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Generalized Newtonian fluid flow through a porous medium

V.J. Ervin [a,*], Hyesuk Lee [b,1], A.J. Salgado [c]

[a] Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, USA
[b] Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, USA
[c] Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA

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ABSTRACT

We present a model for generalized Newtonian fluid flow through a porous medium. In the model the dependence of the fluid viscosity on the velocity is replaced by a dependence on a smoothed (locally averaged) velocity. With appropriate assumptions on the smoothed velocity, existence of a solution to the model is shown. Two examples of smoothing operators are presented in the appendices. A numerical approximation scheme is presented and an a priori error estimate derived. A numerical example is given illustrating the approximation scheme and the a priori error estimate.

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1. Introduction

Of interest in this article is the modeling and approximation of generalized Newtonian fluid flow through a porous medium. Darcy's modeling equations for a steady-state fluid flow through a porous medium, Ω, are

$$\nu_{eff}\, K^{-1}\, \mathbf{u} + \nabla p = 0, \text{ in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega, \tag{1.2}$$

where $\mathbf{u}$ and p denote the velocity and pressure of the fluid, respectively. $K(\mathbf{x})$ in (1.1) represents the permeability of the medium at $\mathbf{x} \in \Omega$, which is assumed to be a symmetric, positive definite tensor. As our investigations are not concerned with K, we assume that K is of the form $k(\mathbf{x})\mathbf{I}$ where $k(\mathbf{x})$ is a Lipschitz continuous, positive, bounded and bounded away from zero, scalar function. ν_{eff} in (1.1) represents the effective viscosity of the fluid.

* Corresponding author.
E-mail addresses: vjervin@clemson.edu (V.J. Ervin), hklee@mail.clemson.edu (H. Lee), asalgad1@utk.edu (A.J. Salgado).
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In the case of a Newtonian fluid in a porous medium we have that ν_{eff} is a positive constant. For a generalized Newtonian fluid various models for ν_{eff} have been proposed in the literature. (See for example, [14,16] and the references cited therein.) Based upon dimensional analysis most models assume that ν_{eff} is a function of $|\mathbf{u}|/(\sqrt{|K|}m_c)$, where m_c is a constant related to the internal structure of the porous media. Models for ν_{eff} include [14,16]:

$$\text{Power Law Model:}\quad \nu_{eff}(|\mathbf{u}|) = c_\nu|\mathbf{u}|^{r-2}, \qquad \text{Cross Model:}\quad \nu_{eff}(|\mathbf{u}|) = \nu_\infty + \frac{\nu_0 - \nu_\infty}{1 + c_\nu|\mathbf{u}|^{2-r}}, \tag{1.3}$$

where c_ν, ν_0, ν_∞ and r are fluid dependent constants. For shear thinning fluids $1 < r < 2$. (In modeling the viscosity of shear thinning fluids the Power Law model suffers the criticism that as $|\mathbf{u}| \to 0$ $\nu_{eff} \to \infty$.)

For the case of a Newtonian fluid (1.1), (1.2) are well studied. The two standard approaches in analyzing (1.1), (1.2) are: (i) study (1.1), (1.2) as a mixed formulation problem for $\mathbf{u}$ and p (either $(\mathbf{u}, p) \in H_{div}(\Omega) \times L^2(\Omega)$, or $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$), or (ii) use (1.2) to eliminate $\mathbf{u}$ in (1.1) to obtain a generalized Laplace's equation for p.

For generalized Newtonian fluids, with $\nu_{eff} = \nu_{eff}(|\mathbf{u}|)$, assumptions are required on ν_{eff} in order to establish existence and uniqueness of solutions. Typical assumptions are uniform continuity of $\nu_{eff}(|\mathbf{u}|)\mathbf{u}$ and strong monotonicity of $\nu_{eff}(|\mathbf{u}|)$ [8,9,11], i.e., there exists $C > 0$ such that

$$|\nu_{eff}(|\mathbf{u}|)\mathbf{u} - \nu_{eff}(|\mathbf{v}|)\mathbf{v}| \le C|\mathbf{u} - \mathbf{v}|, \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \tag{1.4}$$

$$(\nu_{eff}(|\mathbf{u}|)\mathbf{u} - \nu_{eff}(|\mathbf{v}|)\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \ge C(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}), \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d. \tag{1.5}$$

A more general setting where the fluid rheology is defined implicitly has been analyzed in [6,7]. The case where the fluid viscosity depends on the shear rate and pressure has been studied in [15,13]. For both of these cases additional structure beyond (1.4) and (1.5) is required in order to establish existence and uniqueness of a solution.

A nonlinear Darcy fluid flow problem, with a permeability dependent upon the pressure was investigated by Azaïez, Ben Belgacem, Bernardi, and Chorfi [2], and Girault, Murat, and Salgado [12]. For a Lipschitz continuous permeability function, bounded above and bounded away from zero, existence of a solution $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$ was established. Important in handling the nonlinear permeability function, in establishing existence of a solution, was the property that $p \in H^1(\Omega)$. In [2] the authors also investigated a spectral numerical approximation scheme for the nonlinear Darcy problem, assuming an axisymmetric domain Ω. A convergence analysis for the finite element discretization of that problem was given in [12].

Our interest in this paper is in relaxing the assumptions (1.4) and (1.5). Specifically, our interest is assuming that $\nu_{eff}(\cdot)$ is only Lipschitz continuous and both bounded above and bounded away from zero. However, relaxing the conditions (1.4) and (1.5) requires us to make an additional assumption regarding the argument of $\nu_{eff}(\cdot)$. In order to obtain a modeling system of equations for which a solution can be shown to exist, we replace $\mathbf{u}$ in $\nu_{eff}(|\mathbf{u}|)$ by a *smoothed* velocity, $\mathbf{u}^s$. The approach of regularizing the model with the introduction of $\mathbf{u}^s$ is, in part, motivated by the fact that the Darcy fluid flow equations can be derived by *averaging*, e.g. volume averaging [19], homogenization [1], or mixture theory [17].

Presented in Appendices A and B are two smoothing operators for $\mathbf{u}$. One is a local averaging operator, whereby $\mathbf{u}^s(\mathbf{x})$ is obtained by averaging $\mathbf{u}$ in a neighborhood of $\mathbf{x}$. The second smoothing operator, which is nonlocal, computes $\mathbf{u}^s(\mathbf{x})$ using a differential filter applied to $\mathbf{u}$. That is, $\mathbf{u}^s$ is given by the solution to an elliptic differential equation whose right hand side is $\mathbf{u}$. For establishing the existence of a solution to (1.1)–(1.2), the key property of the smoothing operators is that they transform a weakly convergent sequence in $L^2(\Omega)$ into a sequence which converges strongly in $L^\infty(\Omega)$.

For the mathematical analysis of this problem it is convenient to have homogeneous boundary conditions. This is achieved by introducing a suitable change of variables. For example, assuming $\partial\Omega = \Gamma_{in} \cup \Gamma \cup \Gamma_{out}$, in the case the specified boundary conditions are

$$\mathbf{u} \cdot (-\mathbf{n}) = g_{in} \text{ on } \Gamma_{in}, \qquad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \qquad p = p_{out} \text{ on } \Gamma_{out},$$

we introduce functions $\mathbf{b}(\mathbf{x})$ and $p_b(\mathbf{x})$ defined on Ω satisfying

$$\begin{aligned} \nabla \cdot \mathbf{b} &= 0, \text{ in } \Omega, & \nabla \cdot \nabla p_b &= 0, \text{ in } \Omega, \\ \mathbf{b} \cdot \mathbf{n} &= -g_{in}, \text{ on } \Gamma_{in}, & p_b &= p_{out}, \text{ on } \Gamma_{out}, \\ \mathbf{b} \cdot \mathbf{t}_i &= 0, \text{ on } \Gamma_{in}, & \tfrac{\partial p_b}{\partial \mathbf{n}} &= 0, \text{ on } \partial\Omega \backslash \Gamma_{out}, \\ \mathbf{b} &= \mathbf{0}, \text{ on } \partial\Omega \backslash \Gamma_{in}, \end{aligned}$$

where t_i, $i = 1, \ldots, (d-1)$ denotes an orthogonal set of tangent vectors on Γ_{in}.

(In case the pressure is specified on the inflow boundary Γ_{in}, then $\mathbf{b} = \mathbf{0}$, and the definition of p_b is appropriately modified.)

With the change of variables: $\mathbf{u} = \mathbf{u}_0 + \mathbf{b}$ and $p = p_0 + p_b$, and subsequent relabeling $\mathbf{u}_0 = \mathbf{u}$, $p_0 = p$ and $\mathbf{f} = -\nabla p_b$ we obtain the following system of modeling equations:

$$\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u} + \beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b} + \nabla p = \mathbf{f}, \text{ in } \Omega, \tag{1.6}$$

$$\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega, \tag{1.7}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \text{ on } \Gamma_{in} \cup \Gamma, \tag{1.8}$$

$$p = 0, \text{ on } \Gamma_{out}, \tag{1.9}$$

where $\beta(|\mathbf{u}^s + \mathbf{b}|) = \nu_{\mathit{eff}}(|\mathbf{u}^s + \mathbf{b}|)\, k^{-1}$.

In the next section we show that, under suitable assumptions on $\beta(\cdot)$ and $\mathbf{u}^s$, there exists a unique solution to (1.6)–(1.9). An approximation scheme is presented in Section 3, and an a priori error estimate derived. Numerical examples illustrating the approximation scheme and the a priori error estimate are presented in Section 4.

2. Existence and uniqueness

In this section we investigate the existence and uniqueness of solutions to the nonlinear system of equations (1.6)–(1.9). We assume that $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a convex polyhedral domain and for vectors in $\mathbb{R}^d$ $|\cdot|$ denotes the Euclidean norm.

Throughout, we use C to denote a generic nonnegative constant, independent of the mesh parameter h, whose actual value may change from line to line in the analysis.

We make the following assumptions on $\beta(\cdot)$ and $\mathbf{u}^s$.

