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Spectral deformations and exponential decay of eigenfunctions for the Neumann Laplacian on manifolds with quasicylindrical ends

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ABSTRACT

We study spectral properties of the Neumann Laplacian on manifolds with quasicylindrical ends. In particular, we prove exponential decay of the non-threshold eigenfunctions and show that the eigenvalues can accumulate only at thresholds of the absolutely continuous spectrum and only from below. The non-threshold eigenvalues are also discrete eigenvalues of a non-selfadjoint operator.

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1. Introduction and statement of results

We consider manifolds with dilation analytic quasicylindrical ends and non-compact smooth boundary. These manifolds correspond to a class of long-range geometric perturbations of manifolds with cylindrical ends. For mild geometric perturbations spectral properties and eigenfunction asymptotics of elliptic problems are essentially well known, e.g. [2,4,6,8,14–17]. The perturbations we consider here are so strong that eigenfunctions do not admit asymptotic expansions at the ends and can only be represented as a sum of norm convergent operator series plus a remainder, e.g. [15]. Thus the asymptotic theory per se does not give any information on decay of eigenfunctions. Besides, examples demonstrate that eigenvalues can accumulate at finite values of the spectral parameter.

We study the spectrum of the Neumann Laplacian and prove exponential decay of eigenfunctions by the method of spectral deformations originating from the theory of resonances for N-body quantum scattering, see e.g. [3,7,19]. Spectral deformations (also known as *complex scaling* and *analytic dilations*) are also traditionally used in numerical analysis [3]. In 1994 a modification of this approach was reinvented by

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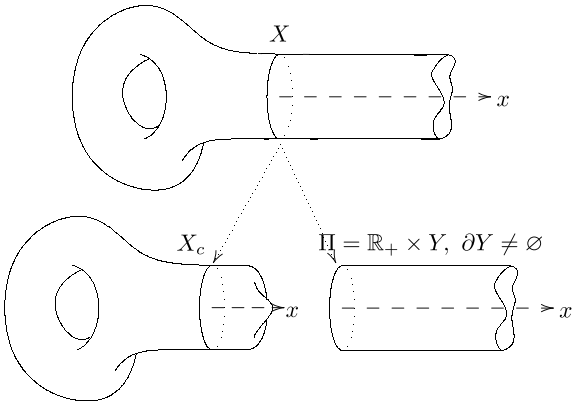


Fig. 1. Representation $X = X_c \cup \Pi$.

Bérenger [1] under the name of Perfectly Matched Layer (PML) method and became very popular since then. In our study of the PML method for the Neumann Laplacian [10] we already developed spectral deformations on manifolds with quasicylindrical ends to some extent: we introduced proper geometric deformations in quasicylindrical ends (or, equivalently, constructed infinite PMLs in the quasicylindrical ends) and studied corresponding deformations of the essential spectrum. As the main results of this work we prove that 1) The eigenvalues are of finite multiplicity and can accumulate only at thresholds of the continuous spectrum and only from below (Assertion 8 of Theorem 1 below); 2) Any non-threshold eigenfunction decays with an exponential rate that depends on the distance from the corresponding eigenvalue to the next threshold of continuous spectrum (Assertion 9 of Theorem 1 below).

Let Y be a compact n-dimensional manifold with smooth boundary ∂Y. Denote by Π the semi-cylinder defined as the Cartesian product $\mathbb{R}_+ \times Y$, where $\mathbb{R}_+$ is the positive semi-axis. Consider an oriented $(n+1)$-dimensional manifold X representable in the form $X = X_c \cup \Pi$, where X_c is a compact manifold with smooth boundary, see Fig. 1. We assume that the boundary ∂X of X is smooth.

Let $\mathfrak{g}$ be a Riemannian metric on X. We identify the cotangent bundle $\mathrm{T}^*\Pi$ with the tensor product $\mathrm{T}^*\mathbb{R}_+ \otimes \mathrm{T}^*Y$ via the natural isomorphism induced by the product structure on Π. Then

$$\mathfrak{g}\restriction_\Pi = \mathfrak{g}_0 dx^2 + 2\mathfrak{g}_1 dx + \mathfrak{g}_2, \quad \mathfrak{g}_k(x) \in C^\infty \mathrm{T}^*Y^{\otimes k},\ x \in \mathbb{R}_+. \tag{1}$$

Let $\mathbb{C}\mathrm{T}^*Y^{\otimes k}$ stand for the tensor power of the complexified cotangent bundle $\mathbb{C}\mathrm{T}^*Y$ with fibers $\mathbb{C}\mathrm{T}^*_yY = \mathrm{T}^*_yY \otimes \mathbb{C}$. In what follows C^m stands for sections of complexified bundles. We equip the space $C^1\mathrm{T}^*Y^{\otimes k}$ with the norm

$$\|\cdot\|_{\mathfrak{e}} = \max_{y\in Y}\big(|\cdot|_{\mathfrak{e}}(y) + |\nabla \cdot|_{\mathfrak{e}}(y)\big), \tag{2}$$

where $\mathfrak{e}$ is a Riemannian metric on Y, $|\cdot|_{\mathfrak{e}}(y)$ is the norm in $\mathbb{C}\mathrm{T}^*_yY^{\otimes k}$, and $\nabla : C^1\mathrm{T}^*Y^{\otimes k} \to C^0\mathrm{T}^*Y^{\otimes k+1}$ is the Levi-Civita connection on $(Y, \mathfrak{e})$. By taking a different metric $\mathfrak{e}$ on Y one gets an equivalent norm $\|\cdot\|_{\mathfrak{e}}$.

Definition 1. We say that $(X, \mathfrak{g})$ is a manifold with dilation analytic quasicylindrical end $(\Pi, \mathfrak{g}\restriction_\Pi)$, if the following conditions hold:

i. The coefficients $\mathbb{R}_+ \ni x \mapsto \mathfrak{g}_k(x) \in C^\infty \mathrm{T}^*Y^{\otimes k}$ in (1) extend by analyticity from the semi-axis $\mathbb{R}_+$ to the sector $\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha < \pi/4\}$.
ii. The values $\|\mathfrak{g}_0(z) - 1\|_{\mathfrak{e}}$, $\|\mathfrak{g}_1(z)\|_{\mathfrak{e}}$, and $\|\mathfrak{g}_2(z) - \mathfrak{h}\|_{\mathfrak{e}}$ converge to zero uniformly in $z \in \mathbb{S}_\alpha$ as $|z| \to \infty$, where $\mathfrak{h}$ is a metric on Y.

Let $r > 0$ be a sufficiently large fixed number and $\mathsf{s} \in C^\infty(\mathbb{R})$ possess the properties:

$$\begin{aligned} &\mathsf{s}(x) = 0 \text{ for all } x \leqslant r, \\ &0 \leqslant \mathsf{s}'(x) \leqslant 1 \text{ for all } x \in \mathbb{R}, \text{ and } \mathsf{s}'(x) = 1 \text{ for } x > R > r; \end{aligned} \tag{3}$$

here $\mathsf{s}' = \partial\mathsf{s}/\partial x$. For $\lambda \in (-1, 1)$ the diffeomorphism $\varkappa_\lambda(x, y) = (x + \lambda \mathsf{s}_r(x), y)$ scales the semi-cylinder Π along its axis. We extend $\varkappa_\lambda$ to X by identity and denote by g_λ the pullback of the metric g by the diffeomorphism $\varkappa_\lambda$. Clearly, $\mathsf{g} \equiv \mathsf{g}_0$.

