad \text{a.e. in } x \in \mathbb{R}^3. \tag{3.22}$$

Hence,

$$\|u_n\|_E^2 + \int_{\mathbb{R}^3} \phi_{u_n} {u_n}^2 \, dx \leqslant C, \qquad u_n(x) \geqslant 0 \quad \text{a.e. in } x \in \mathbb{R}^3.$$

Moreover, (2.20) implies that (3.10) holds with $\lambda_n \equiv \lambda > 0$.

If $\{u_n\}$ does not converge to 0 in $L^6(\mathbb{R}^3)$, by Theorem 3.1 there exist $\{\tilde{z}_n\} = \{(0, z_n)\} \subset \mathbb{R}^2 \times \mathbb{R}$ and a nonnegative function $w \in E \setminus \{0\}$ such that

$$w_n = T_{1,\tilde{z}_n} u_n \xrightharpoonup{n} w \quad \text{weakly in } E, \tag{3.23}$$
$$\int_{\mathbb{R}^3} \left[\nabla w \nabla \varphi + \left(\frac{1}{|y|^\alpha} + \lambda\right) w\varphi\right] dx + \int_{\mathbb{R}^3} \phi_w(x) w\varphi \, dx = \int_{\mathbb{R}^3} w^p \varphi \, dx \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^3 \setminus T),$$

i.e., w is a weak solution of (1.1) in E. Moreover, $w \in C^2(\mathbb{R}^3 \setminus T)$ and

$$\|w\|_E + \int_{\mathbb{R}^3} \phi_w w^2 \, dx \leqslant C. \tag{3.24}$$

Now, we claim that $w \in H$. In fact, by (3.3) and (3.23), we know that $\|w_n\|_H = \|u_n\|_H$ and $\|w_n\|_H$ is bounded, so there exists $w^* \in H$ such that

$$w_n \xrightharpoonup{n} w^* \quad \text{weakly in } H \quad \text{and} \quad w_n(x) \xrightarrow{n} w^*(x), \quad \text{a.e. in } x \in \mathbb{R}^3. \tag{3.25}$$

On the other hand, (3.23) implies that

$$w_n(x) \xrightarrow{n} w(x), \quad \text{a.e. in } x \in \mathbb{R}^3.$$

This and (3.25) show that $w = w^* \in H$. Moreover, if $\lambda \in (0,1]$, Lemma 2.6 shows that there exists $M > 0$ independent of $\lambda \in (0,1]$ such that (3.22) holds with $C = M$, then (3.24) holds with $C = M$. Hence, to complete the proof of Theorem 1.1, we only need to prove that $\{u_n\}$ cannot converge to 0 in $L^6(\mathbb{R}^3)$. For $r \in (2,6)$, by Hölder inequality we have

$$\int_{\mathbb{R}^3} |u_n|^r\,dx = \int_{\mathbb{R}^3} |u_n|^{\frac{2}{q}} |u_n|^{\frac{6}{q'}}\,dx \leqslant |u_n|_2^{\frac{2}{q}} |u_n|_6^{\frac{6}{q'}}$$

where $q = \frac{4}{6-r} > 1$, $q' = \frac{q}{q-1} = \frac{4}{r-2} > 1$. Hence, if $u_n \xrightarrow{n} 0$ in $L^6(\mathbb{R}^3)$, then $u_n \xrightarrow{n} 0$ in $L^r(\mathbb{R}^3)$ for $r \in (2,6)$, this and (1.8) imply that $\int_{\mathbb{R}^3} \phi_{u_n}(x) {u_n}^2\,dx \xrightarrow{n} 0$. Therefore, by (2.20) we have that, for $p \in (2,5)$,

$$\begin{aligned} c &= \lim_{n\to\infty}\left[I(u_n) - \frac{1}{2}I'(u_n)u_n\right] \\ &= \lim_{n\to\infty}\left[-\frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}(x){u_n}^2\,dx + \frac{p-3}{2(p+1)}\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx\right] = 0, \end{aligned}$$

but, this is impossible since $c > 0$. □

4. Existence for $\lambda = 0$: Proof of Theorem 1.2

In order to prove our Theorem 1.2, we need some further lemmas.