Assumptions on $\beta(\cdot)$:

A$\boldsymbol{\beta}$1 : $\beta(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$,

A$\boldsymbol{\beta}$2 : $0 < \beta_{min} \leq \beta(s) \leq \beta_{max}, \ \forall s \in \mathbb{R}^+$,

A$\boldsymbol{\beta}$3: β is Lipschitz continuous, $|\beta(s_1) - \beta(s_2)| \leq C_\beta |s_1 - s_2|$.

Assumptions on $\mathbf{u}^s$:

Au^{s}1: For $\mathbf{u} \in L^2(\Omega)$, $\|\mathbf{u}^s\|_{L^\infty(\Omega)} \leq C_s \|\mathbf{u}\|_{L^2(\Omega)}$,

Au^{s}2: For $\{\mathbf{u}_n\}_{n=1}^\infty \subset L^2(\Omega)$, with $\mathbf{u}_n$ converging weakly to $\mathbf{u} \in L^2(\Omega)$, then $\{\mathbf{u}_n^s\}_{n=1}^\infty$ converges to $\mathbf{u}^s$ in $L^\infty(\Omega)$,

Au^{s}3: The mapping $\mathbf{u} \mapsto \mathbf{u}^s$ is linear.

Weak formulation of (1.6)–(1.9)

Let $X = \{\mathbf{v} \in H_{div}(\Omega) : \mathbf{v}\cdot\mathbf{n} = 0, \text{ on } \Gamma_{in} \cup \Gamma\}$. We use

$$(f,g) := \int_\Omega f \cdot g \, d\Omega, \quad \text{and} \quad \|f\| := (f,f)^{1/2}$$

to denote the L^2 inner product and the L^2 norm over Ω, respectively, for both scalar and vector valued functions. Additionally, we introduce the norm

$$\|\mathbf{v}\|_X = \left(\int_\Omega (\nabla\cdot\mathbf{v}\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v})\, d\Omega \right)^{1/2}.$$

Remark: For $\mathbf{v} \in H_{div}(\Omega)$ it follows that $\mathbf{v}\cdot\mathbf{n} \in H^{-1/2}(\partial\Omega)$. For the interpretation of the condition $\mathbf{v}\cdot\mathbf{n} = 0$ on $\Gamma_{in} \cup \Gamma$ see [10,18].

We restate (1.6)–(1.9) as: *Given* $\mathbf{b}, \mathbf{f} \in L^2(\Omega)$, *find* $(\mathbf{u}, p) \in X \times L^2(\Omega)$, *such that for all* $\mathbf{v} \in X$ *and* $q \in L^2(\Omega)$

$$(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \tag{2.1}$$

$$(q, \nabla\cdot\mathbf{u}) = 0. \tag{2.2}$$

For the spaces X and $L^2(\Omega)$ we have the following inf–sup condition

$$\inf_{q\in L^2(\Omega)} \sup_{\mathbf{v}\in X} \frac{(q, \nabla\cdot\mathbf{v})}{\|q\|\|\mathbf{v}\|_X} \geq c_0 > 0. \tag{2.3}$$

We begin by establishing boundedness of any solution to (2.1)–(2.2).

Lemma 2.1. *Any solution* $(\mathbf{u}, p) \in X \times L^2(\Omega)$ *to (2.1)–(2.2) satisfies*

$$\|\mathbf{u}\|_X + \|p\| \leq C\left(\|\mathbf{b}\| + \|\mathbf{f}\|\right). \tag{2.4}$$

Proof. From (2.2) and that $\nabla\cdot X \subset L^2(\Omega)$ we have that any solution $\mathbf{u}$ to (2.1)–(2.2) satisfies

$$\|\nabla\cdot\mathbf{u}\| = 0. \tag{2.5}$$

With the choice $\mathbf{v} = \mathbf{u}$, $q = p$, subtracting (2.2) from (2.1), and using assumption **Aβ2** yields

$$\begin{aligned}(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}, \mathbf{u}) &= -(\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}, \mathbf{u}) + (\mathbf{f}, \mathbf{u}),\\ \beta_{min}\|\mathbf{u}\|^2 &\leq \beta_{max}\|\mathbf{b}\|\|\mathbf{u}\| + \|\mathbf{f}\|\|\mathbf{u}\|.\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) we obtain the stated bound for $\mathbf{u}$. The estimate for p is obtained using the inf–sup condition (2.3).

$$\|p\| \leq \frac{1}{c_0}\sup_{\mathbf{v}\in X}\frac{(p,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X} = \frac{1}{c_0}\sup_{\mathbf{v}\in X}\frac{(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{u},\mathbf{v}) + (\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{b},\mathbf{v}) - (\mathbf{f},\mathbf{v})}{\|\mathbf{v}\|_X}$$

$$\leq \frac{1}{c_0}\left(\|\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{u}\|+\|\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{b}\|+\|\mathbf{f}\|\right)$$
$$\leq \frac{1}{c_0}\left(\beta_{max}\left(\|\mathbf{u}\|+\|\mathbf{b}\|\right)+\|\mathbf{f}\|\right),$$

from which the stated bound follows. □

$$\text{Define}\quad Z=\left\{\mathbf{v}\in X:(q,\nabla\cdot\mathbf{v})=0,\ \forall q\in L^2(\Omega)\right\}.$$

Because of the inf–sup condition (2.3), the weak formulation (2.1)–(2.2) can be equivalently stated as: *Given* $\mathbf{b},\mathbf{f}\in L^2(\Omega)$*, find* $\mathbf{u}\in Z$*, such that for all* $\mathbf{v}\in Z$

$$(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{u},\mathbf{v})+(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{b},\mathbf{v})=(\mathbf{f},\mathbf{v}). \tag{2.7}$$

Remark: For $\mathbf{v}\in Z$, $\|\mathbf{v}\|_X=\|\mathbf{v}\|$, as $\|\nabla\cdot\mathbf{v}\|=0$.

To establish the existence of a solution to (2.7) we use the Leray–Schauder fixed point theorem. To do this we show that a solution to (2.7) is a fixed point of a compact mapping Φ.

Theorem 2.1. *For* $\beta(\cdot)$ *and* $\mathbf{u}^s$ *satisfying assumptions* $\mathbf{A\beta 1}$–$\mathbf{A\beta 3}$ *and* $\mathbf{Au^s1}$–$\mathbf{Au^s2}$*, respectively, there exists a solution* $\mathbf{u}$ *to (2.7).*

Proof. Let $\Phi: Z\longrightarrow Z$ be defined by $\Phi(\mathbf{u})=\mathbf{w}$, where $\mathbf{w}$ satisfies

$$(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{w},\mathbf{v})+(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{b},\mathbf{v})=(\mathbf{f},\mathbf{v}). \tag{2.8}$$

That Φ is well defined follows from $\mathbf{A\beta 2}$ and the Lax–Milgram theorem.

To show that Φ is a compact operator, let $\{\mathbf{u}_n\}_{n=1}^{\infty}$ denote a bounded sequence in Z. From $\{\mathbf{u}_n\}_{n=1}^{\infty}$ we can extract a subsequence, which we again denote as $\{\mathbf{u}_n\}_{n=1}^{\infty}$, such that $\{\mathbf{u}_n\}_{n=1}^{\infty}$ converges weakly to $\mathbf{u}\in Z$. For $\mathbf{w}_n=\Phi(\mathbf{u}_n)$, using (2.8)

$$\begin{aligned}(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{w},\mathbf{v})-(\beta(|\mathbf{u}_n^s+\mathbf{b}|)\mathbf{w}_n,\mathbf{v}) &= -(\beta(|\mathbf{u}^s+\mathbf{b}|)\mathbf{b},\mathbf{v})+(\beta(|\mathbf{u}_n^s+\mathbf{b}|)\mathbf{b},\mathbf{v})\\ \iff\quad (\beta(|\mathbf{u}_n^s+\mathbf{b}|)(\mathbf{w}-\mathbf{w}_n),\mathbf{v}) &= -\left((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_n^s+\mathbf{b}|))\,\mathbf{w},\mathbf{v}\right)\\ &\quad -\left((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_n^s+\mathbf{b}|))\,\mathbf{b},\mathbf{v}\right).\end{aligned}$$

With $\mathbf{v}=\mathbf{w}-\mathbf{w}_n$, and using $\mathbf{A\beta 2}$ and $\mathbf{A\beta 3}$

$$\begin{aligned}\beta_{min}\|\mathbf{w}-\mathbf{w}_n\|^2 &\leq \|C_\beta\left(|\mathbf{u}^s+\mathbf{b}|-|\mathbf{u}_n^s+\mathbf{b}|\right)|\,\mathbf{w}\|\|\mathbf{w}-\mathbf{w}_n\|+\|C_\beta\left(|\mathbf{u}^s+\mathbf{b}|-|\mathbf{u}_n^s+\mathbf{b}|\right)|\,\mathbf{b}\|\|\mathbf{w}-\mathbf{w}_n\|\\ &\leq \|C_\beta|\mathbf{u}^s-\mathbf{u}_n^s|\mathbf{w}\|\|\mathbf{w}-\mathbf{w}_n\|+\|C_\beta|\mathbf{u}^s-\mathbf{u}_n^s|\mathbf{b}\|\|\mathbf{w}-\mathbf{w}_n\|\\ &\leq C_\beta\sqrt{d}\|\mathbf{u}^s-\mathbf{u}_n^s\|_{L^\infty(\Omega)}\|\mathbf{w}\|\|\mathbf{w}-\mathbf{w}_n\|+C_\beta\sqrt{d}\|\mathbf{u}^s-\mathbf{u}_n^s\|_{L^\infty(\Omega)}\|\mathbf{b}\|\|\mathbf{w}-\mathbf{w}_n\|\end{aligned}$$

$$\Rightarrow\quad \|\mathbf{w}-\mathbf{w}_n\|_X=\|\mathbf{w}-\mathbf{w}_n\|\leq\frac{C_\beta\sqrt{d}}{\beta_{min}}\|\mathbf{u}^s-\mathbf{u}_n^s\|_{L^\infty(\Omega)}\left(\|\mathbf{w}\|+\|\mathbf{b}\|\right),$$

from which, with $\mathbf{Au^s2}$, we can conclude that Φ is a compact operator.