Consider the sesquilinear quadratic form

$$\mathsf{q}_\lambda[u, u] = \big(du, d(\varrho_{\bar\lambda}\, u)\big)_\lambda, \quad u \in C_c^\infty(X),\ \lambda \in (-1, 1),$$

where $(\cdot,\cdot)_\lambda$ is the global inner product on (X, g_λ), $d : C_c^\infty(X) \to C_c^\infty \mathrm{T}^*\mathcal{M}$ is the exterior derivative, and $\varrho_\lambda \in C^\infty(X)$ is such that $\varrho_\lambda \,\mathrm{dvol}_\lambda = \mathrm{dvol}_0$. The form q_λ extends to an analytic (in the sense of Kato [12]) family of unbounded sectorial quadratic forms $\mathcal{O}_\alpha \ni \lambda \mapsto \mathsf{q}_\lambda$ in $L^2(X)$ with domain $H^1(X)$ and a common sector [10, Lemma 1]. Here $\mathcal{O}_\alpha$ is the disk

$$\mathcal{O}_\alpha = \{\lambda \in \mathbb{C} : |\lambda| < \sin\alpha\}$$

with the same α as in Definition 1, $L^2(X) \equiv L^2(X, \mathrm{dvol}_0)$, and the Sobolev space $H^1(X)$ is the completion of the set $C_c^\infty(X)$ with respect to the norm $\sqrt{(du, du)_0 + (u, u)_0}$.

The family $\mathcal{O}_\alpha \ni \lambda \mapsto \mathsf{q}_\lambda$ uniquely determines an analytic family $\mathcal{O}_\alpha \ni \lambda \mapsto {}^\lambda\Delta$ of unbounded m-sectorial operators [12,19]. (Here and elsewhere m-sectorial means that the numerical range $\{(Au, u) : u \in \mathscr{D}(A)\}$ and the spectrum $\sigma(A)$ of a closed unbounded operator A in $L^2(X)$ with the domain $\mathscr{D}(A)$ are both in some sector $\{\mu \in \mathbb{C} : |\arg(\mu + c)| \leqslant \vartheta < \pi/2, c > 0\}$.) For $\lambda \in \mathcal{O}_\alpha$ we have

$$({}^\lambda\Delta u, v)_0 = \mathsf{q}_\lambda[u, v], \quad u \in \mathscr{D}({}^\lambda\Delta), v \in H^1(X). \tag{4}$$

In particular, $({}^0\Delta u, v)_0 = (du, dv)_0$ and ${}^0\Delta$ is the selfadjoint Neumann Laplacian. Note that ${}^\lambda\Delta$ with non-real $\lambda \in \mathcal{O}_\alpha$ is a non-selfadjoint operator modeling an infinite perfectly matched layer on $(r, \infty) \times Y$ (or, equivalently, it is the Neumann Laplacian deformed by means of analytic dilations or complex scaling in the ends, or, equivalently, an auxiliary operator describing spectral deformations). The domain $\mathscr{D}({}^\lambda\Delta)$ of ${}^\lambda\Delta$ consists of the functions $u \in H^2(X)$ satisfying different boundary conditions on ∂X for different values of λ, see [10] for details. By definition, we say that the eigenvalues of the Neumann Laplacian Δ_Y on $(Y, \mathfrak{h})$ are the thresholds of ${}^\lambda\Delta$; here the metric $\mathfrak{h}$ on Y is the same as in Definition 1. As is well-known, the spectrum $\sigma(\Delta_Y)$ of Δ_Y consists of infinitely many isolated non-negative eigenvalues.

In order to formulate our results, we also need to define a sufficiently large set of analytic vectors. Introduce the algebra $\mathscr{E}$ of all entire functions $\mathbb{C} \ni z \mapsto F(z, \cdot) \in C^\infty(Y)$ such that the value $\|F(z, \cdot)\|_{L^2(Y)}$ decays faster than any inverse power of $\Re z$ as z goes to infinity in any sector $|\Im z| \leqslant (1 - \epsilon)\Re z$ with $\epsilon > 0$. For example, the functions $F(z, y) = e^{-\gamma z^2} P(z, y)$, where $\gamma > 0$ and $P(z, y)$ is an arbitrary polynomial in z with coefficients in $C^\infty(Y)$, are in $\mathscr{E}$. By definition, $f \in L^2(X)$ is an analytic vector if there exists $F \in \mathscr{E}$ such that $f(x, y) = F(x, y)$ for $(x, y) \in \Pi$. For any f in the set $\mathcal{A}$ of all analytic vectors we define the analytic function $\mathcal{O}_\alpha \ni \lambda \mapsto f_\lambda \in L^2(X)$ by setting $f_\lambda = f$ on $X \setminus \Pi$ and $f_\lambda(x, y) = f(x + \lambda \mathsf{s}_r(x), y)$ for $(x, y) \in \Pi$. The set $\varkappa_\lambda[\mathcal{A}] = \{f_\lambda : f \in \mathcal{A}\}$ is dense in $L^2(X)$ for any $\lambda \in \mathcal{D}_\alpha$, e.g. [7, Theorem 3].

Now we are in position to formulate the main results of this work.

Theorem 1. *The following assertions are valid.*

1. *The spectrum $\sigma({}^\lambda\Delta)$ of ${}^\lambda\Delta$ does not depend on $\mathsf{s}(x)$ satisfying (3).*

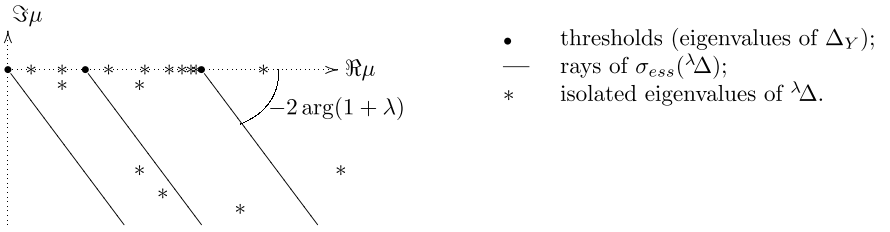


Fig. 2. Spectral portrait of ${}^{\lambda}\Delta$, $\Im\lambda > 0$.

2. *$\mu \in \mathbb{C}$ is a point of the essential spectrum $\sigma_{ess}({}^{\lambda}\Delta)$, if and only if $\mu - (1+\lambda)^{-2}\xi^2 \in \sigma(\Delta_Y)$ for some $\xi \in \mathbb{R}$.*
3. *$\sigma({}^{\lambda}\Delta) = \sigma_{ess}({}^{\lambda}\Delta) \cup \sigma_d({}^{\lambda}\Delta)$, where $\sigma_d({}^{\lambda}\Delta)$ stands for the discrete spectrum of ${}^{\lambda}\Delta$.*
4. *Let $\mu \in \sigma_d({}^{\lambda}\Delta)$. If λ changes continuously in the disk $\mathcal{O}_\alpha$ so that $\mu \notin \sigma_{ess}({}^{\lambda}\Delta)$, then the point μ remains in $\sigma_d({}^{\lambda}\Delta)$.*
5. *For any $f, g \in \mathcal{A}$ the analytic function $\mathbb{C} \setminus \overline{\mathbb{R}_+} \ni \mu \mapsto \big(({}^{0}\Delta - \mu)^{-1} f, g\big)_0$ has a meromorphic continuation to the set $\mathbb{C} \setminus \sigma_{ess}({}^{\lambda}\Delta)$. Moreover, μ is a pole of the meromorphic continuation for some $f, g \in \mathcal{A}$, if and only if $\mu \in \sigma_d({}^{\lambda}\Delta)$.*
6. *A point $\mu \in \mathbb{R} \setminus \sigma(\Delta_Y)$ is an (embedded) eigenvalue of the Neumann Laplacian ${}^{0}\Delta$, if and only if $\mu \in \sigma_d({}^{\lambda}\Delta)$ with non-real $\lambda \in \mathcal{O}_\alpha$.*
7. *The Neumann Laplacian ${}^{0}\Delta$ has no singular continuous spectrum.*
8. *The eigenvalues of ${}^{0}\Delta$ are of finite multiplicity and can accumulate only at thresholds and only from below.*
9. *As $x \to +\infty$ any eigenfunction Ψ corresponding to a non-threshold eigenvalue μ of ${}^{0}\Delta$ meets the estimate*

$$\|\Psi(x)\|_{L^2(Y)} = O(e^{-\gamma x}), \quad \gamma < \min_{\nu \in \sigma(\Delta_Y) : \nu > \mu} \sqrt{\nu - \mu}. \tag{5}$$

