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Asymptotic formulae for generalized Freud polynomials ☆

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ABSTRACT

We establish the Mehler–Heine type formulae for orthonormal polynomials with respect to generalized Freud weights. Using this type of asymptotics, we can give estimates of the value at the origin of these polynomials and of all their derivatives as well as the asymptotic behavior of the corresponding zeros.

1. Introduction

The theory of orthogonal polynomials is a major topic in Approximation Theory. Since the nineteenth century they have been studied widely, especially when they are orthogonal with respect to a standard inner product, i.e., an inner product $(\cdot,\cdot)$ defined in a pre-Hilbert space H containing the space of the polynomials $\mathbb{P}$ such that $(xf,g)=(f,xg)$, for all $f,g\in H$. A very important case of this type of inner products refers to the ones generated by means of weight functions. Thus, if we consider $W(x)$ a weight function on an interval $I\subseteq\mathbb{R}$, then we can construct an inner product as

$$(f,g)=\int_I f(x)g(x)W(x)dx,\quad \text{where } f,g\in L_W^2:=\left\{f:\int_I f^2(x)W(x)dx<\infty\right\}.$$

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In this paper, we consider the exponential weights $W_\alpha(x) = \exp(-c|x|^\alpha)$, $\alpha > 1$, on the real line where $c > 0$ is a normalization constant. The orthogonal polynomials with respect to the inner product

$$(f,g) = \int_{\mathbb{R}} f(x)g(x)W_\alpha^2(x)dx,$$

are so-called Freud orthogonal polynomials. The literature on this topic has been very wide since the sixties when G. Freud started to study these weights, though it is mandatory to cite two very nice and deep books: one by A.L. Levin and D.S. Lubinsky [6] and the other one by E.B. Saff and V. Totik [9]. About the asymptotics of the corresponding orthogonal polynomials, the first results correspond to the cases $\alpha = 4$ with $c = 1/2$ [8] and $\alpha = 6$ with $c = 1/12$ [10]. Later, using the powerful Riemann–Hilbert method, several authors have given precise asymptotic results (see the survey [13] and the references therein).

From now on, we choose $c = 1$, i.e.

$$W_\alpha(x) = \exp\bigl(-|x|^\alpha\bigr), \quad \alpha > 1. \tag{1}$$

We denote by $(p_n)_n$ the sequence of orthonormal polynomials with respect to $W_\alpha^2(x)$, $p_n(x) = \gamma_n x^n + \dots$, with $\gamma_n > 0$. These weights can be generalized considering $W_{\alpha,m}(x) = x^m \exp(-|x|^\alpha)$ with $\alpha > 1$ and $m \in \mathbb{N} \cup \{0\}$. Thus, the functions

$$W_{\alpha,m}^2(x) = x^{2m} \exp\bigl(-2|x|^\alpha\bigr), \quad \alpha > 1,\ m \in \mathbb{N} \cup \{0\}, \tag{2}$$

are weights on the real line. We denote by $(p_n^{[m]})_n$ the sequence of orthonormal polynomials with respect to (2), $p_n^{[m]}(x) = \gamma_n^{[m]} x^n + \dots$, with $\gamma_n^{[m]} > 0$. Clearly, when $m = 0$ we have the Freud polynomials, i.e., $p_n^{[0]} = p_n$, for all n. Thus, the polynomials $p_n^{[m]}$ are so-called *generalized Freud orthonormal polynomials*. These polynomials belong to a wider class of weights on the real line given by $x^{2m} \exp(-Q(x))$ with Q belonging to the class $\mathcal{F}(C^2+)$ (see [6] to get more information about this and other classes). In more general frameworks, asymptotic properties of these polynomials have been obtained, for example, in [6] or in [14] using powerful techniques.

The main aim of this paper is to establish the Mehler–Heine type asymptotics of the sequence $(p_n^{[m]})_n$, and as an immediate consequence we deduce the asymptotic behavior of the corresponding zeros. Besides, this formula also permits to obtain asymptotic estimates of $(p_n^{[m]})^{(j)}(0)$ with $j = 0, \dots, n$.

The structure of the paper is the following. In Section 2, we establish the Mehler–Heine type formulae for the sequence $(p_n)_n$ on compact subsets of the complex plane. In addition, we obtain estimates for $(p_n)^{(j)}(0)$, $j = 0, \dots, n$, when $n \to \infty$. In Section 3, we give our main result about the Mehler–Heine asymptotics of the generalized Freud orthonormal polynomials and their consequences on the asymptotic behavior of $(p_n^{[m]})^{(j)}(0)$ and on the zeros of this family of orthogonal polynomials.

Throughout the paper we use the notation $x_n \simeq y_n$, when $n \to \infty$, meaning $\lim_{n\to\infty} x_n/y_n = 1$.

2. Freud orthonormal polynomials

As we have commented in the Introduction, in this section we will establish the Mehler–Heine type asymptotics of the polynomials p_n. Thus, we have

Theorem 1. *Let $(p_n)_n$ be the sequence of Freud orthonormal polynomials with respect to the weight function $W_\alpha^2(x)$ defined by (1). Then, the polynomials p_n satisfy the following Mehler–Heine type formulae*

$$\lim_{n\to\infty} (-1)^n a_{2n}^{1/2} p_{2n}\left(\frac{z}{b_{2n}}\right) = \sqrt{\frac{2}{\pi}} \cos z,$$

$$\lim_{n\to\infty} (-1)^n a_{2n+1}^{1/2} p_{2n+1}\left(\frac{z}{b_{2n+1}}\right) = \sqrt{\frac{2}{\pi}} \sin z,$$

both uniformly on compact subsets of $\mathbb{C}$, *where*

$$a_n = (c_\alpha n)^{1/\alpha} \quad \text{and} \quad b_n = \frac{\alpha}{\alpha-1}\frac{n}{a_n}, \tag{3}$$

with

$$c_\alpha = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha/2)}{\Gamma((\alpha+1)/2)}. \tag{4}$$

Note that, for each n, a_n is the well-known n-th Mhaskar–Rakhmanov–Saff number. This number is defined as the positive solution of an integral equation, and for the weight (1) it can be computed explicitly obtaining the value given by (3)–(4) (see, for example, [2, Proposition 10.1.1] or the seminal paper [7]). For general exponential weights, the constant b_n was introduced in [2, Eq. (1.2.2)]. This constant plays an important role in Melher–Heine type asymptotics as well as in polynomial approximation with exponential weights.