For $r=\frac{\beta_{max}}{\beta_{min}}\left(\|\mathbf{b}\|+\|\mathbf{f}\|\right)$, from Lemma 2.1 we have that $\|\Phi(\mathbf{u})\|\leq r$, $\forall\mathbf{u}\in Z$. Then, applying the Leray–Schauder fixed point theorem [20] we obtain that there exists a $\mathbf{u}\in Z$ such that $\mathbf{u}=\Phi(\mathbf{u})$. □

Under small data conditions we have the following theorem guaranteeing uniqueness of solutions to (2.7).

Theorem 2.2. *With the stated assumptions* **Aβ1**–**Aβ3** *and* **Au^{s}1**–**Au^{s}2**, *and the condition that* $\|\mathbf{b}\| \leq \max\left\{\beta_{min}/\beta_{max}, \beta_{min}/(C_\beta\sqrt{d}\,C_s)\right\}$, *if a solution* $\mathbf{u}$ *to (2.7) exists satisfying*

$$\|\mathbf{u}\| < \max\left\{\frac{\beta_{min}}{\beta_{max}}, \frac{\beta_{min}}{C_\beta\sqrt{d}\,C_s}\right\} - \|\mathbf{b}\|, \tag{2.9}$$

then there is no other solution to (2.7).

Proof. Suppose that both $\mathbf{u}$ and $\mathbf{w} \in Z$ satisfy (2.7), i.e., together with (2.7) we have that

$$(\beta(|\mathbf{w}^s + \mathbf{b}|)\mathbf{w}, \mathbf{v}) + (\beta(|\mathbf{w}^s + \mathbf{b}|)\mathbf{b}, \mathbf{v}) - (\mathbf{f}, \mathbf{v}) = 0, \ \forall \mathbf{v} \in Z. \tag{2.10}$$

With $\mathbf{v} = \mathbf{u} - \mathbf{w}$, subtracting (2.10) from (2.7) and using the bounds for $\beta(\cdot)$ we obtain

$$\begin{aligned}(\beta(|\mathbf{w}^s + \mathbf{b}|)(\mathbf{u} - \mathbf{w}), (\mathbf{u} - \mathbf{w})) + ((\beta(|\mathbf{u}^s + \mathbf{b}|) - \beta(|\mathbf{w}^s + \mathbf{b}|))\mathbf{u}, (\mathbf{u} - \mathbf{w}))& \\ + ((\beta(|\mathbf{u}^s + \mathbf{b}|) - \beta(|\mathbf{w}^s + \mathbf{b}|))\mathbf{b}, (\mathbf{u} - \mathbf{w})) = 0& \qquad (2.11)\\ \Rightarrow \quad \beta_{min}\|\mathbf{u} - \mathbf{w}\|^2 \leq \beta_{max}\|\mathbf{u}\|\|\mathbf{u} - \mathbf{w}\| + \beta_{max}\|\mathbf{b}\|\|\mathbf{u} - \mathbf{w}\|& \\ \Rightarrow \quad (\beta_{min} - \beta_{max}(\|\mathbf{u}\| + \|\mathbf{b}\|))\|\mathbf{u} - \mathbf{w}\| \leq 0& \qquad (2.12)\\ \Rightarrow \quad \mathbf{w} = \mathbf{u}, \text{ provided} \quad \|\mathbf{u}\| < \frac{\beta_{min}}{\beta_{max}} - \|\mathbf{b}\|.& \qquad (2.13)\end{aligned}$$

Alternatively, from (2.11), using **Aβ3** and **Au^{s}1**,

$$\begin{aligned}\beta_{min}\|\mathbf{u} - \mathbf{w}\|^2 &\leq C_\beta\sqrt{d}\|\mathbf{u}^s - \mathbf{w}^s\|_{L^\infty(\Omega)}\|\mathbf{u}\|\|\mathbf{u} - \mathbf{w}\| + C_\beta\sqrt{d}\|\mathbf{u}^s - \mathbf{w}^s\|_{L^\infty(\Omega)}\|\mathbf{b}\|\|\mathbf{u} - \mathbf{w}\| \\ &\leq C_\beta\sqrt{d}\,C_s\left(\|\mathbf{u}\| + \|\mathbf{b}\|\right)\|\mathbf{u} - \mathbf{w}\|^2 \\ \Rightarrow \quad (\beta_{min} - C_\beta\sqrt{d}\,C_s(\|\mathbf{u}\| + \|\mathbf{b}\|))\|\mathbf{u} - \mathbf{w}\|^2 &\leq 0 \\ \Rightarrow \quad \mathbf{w} = \mathbf{u}, \text{ provided} \quad \|\mathbf{u}\| &< \frac{\beta_{min}}{C_\beta\sqrt{d}\,C_s} - \|\mathbf{b}\|. \qquad \square\end{aligned}$$

3. Finite element approximation

In this section we investigate the finite element approximation to $(\mathbf{u}, p)$ satisfying (1.6)–(1.9).

Let T_h be a triangulation of Ω made of triangles (in $\mathbb{R}^2$) or tetrahedrons (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\overline{\Omega} = \cup_{K\in T_h}\overline{K}.$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \leq h_K \leq c_2\rho_K,$$

where h_K is the diameter of triangle (tetrahedron) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K\in T_h} h_K$. For $k \in \mathbb{N}$, let $P_k(A)$ denote the space of polynomials on A of degree no greater than k, and $RT_k(T_h)$ the (Piola) affine transformation of the Raviart–Thomas elements of order k on the unit triangle. We define the finite element spaces X_h, X_h^s and Q_h as follows.

$$X_h := \{RT_k(T_h) \cap X\}, \tag{3.1}$$

$$X_h^s := \left\{\mathbf{v} \in X \cap C^0(\overline{\Omega}) : \mathbf{v}|_K \in P_l(K),\ \forall K \in T_h\right\}, \tag{3.2}$$

$$Q_h := \left\{q \in L^2(\Omega) : q|_K \in P_k(K),\ \forall K \in T_h\right\}. \tag{3.3}$$

$$\text{Additionally, let} \quad Z_h := \{\mathbf{v} \in X_h : (q, \mathbf{v}) = 0,\ \forall q \in Q_h\}. \tag{3.4}$$

Note that as $\nabla \cdot X_h \subset Q_h$, for $\mathbf{v} \in Z_h$ we have that $\|\nabla \cdot \mathbf{v}\| = 0$, thus $\|\mathbf{v}\|_X = \|\mathbf{v}\|$.

For X_h and Q_h defined in (3.1) and (3.3), the following discrete inf–sup condition is satisfied

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\|_Q \|\mathbf{v}\|_X} \geq c_0 > 0. \tag{3.5}$$

With X_h, Z_h, Q_h defined above, we have the following approximation properties [5,4]. For $\mathbf{u} \in Z \cap H^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$

$$\inf_{\mathbf{v} \in Z_h} \|\mathbf{u} - \mathbf{v}\|_X = \inf_{\mathbf{v} \in Z_h} \|\mathbf{u} - \mathbf{v}\| \leq C \inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\| = C\, h^{k+1} \|\mathbf{u}\|_{H^{k+1}(\Omega)}, \tag{3.6}$$

$$\inf_{q \in Q_h} \|p - q\| \leq C\, h^{k+1} \|p\|_{H^{k+1}(\Omega)}. \tag{3.7}$$

The approximation scheme we investigate is: *Given* $\mathbf{b}, \mathbf{f} \in L^2(\Omega)$, *determine* $(\mathbf{u}_h, p_h) \in X_h \times Q_h$, *satisfying*

$$(\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) + (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in X_h, \tag{3.8}$$

$$(q, \nabla \cdot \mathbf{u}_h) = 0, \ \forall q \in Q_h. \tag{3.9}$$

Regarding $\mathbf{u}_h^s$, note that applying a smoother to a function $\mathbf{v} \in X_h$ (typically) does not result in $\mathbf{v}^s \in X_h^s$. Therefore, we let $\tilde{\mathbf{u}}_h^s \in H^{l+1}(\Omega) \cap C^0(\Omega)$ denote the result of the smoother applied to $\mathbf{u}_h$, and define

$$\mathbf{u}_h^s(x) = I_h \tilde{\mathbf{u}}_h^s(x), \tag{3.10}$$

where $I_h : C^0(\Omega) \longrightarrow X_h^s$ denotes an interpolation operator.

We assume that the smoothed velocity $\tilde{\mathbf{u}}_h^s$ is sufficiently regular such that there exists a constant dependent on $\tilde{\mathbf{u}}_h^s$, $C_{\tilde{\mathbf{u}}_h^s}$ such that

$$\|\tilde{\mathbf{u}}_h^s - I_h \tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)} \leq C_{\tilde{\mathbf{u}}_h^s} h^{l+1}. \tag{3.11}$$

The precise dependence of $C_{\tilde{\mathbf{u}}_h^s}$ on $\tilde{\mathbf{u}}_h^s$ will depend on the particular smoother used.

The existence, uniqueness, and boundedness of the solutions $(\mathbf{u}_h^n, p_h^n)$ to (3.8)–(3.9) are established in a completely analogous manner as for the continuous problem.

Corollary 3.1. *(See Lemma 2.1.) Any solution* $(\mathbf{u}, p) \in X_h \times Q_h$ *to (3.8)–(3.9) satisfies*

$$\|\mathbf{u}_h\|_X + \|p_h\| \leq C\left(\|\mathbf{b}\| + \|\mathbf{f}\|\right). \tag{3.12}$$

Corollary 3.2. *(See Theorem 2.1.) For* $\beta(\cdot)$ *and* $\mathbf{u}_h^s$ *satisfying assumptions* **Aβ1**–**Aβ3** *and* **Au^{s}1**–**Au^{s}2**, *respectively, there exists a solution* $(\mathbf{u}_h, p_h)$ *to (3.8)–(3.9).*

Proof. The existence of $\mathbf{u}_h$ is established as that for $\mathbf{u}$ in Theorem 2.1. The existence of p_h then follows from the discrete inf–sup condition (3.5). □

In the next lemma we present the a priori error estimate for the approximation given by (3.8)–(3.9).