Lemma 4.1. *Let $N \geqslant 3$ and Ω be a domain (bounded or unbounded) in $\mathbb{R}^N$, let $\Gamma \subset \Omega$ be a closed Manifold with $\operatorname{codim}\Gamma = k \geqslant 2$. Then $C_0^\infty(\Omega \setminus \Gamma)$ is dense in $H_0^1(\Omega)$.*

Proof. For each $u \in H_0^1(\Omega) \cap C_0^\infty(\Omega \setminus \Gamma)^\perp$ and $\tilde{\varphi} \in C_0^\infty(\Omega \setminus \Gamma)$, we have

$$\langle u, \tilde{\varphi}\rangle_{H_0^1(\Omega)} = 0. \tag{4.1}$$

Since $C_0^\infty(\Omega \setminus \Gamma)$ is dense in $H_0^1(\Omega \setminus \Gamma)$, it follows that

$$\langle u, \psi\rangle_{H_0^1(\Omega)} = 0 \quad \text{for any } \psi \in H_0^1(\Omega \setminus \Gamma). \tag{4.2}$$

It is true that $C_0^\infty(\Omega \setminus \Gamma)$ is dense in $H_0^1(\Omega)$ if $C_0^\infty(\Omega \setminus \Gamma)^\perp \cap H_0^1(\Omega) = \{0\}$. Hence, we only need to show that (4.1) holds for all $\tilde{\varphi} \in C_0^\infty(\Omega)$ as follows.

For any $\varphi \in C_0^\infty(\Omega)$, let $\Omega_0 = \operatorname{supp}\varphi$. If $\Omega_0 \cap \Gamma = \emptyset$, then $\varphi \in C_0^\infty(\Omega \setminus \Gamma)$ and (4.1) holds with $\tilde{\varphi} = \varphi$. Otherwise, $\Omega_0 \cap \Gamma \neq \emptyset$, setting $\Gamma_0 = \Omega_0 \cap \Gamma$, and taking $d > 0$ small enough such that $\Gamma_d := \{x \in \Omega\colon\ dist(x, \Gamma_0) < d\} \subset \Omega$. Let

$$\psi_d(x) := \begin{cases} \frac{dist(x,\Gamma_{2d})}{d}, & x \in \Gamma_{3d}, \\ 1, & x \in \Omega \setminus \Gamma_{3d}, \end{cases}$$

then $\psi_d(x) \in C^{0,1}(\Omega)$ and $\|\psi_d\|_{C^{0,1}(\Omega)} \leqslant \frac{1}{d}$. Let $\varphi_d := \varphi(1-\psi_d)$, we have $\varphi\psi_d \in H_0^1(\Omega \setminus \Gamma)$ and $\varphi_d \in H_0^1(\Gamma_{3d})$. It follows from (4.2) that

$$\begin{aligned}\langle u,\varphi\rangle_{H^1} &= \langle u,\varphi_d+\varphi\psi_d\rangle_{H^1} = \langle u,\varphi_d\rangle_{H^1} + \langle u,\varphi\psi_d\rangle_{H^1}\\ &= \langle u,\varphi_d\rangle_{H^1} \leqslant \|u\|_{H^1(\Gamma_{3d})}\|\varphi_d\|_{H^1(\Gamma_{3d})}.\end{aligned} \tag{4.3}$$

By the definition of φ_d, we have

$$|\varphi_d|^2_{L^2(\Gamma_{3d})} = \int_{\Gamma_{3d}} \varphi_d^2\,dx \leqslant 4|\varphi|^2_{L^\infty(\Omega)}|\Gamma_{3d}| \xrightarrow{d\to 0} 0, \tag{4.4}$$

$$\begin{aligned}|\nabla\varphi_d|^2_{L^2(\Gamma_{3d})} &= \int_{\Gamma_{3d}} |\nabla\varphi_d|^2\,dx \leqslant C\|\varphi\|^2_{C^1(\Omega)}|\Gamma_{3d}|\left(1+\frac{1}{d^2}\right)\\ &\overset{codim\,\Gamma=k}{\leqslant} Cd^{k-2} \leqslant C, \quad \text{since } k\geqslant 2.\end{aligned} \tag{4.5}$$

So, $|\Gamma_{3d}| \xrightarrow{d\to 0} 0$ implies that

$$\|u\|_{H^1(\Gamma_{3d})} \xrightarrow{d\to 0} 0. \tag{4.6}$$

It follows from (4.3) to (4.6) that (4.1) holds for all $\tilde\varphi \in C_0^\infty(\Omega)$. □

Lemma 4.2. *Under the same assumptions as Lemma 4.1, then* $\{\varphi\in H_0^1(\Omega\setminus\Gamma)\colon\ \varphi(x)\geqslant 0\}$ *is dense in* $\{\varphi\in H_0^1(\Omega)\colon\ \varphi(x)\geqslant 0\}$.