Let us discuss the spectral deformations described in Theorem 1. The spectral portrait of the operator ${}^{\lambda}\Delta$ is depicted on Fig. 2. As λ varies, the rays of $\sigma_{ess}({}^{\lambda}\Delta)$ rotate about the thresholds $\sigma(\Delta_Y)$ of the Neumann Laplacian. By Assertion *4* all discrete eigenvalues of ${}^{\lambda}\Delta$ belong to the sector $|\arg\mu| < 2\alpha$. As λ varies, they remain unchanged until they are covered by one of the rotating rays of $\sigma_{ess}({}^{\lambda}\Delta)$. The rotating rays can also uncover new eigenvalues. The operator ${}^{\lambda}\Delta$ with $\Im\lambda \geqslant 0$ (resp. $\Im\lambda \leqslant 0$) does not have eigenvalues in the half-plane $\Im\mu > 0$ (resp. $\Im\mu < 0$). (Indeed, Assertion *4* implies that any eigenvalue μ, $\Im\mu > 0$ (resp. $\Im\mu < 0$), of ${}^{\lambda}\Delta$ with $\Im\lambda \geqslant 0$ (resp. $\Im\lambda \leqslant 0$) is also a non-real eigenvalue of the selfadjoint Neumann Laplacian ${}^{0}\Delta$.) By Assertion *6* the non-threshold embedded eigenvalues of ${}^{0}\Delta$ are also the real discrete eigenvalues of the non-selfadjoint operator ${}^{\lambda}\Delta$, $\Im\lambda \neq 0$. Since any point $\mu \in \mathbb{R} \setminus \sigma(\Delta_Y)$ can be separated from $\sigma_{ess}({}^{\lambda}\Delta)$ by a small variation of λ, the embedded eigenvalues of ${}^{0}\Delta$ have no accumulation points, except possibly for the thresholds (the eigenvalues of Δ_Y). In fact, the eigenvalues of ${}^{0}\Delta$ are of finite multiplicity and can accumulate at thresholds only from below, see Assertion *8*. By Assertion *9* any eigenfunction corresponding to a non-threshold embedded eigenvalue μ of ${}^{0}\Delta$ decays at infinity with an exponential rate. The rate depends on the distance from μ to the next threshold of ${}^{0}\Delta$.

We also note that the resonances of the Neumann Laplacian are naturally identified with the non-real eigenvalues of ${}^{\lambda}\Delta$, by Assertion *5* they correspond to the non-real poles of the meromorphic continuation of all resolvent matrix elements $\mu \mapsto \big(({}^{0}\Delta - \mu)^{-1} f, g\big)$, $f, g \in \mathcal{A}$.

Similar results for the stationary Schrödinger operator in $\mathbb{R}^n$ are known from the theory of resonances for N-body quantum scattering problem, e.g. [3,7,18,19] and references therein; for a related study of stability and convergence of the PML method see [13] and references there.

For the (Euclidean) Dirichlet Laplacian in domains with quasicylindrical ends Assertions *1*–*7* were proved in [11], see also [9] for a related study of the PML method. In fact, in contrast to the case of Neumann

Laplacian on $(X, \mathfrak{g})$, complex dilations do not affect the domain of Dirichlet Laplacians, which makes it simpler to prove Theorem 1 for the Dirichlet Laplacian, where the thresholds are now given by the spectrum $\sigma(\Delta_Y)$ of the Dirichlet Laplacian Δ_Y on $(Y, \mathfrak{h})$. We also note that Assertion *6* gives an alternative to the widely used approach [5], where a certain symmetry is required for construction of a boundary value problem whose discrete eigenvalue is also an embedded eigenvalue of the Laplacian. By Assertion *6* the non-threshold embedded eigenvalues can also be considered as discrete eigenvalues of (unfortunately non-selfadjoint) boundary value problem corresponding to ${}^{\lambda}\Delta$.

Let us give an example of a long-range perturbation leading to accumulation of eigenvalues at thresholds. Let $(Z, \mathfrak{h})$ be a smooth compact n-dimensional manifold with boundary. Consider the infinite cylinder $X := \mathbb{R} \times Z$ endowed with the metric $\mathfrak{g}(x) = dx^2 + f(x)^{4/n}\mathfrak{h}$, where f is a positive smooth function of $x \in \mathbb{R}$, such that $f(x)^{4/n} = 1 + |x|^{-\delta}$ for $|x| \geqslant c > 0$ and $\delta \in (0, 2]$. Then $(X, \mathfrak{g})$ satisfies all our assumptions (Y consists of two copies of Z). Separation of variables shows that μ is an eigenvalue of the Neumann Laplacian ${}^{0}\Delta = \frac{1}{f}(-d^2/dx^2 + f''/f + f^{-4/n}\Delta_Z)f$, where Δ_Z is the Neumann Laplacian on $(Z, \mathfrak{h})$, if and only if $\mu - \nu$ is an eigenvalue of the Schrödinger operator $-d^2/dx^2 + f''/f + (f^{-4/n} - 1)\nu$ in $L^2(\mathbb{R})$ for some $\nu \in \sigma(\Delta_Z)$. The minimax principle implies that for all sufficiently large ν the discrete eigenvalues of the Schrödinger operator accumulate at zero from below, cf. [19, Theorem XIII.6]. Thus the eigenvalues of ${}^{0}\Delta$ accumulate at each sufficiently large threshold ν. Some examples of accumulations can also be found in [4].

The subsequent sections are devoted to the proof of Theorem 1. In Section 2 we prove Assertions 1–7. The proof of Assertions 8 and 9 requires some additional constructions, which are done in Section 3. Then, in Section 4, we prove three lemmas on exponential decay of eigenfunctions and accumulation of eigenvalues. Finally, in Section 5, we prove Assertions 8 and 9 and complete the proof of Theorem 1.

2. Proof of Theorem 1: Assertions 1–7

2. For the proof of Assertion *2* we refer to [10] as it is exactly the statement of [10, Prop. 1], where one should set $\beta = 0$.

The proof of Assertions *1* and *3–7* is based on ideas of the theory of resonances for N-body quantum scattering problem, e.g. [3,7,11,19]. We outline the proof below.

3. Let $\lambda \in \mathcal{O}_\alpha$ be fixed. Recall that μ is a point of $\sigma_{ess}({}^{\lambda}\Delta_\beta)$ if the bounded operator ${}^{\lambda}\Delta - \mu : \mathscr{D}({}^{\lambda}\Delta) \to L^2(X)$ is not Fredholm; here $\mathscr{D}({}^{\lambda}\Delta)$ is a Hilbert space equipped the norm $\sqrt{\|\cdot\|^2 + \|{}^{\lambda}\Delta \cdot\|^2}$. The m-sectorial operator ${}^{\lambda}\Delta$ has a regular point in the set $\mathbb{C} \setminus \sigma_{ess}({}^{\lambda}\Delta)$. Therefore ${}^{\lambda}\Delta - \mu : \mathscr{D}({}^{\lambda}\Delta) \to L^2(X)$ is a holomorphic Fredholm operator function of $\mu \in \mathbb{C} \setminus \sigma_{ess}({}^{\lambda}\Delta)$ and thus can have only isolated eigenvalues of finite algebraic multiplicity in $\mathbb{C} \setminus \sigma_{ess}({}^{\lambda}\Delta)$, see e.g. [14, Proposition A.8.4].