Lemma 3.1. *For* $(\mathbf{u},p) \in H^{k+1}(\Omega)\cap X \times H^{k+1}(\Omega)$ *satisfying (2.1)–(2.2),* $(\mathbf{u}_h,p_h)$ *satisfying (3.8)–(3.9), and* $\mathbf{u}$ *satisfying the small data condition*

$$C_\beta\sqrt{d}\,C_s(\|\mathbf{u}\|+\|\mathbf{b}\|) < \beta_{min}, \tag{3.13}$$

and assuming that $C_{\tilde{\mathbf{u}}_h^s}$ *given in (3.11) is bounded by a constant* $C_{\mathbf{u}}$*, we have that there exists* $C>0$ *such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_X + \|p-p_h\| \le C\left(h^{k+1}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + h^{k+1}\|p\|_{H^{k+1}(\Omega)} + C_{\mathbf{u}}h^{l+1}\right). \tag{3.14}$$

Remark: The condition (3.13) guarantees uniqueness of the solution to (3.8)–(3.9), see Theorem 2.2.

Proof. We have that the solutions $\mathbf{u}_h$ and $\mathbf{u}$ to (3.8)–(3.9) and (2.1)–(2.2), respectively, satisfy the following equations for all $\mathbf{v}\in Z_h$:

$$(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{u}_h,\mathbf{v}) + (\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{b},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \tag{3.15}$$

and

$$\begin{aligned}(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{u},\mathbf{v}) + (\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{b},\mathbf{v}) = (\mathbf{f},\mathbf{v}) &- ((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{u},\mathbf{v})\\ &- ((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{b},\mathbf{v}).\end{aligned} \tag{3.16}$$

With $\mathbf{e}=\mathbf{u}-\mathbf{u}_h$, subtracting equations (3.15) and (3.16) we obtain

$$\begin{aligned}(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{e},\mathbf{v}) = &-((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{u},\mathbf{v})\\ &-((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{b},\mathbf{v}),\ \forall\mathbf{v}\in Z_h.\end{aligned} \tag{3.17}$$

For $\mathbf{U}\in Z_h$, let $\mathbf{e}=(\mathbf{u}-\mathbf{U})+(\mathbf{U}-\mathbf{u}_h):=\mathbf{\Lambda}+\mathbf{E}$. Then, for $\mathbf{v}=\mathbf{E}$, (3.17) becomes

$$\begin{aligned}(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{E},\mathbf{E}) = &-(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{\Lambda},\mathbf{E}) - ((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{u},\mathbf{E})\\ &-((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{b},\mathbf{E}).\end{aligned} \tag{3.18}$$

Next we bound each of the terms in (3.18).

$$(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{E},\mathbf{E}) \ge \beta_{min}\|\mathbf{E}\|^2, \tag{3.19}$$

$$-(\beta(|\mathbf{u}_h^s+\mathbf{b}|)\mathbf{\Lambda},\mathbf{E}) \le \beta_{max}\|\mathbf{\Lambda}\|\|\mathbf{E}\| \le \epsilon_1\|\mathbf{E}\|^2 + \frac{1}{4\epsilon}\beta_{max}^2\|\mathbf{\Lambda}\|^2, \tag{3.20}$$

$$\begin{aligned}-((\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{u},\mathbf{E}) &\le \|\,(\beta(|\mathbf{u}^s+\mathbf{b}|)-\beta(|\mathbf{u}_h^s+\mathbf{b}|))\,\mathbf{u}\|\|\mathbf{E}\|\\ &\le C_\beta\sqrt{d}\|\mathbf{u}^s-\mathbf{u}_h^s\|_{L^\infty(\Omega)}\|\mathbf{u}\|\|\mathbf{E}\|\\ &\le C_\beta\sqrt{d}\left(\|\mathbf{u}^s-\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}+\|\tilde{\mathbf{u}}_h^s-\mathbf{u}_h^s\|_{L^\infty(\Omega)}\right)\|\mathbf{u}\|\|\mathbf{E}\|\\ &\le C_\beta\sqrt{d}\left(C_s\|\mathbf{u}-\mathbf{u}_h\|+\|\tilde{\mathbf{u}}_h^s-I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}\right)\|\mathbf{u}\|\|\mathbf{E}\|\\ &\le C_\beta\sqrt{d}\left(C_s\left(\|\mathbf{\Lambda}\|+\|\mathbf{E}\|\right)+\|\tilde{\mathbf{u}}_h^s-I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}\right)\|\mathbf{u}\|\|\mathbf{E}\|\\ &\le C_\beta\sqrt{d}\,C_s\|\mathbf{u}\|\|\mathbf{E}\|^2+\epsilon_2\|\mathbf{E}\|^2+\frac{1}{2\epsilon_2}C_\beta^2\,d\|\mathbf{u}\|^2\left(C_s^2\|\mathbf{\Lambda}\|^2+\|\tilde{\mathbf{u}}_h^s-I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}^2\right).\end{aligned} \tag{3.21}$$

A similar bound to that given in (3.21) holds for the third term on the right hand side of (3.18). Combining the estimates (3.19)–(3.21) with (3.18) we have

$$\left(\beta_{min} - \epsilon_1 - C_\beta\sqrt{d}\,C_s(\|\mathbf{u}\| + \|\mathbf{b}\|) - 2\epsilon_2\right)\|\mathbf{E}\|^2$$
$$\leq \left(\frac{1}{4\,\epsilon_1}\beta_{max}^2 + \frac{1}{2\,\epsilon_2}C_\beta^2\,d\,C_s^2(\|\mathbf{u}\|^2 + \|\mathbf{b}\|^2)\right)\|\mathbf{\Lambda}\|^2 + \frac{1}{2\,\epsilon_2}C_\beta^2\,d\left(\|\mathbf{u}\|^2 + \|\mathbf{b}\|^2\right)\right)\|\tilde{\mathbf{u}}_h^s - I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}^2. \quad (3.22)$$

Hence, in view of the stated hypothesis (3.13), there exists $C > 0$ such that $\|\mathbf{E}\| \leq C\big(\|\mathbf{\Lambda}\| + \|\tilde{\mathbf{u}}_h^s - I_h\tilde{\mathbf{u}}_h^s\|_{L^\infty(\Omega)}\big)$. Finally, from the triangle inequality and (3.6) we have that

$$\|\mathbf{u} - \mathbf{u}_h\|_X = \|\mathbf{u} - \mathbf{u}_h\| \leq \|\mathbf{\Lambda}\| + \|\mathbf{E}\| \leq C\left(h^{k+1}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + C_{\mathbf{u}}\,h^{l+1}\right). \quad (3.23)$$

To obtain the error estimate for the pressure, let $P \in Q_h$. Then, from (3.5) we have that there exists $\mathbf{v} \in X_h$ such that

$$c_0\|P - p_h\| \leq \frac{(P - p_h, \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X} = \frac{(P, \nabla\cdot\mathbf{v}) - (p_h, \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_X}.$$

Using (3.8) and (2.1) we obtain

$$\begin{aligned}
c_0\|\mathbf{v}\|_X\|P - p_h\| &\leq (P, \nabla\cdot\mathbf{v}) + (\mathbf{f}, \mathbf{v}) - (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h, \mathbf{v}) - (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b}, \mathbf{v}) \\
&= (P, \nabla\cdot\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{u}, \mathbf{v}) + (\beta(|\mathbf{u}^s + \mathbf{b}|)\mathbf{b}, \mathbf{v}) \\
&\quad - (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{u}_h, \mathbf{v}) - (\beta(|\mathbf{u}_h^s + \mathbf{b}|)\mathbf{b}, \mathbf{v}) \\
&= (P - p, \nabla\cdot\mathbf{v}) + (\beta(|\mathbf{u}_h^s + \mathbf{b}|)(\mathbf{u} - \mathbf{u}_h), \mathbf{v}) + ((\beta(|\mathbf{u}^s + \mathbf{b}|) - \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{u}, \mathbf{v}) \\
&\quad + ((\beta(|\mathbf{u}^s + \mathbf{b}|) - \beta(|\mathbf{u}_h^s + \mathbf{b}|))\,\mathbf{b}, \mathbf{v}) \\
\Rightarrow \quad c_0\|P - p_h\| &\leq \|P - p\| + \beta_{max}\|\mathbf{u} - \mathbf{u}_h\| \\
&\quad + C_\beta\sqrt{d}\left(C_s\|\mathbf{u} - \mathbf{u}_h\| + \|\tilde{\mathbf{u}}_h^s - I_h\tilde{\mathbf{u}}_h\|_{L^\infty(\Omega)}\right)(\|\mathbf{u}\| + \|\mathbf{b}\|).
\end{aligned}$$

Using the triangle inequality, (2.4), (3.7), (3.11) and (3.23) we obtain the stated estimate for $\|p - p_h\|$. □

Remark: The $L^\infty(\Omega)$ norm used for the term $(\tilde{\mathbf{u}}_h^s - I_h\tilde{\mathbf{u}}_h^s)$ and the $L^2(\Omega)$ norm used for $\mathbf{u}$ and $\mathbf{b}$ in (3.22) may be interchanged, assuming that the functions $\mathbf{u}$ and $\mathbf{b}$ are sufficiently regular.

4. Numerical computations

In this section we present three numerical examples to illustrate the numerical approximation scheme (3.8)–(3.9). Examples 1 and 2, for which we have an exact solution, are chosen to investigate the derived a priori error estimate for the approximation (3.14), and the dependence of the approximation on the interpolant of the smoother. Example 3 is chosen to demonstrate the difference between the approximations obtained from using a smoothed and an unsmoothed velocity in the effective viscosity function.