Proof. Lemma 4.1 shows that for any $u(x)\in H_0^1(\Omega)$, there exist $\{\varphi_n(x)\}\subset C_0^\infty(\Omega\setminus\Gamma)$ such that

$$\|\varphi_n - u\|_{H^1(\Omega)} \xrightarrow{n} 0.$$

This lemma is proved if we have

$$\big\||\varphi_n|-|u|\big\|_{H^1(\Omega)} \xrightarrow{n} 0,$$

which is clear by using the following two facts,

$$\begin{aligned}0 &\leqslant \big|\big(|\varphi_n|-|u|\big)\big|_2^2 = \int_\Omega |\varphi_n|^2+|u|^2-2|\varphi_n||u|\,dx\\ &\leqslant \int_\Omega \big(\varphi_n^2+u^2-2\varphi_n u\big)\,dx = |\varphi_n-u|_2^2 \xrightarrow{n} 0,\\ 0 &\leqslant \big\||\varphi_n|-|u|\big\|^2_{D^{1,2}} = \int_\Omega \big(\big|\nabla|\varphi_n|\big|^2+\big|\nabla|u|\big|^2-2\nabla|\varphi_n|\nabla|u|\big)\,dx\\ &= \int_\Omega \big|\nabla(\varphi_n-u)\big|^2\,dx + 4\int_\Omega \nabla\varphi_n^+\nabla u^- + \nabla\varphi_n^-\nabla u^+\,dx \xrightarrow{n} 0,\end{aligned}$$

here and in what follows, we always mean that

$$w^+(x)=\max\{0,w(x)\} \quad\text{and}\quad w^-(x)=\min\{0,w(x)\}$$

for any function $w(x)$ on $\mathbb{R}^3$. □

Lemma 4.3. *(See Lemma 3.2 of [18].) Let $N \geqslant 3$, $p \in (1, \frac{N+2}{N-2})$ and let $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$ be a nonnegative function such that*

$$\int_{\mathbb{R}^N} \nabla u \nabla\big(h(u)\varphi\big)\,dx \leqslant \int_{\mathbb{R}^N} |u|^{p-1}uh(u)\varphi\,dx, \tag{4.7}$$

holds for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$ and any nonnegative piecewise smooth function h on $[0,+\infty)$ with $h' \in L^\infty(\mathbb{R})$. Then, $u \in L^\infty(\mathbb{R}^N)$ and there exist $C_1 > 0$ and $C_2 > 0$, which depend only on N and p, such that

$$|u|_\infty \leqslant C_1\big(1 + |u|_{2^*}^{C_2}\big)|u|_{2^*}.$$

Lemma 4.4. *For $p > 2$, let $(u,\phi) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a nontrivial nonnegative weak solution of the following problem*

$$\begin{cases} -\Delta u + \mu\phi(x)u \leqslant |u|^{p-1}u, & \mu > 0,\ x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{4.8}$$

Then

$$|u|_\infty > \mu^{\frac{1}{2(p-2)}}.$$

Proof. By assumption, $(u,\phi) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a weak solution of (4.8), then, for any nonnegative function $v \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \nabla u \nabla v\,dx + \mu\int_{\mathbb{R}^3} \phi(x)uv\,dx - \int_{\mathbb{R}^3} |u|^{p-1}uv\,dx \leqslant 0, \tag{4.9}$$

$$\int_{\mathbb{R}^3} \nabla\phi\nabla v\,dx = \int_{\mathbb{R}^3} u^2 v\,dx. \tag{4.10}$$

For $c > 0$, adding $c\int_{\mathbb{R}^3} u^2 v\,dx$ to both sides of (4.9), and using (4.10) we know that, for any $v \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla u \nabla v\,dx + \int_{\mathbb{R}^3} \big[cu^2 - |u|^{p-1}u\big]v\,dx + \mu\int_{\mathbb{R}^3} \phi(x)uv\,dx \leqslant c\int_{\mathbb{R}^3} \nabla\phi\nabla v\,dx. \tag{4.11}$$

For the above $c > 0$, taking $\epsilon > 0$ small, we set

$$w_1(x) = \big(u(x) - c\phi(x) - \epsilon\big)^+ \quad \text{and} \quad \Omega_1 = \big\{x \in \Omega:\ w_1(x) > 0\big\}. \tag{4.12}$$

It is easy to see that $u(x) \xrightarrow{|x|\to+\infty} 0$ and $\phi(x) \geqslant 0$ a.e. $x \in \mathbb{R}^3$, then $w_1 \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$ and $u(x)|_{\Omega_1} > c\phi(x) > 0$. Taking $v(x) = w_1(x)$ in (4.11), we see that

$$\int_{\Omega_1} \nabla u \nabla w_1\,dx + \int_{\Omega_1} \big[cu^2 - |u|^{p-1}u\big]w_1\,dx \leqslant c\int_{\Omega_1} \nabla\phi\nabla w_1\,dx. \tag{4.13}$$