5. For $\lambda \in (-1, 1)$, $\mu < 0$, and $f \in L^2(X)$ we have

$$({}^{0}\Delta - \mu)^{-1} f = \big(({}^{\lambda}\Delta - \mu)^{-1}(f \circ \varkappa_\lambda)\big) \circ \varkappa_\lambda^{-1}.$$

Using the identity $(h \circ \varkappa_\lambda^{-1}, g)_0 = (h, g \circ \varkappa_\lambda)_\lambda$ we rewrite the last equality in the form

$$\big(({}^{0}\Delta - \mu)^{-1} f, g\big)_0 = \big(({}^{\lambda}\Delta - \mu)^{-1} f_\lambda, g_{\overline{\lambda}}\big)_\lambda, \tag{6}$$

where $f_\lambda := f \circ \varkappa_\lambda$, $g_\lambda := g \circ \varkappa_\lambda$, and $f, g \in L^2(X)$.

We implement the Aguilar–Balslev–Combes argument to the equality (6) with arbitrary f and g in the set $\mathcal{A}$ of analytic vectors. Then the Right Hand Side (RHS) of (6) provides the Left Hand Side (LHS) with a meromorphic continuation in μ across $\sigma_{ess}({}^{0}\Delta)$.

In more detail, take some $\mu << -1$ that is outside of the sector of ${}^{\lambda}\Delta$ for all $\lambda \in \mathcal{O}_\alpha$. Then the resolvent $({}^{\lambda}\Delta - \mu)^{-1}$ is an analytic function of $\lambda \in \mathcal{O}_\alpha$ and the equality (6) with $f, g \in \mathcal{A}$ extends by analyticity to all $\lambda \in \mathcal{O}_\alpha$. Fix some non-real $\lambda \in \mathcal{O}_\alpha$ and consider the LHS and the RHS as functions of μ. The LHS is

analytic in $\mu \in \mathbb{C} \setminus \sigma({}^0\Delta)$, where $\sigma({}^0\Delta) = \overline{\mathbb{R}}_+$. The RHS is a meromorphic function of $\mu \in \mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$ which provides the LHS with a meromorphic continuation from $\mu \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ across the intervals between the thresholds $\sigma(\Delta_Y)$ of ${}^0\Delta$ to the strips between the rays of $\sigma_{ess}({}^\lambda\Delta)$, cf. Fig. 2. Clearly, this continuation can have poles only at points of $\sigma_d({}^\lambda\Delta)$. Conversely, let P stand for the Riesz projection of ${}^\lambda\Delta$ at $\mu_0 \in \sigma_d({}^\lambda\Delta)$ (i.e. P is the residue of $({}^\lambda\Delta - \mu)^{-1}$ at the pole μ_0 and the kernel $\ker({}^\lambda\Delta - \mu_0) \neq \{0\}$ is in the range of P). Since the set $\varkappa_\lambda[\mathcal{A}]$ is dense in $L^2(X)$, we have $\left(\mathsf{P}f_\lambda, g_{\overline{\lambda}}\right)_\lambda \neq 0$ for some $f, g \in \mathcal{A}$. Thus μ_0 is a pole.

1. The LHS does not depend on s and therefore its meromorphic continuation and its poles cannot depend on s. Then $\sigma_d({}^\lambda\Delta)$ and $\sigma_{ess}({}^\lambda\Delta)$ do not depend on s by Assertions *5* and *2* respectively. Finally, the same applies to $\sigma({}^\lambda\Delta)$ due to Assertion *3*.

4. Let $\mu \in \sigma_d({}^\lambda\Delta)$. As λ changes continuously in the disk $\mathcal{O}_\alpha$ and $\mu \notin \sigma_{ess}({}^\lambda\Delta)$, the RHS provides the LHS with a unique meromorphic continuation to a neighborhood of μ. Therefore μ remains a pole of the meromorphic continuation, which is equivalent to the inclusion $\mu \in \sigma_d({}^\lambda\Delta)$ by Assertion *5*.

6. Let λ be a non-real number in $\mathcal{O}_\alpha$. Consider the projection

$$\mathsf{P} = \text{s-}\lim_{\epsilon \downarrow 0} i\epsilon({}^0\Delta - \mu_0 - i\epsilon)^{-1}$$

onto the eigenspace of the selfadjoint operator ${}^0\Delta$. Suppose that $\mu_0 \in \mathbb{R}$ and $\mu_0 \notin \sigma({}^\lambda\Delta)$ (then $\mu_0 \notin \sigma(\Delta_Y)$). For $f, g \in \mathcal{A}$ the RHS is an analytic function of μ in a neighborhood of μ_0 and $(\mathsf{P}f, g)_0 = 0$. Then $\ker({}^0\Delta - \mu_0) = \{0\}$ as $\mathsf{P} = 0$ due to the density of $\mathcal{A}$ in $L^2(X)$.

Conversely, let $\mu_0 \in \mathbb{R} \cap \sigma_d({}^\lambda\Delta)$. Then μ_0 is a pole of the resolvent $({}^\lambda\Delta - \mu)^{-1}$. Since $\varkappa_\lambda[\mathcal{A}]$ is dense in $L^2(X)$, there exist $f, g \in \mathcal{A}$, such that μ_0 is a pole of the RHS. The equality (6) gives $(\mathsf{P}f, g)_0 \neq 0$, and hence $\ker({}^0\Delta - \mu_0) \neq \{0\}$.

7. The RHS of (6) with $\Im\lambda > 0$ (and hence the LHS) is densely defined and has limits at the points $\mathbb{R} \setminus \sigma({}^\lambda\Delta)$ as μ tends to these points from $\mathbb{C}^+$. Thus ${}^0\Delta$ has no singular continuous spectrum, e.g. [19, Theorem XII.20].

3. Conjugated operator

In order to prove Assertions *8* and *9* of Theorem 1, we need to conjugate the operator ${}^\lambda\Delta$ with an exponent. Consider the function s from (3) as a function of $x \in \mathbb{R}_+$ on the semi-cylinder Π, and extend it to a function in $C^\infty(X)$ by setting $\mathsf{s}\restriction_{X\setminus\Pi} \equiv 0$. For $\beta \in \mathbb{C}$ we introduce the conjugated operator ${}^\lambda\Delta_\beta = e^{\beta\mathsf{s}}\,{}^\lambda\Delta\, e^{-\beta\mathsf{s}}$ on functions $\{u \in C_c^\infty(X) : e^{-\beta\mathsf{s}}u \in \mathscr{D}({}^\lambda\Delta)\}$, where $e^{\beta\mathsf{s}}$ is the operator of multiplication by the exponent. The Friedrichs extension of ${}^\lambda\Delta_\beta$ is an m-sectorial operator and μ belongs to its essential spectrum $\sigma_{ess}({}^\lambda\Delta_\beta)$ if and only if

$$\mu - (1+\lambda)^{-2}(\xi - i\beta)^2 \in \sigma(\Delta_Y)$$

for some $\xi \in \mathbb{R}$, see [10, Prop. 1]. The spectrum $\sigma_{ess}({}^\lambda\Delta_\beta)$ is depicted on Fig. 3. In the case $\beta = 0$ the parabolas collapse to the dashed rays and we obtain the essential spectrum of ${}^\lambda\Delta \equiv {}^\lambda\Delta_0$, cf. Theorem 1.2 and Fig. 2.