Example 1 and Example 2. Let $\Omega = (-1, 1) \times (0, 1)$, $\beta(s) = v_\infty + (v_0 - v_\infty)/(1 + ks^{2-r})$, with parameters $v_\infty = 1$, $v_0 = 5$, $k = 1$, and $r = 1/2$. ($\beta(\cdot)$ represents the Cross model for the effective viscosity for a generalized Newtonian fluid.) The true solution $\mathbf{u}$ and p are taken to be

$$\mathbf{u}(x, y) = \begin{bmatrix}\sin(\pi x)\cos(\pi y)\\ \cos(\pi x)\sin(\pi y)\end{bmatrix}, \qquad p(x, y) = xy. \quad (4.1)$$

For this choice of $\mathbf{u}$, $\nabla\cdot\mathbf{u} \neq 0$, hence a right hand side function is added to (3.9). The boundary conditions used are $\mathbf{u}\cdot\mathbf{n}$ along $\{1\}\times(0, 1)$, $(-1, 1)\times\{1\}$, $\{-1\}\times(0, 1)$, with $p = 0$ weakly imposed along $(-1, 1)\times\{0\}$. A computation mesh corresponding to mesh parameter $h = 1/4$ is presented in Fig. 4.1. Plots of $\beta(|\mathbf{u}|)$, $\mathbf{u}$ and p are given in Figs. 4.2, 4.3 and 4.4, respectively.

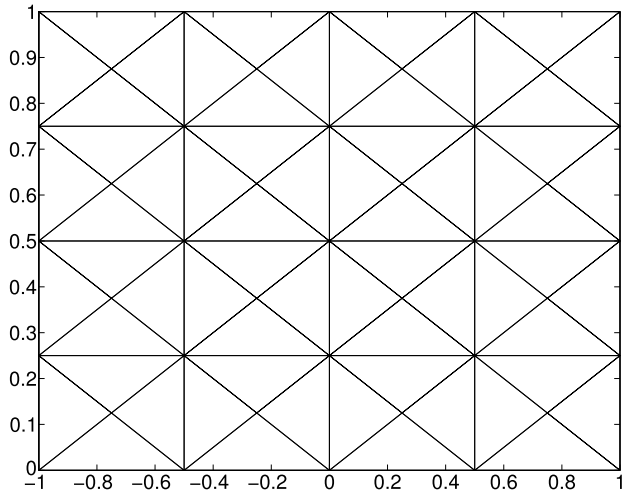


Fig. 4.1. Computational mesh, $h = 1/4$, Example 1.

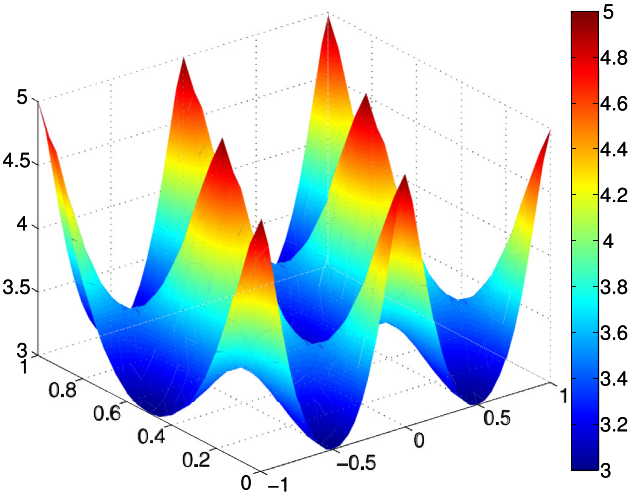


Fig. 4.2. Plot of $\beta(|\mathbf{u}|)$, Example 1.

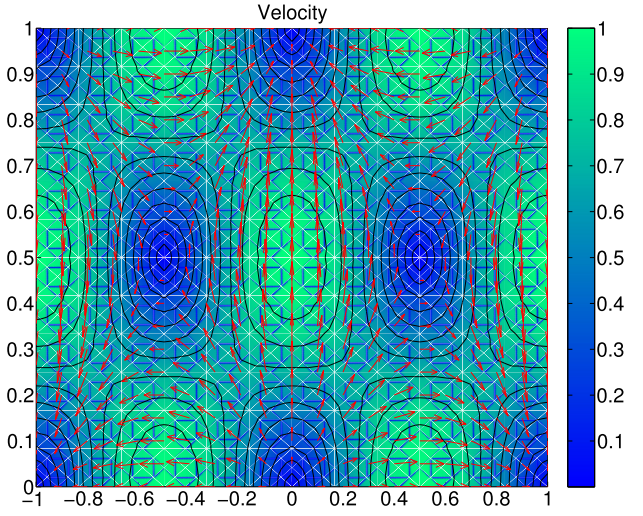


Fig. 4.3. Velocity flow field $\mathbf{u}$, Example 1.

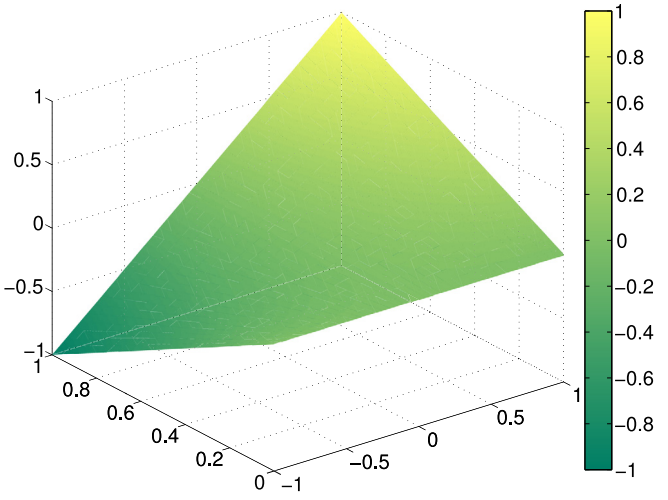


Fig. 4.4. Pressure function p, Example 1.

Table 4.1
Example 1, $\mathbf{u}_h^s$ a quadratic interpolant of $\tilde{\mathbf{u}}_h^s$.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}$	Cvg. rate	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. rate
			$X_h = RT_0$ $Q_h = discP_0$			
1/4	3.543E−01	0.98	1.274E+00	0.97	9.212E−2	1.29
1/6	2.376E−01	0.98	8.589E−01	0.99	5.464E−2	1.10
1/8	1.790E−01	1.00	6.468E−01	0.99	3.981E−2	1.08
1/10	1.433E−01	1.00	5.184E−01	0.99	3.131E−2	1.05
1/12	1.195E−01		4.325E−01		2.588E−2	
Pred.		1.0		1.0		1.0
			$X_h = RT_1$ $Q_h = discP_1$			
1/4	5.645E−02	1.94	2.020E−01	1.97	5.680E−03	2.80
1/6	2.574E−02	1.98	9.089E−02	1.99	1.824E−03	2.44
1/8	1.456E−02	1.99	5.134E−02	1.99	9.049E−04	2.30
1/10	9.344E−03	1.99	3.292E−02	1.99	5.419E−04	2.21
1/12	6.495E−03		2.289E−02		3.619E−04	
Pred.		2.0		2.0		2.0
			$X_h = RT_2$ $Q_h = discP_2$			
1/4	6.661E−03	3.09	2.268E−02	2.97	9.877E−04	3.98
1/6	1.905E−03	3.06	6.788E−03	2.99	1.966E−04	3.94
1/8	7.905E−04	3.02	2.874E−03	2.99	6.328E−05	4.02
1/10	4.028E−04	3.02	1.474E−03	3.00	2.578E−05	3.98
1/12	2.321E−04		8.537E−04		1.247E−05	
Pred.		3.0		3.0		3.0

Example 1. For $\mathbf{u}_h^s$, the interpolate of $\tilde{\mathbf{u}}_h^s$ (the *smoothed* function of $\mathbf{u}_h$), we compute a continuous, piecewise quadratic, velocity by taking a simple average of $\mathbf{u}_h$ at the nodal points of $\mathbf{u}_h^s$. Computations were performed using $RT_0 - discP_0$, $RT_1 - discP_1$, and $RT_2 - discP_2$ elements for the velocity and pressure. (By RT_k we are referring to Raviart–Thomas elements of degree k, and $discP_k$ refers to the space of discontinuous scalar functions which are polynomials of degree less than or equal to k on each triangle in the triangulation.) The results, together with the experimental convergence rates are presented in Table 4.1. The experimental convergence rates are consistent with those predicted by (3.14) for $l = 2$. (Regarding the $O(h^4)$ experimental convergence rate for the pressure using $RT_2 - discP_2$ elements, note that the true solution for the pressure lies in the $discP_2$ approximation space.)