However, for all $x \in \Omega_1$ we have $cu^2 - |u|^{p-1}u \geqslant 0$ if $c = \delta^{p-2}$ with $\delta = |u|_\infty$. Then, let $c = \delta^{p-2}$ and (4.13) implies that

$$\int_{\Omega_1} \nabla u \nabla w_1 \, dx - c \int_{\Omega_1} \nabla \phi \nabla w_1 \, dx \leqslant 0,$$

that is,

$$\int_{\Omega_1} \nabla\big(u - \delta^{p-2}\phi\big) \nabla w_1 \, dx = \int_{\Omega_1} |\nabla w_1|^2 \, dx = 0. \tag{4.14}$$

Hence, either $|\Omega_1| = 0$ or $w_1|_{\Omega_1} \equiv$ constant, this means that $u(x) \leqslant \delta^{p-2}\phi(x) + \epsilon$ a.e. $x \in \mathbb{R}^3$. For $\epsilon \to 0$, we have

$$u(x) \leqslant \delta^{p-2}\phi(x), \quad \text{a.e. in } x \in \mathbb{R}^3. \tag{4.15}$$

To prove that $|u|_\infty > \mu^{\frac{1}{2(p-2)}}$, we let $v = u$ in (4.9), it follows that

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \mu \int_{\mathbb{R}^3} \phi(x) u^2 \, dx - \int_{\mathbb{R}^3} u^{p+1} \, dx \leqslant 0,$$

that is,

$$\mu \int_{\mathbb{R}^3} \phi(x)|u|^2 \, dx \leqslant \int_{\mathbb{R}^3} |u|^{p+1} \, dx.$$

This and (4.15) show that

$$\int_{\mathbb{R}^3} \big(u^{p-2} - \mu\delta^{2-p}\big) u^3 \, dx \geqslant 0.$$

Hence, $\delta^{p-2} \geqslant \mu\delta^{2-p}$ by $p > 2$. On the other hand, by $u \not\equiv 0$ we know that $\delta > 0$. Then $|u|_\infty = \delta \geqslant \mu^{\frac{1}{2(p-2)}}$. □

Proof of Theorem 1.2. By Theorem 1.1, we know that, for each $\lambda \in (0, 1)$, problem (1.1) has nonnegative solution $u_\lambda \in H \setminus \{0\}$ such that $\|u_\lambda\|_E + \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 \, dx \leqslant M$ and (3.10) holds with $u_n = u_\lambda$ and $\lambda_n = \lambda$. Since $u_\lambda \geqslant 0$, it follows from (3.10) that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla \varphi \, dx + \int_{\mathbb{R}^3} \phi_{u_\lambda}(x) u_\lambda \varphi \, dx \leqslant \int_{\mathbb{R}^3} u_\lambda^p \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty\big(\mathbb{R}^3 \setminus T\big),\ \varphi \geqslant 0.$$

This and Lemma 4.2 show that, for all $v \in H^1(\mathbb{R}^3)$ with $v \geqslant 0$,

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla v \, dx + \int_{\mathbb{R}^3} \phi_{u_\lambda}(x) u_\lambda v \, dx \leqslant \int_{\mathbb{R}^3} u_\lambda^p v \, dx, \tag{4.16}$$

this means (4.8) holds with $u = u_\lambda$ and $\mu = 1$. Hence, Lemma 4.4 gives that

$$|u_\lambda|_\infty \geqslant 1 \quad \text{for all } \lambda > 0. \tag{4.17}$$

Meanwhile, for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^3)$ and any nonnegative piecewise smooth function h on $[0, +\infty)$, we see that $h(u_\lambda)\varphi \in H^1(\mathbb{R}^3)$. Let $v = h(u_\lambda)\varphi$ in (4.16), it follows that (4.7) holds with $u = u_\lambda$ and $N = 3$. Hence, by Lemma 4.3, we have

$$|u_\lambda|_\infty \leqslant C_1\big(1 + |u_\lambda|_6^{C_2}\big)|u_\lambda|_6. \tag{4.18}$$