In the next lemma we show that the eigenfunctions of $\sigma_d({}^\lambda\Delta)$ are exponentially decaying at infinity in the mean; this does not yet give any information on the eigenfunctions of the Neumann Laplacian as $\sigma_d({}^0\Delta) = \varnothing$.

Lemma 1. *For $\mu \in \sigma_d({}^\lambda\Delta)$ and $\Psi \in \ker({}^\lambda\Delta - \mu)$ we have $e^{\beta\mathsf{s}}\Psi \in \ker({}^\lambda\Delta_\beta - \mu)$ with some $\beta > 0$.*

Proof. It is easy to see that the operators $\mathbb{C} \ni \beta \mapsto {}^\lambda\Delta_\beta$ define an analytic family of quadratic forms, see [10, Lemma 2] for details. Thus in a small disk $\{\beta \in \mathbb{C} : |\beta| < \epsilon\}$ there exist functions $\mu_1, \dots, \mu_k$ with

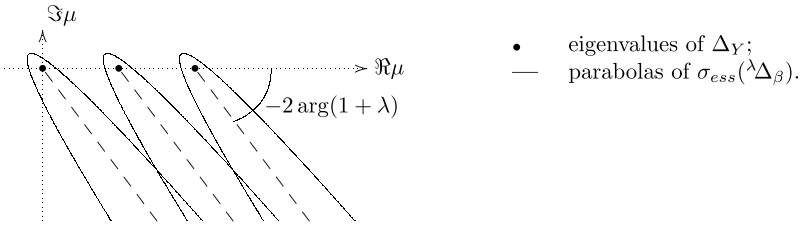


Fig. 3. Essential spectrum of ${}^{\lambda}\Delta_\beta$ for $\Im\lambda > 0$ and $\beta \gtrless 0$.

at worst algebraic branching point at $\beta = 0$, such that the spectrum of ${}^{\lambda}\Delta_\beta$ in a small neighborhood $\mathcal{U}$ of $\mu \in \sigma_d({}^{\lambda}\Delta)$ consists of the isolated eigenvalues $\mu_1(\beta), \dots, \mu_k(\beta)$. Moreover, in the same disk $|\beta| < \epsilon$ we have the analytic function

$$\beta \mapsto \mathsf{P}(\beta) = \frac{1}{2\pi i} \oint_{\partial\mathcal{U}} ({}^{\lambda}\Delta_\beta - \zeta)^{-1}\, d\zeta,$$

whose values are the projections onto the generalized eigenspace of ${}^{\lambda}\Delta_\beta$ associated with the eigenvalues $\mu_1(\beta), \dots, \mu_k(\beta)$, e.g. [12,19]. For all $\beta \in i\mathbb{R}$ and $u \in \mathscr{D}({}^{\lambda}\Delta)$ we have $e^{-\beta \mathsf{s}}\, {}^{\lambda}\Delta_\beta\, e^{\beta \mathsf{s}} u = {}^{\lambda}\Delta u$. Therefore $\mu_j(\beta) = \mu$ and

$$\mathsf{P}(\beta) e^{\beta \mathsf{s}} f = e^{\beta \mathsf{s}} \mathsf{P}(0) f, \quad f \in C_0^\infty(X), \tag{7}$$

for all $\beta \in i\mathbb{R}$. By analyticity these equalities extend to the disk $|\beta| < \epsilon$. Since the set $C_0^\infty(X)$ is dense in $L^2(X)$ and the range of $\mathsf{P}(0)$ is finite dimensional, for any $\Psi \in \ker({}^{\lambda}\Delta - \mu)$ we have $\Psi = \mathsf{P}(0) f$ with some $f \in C_0^\infty(X)$. Due to (7) the function $i\mathbb{R} \ni \beta \mapsto e^{\beta \mathsf{s}} \Psi \in L^2(X)$ extends by analyticity to the disk $|\beta| < \epsilon$. Clearly, $e^{\beta \mathsf{s}} \Psi \in \ker({}^{\lambda}\Delta_\beta - \mu)$. It can also be shown that as β varies continuously in a neighborhood of zero the equality (7) and the inclusions $\mu \in \sigma_d({}^{\lambda}\Delta_\beta)$ and $e^{\beta \mathsf{s}} \Psi \in \ker({}^{\lambda}\Delta_\beta - \mu)$ remain valid as long as the parabolas of $\sigma_{ess}({}^{\lambda}\Delta_\beta)$ do not cover the point μ. □

4. Exponential decay of eigenfunctions and accumulation of eigenvalues

We first prove that any non-threshold eigenfunction is of some exponential decay at infinity in Lemma 2 below. Then we refine this result in Lemma 3. We complete this section with Lemma 4, where as a consequence of eigenfunctions decay estimates we deduce our results on accumulation of embedded eigenvalues of the Neumann Laplacian ${}^{0}\Delta$.

Lemma 2. *Let $\Psi \in \ker({}^{0}\Delta - \mu)$ with $\mu \notin \sigma(\Delta_Y)$. Then 1. the function $\lambda \mapsto \Psi_\lambda := \Psi \circ \varkappa_\lambda \in \ker({}^{\lambda}\Delta - \mu)$ extends by analyticity from real to all $\lambda \in \mathcal{O}_\alpha$; 2. the estimate $\|\Psi(x)\|_{L^2(Y)} = O(e^{-\gamma x})$ holds with some $\gamma > 0$.*

Proof. *1.* Let $\lambda \in \mathcal{O}_\alpha$ be real. Then the Neumann Laplacian ${}^{\lambda}\Delta$ on (X, g_λ) and the projection

$$\mathsf{P}_\lambda = \text{s-}\lim_{\epsilon \downarrow 0} i\epsilon ({}^{\lambda}\Delta - \mu + i\epsilon)^{-1}$$

onto its eigenspace are selfadjoint with respect to the inner product $(\cdot,\cdot)_\lambda$ in $L^2(X)$. For non-real $\lambda \in \mathcal{O}_\alpha$ we define the projection P_λ onto the eigenspace of the non-selfadjoint operator ${}^{\lambda}\Delta$ associated with real eigenvalue $\mu \in \sigma_d({}^{\lambda}\Delta)$ as the Riesz projection; i.e. as the first order residue of the resolvent $({}^{\lambda}\Delta - \zeta)^{-1}$ at the simple pole μ. The resolvent $({}^{\lambda}\Delta - \zeta)^{-1}$ is an analytic function of two variables on the set $\{(\lambda, \zeta) :$

$\lambda \in \mathcal{O}_\alpha, \mu \in \mathbb{C} \setminus \sigma({}^\lambda\Delta)\}$, e.g. [19, Theorem XII.7]. As a consequence, the Riesz projection P_λ is an analytic function of $\lambda \in \mathcal{O}_\alpha \setminus \mathbb{R}$. For $f, g \in \mathcal{A}$ from the equality (6) we get

$$\big(\mathsf{P}_0 f, g\big)_0 = \big(\mathsf{P}_\lambda f_\lambda, g_{\bar\lambda}\big)_\lambda, \quad \lambda \in \mathcal{O}_\alpha. \tag{8}$$

Hence for any real λ_0 and non-real λ in the disk $\mathcal{O}_\alpha$ we have

$$\big(\mathsf{P}_\lambda f_\lambda - \mathsf{P}_{\lambda_0} f_{\lambda_0}, g_{\lambda_0}\big)_{\lambda_0} = \big(\mathsf{P}_\lambda f_\lambda, g_{\lambda_0}/\varrho_{\lambda_0} - g_{\bar\lambda}/\varrho_{\bar\lambda}\big)_0. \tag{9}$$