Example 2. In order to investigate the dependence of the approximation on the interpolant of the smoother, in this case we take $\mathbf{u}_h^s$ to be a continuous, piecewise linear function, obtained by taking a simple average

Table 4.2
Example 2, $\mathbf{u}_h^s$ a linear interpolant of $\tilde{\mathbf{u}}_h^s$.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$	Cvg. rate	$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}$	Cvg. rate	$\|p-p_h\|_{L^2(\Omega)}$	Cvg. rate
			$X_h = RT_1$ $Q_h = discP_1$			
1/4	6.744E−2	1.88	2.020E−01	1.97	2.420E−2	1.99
1/6	3.150E−2	1.93	9.089E−02	1.99	1.079E−2	2.06
1/8	1.808E−2	1.95	5.134E−02	1.99	5.960E−3	2.01
1/10	1.170E−2	1.97	3.292E−02	1.99	3.802E−3	2.01
1/12	8.169E−3		2.289E−02		2.634E−3	
Pred.		2.0		2.0		2.0
			$X_h = RT_2$ $Q_h = discP_2$			
1/4	3.635E−2	1.84	2.268E−02	2.97	2.770E−2	1.17
1/6	1.727E−2	1.97	6.788E−03	2.99	1.727E−2	3.15
1/8	9.804E−3	1.97	2.874E−03	2.99	6.984E−3	2.00
1/10	6.310E−3	1.97	1.474E−03	3.00	4.473E−3	2.00
1/12	4.404E−3		8.537E−04		3.107E−3	
Pred.		2.0		2.0		2.0

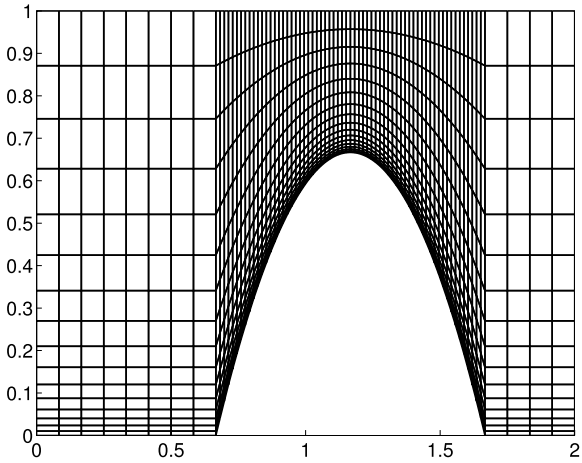


Fig. 4.5. Domain and mesh for Example 3.

of $\tilde{\mathbf{u}}_h^s$ at the vertices of the triangles in the triangulations. The results obtained using $RT_1 - discP_1$, and $RT_2 - discP_2$ approximating elements are presented in Table 4.2. In this case ($l = 1$) we observe optimal convergence for $RT_1 - discP_1$ (and $RT_0 - discP_0$, results not included). However, the experimental convergence rates for the $RT_2 - discP_2$ approximation is limited to 2 for the velocity and pressure, consistent with (3.14).

Example 3. In this example we consider a flow around a parabolic obstacle, whose domain and computational mesh are illustrated in Fig. 4.5. We assume that the fluid is a shear thinning fluid with an effective viscosity function, depicted in Fig. 4.6, given by

$$\beta(s) = \begin{cases} 50, & 0 \le s < 7, \\ 10(5-(s-7)), & 7 \le s < 8, \\ 40, & 8 \le s < 11, \\ 10(4-(s-11)), & 11 \le s < 12, \\ 30, & s \ge 12. \end{cases}$$

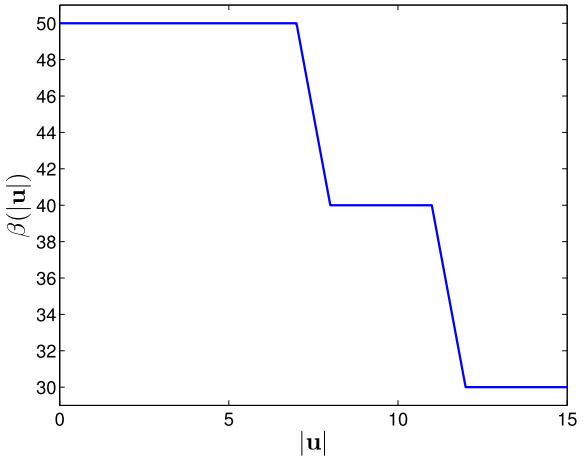


Fig. 4.6. Plot of $\beta(|\mathbf{u}|)$, Example 3.

The fluid's inflow velocity was specified along the boundary $y = 1$, $0 \le x \le 2$, as:

$$\mathbf{u}_{in} = [0, -10x(2-x)]^T, \ 0 \le x \le 2.$$

Along the edges of the domain, and along the boundary of the parabolic region, the no flow boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ was imposed. For the outflow boundaries $y = 0$, $0 \le x \le 2/3$, and $y = 0$, $5/3 \le x \le 2$, the boundary condition $p = 0$ was used.

Computations were performed using the deal.ii [3] software, with the velocity approximated using RT_1 elements and the pressure using $discP_2$.

The smoothed velocity was obtained via differential smoothing by solving the elliptic problem

$$-\delta \Delta \mathbf{u}_h^s + \mathbf{u}_h^s = \mathbf{u}_h, \tag{4.2}$$

with $\delta = 0.05$, and boundary conditions matching those for $\mathbf{u}$.

The approximating nonlinear system of equations was solved iteratively until the l^2 norm between successive vectors representing the numerical approximations was less than 1×10^{-9}.

The results for the flow field and the magnitude of the velocity are shown in Figs. 4.7 and 4.8.

Captured in the zoomed images is the effect of using the smoothed velocity in the effective viscosity function. The approximation computed with the smoothed velocity results in a (physical) smooth velocity profile, whereas the approximation computed without using a smooth velocity in the effective viscosity function has sharp transition regions for the magnitude of the velocity. (In addition existence of a solution to the problem without using a smoothed velocity in the effective viscosity function is not guaranteed.)

Acknowledgments

The authors gratefully acknowledge the assistance of Mr. Javier Ruiz-Ramirez with the numerical computations in Example 3.

Appendix A. Example of a local smoothing function

In this appendix we give an example of a local smoothing function which satisfies properties $\mathbf{Au^s1}$ and $\mathbf{Au^s2}$ presented in Section 2. The smoothing function is a simple averaging operator. We use the term *domain* to refer to an open connected set in $\mathbb{R}^n$.

For simplicity we present the case for a scalar function $u(\mathbf{x})$. For a vector valued function the smoother is simply applied to each of the coordinate functions.

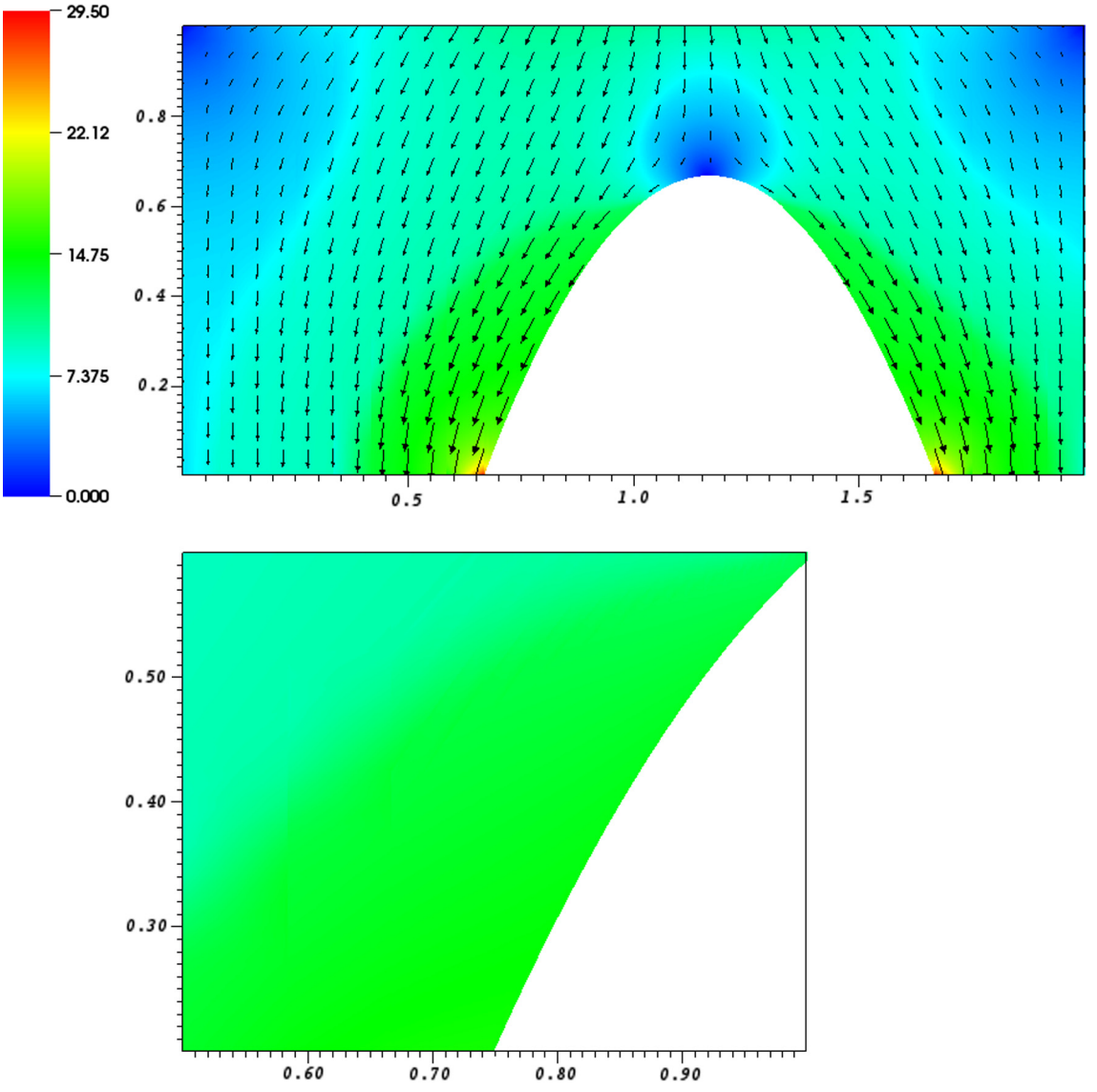


Fig. 4.7. Numerical approximation of the velocity using the differential smoother (4.2).

Let Ω denote a bounded domain in $\mathbb{R}^n$ and $\mathcal{L}(\Omega)$ the Lebesgue measurable sets in Ω. Let $\delta > 0$ denote the (fixed) volume measure over which we average a function to obtain its *smoothed* value. For $\mathbf{x} \in \Omega$ the typical averaging volume which comes to mind is $B(\mathbf{x}, r_\delta)$, where $B(\mathbf{x}, r_\delta)$ denotes the ball centered at $\mathbf{x}$ of radius r_δ having volume δ. As δ is fixed the difficulty in using $B(\mathbf{x}, r_\delta)$ arises for points whose distance from $\partial\Omega$ is less than r_δ. This requires us to consider averaging volumes other than balls. Namely, for each point $\mathbf{x} \in \Omega$ we associate a domain $V(\mathbf{x})$ having a volume of δ. We require that the association of $\mathbf{x}$ with $V(\mathbf{x})$ be continuous. This continuity is formally described in the next paragraph.