To prove this result we use the asymptotic formulae for the polynomials p_n obtained by Kriecherbauer and McLaughlin in [5] using a Riemann–Hilbert method. In that asymptotic behavior there appears a function, namely Ψ_α. Next, we show some properties of that function which play an important role in the proof of Theorem 1.

Lemma 1. *Let* $\Omega = \{z \in \mathbb{C};\ \operatorname{Re} z > 0, z \notin [1,\infty)\}$ *and the function* $\Psi_\alpha : \Omega \to \mathbb{C}$ *defined by*

$$\Psi_\alpha(z) := \frac{\alpha}{\pi} z^{\alpha-1} \int_1^{1/z} \frac{u^{\alpha-1}}{\sqrt{u^2-1}}\, du,$$

where we take the principal branch of the logarithm function and $\alpha > 1$. *Then, the following properties are satisfied:*

(a) Ψ_α *is a holomorphic function on* Ω, *namely* $\Psi_\alpha \in H(\Omega)$.
(b) Ψ_α *has a continuous extension to* $(D(0,1) \cap \{\operatorname{Re} z \geq 0\}) \cup \{1\}$.
(c) $\int_0^z \Psi_\alpha(y)dy = \int_1^z \Psi_\alpha(y)dy + 1/2$, $z \in (D(0,1) \cap \{\operatorname{Re} z \geq 0\}) \cup \{1\}$.
(d) $\int_0^z \Psi_\alpha(y)dy = \Psi_\alpha(0)z + O(z^{\min\{\alpha,3\}})$, $z \to 0$.

Proof. (a) Fixed $z \in \Omega$, let $\gamma : [0,1] \to \mathbb{C}$ be a complex function defined by

$$\gamma(t) := \gamma_z(t) = 1 + t\left(\frac{1}{z} - 1\right), \quad t \in [0,1].$$

Hence

$$\Psi_\alpha(z) = \frac{\alpha}{\pi} z^{\alpha-1}\left(\frac{1}{z} - 1\right) \int_0^1 g(t,z)dt$$

where

$$g(t,z) = \frac{\gamma(t)^{\alpha-1}}{\sqrt{\gamma(t)^2-1}}.$$

To prove that $\Psi_\alpha \in H(\Omega)$, it suffices to show that $G(z) = \int_0^1 g(t,z)dt \in H(\Omega)$, because $z^{\alpha-1} \in H(\Omega)$.
First, we define the function $G_\epsilon(z)$ for any $\epsilon > 0$ as

$$G_\epsilon(z) := \int_\epsilon^1 g(t,z)dt,$$

which verifies the following properties:

i) Fixed $t \in [\epsilon, 1]$, $g(t,z) \in H(\Omega)$ because if $z \in \Omega$, then $1/z \notin (0,1]$, $\operatorname{Re}(1/z) > 0$, and the function $\sqrt{u^2-1}$ is holomorphic on $\mathbb{C} \setminus (i\mathbb{R} \cup [-1,1])$.

ii) $G_\epsilon(z)$ is a continuous function on Ω. To check this, it can be used Lebesgue's dominated convergence theorem since $|g(t,z)| \le M \in L^1([\epsilon,1])$ for $z \in K$, where K is an arbitrary compact subset of Ω.

iii) Applying Morera's Theorem, $G_\epsilon(z)$ is holomorphic on Ω. Indeed, let Δ be a closed triangle in Ω. Then, using Fubini's Theorem and the fact that $g(t,z) \in H(\Omega)$, we get

$$\int_{\partial\Delta} G_\epsilon(z)dz = \int_\epsilon^1 \left(\int_{\partial\Delta} g(t,z)dz \right) dt = 0.$$

Now, to establish that the function $G(z) \in H(\Omega)$ we are going to prove that $\lim_{\epsilon\to 0} G_\epsilon(z) = G(z)$, uniformly on compact subsets of Ω.

Let K be a fixed compact set in Ω. Using Lebesgue's dominated convergence theorem again, we have

$$\sup_{z\in K} \left| G_\epsilon(z) - G(z) \right| \le \int_0^1 \sup_{z\in K} \left\{ \left| g(t,z) \right| \right\} \left| \chi_{[\epsilon,1]}(t) - 1 \right| dt \to 0, \quad \epsilon \to 0,$$

since $|g(t,z)| \le Ct^{-1/2} \in L^1([0,1])$ for all $z \in K$, where C is a positive constant which depends only on K.

(b) Performing a change of variable in the integral which appears in the definition of Ψ_α, the restriction of this function to $(0,1)$ takes the form

$$\Psi_\alpha(x) = \frac{\alpha}{\pi} \int_x^1 \frac{t^{\alpha-1}}{\sqrt{t^2 - x^2}}\, dt.$$

Next, handling the above integral we get another useful expression for the function Ψ_α. Let $x \in (0,1)$, then

$$\begin{aligned}
\frac{\pi}{\alpha}\Psi_\alpha(x) &= \int_x^1 \frac{t^{\alpha-2}}{\sqrt{1-\frac{x^2}{t^2}}} dt = x^{\alpha-1} \int_x^1 s^{-\alpha}\left(1-s^2\right)^{-1/2} ds \\
&= x^{\alpha-1} \lim_{\epsilon\to 0} \int_x^{1-\epsilon} s^{-\alpha}\left(1-s^2\right)^{-1/2} ds = x^{\alpha-1} \lim_{\epsilon\to 0} \int_x^{1-\epsilon} s^{-\alpha} \sum_{n=0}^\infty \binom{-1/2}{n} (-1)^n s^{2n} ds \\
&= x^{\alpha-1} \lim_{\epsilon\to 0} \sum_{n=0}^\infty \binom{-1/2}{n} (-1)^n \frac{(1-\epsilon)^{2n-\alpha+1}}{2n-\alpha+1} - \sum_{n=0}^\infty \binom{-1/2}{n} \frac{(-1)^n x^{2n}}{2n-\alpha+1} \\
&= x^{\alpha-1} \sum_{n=0}^\infty \binom{-1/2}{n} \frac{(-1)^n}{2n-\alpha+1} - \sum_{n=0}^\infty \binom{-1/2}{n} \frac{(-1)^n x^{2n}}{2n-\alpha+1},
\end{aligned}$$

where the last two equalities hold because the radius of convergence of the series is equal to 1 and the series converges at $x = 1$.