Here the right hand side tends to zero as $\lambda \to \lambda_0$ because $g_{\lambda_0}/\varrho_{\lambda_0}$ tends to $g_{\bar\lambda}/\varrho_{\bar\lambda}$ in $L^2(X)$ while the $L^2(X)$-norm of $\mathsf{P}_\lambda f_\lambda$ remains bounded. Indeed, thanks to (8) we have

$$\|\mathsf{P}_\lambda f_\lambda\| = \sup_{g\in\mathcal{A}\setminus\{0\}} \frac{\big|\big(\mathsf{P}_\lambda f_\lambda, g_{\bar\lambda}\big)_0\big|}{\|g_{\bar\lambda}\|} = \sup_{g\in\mathcal{A}\setminus\{0\}} \frac{\big|\big(\mathsf{P}_{\lambda_0} f_{\lambda_0}, g_{\lambda_0}\big)_0\big|}{\|g_{\bar\lambda}\|} \leqslant \|\mathsf{P}_{\lambda_0} f_{\lambda_0}\| \sup_{g\in\mathcal{A}\setminus\{0\}} \frac{\|g_{\lambda_0}\|}{\|g_{\bar\lambda}\|}.$$

Thus the left hand side of (9) tends to zero. Due to the density of $\varkappa_{\lambda_0}[\mathcal{A}]$ this implies that $\mathsf{P}_\lambda f_\lambda$ weakly converges to $\mathsf{P}_{\lambda_0} f_{\lambda_0}$ as $\lambda \to \lambda_0$, e.g. [12, Lemma III.1.31]. Therefore $\lambda \mapsto \mathsf{P}_\lambda f_\lambda \in L^2(X)$ is weakly (and therefore strongly) analytic in $\mathcal{O}_\alpha$ for any $f \in \mathcal{A}$. We have $\big(\mathsf{P}_0 f\big) \circ \varkappa_\lambda = \mathsf{P}_\lambda f_\lambda$ (first for real, and then by analyticity) for all $\lambda \in \mathcal{O}_\alpha$. Since the range $\operatorname{Ran}\mathsf{P}_0 = \ker({}^0\Delta - \mu)$ is finite dimensional, for any $\Psi \in \ker({}^0\Delta - \mu)$ there exists $f \in \mathcal{A}$ such that $\Psi = \mathsf{P}_0 f$. The equality $\Psi \circ \varkappa_\lambda = \mathsf{P}_\lambda f_\lambda$ provides $\lambda \mapsto \Psi_\lambda \in L^2(X)$ with analytic continuation to $\mathcal{O}_\alpha$. It remains to note that $\operatorname{Ran}\mathsf{P}_\lambda = \ker({}^\lambda\Delta - \mu)$ as $\mu \in \sigma_d({}^\lambda\Delta) \cap \mathbb{R}$ is a simple pole of the resolvent $({}^\lambda\Delta - \zeta)^{-1}$.

2. First we show that $\lambda \mapsto \Psi_\lambda$ is also analytic as a function from $\mathcal{O}_\alpha$ to $H^1(X)$. As is known [12, Chapter VII.4], the family $\lambda \mapsto \mathsf{q}_\lambda$ satisfies

$$b\mathsf{q}_0[u,u] \leqslant \Re\mathsf{q}_\lambda[u,u] + a\|u\|, \ \ |\mathsf{q}_\lambda[u,v] - \mathsf{q}_\varsigma[u,v]| \leqslant C_{\lambda,\varsigma}\|u\|_{H^1(X)}\|v\|_{H^1(X)},$$

where the constants $a, b > 0$, and $C_{\lambda,\varsigma}$ are bounded uniformly in $u, v \in H^1(X)$ and $\lambda, \varsigma \in \mathcal{O}_\alpha$; moreover, $C_{\lambda,\varsigma} \to 0$ as $|\lambda - \varsigma| \to 0$. We also note that $\mathsf{q}_\lambda[\Psi_\lambda, v] = \mu(\Psi_\lambda, v)_\lambda$ and $|(\Psi_\lambda, \Psi_\lambda)_\lambda| \leqslant c\|\Psi_\lambda\|^2$. As a consequence we obtain

$$\begin{aligned}
\|\Psi_\lambda\|^2_{H^1(X)} &= \mathsf{q}_0[\Psi_\lambda, \Psi_\lambda] + \|\Psi_\lambda\|^2 \leqslant (b^{-1}(c\mu + a) + 1)\|\Psi_\lambda\|^2,\\
b\|\Psi_\lambda - \Psi_\varsigma\|^2_{H^1(X)} &- (a + b + c\mu)\|\Psi_\lambda - \Psi_\varsigma\|^2\\
&\leqslant \Re\mathsf{q}_\lambda[\Psi_\lambda - \Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] - \mu\Re(\Psi_\lambda - \Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma)_\lambda\\
&\leqslant \big|\mathsf{q}_\lambda[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] - \mu(\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma)_\lambda - \mathsf{q}_\varsigma[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] + \mu(\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma)_\varsigma\big|\\
&\leqslant \big|\mathsf{q}_\lambda[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma] - \mathsf{q}_\varsigma[\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma]\big| + \mu\big|\big((\varrho_\lambda - \varrho_\varsigma)\Psi_\varsigma, \Psi_\lambda - \Psi_\varsigma\big)\big|\\
&\leqslant \mathrm{c}_{\lambda,\varsigma}(\mu)\|\Psi_\varsigma\|_{H^1(X)}\|\Psi_\lambda - \Psi_\varsigma\|_{H^1(X)},
\end{aligned}$$

where $c_{\lambda,\varsigma}(\mu)$ tends to zero as $|\lambda - \varsigma| \to 0$. This shows that $\lambda \mapsto \Psi_\lambda \in H^1(X)$ is continuous. Therefore it is also analytic by the Morera's theorem.

For any $x \in \mathbb{R}_+$ the trace $\Psi_\lambda(x)$ of Ψ_λ satisfies $\|\Psi_\lambda(x)\|_{L^2(Y)} \leqslant c\|\Psi_\lambda\|_{H^1(X)}$. Hence the function $\mathcal{O}_\alpha \ni \lambda \mapsto \Psi_\lambda(x) \in L^2(Y)$ is analytic. Let us consider $\Psi_\lambda(x) = \Psi(x + \lambda\mathsf{s}(x))$ as an analytic function of $z = x + \lambda\mathsf{s}(x)$ in the complex conical neighborhood

$$\mathcal{S}_\epsilon = \{x + \lambda\mathsf{s}(x) \in \mathbb{C} : x \in \mathbb{R}_+, \mathsf{s}(x) \neq 0, |\lambda| \leqslant \epsilon\}, \quad \epsilon < \sin\alpha,$$

of infinity. It is uniformly bounded in the following sense:

$$\int\limits_{\mathfrak{L}_\lambda} \|\Psi(z)\|^2_{L^2(Y)}\,|dz| = \int\limits_0^\infty \|\Psi_\lambda(x)\|^2_{L^2(Y)}|1+\lambda\mathsf{s}'(x)|\,dx \leqslant 2\|\Psi_\lambda\|^2 \leqslant C, \tag{10}$$

where $\mathfrak{L}_\lambda$ is the curve $\{x+\lambda\mathsf{s}(x) : x\in\mathbb{R}_+\}$. By Lemma 1 for any non-real $\lambda\in\mathcal{O}_\alpha$ we have $\mu\in\sigma_d({}^\lambda\Delta)$ and $e^{\beta(\lambda)\mathsf{s}}\Psi_\lambda\in L^2(X)$ with some $\beta(\lambda)>0$. Therefore

$$\int\limits_{\mathfrak{L}_\lambda} \|e^{\beta(\lambda)z/2}\Psi(z)\|^2_{L^2(Y)}\,|dz| \leqslant C(\lambda)<\infty, \quad \beta(\lambda)>0, |\lambda|\leqslant\epsilon, \Im\lambda\neq 0. \tag{11}$$

Now the statement follows from a variant of the Phragmén–Lindelöf principle, which says that any analytic function $\mathcal{S}_\epsilon\ni z\mapsto\Psi(z)\in L^2(Y)$ satisfying (10) and (11) also meets the estimate $\|\Psi(x)\|_{L^2(Y)}=O(e^{-\gamma x})$ with some $\gamma>0$.