Let ν denote the Lebesgue measure in $\mathbb{R}^n$. For $S_1, S_2 \in \mathcal{L}(\Omega)$, introduce the metric $d(S_1, S_2)$ defined by

$$d(S_1, S_2) := \nu(S_1 \triangle S_2), \text{ where } S_1 \triangle S_2 := (S_1 \backslash S_2) \cup (S_2 \backslash S_1). \tag{A.1}$$

Now, let $V : \overline{\Omega} \longrightarrow \mathcal{L}(\Omega)$ satisfy: (i) $V(\mathbf{x})$ is a domain with $\nu(V(\mathbf{x})) = \delta$ for all $\mathbf{x} \in \Omega$, and (ii) $d(V(\mathbf{x}), V(\mathbf{y})) = \nu(V(\mathbf{x}) \triangle V(\mathbf{y})) \leq C_V |\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \Omega$, where C_V a fixed constant. For convenience we denote the domain $V(\mathbf{x})$ as $V_{\mathbf{x}}$.

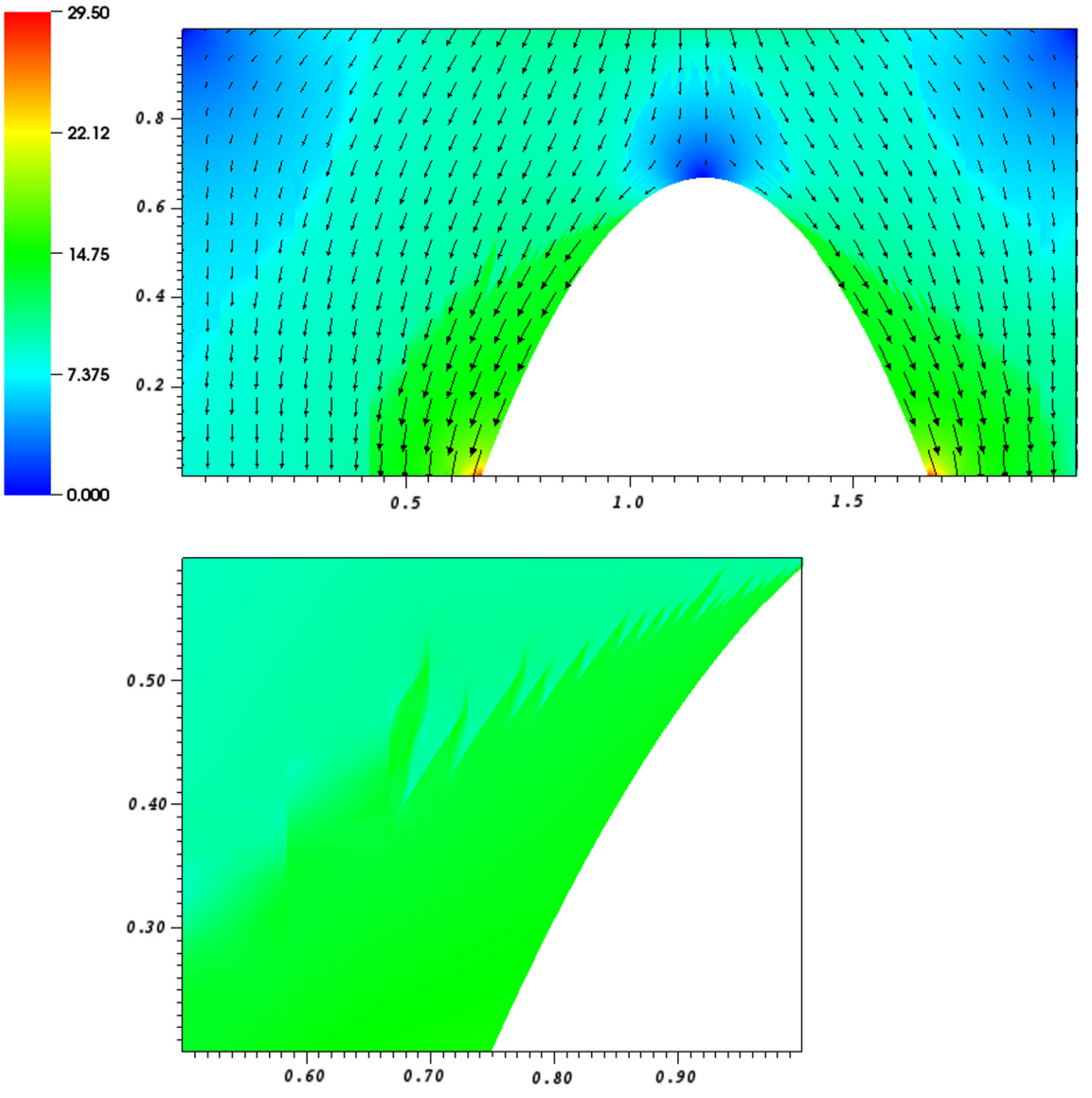


Fig. 4.8. Numerical approximation of the velocity without smoothing.

Definition *(Local smoothing operator).* For $u \in L^2(\Omega)$, define u^s as

$$u^s(\mathbf{x}) = \frac{1}{\delta} \int_{V_{\mathbf{x}}} u(\mathbf{z})\, d\Omega. \tag{A.2}$$

We have the following properties for $u^s(\mathbf{x})$.

Lemma A.1. *For $u \in L^2(\Omega)$, u^s defined by (A.2) satisfies the following properties.*

(i) $\|u^s\|_{L^\infty(\Omega)} \leq \delta^{-1/2} \|u\|_{L^2(\Omega)}$.
(ii) $u^s : \overline{\Omega} \longrightarrow \mathbb{R}$ *is uniformly continuous.*
(iii) *Suppose that $\{u_n\}_{n=1}^\infty \subset L^2(\Omega)$ and that u_n converges weakly to $u \in L^2(\Omega)$. Then $\{u_n^s\}_{n=1}^\infty$ converges to u^s in $L^\infty(\Omega)$.*

Proof. Let $1_S \in L^2(\Omega)$ denote the characteristic function of the domain S. From (A.2), for $\mathbf{x} \in \Omega$

$$\begin{aligned} u^s(\mathbf{x}) &= \frac{1}{\delta} \int\limits_{V_{\mathbf{x}}} u(\mathbf{z})\, d\Omega = \frac{1}{\delta} \int\limits_{\Omega} 1_{V_{\mathbf{x}}}\, u(\mathbf{z})\, d\Omega \\ &\leq \frac{1}{\delta} \left(\int\limits_{\Omega} (1_{V_{\mathbf{x}}})^2 \, d\Omega \right)^{1/2} \left(\int\limits_{\Omega} u(\mathbf{z})^2 \, d\Omega \right)^{1/2} \\ &= \delta^{-1/2} \|u\|_{L^2(\Omega)}, \end{aligned}$$

which establishes (i).

For $\mathbf{x}, \mathbf{y} \in \Omega$,

$$\begin{aligned} |u^s(\mathbf{x}) - u^s(\mathbf{y})| &\leq \frac{1}{\delta} \int\limits_{\Omega} \left| 1_{V_{\mathbf{x}}} - 1_{V_{\mathbf{y}}} \right| |u(\mathbf{z})|\, d\Omega \\ &= \frac{1}{\delta} \left(\int\limits_{\Omega} \left(1_{V_{\mathbf{x}}} - 1_{V_{\mathbf{y}}} \right)^2 \, d\Omega \right)^{1/2} \left(\int\limits_{\Omega} u(\mathbf{z})^2 \, d\Omega \right)^{1/2} \\ &= \frac{1}{\delta} \|u\|_{L^2(\Omega)}\, d(V(\mathbf{x}), V(\mathbf{y}))^{1/2} \\ &= \frac{C_V^{1/2}}{\delta} \|u\|_{L^2(\Omega)} |\mathbf{x} - \mathbf{y}|^{1/2}, \end{aligned}$$

which establishes the uniform continuity of u^s. As u^s is bounded on Ω then u^s can be continuously extended to $\partial\Omega$.

To establish (iii), as $\{u_n\}$ converges weakly, let $\sup_n \|u_n\| = M < \infty$. In addition, for $\epsilon > 0$, $\sigma = \left(\epsilon / (6\, M\, C_V^{1/2}) \right)^2$, let $\{\mathbf{z}_i\}_{i=1}^N$ denote a σ-net of $\overline{\Omega}$, i.e., for all $\mathbf{x} \in \Omega$ there exists an $i_{\mathbf{x}} \in \{1, 2, \ldots, N\}$ such that $|\mathbf{x} - \mathbf{z}_{i_{\mathbf{x}}}| < \sigma$.