So, (4.17) and (4.18) imply that u_λ does not converge to 0 in $L^6(\mathbb{R}^3)$ as $\lambda \to 0$, then Theorem 3.1 shows that there exists a nonnegative function $u \in E$, $u \not\equiv 0$, such that,

$$\int\limits_{\mathbb{R}^3} \nabla u \nabla \varphi + \frac{u\varphi}{|y|^\alpha}\,dx + \int\limits_{\mathbb{R}^3} \phi_u(x) u\varphi\,dx = \int\limits_{\mathbb{R}^3} u^p \varphi\,dx, \quad \text{for all } \varphi \in C_0^\infty\big(\mathbb{R}^3 \setminus T\big).$$

Moreover, $u \in C^2(\mathbb{R}^3 \setminus T)$. $\square$

Acknowledgments

This work was supported by NSFC (11071245, 11171339 and 11201486). The second author would like to thank IMS-CUHK for its support during his visit in 2012, where part of this work was carried out. Thanks to Professor C.A. Stuart for some helpful discussions. Both authors thank also the referees for their very useful comments and suggestions upon which the paper were revised.

References

[1] C.O. Alves, M.A.S. Souto, S.H.M. Soares, Schrödinger–Poisson equations without Ambrosetti–Rabinowitz condition, J. Math. Anal. Appl. 377 (2011) 584–592.
[2] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger–Poisson problem, Commun. Contemp. Math. 10 (2008) 391–404.
[3] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger–Maxwell equations, J. Math. Anal. Appl. 345 (2008) 90–108.
[4] A. Azzollini, P. d'Avenia, A. Pomponio, On the Schrödinger Maxwell equations under the effect of a general nonlinear term, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010) 779–791.
[5] M. Badiale, V. Benci, S. Rolando, A nonlinear elliptic equation with singular potential and applications to nonlinear field equations, J. Eur. Math. Soc. (JEMS) 9 (2007) 355–381.
[6] M. Badiale, V. Benci, S. Rolando, Three dimensional vortices in the nonlinear wave equation, Boll. Unione Mat. Ital. 2 (2009) 105–134.
[7] V. Benci, D. Fortunato, Three dimensional vortices in abelian gauge theories, Nonlinear Anal. 70 (2009) 4402–4421.
[8] V. Benci, D. Fortunato, A minimization method and applications to the study of solitons, Nonlinear Anal. 75 (2012) 4398–4421.
[9] V. Benci, N. Visciglia, Solitary waves with non-vanishing angular momentum, Adv. Nonlinear Stud. 3 (2003) 151–160.
[10] G.M. Coclite, A multiplicity result for the nonlinear Schrödinger–Maxwell equations, Commun. Appl. Anal. 7 (2003) 417–423.
[11] T. D'Aprile, On a class of solutions with non-vanishing angular momentum for nonlinear Schrödinger equations, Differential Integral Equations 16 (2003) 349–384.
[12] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein–Gordon–Maxwell equations, Adv. Nonlinear Stud. 4 (2004) 307–322.
[13] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 893–906.
[14] P. D'Avenia, Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with maxwell equations, Adv. Nonlinear Stud. 2 (2002) 177–192.
[15] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.
[16] M. Guida, M. Badiale, S. Rolando, Elliptic equations with decaying cylindrical potentials and power-type nonlinearities, Adv. Difference Equ. 12 (2007) 1321–1362.
[17] Y.S. Jiang, H.S. Zhou, Bound states for a stationary nonlinear Schrödinger–Poisson systems with sign-changing potential in $\mathbb{R}^3$, Acta Math. Sci. Ser. B 29 (2009) 1095–1104.
[18] Y.S. Jiang, H.S. Zhou, Schrödinger–Poisson system with steep potential well, J. Differential Equations 251 (2011) 582–608.
[19] E.H. Lieb, M. Loss, Analysis, Grad. Stud. Math., vol. 14, Amer. Math. Soc., Providence, RI, 1997.
[20] D. Mugnai, The Schrödinger–Poisson system with positive potential, Comm. Partial Differential Equations 36 (2011) 1099–1117.

[21] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237 (2006) 655–674.
[22] S. Solimini, A note on compactness-type properties with respect to Lorentz norms of bounded subsets of a Sobolev space, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (1995) 319–337.
[23] Z.P. Wang, H.S. Zhou, Positive solution for a nonlinear stationary Schrödinger–Poisson system in $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. 18 (2007) 809–816.
[24] M. Willem, Minimax Theorems, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.
[25] M.B. Yang, Z.F. Shen, Y.H. Ding, Multiple semiclassical solutions for the nonlinear Maxwell–Schrödinger system, Nonlinear Anal. 71 (2009) 730–739.
[26] L.G. Zhao, F.K. Zhao, On the existence of solutions for the Schrödinger–Poisson equations, J. Math. Anal. Appl. 346 (2008) 155–169.