Indeed, the function $e^{-\delta x^2+\gamma x}\Psi(x)$, $\delta>0$, can be represented via the Cauchy integral along a small circle around x. The uniform estimate (10) allows to change the contour of integration so that

$$e^{-\delta x^2+\gamma x}\Psi(x) = \frac{1}{2\pi i}\Big(\int\limits_{\mathfrak{L}_{i\epsilon}} - \int\limits_{\mathfrak{L}_{-i\epsilon}}\Big)\frac{e^{-\delta z^2+\gamma z}\Psi(z)}{x-z}\,dz. \tag{12}$$

Both integrals in (12) are absolutely convergent in $L^2(Y)$ because of (10). Moreover, thanks to (11), for a small $\gamma>0$ these integrals are bounded in $L^2(Y)$ uniformly in $\delta>0$ and large $x>0$. Hence the same is true for the left hand side of (12). This justifies the Phragmén–Lindelöf principle and completes the proof. □

Lemma 3 *(Refined exponential decay). Let $\Psi\in\ker({}^0\Delta-\mu)$ with $\mu\notin\sigma(\Delta_Y)$. Then (5) is valid.*

Proof. Consider the continuous operator

$$\{\Delta_Y-(\partial_x-\beta)^2-\mu,\partial_\eta\} : H^2(\mathbb{R}\times Y)\to L^2(\mathbb{R}\times Y)\times H^{1/2}(\mathbb{R}\times\partial Y) \tag{13}$$

of the non-homogeneous Neumann problem on $(\mathbb{R}\times Y, dx\otimes dx+\mathfrak{h})$. Here ∂_η is the operator of the Neumann boundary conditions, the space $H^\ell(\mathbb{R}\times Y)$ is introduced as the completion of the set $C_c^\infty(\mathbb{R}\times Y)$ in the norm $\Big(\int_{\mathbb{R}}\sum_{k\leqslant\ell}\|\partial_x^k u\|^2_{H^{\ell-k}(Y)}\Big)^{1/2}$, and $H^{1/2}(\mathbb{R}\times\partial Y)$ is the space of traces of the functions in $H^1(\mathbb{R}\times Y)$. (Note that due to stabilization of $\mathfrak{g}$ to $dx^2+\mathfrak{h}$ at infinity the $H^\ell(\mathbb{R}\times Y)$- and $H^\ell(X)$-norms of the functions supported in Π are equivalent, see [10] for details.) Applying the Fourier transform $\mathcal{F}_{x\mapsto\xi}$ we pass from (13) to the operator $\{\Delta_Y-(i\xi-\beta)^2-\mu,\partial_\eta\}$ of the non-homogeneous Neumann problem on $(Y,\mathfrak{h})$. Suppose that for any $\xi\in\mathbb{R}$ the number $\mu-(\xi-i\beta)^2$ is not an eigenvalue of the Neumann Laplacian Δ_Y on $(Y,\mathfrak{h})$ (or, equivalently, $\mu\notin\sigma_{ess}({}^0\Delta_\beta)$). Then the operator (13) realizes an isomorphism, see e.g. [15, Theorem 5.2.2] or [14, Theorem 2.4.1] for details.

Take a cutoff function $\chi\in C^\infty(\mathbb{R})$, such that $0\leqslant\chi(x)\leqslant 1$, $\chi(x)=0$ for $x\leqslant 1$, and $\chi(x)=1$ for $x\geqslant 2$. Consider the infinite cylinder $\mathbb{R}\times Y$ endowed with the compound metric $\mathfrak{g}_T := (1-\chi_T)\big(dx^2+\mathfrak{h}\big)+\chi_T\mathfrak{g}$, where $\chi_T(x)=\chi(x-T)$ and T is a sufficiently large positive number. The metric $\mathfrak{g}_T$ is well-defined, because $\mathfrak{g}$ stabilizes at infinity to $dx^2+\mathfrak{h}$ by Definition 1. Let Δ^T be the Laplacian on $(\mathbb{R}\times Y,\mathfrak{g}_T)$, by ∂_η^T we denote the corresponding operator of the Neumann boundary conditions on $\partial Y\times\mathbb{R}$. Introduce the conjugated operator $\Delta^T_\beta = e^{\beta x}\Delta^T e^{-\beta x}$, where $e^{-\beta x}$ is the operator of multiplication by the exponent. The continuous operator

$$\{\Delta^T_\beta-\mu, e^{\beta x}\,\partial^T_\eta\, e^{-\beta x}\} : H^2(\mathbb{R}\times Y)\to L^2(\mathbb{R}\times Y)\times H^{1/2}(\mathbb{R}\times\partial Y) \tag{14}$$

coincides with $\{\Delta_Y - (\partial_x - \beta)^2 - \mu, \partial_\eta\}$ on $(-\infty, T) \times Y$. Thanks to stabilization of $\mathfrak{g}$ to $dx^2 + \mathfrak{h}$ at infinity the estimate

$$\|(\Delta_\beta^T - \Delta_Y + (\partial_x - \beta)^2)u\|_{L^2(\mathbb{R}\times Y)} + \|\partial_\eta u - e^{\beta x}\,\partial_\eta^T\, e^{-\beta x}u\|_{H^{1/2}(\mathbb{R}\times\partial Y)} \leqslant C(T)\|u\|_{H^2(\mathbb{R}\times Y)} \tag{15}$$

is valid, where the constant $C(T)$ is independent of $u \in H^2(\mathbb{R} \times Y)$ and $\beta \in \mathcal{C}$, where $\mathcal{C}$ is any compact subset of $\mathbb{C}$. Moreover, $C(T) \to 0$ as $T \to +\infty$.

Let us show that for any small $\delta > 0$ there exists a sufficiently large $T > 0$ such that the inverse of (14) is an analytic function of β on the compact set

$$\mathcal{C}_\delta = \{\beta \in \mathbb{C} : \delta \leqslant \Re\beta \leqslant \min_{\nu\in\sigma(\Delta_Y):\nu>\mu} \sqrt{\nu - \mu} - \delta, |\Im\beta| \leqslant 1\}.$$

For $\beta \in \mathcal{C}_\delta$ the operator (13) yields an isomorphism. Thanks to (15) there exists a sufficiently large $T(\delta) > 0$ such that for all $T \geqslant T(\delta)$ the operator (14) is a small in norm perturbation of (13) and thus yields an isomorphism. This implies that its inverse is an analytic function of $\beta \in \mathcal{C}_\delta$.