Now, we consider the function

$$F(z) = z^{\alpha-1}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n}{2n-\alpha+1} - \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n z^{2n}}{2n-\alpha+1},$$

which is analytic on $D(0,1)\setminus(-1,0]$. So, taking into account that $\Psi_\alpha \in H(\Omega)$, the analytic continuation principle yields

$$\Psi_\alpha(z) = \frac{\alpha}{\pi}F(z), \quad z \in D(0,1)\cap\Omega,$$

that is, for $z \in D(0,1)\cap\{\operatorname{Re} z > 0\}$, we have obtained

$$\Psi_\alpha(z) = Az^{\alpha-1} - \frac{\alpha}{\pi}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n z^{2n}}{2n-\alpha+1}, \tag{5}$$

where $A = \frac{\alpha}{\pi}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{(-1)^n}{2n-\alpha+1}$. Thus, we can extend the definition of the function Ψ_α to $(D(0,1)\cap\{\operatorname{Re} z \geq 0\})\cup\{1\}$ being

$$\Psi_\alpha(0) := \frac{\alpha}{\pi}\frac{1}{\alpha-1}. \tag{6}$$

(c) Since $F(z)$ is holomorphic on $D(0,1)\setminus(-1,0]$ and continuous at $z = 0$ and $z = 1$, it can be deduced that

$$\int_{[0,1]\cup[1,z]\cup[z,0]} F(u)\,du = 0,$$

and therefore, for all $z \in D(0,1)\cap\{\operatorname{Re} z \geq 0\}$, we have

$$\int_{[0,1]\cup[1,z]\cup[z,0]} \Psi_\alpha(u)\,du = 0.$$

To conclude, it suffices to observe that applying Fubini's Theorem we get

$$\int_0^1 \Psi_\alpha(x)\,dx = \frac{\alpha}{\pi}\int_0^1 t^{\alpha-1}\left(\int_0^t \frac{1}{\sqrt{t^2-x^2}}\,dx\right)dt = \frac{\alpha}{2}\int_0^1 t^{\alpha-1}dt = 1/2.$$

(d) Considering the expression of $\Psi_\alpha(z)$ given by (5) and since the series which appears converges uniformly on compact subsets of $D(0,1)$, we obtain

$$\begin{aligned}\int_0^z \Psi_\alpha(u)\,du &= \int_0^z \left(Au^{\alpha-1} - \frac{\alpha}{\pi}\frac{1}{1-\alpha} - \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\binom{-1/2}{n}\frac{(-1)^n u^{2n}}{2n-\alpha+1}\right)du \\ &= A\frac{z^\alpha}{\alpha} + \Psi_\alpha(0)z - \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\binom{-1/2}{n}\frac{(-1)^n z^{2n+1}}{(2n-\alpha+1)(2n+1)},\end{aligned}$$

and the result follows. □

Proof of Theorem 1. Taking into account the symmetries

$$p_n(-z) = (-1)^n p_n(z), \qquad p_n(\bar{z}) = \overline{p_n(z)},$$

we only need to prove the Mehler–Heine type formulae in the first quadrant of the complex plane, that is in $\{z \in \mathbb{C}; \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$.

In [5, Theorem 1.16], the authors give the strong asymptotics of polynomials orthogonal with respect to the Freud weights $\widetilde{w}_\alpha = \exp(-k_\alpha |x|^\alpha)$ where $k_\alpha = 2c_\alpha$, with $\alpha > 0$.

Denoting by $\widetilde{p}_n(z)$ the orthonormal polynomials with respect to the weight function $\widetilde{w}_\alpha$, it can be obtained that the relation between the two families of orthonormal polynomials $\widetilde{p}_n$ and p_n is

$$n^{1/2\alpha} \widetilde{p}_n\big(n^{1/\alpha} z\big) = a_n^{1/2} p_n(a_n z).$$

Then, from Theorem 1.16(v) it follows that given $\alpha > 1$, there exists a $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ and for $z \in D_\delta = \{z; |z| \leq \delta, \operatorname{Re} z \geq 0, \operatorname{Im} z \geq 0\}$ the following asymptotic estimation for the polynomials p_n holds:

$$\begin{aligned} a_n^{1/2} p_n(a_n z) = {} & \sqrt{\frac{2}{\pi}} \exp\big(a_n^\alpha z^\alpha\big)(1-z)^{-1/4}(1+z)^{-1/4} \\ & \times \left\{ \cos\left(n\pi \int_1^z \Psi_\alpha(y) dy + \frac{1}{2} \arcsin z \right) \big(1 + o(1)\big) \right. \\ & \left. + \sin\left(n\pi \int_1^z \Psi_\alpha(y)\, dy - \frac{1}{2} \arcsin z \right) o(1) \right\}, \quad n \to \infty, \end{aligned} \tag{7}$$

where Ψ_α is the function defined in Lemma 1 and the error terms on the right-hand side are uniform in $z \in D_\delta$.

Next, we consider a fixed compact set K in the first quadrant of $\mathbb{C}$ and $M = M_K$ a constant such that $|z| \leq M$ for all $z \in K$. Writing

$$z = a_n b_n w,$$

it follows that

$$\begin{aligned} |a_n w| &= \left| \frac{z}{b_n} \right| \leq \frac{M}{b_n} \longrightarrow 0, \quad n \to \infty, \\ |w| &= \left| \frac{z}{a_n b_n} \right| \leq \frac{M(\alpha - 1)}{\alpha n} \longrightarrow 0, \quad n \to \infty. \end{aligned} \tag{8}$$

Thus, if $z \in K$ we can assure that $|w| \leq \delta_0 < 1$ for n large enough.

Hence, from (7) and Lemma 1(c), we obtain

$$\begin{aligned} a_n^{1/2} p_n\left(\frac{z}{b_n}\right) = {} & \sqrt{\frac{2}{\pi}} \exp\big(a_n^\alpha w^\alpha\big)(1-w)^{-1/4}(1+w)^{-1/4} \\ & \times \left\{ \cos\left(n\pi \int_0^w \Psi_\alpha(y) dy - n\frac{\pi}{2} + \frac{1}{2} \arcsin w \right) \big(1 + o(1)\big) \right. \\ & \left. + \sin\left(n\pi \int_0^w \Psi_\alpha(y)\, dy - n\frac{\pi}{2} - \frac{1}{2} \arcsin w \right) o(1) \right\}, \quad n \to \infty. \end{aligned}$$

Besides, using the asymptotic estimates:

$$\begin{aligned}\exp\bigl(a_n^\alpha w^\alpha\bigr) &= 1 + O\bigl((a_n w)^\alpha\bigr) = 1 + O(a_n w), \quad a_n w \to 0,\\ (1-w)^{-1/4}(1+w)^{-1/4} &= 1 + O\bigl(w^2\bigr), \quad w \to 0,\\ \frac{1}{2}\arcsin w &= \frac{1}{2}w + O\bigl(w^3\bigr), \quad w \to 0,\end{aligned}$$

and Lemma 1(d), we have

$$\begin{aligned} a_n^{1/2} p_n\left(\frac{z}{b_n}\right) &= \sqrt{\frac{2}{\pi}}\bigl(1+o(1)\bigr)\bigl(1+O\bigl(w^2\bigr)\bigr)\\ &\quad \times \left\{\cos\left(n\pi\Psi_\alpha(0)w + O\bigl(nw^{\min\{\alpha,3\}}\bigr) - n\frac{\pi}{2} + o(1)\right)\bigl(1+o(1)\bigr)\right.\\ &\quad \left. + \sin\left(n\pi\Psi_\alpha(0)w + O\bigl(nw^{\min\{\alpha,3\}}\bigr) - n\frac{\pi}{2} + o(1)\right)o(1)\right\}, \quad n \to \infty.\end{aligned}$$

Now, since $\alpha > 1$, from (8) we get $O(nw^{\min\{\alpha,3\}}) = o(1)$, and from (6), we have

$$n\pi\Psi_\alpha(0)w = a_n b_n w = z,$$

so we obtain

$$\begin{aligned} a_n^{1/2} p_n\left(\frac{z}{b_n}\right) &= \sqrt{\frac{2}{\pi}}\bigl(1+o(1)\bigr)\left\{\cos\left(z - n\frac{\pi}{2} + o(1)\right)\right.\\ &\quad \left. + \left[\cos\left(z - n\frac{\pi}{2} + o(1)\right) + \sin\left(z - n\frac{\pi}{2} + o(1)\right)\right]o(1)\right\}, \quad n \to \infty.\end{aligned}$$

Finally, observe that since the functions sine and cosine are bounded on any compact set $K \subset \mathbb{C}$, the following relations

$$\begin{aligned}\cos\left(z - n\frac{\pi}{2} + o(1)\right) &= \cos\left(z - n\frac{\pi}{2}\right) + o(1), \quad n \to \infty,\\ \sin\left(z - n\frac{\pi}{2} + o(1)\right) &= \sin\left(z - n\frac{\pi}{2}\right) + o(1), \quad n \to \infty,\end{aligned}$$

hold. Thus,

$$\begin{aligned} a_n^{1/2} p_n\left(\frac{z}{b_n}\right) &= \sqrt{\frac{2}{\pi}}\bigl(1+o(1)\bigr)\left[\cos\left(z - n\frac{\pi}{2}\right) + o(1)\right]\\ &= \sqrt{\frac{2}{\pi}}\cos\left(z - n\frac{\pi}{2}\right) + o(1), \quad n \to \infty,\end{aligned}$$

and the proof is concluded. □

Remark 1. This result in a more general setting was obtained by Ganzburg for compact subsets of $\mathbb{R}$ (see [2, Theorem 8.3.2]).

Next, we claim that a more general version of Theorem 1 can be deduced using Cauchy's integral formula.

Corollary 1. *Let $(p_n)_n$ be the sequence of Freud orthonormal polynomials with respect to the weight function $W_\alpha^2(x)$ defined by (1). Then, the polynomials p_n satisfy the following Mehler–Heine type formulae*

$$\lim_{n\to\infty} (-1)^n a_{2n}^{1/2} p_{2n}\left(\frac{z}{b_{2n+j}}\right) = \sqrt{\frac{2}{\pi}} \cos z, \tag{9}$$

and

$$\lim_{n\to\infty} (-1)^n a_{2n+1}^{1/2} p_{2n+1}\left(\frac{z}{b_{2n+j}}\right) = \sqrt{\frac{2}{\pi}} \sin z, \tag{10}$$

both uniformly on compact subsets of $\mathbb{C}$ and for every $j \in \mathbb{Z}$.

Proof. In a more general setting, we will prove that if $(f_n)_n$ is a sequence of holomorphic functions on $\mathbb{C}$ and $(b_n)_n$ is a sequence of complex numbers satisfying $\lim_{n\to\infty} \frac{b_n}{b_{n+j}} = 1$ for every integer $j \in \mathbb{Z}$ such that $(f_n(z/b_n))_n$ converges to a function f uniformly on compact subsets of $\mathbb{C}$, then

$$\lim_{n\to\infty} f_n\left(\frac{z}{b_{n+j}}\right) = f(z)$$

uniformly on compact subsets of $\mathbb{C}$ for every integer $j \in \mathbb{Z}$.