For $\varphi_T = \chi_{T+2}$ and $\{\beta \in \mathcal{C}_\delta : \Re\beta < \gamma\}$, where $\gamma > 0$ is the same as in Lemma 2, we have

$$e^{\beta x}\varphi_T\Psi = \{\Delta_\beta^T - \mu, e^{\beta x}\,\partial_\eta^T\, e^{-\beta x}\}^{-1} e^{\beta x}\{[\Delta^T, \varphi_T]\chi_T\Psi, [\partial_\eta^T, \varphi_T]\chi_T\Psi\}. \tag{16}$$

(Here $\varphi_T\Psi$ and $\chi_T\Psi$ are extended from their supports in Π to $\mathbb{R} \times Y$ by zero and $[a, b] = ab - ba$.) The equality (16) extends by analyticity to all $\beta \in \mathcal{C}_\delta$. As δ is arbitrarily small, this establishes the inclusion $\varphi_T e^{\beta x}\Psi \in H^2(\mathbb{R}\times Y)$ for $\Re\beta < \min_{\nu\in\sigma(\Delta_Y):\nu>\mu} \sqrt{\nu - \mu}$. Now we conclude that Ψ meets (5) as $\|u(x)\|_{L^2(Y)} \leqslant C\|u\|_{H^2(\mathbb{R}\times Y)}$. □

Lemma 4 *(Accumulation of eigenvalues). Let $\{\mu_k\}_{k=1}^\infty$ be a sequence of eigenvalues of the Neumann Laplacian ${}^0\Delta$ such that $\mu_k \to \mu < \infty$ as $k \to \infty$, and $\mu_k \neq \mu$. Then the corresponding eigenfunctions Ψ_k satisfy $\|\Psi_k(x)\| \leqslant C_k e^{-\gamma_k x}$ as $x \to +\infty$ with some $\gamma_k > 0$ and $\gamma_k \to 0$ as $k \to \infty$. Moreover, $\mu \in \sigma(\Delta_Y)$ (i.e. μ is a threshold of ${}^0\Delta$), and $\{\mu_k\}_{k=1}^\infty$ accumulates to μ only from below.*

Proof. We have already proved that $\|\Psi_k(x)\|_{L^2(Y)} \leqslant C_k e^{-\gamma_k x}$ with some $\gamma_k > 0$ as $x \to +\infty$ (see Lemma 3) and that μ is a threshold (see Assertion *6* of Theorem 1 and the discussion after the theorem). Let us show that $\gamma_k \to 0$ as $k \to \infty$; here we combine ideas used in the proof of Lemma 3 with the compactness argument due to Perry [18].

Assume the contrary. Then there exists $\beta > 0$, such that $\gamma_k > \beta$ for all $k > 0$. By taking a slightly smaller $\beta > 0$ we can always achieve $\mu \notin \sigma_{ess}({}^0\Delta_\beta)$, cf. Fig. 3. Let $T > 0$ be sufficiently large. The equality

$$\varphi_T e^{\beta x}\Psi_k = \{\Delta_\beta^T - \mu, e^{\beta x}\,\partial_\nu^T\, e^{-\beta x}\}^{-1} e^{\beta x}\{[\Delta^T, \varphi_T]\chi_T\Psi_k + (\mu_k - \mu)\varphi_T\Psi_k, [\partial_\eta^T, \varphi_T]\chi_T\Psi_k\}$$

together with boundedness of the inverse of (14) and the estimates

$$\begin{aligned}\|e^{\beta s}[\Delta^T, \varphi_T]\Psi_k\|^2 + \|e^{\beta s}[\partial_\eta^T, \varphi_T]\Psi_k\|^2_{H^{1/2}(\mathbb{R}\times\partial Y)} &\leqslant c\|\Psi_k\|^2_{H^2(X)} \\ &= c(\|{}^0\Delta\Psi_k\|^2 + \|\Psi_k\|^2) = c(\mu_k^2 + 1)\|\Psi_k\|^2\end{aligned}$$

justifies the uniform in k inequality

$$\|\varphi_T e^{\beta s}\Psi_k\|^2 \leqslant C(\mu_k^2 + 1)\|\Psi_k\|^2 + C(\mu_k - \mu)^2\|\varphi_T e^{\beta s}\Psi_k\|^2, \tag{17}$$

where we extended the functions from their supports to X by zero. Since $1 - C(\mu_k - \mu)^2 > 0$ for all sufficiently large k, from (17) we get

$$\|\varphi_T e^{\beta \mathsf{s}}\Psi_k\|^2 \leqslant C(\mu_k^2+1)\big(1-C(\mu_k-\mu)^2\big)^{-1}\|\Psi_k\|^2 \leqslant Const\|\Psi_k\|^2. \tag{18}$$

For any $\Psi \in L^2(X)$ we have $\|(1-\varphi_T)e^{\beta \mathsf{s}}\Psi\| \leqslant c\|\Psi\|$. Thus (18) leads to the uniform in k estimate $\|e^{\beta \mathsf{s}}\Psi_k\| \leqslant \mathrm{C}\|\Psi_k\|$. This estimate together with the Cauchy–Schwarz inequality gives

$$\|\Psi_k\|^2 \leqslant \|e^{-\beta \mathsf{s}}\Psi_k\|\|e^{\beta \mathsf{s}}\Psi_k\| \leqslant \mathrm{C}\|\Psi_k\|\,\|e^{-\beta \mathsf{s}}\Psi_k\|.$$

Finally, for all sufficiently large k we obtain the uniform estimate

$$\|\Psi_k\|^2_{H^2(X)} = (\mu_k^2+1)\|\Psi_k\|^2 \leqslant Const\|e^{-\beta \mathsf{s}}\Psi_k\|^2. \tag{19}$$

The Sobolev space $H^2(X)$ is compactly embedded into the weighted space $L^2(X, e^{-\beta \mathsf{s}})$. Since compactness of the unit ball implies that the space is finite-dimensional, there can only be a finite number of linearly independent normalized eigenfunctions Ψ_k satisfying (19). Since $\mu_k \to \mu \neq \mu_k$ as $k \to \infty$, we come to a contradiction. Thus $\gamma_k \to 0$ as $k \to \infty$.

In order to complete the proof we notice that for any subsequence of $\{\mu_k\}_{k=1}^{\infty}$ that does not accumulate to some $\mu \in \sigma(\Delta_Y)$ from below we can use Lemma 3 in order to refine the corresponding subsequence of $\{\gamma_k\}_{k=1}^{\infty}$ so that the elements γ_k of the subsequence do not tend to zero as $k \to \infty$. □

5. Proof of Theorem 1: Assertions 8 and 9

8. If Ψ is an eigenfunction of ${}^0\Delta - \mu$, then $e^{-\beta \mathsf{s}}\Psi$ with $\beta > 0$ is an eigenfunction of the non-selfadjoint operator ${}^0\Delta_{-\beta} - \mu$. By taking a suitable $\beta > 0$ we achieve $\mu \notin \sigma_{ess}({}^0\Delta_\beta)$. As a result the operator ${}^0\Delta_{-\beta} - \mu : \mathscr{D}({}^\lambda\Delta_{-\beta}) \to L^2(X)$ is Fredholm. In particular, $\ker({}^0\Delta_{-\beta} - \mu)$ is a finite dimensional space. Hence the eigenvalues of the selfadjoint operator ${}^0\Delta$ are of finite multiplicity. The remaining part of the statement follows from Lemma 4.

9. The statement has been proved in Lemma 3. Note that an argument similar to the one in the proof of Lemma 4 demonstrates that the estimate (5) is sharp in the following sense: any compact subset of $\sigma({}^0\Delta) = \overline{\mathbb{R}_+}$ can contain at most a finite number of eigenvalues μ_k corresponding to eigenfunctions Ψ_k satisfying $\|\Psi_k(x)\|_{L^2(Y)} = O(e^{-\gamma_k x})$ with $\gamma_k > \min_{\nu \in \sigma(\Delta_Y):\nu > \mu_k} \sqrt{\nu - \mu_k}$.

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