Indeed, fixed an integer j and K a compact subset of $\mathbb{C}$, there exist constants $C_j > 1$ and $R > 0$ such that $|\frac{b_n}{b_{n+j}}| \le C_j$ for all n and $K \subset D(0,R)$. Define $F_n(z) := f_n(z/b_n)$, then by Cauchy's integral formula we have

$$\begin{aligned} f_n\left(\frac{z}{b_{n+j}}\right) - f_n\left(\frac{z}{b_n}\right) &= F_n\left(\frac{zb_n}{b_{n+j}}\right) - F_n(z) \\ &= \frac{1}{2\pi i} \int\limits_{\partial D(0,2C_jR)} F_n(w)\left(\frac{1}{w - zb_n/b_{n+j}} - \frac{1}{w-z}\right) dw, \end{aligned}$$

for all $z \in K$. We can observe that

$$\begin{gathered} |w-z| \ge |w| - |z| > 2C_jR - C_jR = C_jR > R, \\ |w - zb_n/b_{n+j}| \ge |w| - |z||b_n/b_{n+j}| > 2C_jR - C_jR = C_jR > R, \end{gathered}$$

and since F_n converges uniformly on compact subsets of $\mathbb{C}$, we deduce that there exists a constant $A_j > 0$ such that $\sup\{|F_n(w)|; |w| = 2C_jR\} \le A_j$ for all n. Thus,

$$\sup_{z\in K} \left| f_n\left(\frac{z}{b_{n+j}}\right) - f_n\left(\frac{z}{b_n}\right)\right| \le 2C_jA_j|b_n/b_{n+j} - 1|$$

and the result follows. □

As a consequence of Theorem 1 we can deduce nice results about asymptotic behavior of Freud orthonormal polynomials p_n at the origin.

Proposition 1. *Let $(p_n)_n$ be the sequence of Freud orthonormal polynomials with respect to the weight function $W_\alpha^2(x)$ defined by (1). We have:*

(a) *Derivatives of p_n at $z = 0$.*

$$p_{2n}^{(2k)}(0) \simeq (-1)^{n+k}\sqrt{\frac{2}{\pi}}a_{2n}^{-1/2}b_{2n}^{2k}, \quad n \to \infty$$
$$p_{2n+1}^{(2k+1)}(0) \simeq (-1)^{n+k}\sqrt{\frac{2}{\pi}}a_{2n+1}^{-1/2}b_{2n+1}^{2k+1}, \quad n \to \infty$$

with $k = 0, \ldots, n$.

(b)

$$\lim_{n\to\infty} \frac{np_{2n}^2(0)}{K_{2n}(0,0)} = \lim_{n\to\infty} \frac{np_{2n}^2(0)}{K_{2n-2}(0,0)} = \lim_{n\to\infty} \frac{np_{2n}^2(0)}{K_{2n+1}(0,0)} = \frac{\alpha - 1}{\alpha}, \tag{11}$$

where $K_n(x, y)$ are the kernel polynomials associated with the polynomials p_n.

Proof. (a) From Theorem 1 and using the Taylor series for the polynomials p_{2n} at the point $z = 0$, we have

$$(-1)^n a_{2n}^{1/2} \sum_{k=0}^{2n} \frac{p_{2n}^{(k)}(0)}{k! b_{2n}^k} z^k \to \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k},$$

uniformly on compact subsets of $\mathbb{C}$, which proves the result for the even case. The odd case can be established in the same way.

(b) Using the Stolz–Cesàro criterion we get,

$$\begin{aligned}\lim_{n\to\infty} \frac{K_{2n}(0,0)}{n^{1-1/\alpha}} &= \lim_{n\to\infty} \frac{p_{2n}^2(0)}{n^{1-1/\alpha}(\frac{(\alpha-1)/\alpha}{n} + O(n^{-2}))} \\ &= \lim_{n\to\infty} \frac{n^{1/\alpha}p_{2n}^2(0)}{\frac{\alpha-1}{\alpha} + O(n^{-1})} = \frac{\alpha}{\alpha - 1}\frac{2}{\pi}(2c_\alpha)^{-1/\alpha}.\end{aligned}$$

To conclude, it is sufficient to observe that by (a)

$$\lim_{n\to\infty} \frac{p_{2n-2}^2(0)}{p_{2n}^2(0)} = 1,$$

and $K_{2n+1}(0, 0) = K_{2n}(0, 0)$ by the symmetry of Freud polynomials p_n.

Then, the equalities in (11) follow. □

3. Mehler–Heine type asymptotics for generalized Freud orthonormal polynomials

Now, we consider the weight functions on the real line

$$W_{\alpha,m}^2(x) = x^{2m}\exp\bigl(-2|x|^\alpha\bigr), \quad \alpha > 1, \ m \in \mathbb{N} \cup \{0\}.$$

As we have commented in the Introduction, we denote by $(p_n^{[m]})_n$ the sequence of orthonormal polynomials with respect to the inner product

$$(f, g) = \int_{\mathbb{R}} f(x)g(x)W_{\alpha,m}^2(x)dx.$$

The leading coefficient of $p_n^{[m]}$ is $\gamma_n^{[m]} > 0$, and we also use the notation $K_n^{[m]}(x,y)$ for the corresponding kernel polynomials.

To reach our objective, i.e. the Mehler–Heine formulae for these polynomials, we need to establish a previous result where we give a relation between the sequences of orthonormal polynomials $(p_n^{[m+1]})_n$ and $(p_n^{[m]})_n$.

Lemma 2. *For $m \geq 0$, we have,*

$$xp_{n-1}^{[m+1]}(x) = \frac{\gamma_{n-1}^{[m+1]}}{\gamma_n^{[m]}}\left[p_n^{[m]}(x) - \frac{p_n^{[m]}(0)}{K_{n-1}^{[m]}(0,0)}K_{n-1}^{[m]}(x,0)\right], \quad n \geq 1. \tag{12}$$

Moreover,

$$\left(\frac{\gamma_n^{[m]}}{\gamma_{n-1}^{[m+1]}}\right)^2 = 1 + \frac{(p_n^{[m]}(0))^2}{K_{n-1}^{[m]}(0,0)} = \frac{K_n^{[m]}(0,0)}{K_{n-1}^{[m]}(0,0)}. \tag{13}$$

Proof. We use a technique already considered in other frameworks (see, for example, [1, Lemma 2.1] or [3, Lemma 3]). Expanding $xp_{n-1}^{[m+1]}(x)$ in terms of the polynomials $p_n^{[m]}$:

$$xp_{n-1}^{[m+1]}(x) = \sum_{j=0}^{n} \alpha_j p_j^{[m]}(x).$$

If $j = 1, \ldots, n-1$, the orthonormality properties of $p_n^{[m]}$ and $p_n^{[m+1]}$ yield

$$\begin{aligned}
\alpha_j &= \int_{\mathbb{R}} xp_{n-1}^{[m+1]}(x)p_j^{[m]}(x)W_{\alpha,m}^2(x)dx \\
&= \int_{\mathbb{R}} p_{n-1}^{[m+1]}(x)\frac{p_j^{[m]}(x) - p_j^{[m]}(0)}{x}W_{\alpha,m+1}^2(x)dx + p_j^{[m]}(0)\int_{\mathbb{R}} xp_{n-1}^{[m+1]}(x)W_{\alpha,m}^2(x)dx \\
&= p_j^{[m]}(0)\int_{\mathbb{R}} xp_{n-1}^{[m+1]}(x)W_{\alpha,m}^2(x)dx.
\end{aligned}$$

For $j = 0$, we obtain the same expression because $p_0^{[m]}(x)$ is a constant.

Therefore

$$xp_{n-1}^{[m+1]}(x) = \frac{\gamma_{n-1}^{[m+1]}}{\gamma_n^{[m]}}p_n^{[m]}(x) + \left(\int_{\mathbb{R}} xp_{n-1}^{[m+1]}(x)W_{\alpha,m}^2(x)dx\right)K_{n-1}^{[m]}(x,0).$$

Evaluating the above expression at $x = 0$, we get

$$\int_{\mathbb{R}} xp_{n-1}^{[m+1]}(x)W_{\alpha,m}^2(x)dx = -\frac{\gamma_{n-1}^{[m+1]}}{\gamma_n^{[m]}}\frac{p_n^{[m]}(0)}{K_{n-1}^{[m]}(0,0)},$$

and so, (12) holds.

To prove (13), it is enough to integrate (12) multiplied by $p_{n-2}^{[m]}(x)W_{\alpha,m+1}^2(x)$, and to apply the well-known Christoffel–Darboux formula

$$K_n^{[m]}(x,y) := \sum_{k=0}^{n} p_k^{[m]}(x)p_k^{[m]}(y) = \frac{\gamma_n^{[m]}}{\gamma_{n+1}^{[m]}}\frac{p_{n+1}^{[m]}(x)p_n^{[m]}(y) - p_n^{[m]}(x)p_{n+1}^{[m]}(y)}{x-y}. \qquad \square \tag{14}$$

Now, we are ready to establish our main result for the polynomials $p_n^{[m]}$. We will use the Bessel functions J_ν of the first kind and order ν, $\nu > -1$, defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{C}.$$

We also claim that $z^{-\nu}J_\nu(z)$ is an entire function which does not vanish at the origin.

Theorem 2. *Let $(p_n^{[m]})_n$ be the sequence of generalized Freud orthonormal polynomials with respect to the weight function $W_{\alpha,m}^2(x)$ defined by (2). Then,*

$$\lim_{n\to\infty} (-1)^n \frac{a_{2n}^{1/2}}{b_{2n}^m} p_{2n}^{[m]}\left(\frac{z}{b_{2n+j}}\right) = z^{-(m-1/2)} J_{m-1/2}(z), \tag{15}$$

and

$$\lim_{n\to\infty} (-1)^n \frac{a_{2n+1}^{1/2}}{b_{2n}^m} p_{2n+1}^{[m]}\left(\frac{z}{b_{2n+j}}\right) = z z^{-(m+1/2)} J_{m+1/2}(z), \tag{16}$$

both uniformly on compact subsets of $\mathbb{C}$, and for every $j \in \mathbb{Z}$.

Proof. We are going to use the principle of mathematical induction to prove the result. For $m = 0$, the polynomials $p_n^{[0]}$ are the Freud orthonormal polynomials. Therefore, the Mehler–Heine type formulae (15)–(16) are the ones (9)–(10) established in Corollary 1 taking into account the following identities

$$\sqrt{\frac{2}{\pi}} \sin z = z^{1/2} J_{1/2}(z), \qquad \sqrt{\frac{2}{\pi}} \cos z = z^{1/2} J_{-1/2}(z).$$

We assume that (15)–(16) are true for a nonnegative integer m fixed and then we will show that they are also true for $m+1$.

Even case. Because of the symmetry of generalized Freud polynomials we have from (13), $\gamma_{2n}^{[m+1]} = \gamma_{2n+1}^{[m]}$, and evaluating (12) at z/b_{2n+j} we obtain

$$\frac{(-1)^n a_{2n}^{1/2}}{b_{2n}^{m+1}} p_{2n}^{[m+1]}\left(\frac{z}{b_{2n+j}}\right) = \frac{(-1)^n a_{2n}^{1/2}}{b_{2n}^m} \frac{b_{2n+j}}{b_{2n}} \frac{\gamma_{2n}^{[m+1]}}{\gamma_{2n+1}^{[m]}} z^{-1} p_{2n+1}^{[m]}\left(\frac{z}{b_{2n+j}}\right).$$

Thus, it is enough to apply that $\lim_{n\to\infty} \frac{b_n}{b_{n+j}} = 1$ and the induction hypothesis to complete the process for this case.

Odd case. Evaluating (12) at z/b_{2n+j} and applying the Christoffel–Darboux formula (14) we get

$$\begin{aligned} \frac{(-1)^n a_{2n+1}^{1/2}}{b_{2n+1}^m b_{2n+j}} p_{2n+1}^{[m+1]}\left(\frac{z}{b_{2n+j}}\right) &= \frac{(-1)^n a_{2n+1}^{1/2}}{b_{2n+1}^m} z^{-1} \frac{\gamma_{2n+1}^{[m+1]}}{\gamma_{2n+2}^{[m]}} \\ &\quad \times \left[p_{2n+2}^{[m]}\left(\frac{z}{b_{2n+j}}\right) + \frac{\gamma_{2n+1}^{[m]}}{\gamma_{2n+2}^{[m]}} \frac{(p_{2n+2}^{[m]}(0))^2}{K_{2n}^{[m]}(0,0)} \frac{b_{2n+j}}{z} p_{2n+1}^{[m]}\left(\frac{z}{b_{2n+j}}\right)\right]. \end{aligned} \tag{17}$$

Next, we analyze the asymptotic behavior of the coefficients which appear on the right-hand side of the above expression.

First, from the induction hypothesis for $p_{2n}^{[m]}$ and since $z^{-\nu}J_\nu(z)|_{z=0} = \frac{1}{2^\nu \Gamma(\nu+1)}$, we obtain the asymptotic behavior of $(p_{2n}^{[m]}(0))^2$ when n tends to infinity

$$\left(p_{2n}^{[m]}(0)\right)^2 \simeq \frac{b_{2n}^{2m}}{2^{2m-1}a_{2n}\Gamma^2(m+1/2)} = \left(\frac{\alpha}{\alpha-1}\right)^{2m} \frac{2(2c_\alpha)^{-\frac{2m+1}{\alpha}}}{\Gamma^2(m+1/2)} n^{2m-\frac{2m+1}{\alpha}}. \tag{18}$$

Now, using (18) and the Stolz–Cesàro criterion, we deduce

$$\begin{aligned}\lim_{n\to\infty} \frac{K_{2n}^{[m]}(0,0)}{n^{(2m+1)(\frac{\alpha-1}{\alpha})}} &= \lim_{n\to\infty} \frac{(p_{2n}^{[m]}(0))^2}{n^{(2m+1)(\frac{\alpha-1}{\alpha})}\left(\frac{(2m+1)(\alpha-1)/\alpha}{n} + O(n^{-2})\right)} \\ &= \left(\frac{\alpha}{\alpha-1}\right)^{2m+1} \frac{2(2c_\alpha)^{-\frac{2m+1}{\alpha}}}{(2m+1)\Gamma^2(m+1/2)}.\end{aligned}$$

From this asymptotic behavior for $K_{2n}^{[m]}(0,0)$ and (18), we get

$$\lim_{n\to\infty} \frac{n(p_{2n}^{[m]}(0))^2}{K_{2n}^{[m]}(0,0)} = \lim_{n\to\infty} \frac{n(p_{2n}^{[m]}(0))^2}{K_{2n-2}^{[m]}(0,0)} = \lim_{n\to\infty} \frac{n(p_{2n}^{[m]}(0))^2}{K_{2n-1}^{[m]}(0,0)} = (2m+1)\frac{\alpha-1}{\alpha}, \tag{19}$$

which is obviously the analog of (11) for the generalized Freud orthonormal polynomials. Furthermore, applying (19) in (13) we have

$$\lim_{n\to\infty} \frac{\gamma_{2n+2}^{[m]}}{\gamma_{2n+1}^{[m+1]}} = 1. \tag{20}$$

We only need to give an estimate for the coefficient $\frac{\gamma_{2n+1}^{[m]}}{\gamma_{2n+2}^{[m]}}$. For $m = 0$, this asymptotic behavior is known. For example, in [9, p. 365] we can find it for the orthogonal polynomials with respect to the weight function $\exp(-2c_\alpha|x|^\alpha)$ with $\alpha > 1$ and c_α given in (4). Thus, from this result we can deduce that

$$\lim_{n\to\infty} n^{-1/\alpha}\frac{\gamma_{2n+1}}{\gamma_{2n+2}} = 2^{(1-\alpha)/\alpha}c_\alpha^{1/\alpha}.$$

Moreover, for $m \geq 1$, using again the symmetry of the generalized Freud polynomials, it follows from (13) that $\gamma_{2n+2}^{[m]} = \gamma_{2n+3}^{[m-1]}$. Then, taking into account (20) we have

$$\frac{\gamma_{2n+1}^{[m]}}{\gamma_{2n+2}^{[m]}} = \frac{\gamma_{2n+1}^{[m]}}{\gamma_{2n+2}^{[m-1]}}\frac{\gamma_{2n+2}^{[m-1]}}{\gamma_{2n+3}^{[m-1]}} \simeq \frac{\gamma_{2n+2}^{[m-1]}}{\gamma_{2n+3}^{[m-1]}}.$$

Applying this process repeatedly, we get

$$\frac{\gamma_{2n+1}^{[m]}}{\gamma_{2n+2}^{[m]}} \simeq \frac{\gamma_{2n+m+1}}{\gamma_{2n+m+2}},$$

and therefore

$$\lim_{n\to\infty} n^{-1/\alpha}\frac{\gamma_{2n+1}^{[m]}}{\gamma_{2n+2}^{[m]}} = 2^{(1-\alpha)/\alpha}c_\alpha^{1/\alpha}. \tag{21}$$

To conclude the proof, we take limits in (17). Thus, using the definition of a_n and b_n given by (3), the asymptotic estimates given in (19), (20), (21), the induction hypothesis and the relation satisfied by the Bessel functions (see [11, §1.71])

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu z^{-1} J_\nu(z),$$

we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{(-1)^n a_{2n+1}^{1/2}}{b_{2n+1}^{m+1}} p_{2n+1}^{[m+1]}\left(\frac{z}{b_{2n+j}}\right) &= -z^{-(m+1/2)} J_{m-1/2}(z) + (2m+1) z^{-(m+3/2)} J_{m+1/2}(z) \\ &= z z^{-(m+3/2)} J_{m+3/2}(z),\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$.

So, we have completed the induction process for the odd case. □

Remark 2. We want to highlight the strong influence of m on the Mehler–Heine asymptotics. For $m = 0$, i.e. for Freud orthonormal polynomials with respect to any weight function of the type $\exp(-2|x|^\alpha)$, $\alpha > 1$, we obtain the same formulae. However, this type of asymptotics changes in a nice way when we introduce the factor x^{2m} on the weight, then the order of the Bessel function varies according to m. Thus, the asymptotic behavior of the corresponding zeros also changes, as it is shown in Corollary 3 and later comments.

From the previous theorem, we deduce the asymptotic behavior of the derivatives of the generalized Freud orthonormal polynomials $p_n^{[m]}$ at the origin.

Corollary 2. *For $m \geq 0$, we have the following estimates for the derivatives of $p_n^{[m]}$ at $x = 0$, when n tends to infinity,*

$$\begin{aligned}\left(p_{2n}^{[m]}\right)^{(2k)}(0) &\simeq \frac{(-1)^{n+k}}{2^{2k+m-1/2}\Gamma(m+k+1/2)} \frac{(2k)!}{k!} a_{2n}^{-1/2} b_{2n}^{m+2k}, \\ \left(p_{2n+1}^{[m]}\right)^{(2k+1)}(0) &\simeq \frac{(-1)^{n+k}}{2^{2k+m+1/2}\Gamma(m+k+3/2)} \frac{(2k+1)!}{k!} a_{2n+1}^{-1/2} b_{2n+1}^{m+2k+1},\end{aligned}$$

with $k = 0, \ldots, n$.

Theorem 2 has also another straightforward consequence about the zeros of the polynomials $p_n^{[m]}$. Since these polynomials are orthonormal with respect to a standard inner product, all their zeros are simple and lie on the real line. Applying Hurwitz's Theorem [11, p. 22] in Theorem 2, we deduce a limit relation between these zeros and the ones of the Bessel functions.

Corollary 3. *Let $x_{n,i}^{[m]}$ be the i-th positive zero of $p_n^{[m]}$ with $i = 1, \ldots, [n/2]$ arranged in an increasing order, i.e. $x_{n,i}^{[m]} < x_{n,i+1}^{[m]}$. Then, for $m \geq 0$ and $\alpha > 1$, we have*

$$\begin{aligned}b_{2n} x_{2n,i}^{[m]} &= j_{m-1/2,i}\big(1 + o(1)\big), \quad n \to \infty, \\ b_{2n+1} x_{2n+1,i}^{[m]} &= j_{m+1/2,i}\big(1 + o(1)\big), \quad n \to \infty,\end{aligned}$$

where $j_{\nu,i}$ denotes the i-th positive zero of the Bessel function J_ν, and b_n is given in (3).

Clearly, the sequence $(b_n)_n$, which involves the Mhaskar–Rakhmanov–Saff numbers, plays a very important role in the asymptotic behavior of these generalized Freud orthonormal polynomials and of their zeros as it has been shown in Theorem 2 and Corollary 3. But, it seems convenient to give in detail the asymptotics of the zero $x_{n,i}^{[m]}$, for i fixed, in terms of powers of n. In the case $m = 0$, very sharp estimates for the zeros of those orthogonal polynomials in terms of the zeros of the Airy function Ai were obtained in [5]. For $m \geq 0$ and $\alpha > 1$, we deduce from Corollary 3

$$x_{n,i}^{[m]} = \frac{\alpha - 1}{\alpha} c_\alpha^{1/\alpha} n^{-\frac{\alpha-1}{\alpha}} j_{m+(-1)^s/2,i}\big(1 + o(1)\big), \quad n \to \infty,$$

where

$$s = \begin{cases} 1, & n \text{ even;} \\ 0, & n \text{ odd,} \end{cases} \tag{22}$$

and c_α is given in (4).

We can also get the following estimate of the distance between two consecutive zeros $x_{n,i-1}^{[m]}$ and $x_{n,i}^{[m]}$, for $2 \leq i \leq [n/2]$, in terms of the Mhaskar–Rakhmanov–Saff numbers a_n given by (3):

$$x_{n,i}^{[m]} - x_{n,i-1}^{[m]} = \frac{\alpha - 1}{\alpha} \frac{a_n}{n} (j_{m+(-1)^s/2,i} - j_{m+(-1)^s/2,i-1})\big(1 + o(1)\big), \quad n \to \infty,$$

where s is given by (22). Using the properties of the Bessel functions (see [12], for example), we can simplify the above result getting the following expression for the distance between consecutive zeros of Freud orthogonal polynomials

$$x_{n,i}^{[0]} - x_{n,i-1}^{[0]} = \frac{(\alpha - 1)\pi}{\alpha} \frac{a_n}{n}\big(1 + o(1)\big), \quad n \to \infty.$$

The above relation also holds for $m = 1$ and n even, but in any other case we have $j_{m+(-1)^s/2,i} - j_{m+(-1)^s/2,i-1} > \pi$, with $\lim_{i\to\infty}(j_{m+(-1)^s/2,i} - j_{m+(-1)^s/2,i-1}) = \pi$.

Note that for $m = 0$ and W being a more general weight function, the spacing of the zeros has been studied in [6]. In addition, we want to remark that for $m \geq 1$, properties about the zeros of these polynomials have already been studied in [4] in a more general framework. Thus, the above results should be compared with Theorem 2.2 in that paper, and it can be noticed that we obtain more precise results in our case.

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