Now,

$$\begin{aligned} |u_n^s(\mathbf{x}) - u^s(\mathbf{x})| &= \left| \int\limits_{V_{\mathbf{x}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| \\ &= \left| \int\limits_{V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega + \int\limits_{V_{\mathbf{x}} \setminus V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| \\ &\leq \left| \int\limits_{V_{\mathbf{z}_{i_{\mathbf{x}}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| + \int\limits_{V_{\mathbf{x}} \triangle V_{\mathbf{z}_{i_{\mathbf{x}}}}} |u_n(\mathbf{y}) - u(\mathbf{y})|\, d\Omega. \end{aligned} \tag{A.3}$$

Since $\{u_n\}$ converges weakly to u in $L^2(\Omega)$, for all $w \in L^2(\Omega)$ there exists N_w such that for $n > N_w$

$$\left| \int\limits_{\Omega} (u_n - u)\, w\, d\Omega \right| < \frac{\epsilon}{3}. \tag{A.4}$$

Let $N_\star = \max_{i=1,2,\ldots,N} \left\{ N_{1_{V_{\mathbf{z}_i}}} \right\}$. Then, for $n > N_\star$

$$\left| \int\limits_{V_{\mathbf{z}_{i_\mathbf{x}}}} (u_n(\mathbf{y}) - u(\mathbf{y}))\, d\Omega \right| = \left| \int\limits_{\Omega} (u_n(\mathbf{y}) - u(\mathbf{y}))\, 1_{V_{\mathbf{z}_i}}\, d\Omega \right| < \frac{\epsilon}{3}.$$

For the second term on the right hand side of (A.3) we have

$$\begin{aligned} \int\limits_{V_\mathbf{x} \triangle V_{\mathbf{z}_{i_\mathbf{x}}}} |u_n(\mathbf{y}) - u(\mathbf{y})|\, d\Omega &\le \left(\int\limits_{V_\mathbf{x} \triangle V_{\mathbf{z}_{i_\mathbf{x}}}} |u_n(\mathbf{y}) - u(\mathbf{y})|^2\, d\Omega \right)^{1/2} \left(\int\limits_{V_\mathbf{x} \triangle V_{\mathbf{z}_{i_\mathbf{x}}}} 1\, d\Omega \right)^{1/2} \\ &\le 2M\, \nu(V_\mathbf{x} \triangle V_{\mathbf{z}_{i_\mathbf{x}}})^{1/2} \\ &\le 2M\, C_V^{1/2} |\mathbf{x} - \mathbf{z}_{i_\mathbf{x}}|^{1/2} \le 2M\, C_V^{1/2}\, \sigma^{1/2} \\ &= \frac{\epsilon}{3}. \end{aligned} \tag{A.5}$$

Thus, from (A.3)–(A.5) it follows that for all $\mathbf{x} \in \Omega$, for $n > N_\star$

$$|u_n^s(\mathbf{x}) - u^s(\mathbf{x})| < \frac{2}{3}\epsilon, \text{ i.e., } \|u_n^s - u^s\|_{L^\infty(\Omega)} < \frac{2}{3}\epsilon < \epsilon. \qquad \square$$

A.1. Regularity of u^s (for $u \in L^\infty(\Omega)$)

If, in place of $u \in L^2(\Omega)$, we have $u \in L^\infty(\Omega)$ then u^s defined by (A.2) is an $H^1(\Omega)$ function. To establish this regularity result we begin by citing a characterization of the $W^{1,p}(\mathbb{R}^n)$ function space.

Theorem A.1. *(See [21], Theorem 2.1.6.) Let $1 < p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and*

$$\left(\int\limits_{\mathbb{R}^n} \left| \frac{u(\mathbf{x}+\mathbf{h}) - u(\mathbf{x})}{|\mathbf{h}|} \right|^p d\mathbf{x} \right)^{1/p} = |\mathbf{h}|^{-1} \|u(\mathbf{x}+\mathbf{h}) - u(\mathbf{x})\|_{L^p(\mathbb{R}^n)} \tag{A.6}$$

remains bounded for all $\mathbf{h} \in \mathbb{R}^n$. $\square$

Theorem A.2. *If $u \in L^\infty(\Omega)$ then, for u^s defined by (A.2), $u^s \in H^1(\Omega)$.*

Proof. In order to apply Theorem A.1 we need to define an extension of u to $\mathbb{R}^n$. Let

$$\tilde{u}(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \notin \Omega, \end{cases} \quad \text{and} \quad \widetilde{V} : \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n)$$

denote an extension of V satisfying properties (i) and (ii) (with Ω replaced by $\mathbb{R}^n$), and additionally that there exist constants $C_1 > 0$ and $C_2 \ge 0$ such that (iii) diameter $(\widetilde{V}(\mathbf{z})) \le C_1$ for all $\mathbf{z} \in \mathbb{R}^n$, and (iv) $\sup_{\mathbf{z} \in \mathbb{R}^n} \inf_{\mathbf{y} \in \widetilde{V}(\mathbf{z})} |\mathbf{z} - \mathbf{y}| \le C_2$.

Let Ω_B denote the bounded set, $\Omega_B := \{\mathbf{x} \in \mathbb{R}^n : \inf_{\mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}| < 1 + C_1 + C_2\} \supset \text{support}(\tilde{u}^s)$. Note that for $\mathbf{x} \in \mathbb{R}^n \backslash \Omega_B$ and $|\mathbf{h}| < 1$, $\tilde{u}^s(\mathbf{x}+\mathbf{h}) = 0$.

Now, for $|\mathbf{h}| \ge 1$,

$$\int\limits_{\mathbb{R}^n} \left| \frac{\tilde{u}^s(\mathbf{x}+\mathbf{h}) - \tilde{u}^s(\mathbf{x})}{|\mathbf{h}|} \right|^2 d\mathbf{x} \le \frac{2}{|\mathbf{h}|^2} \left(\int\limits_{\mathbb{R}^n} (\tilde{u}^s(\mathbf{x}+\mathbf{h}))^2\, d\mathbf{x} + \int\limits_{\mathbb{R}^n} (\tilde{u}^s(\mathbf{x}))^2\, d\mathbf{x} \right)$$

$$\leq \frac{4}{|\mathbf{h}|^2}\int_{\mathbb{R}^n} \left(\tilde{u}^s(\mathbf{x})\right)^2 \, d\mathbf{x} \leq \frac{4}{|\mathbf{h}|^2}\|\tilde{u}^s\|^2_{L^\infty(\Omega_B)}\,\nu(\Omega_B)$$
$$\leq 4\,\nu(\Omega_B)\|\tilde{u}\|^2_{L^\infty(\mathbb{R}^n)} = 4\,\nu(\Omega_B)\|u\|^2_{L^\infty(\Omega)}. \tag{A.7}$$

For $|\mathbf{h}| < 1$,

$$\int_{\mathbb{R}^n}\left|\frac{\tilde{u}^s(\mathbf{x}+\mathbf{h})-\tilde{u}^s(\mathbf{x})}{|\mathbf{h}|}\right|^2 d\mathbf{x} = \frac{1}{|\mathbf{h}|^2}\int_{\Omega_B}\left|\tilde{u}^s(\mathbf{x}+\mathbf{h})-\tilde{u}^s(\mathbf{x})\right|^2 d\mathbf{x}$$
$$= \frac{1}{|\mathbf{h}|^2}\int_{\Omega_B}\left|\frac{1}{\delta}\int_{\Omega_B}\tilde{u}(\mathbf{z})\left(1_{\widetilde{V}_{\mathbf{x}+\mathbf{h}}}(\mathbf{z})-1_{\widetilde{V}_{\mathbf{x}}}(\mathbf{z})\right)d\mathbf{z}\right|^2 d\mathbf{x}$$
$$\leq \frac{1}{|\mathbf{h}|^2}\,\frac{1}{\delta^2}\|\tilde{u}\|^2_{L^\infty(\Omega_B)}\int_{\Omega_B}\left(\int_{\Omega_B}\left|1_{\widetilde{V}_{\mathbf{x}+\mathbf{h}}}(\mathbf{z})-1_{\widetilde{V}_{\mathbf{x}}}(\mathbf{z})\right|d\mathbf{z}\right)^2 d\mathbf{x}$$
$$= \frac{1}{|\mathbf{h}|^2}\,\frac{1}{\delta^2}\|\tilde{u}\|^2_{L^\infty(\Omega_B)}\int_{\Omega_B} d(\widetilde{V}_{\mathbf{x}+\mathbf{h}},\widetilde{V}_{\mathbf{x}})^2\,d\mathbf{x}$$
$$\leq \frac{1}{|\mathbf{h}|^2}\,\frac{1}{\delta^2}\|\tilde{u}\|^2_{L^\infty(\Omega_B)}\,C_V^2|\mathbf{h}|^2\,\nu(\Omega_B)$$
$$= \frac{1}{\delta^2}\,C_V^2\,\nu(\Omega_B)\|\tilde{u}\|^2_{L^\infty(\Omega_B)} = \frac{1}{\delta^2}\,C_V^2\,\nu(\Omega_B)\|u\|^2_{L^\infty(\Omega)}. \tag{A.8}$$

From (A.7) and (A.8), together with Theorem A.1, we obtain that $\tilde{u}^s \in H^1(\mathbb{R}^n)$. As $u^s = \tilde{u}^s|_\Omega$, it then follows that $u^s \in H^1(\Omega)$. □

Appendix B. Example of a differential smoothing function

As an alternative to the local averaging filter discussed in Appendix A, in this appendix we present a differential smoothing filter.

$$\text{Let} \quad X^s = H_0^1(\Omega) = \{\mathbf{v} \in H^1(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\} \subset X. \tag{B.1}$$

Definition *(Differential smoothing operator).* For $\mathbf{u} \in L^2(\Omega)$, define $\mathbf{u}^s \in X^s$ as

$$(\nabla\mathbf{u}^s, \nabla\mathbf{v}) = (\mathbf{u}^s, \mathbf{v}), \ \forall \mathbf{v} \in X^s. \tag{B.2}$$

The well posedness of $\mathbf{u}^s$ follows from an application of the Lax–Milgram theorem. Next we show that this smoothing operation satisfies properties $\mathbf{Au^s1}$ and $\mathbf{Au^s2}$ presented in Section 2.

Lemma B.1. *For $\mathbf{u} \in L^2(\Omega)$, $\mathbf{u}^s$ defined by (B.2) satisfies the following properties.*

(i) $\|\mathbf{u}^s\|_{L^\infty(\Omega)} \leq C\|\mathbf{u}\|_{L^2(\Omega)}$.
(ii) *Suppose that $\{\mathbf{u}_n\}_{n=1}^\infty \subset L^2(\Omega)$, and that $\mathbf{u}_n$ converges weakly to $\mathbf{u} \in L^2(\Omega)$. The $\{\mathbf{u}_n^s\}$ converges to $\mathbf{u}^s$ in $L^\infty(\Omega)$.*

Proof. From (B.2) we have that $\mathbf{u}^s \in X^s$, and as $\mathbf{u} \in L^2(\Omega)$, from the *shift theorem* (together with a sufficiently smooth $\partial\Omega$), it follows that

$$\mathbf{u}^s \in H^2(\Omega) \cap X^s, \text{ with } \|\mathbf{u}^s\|_{H^2(\Omega)} \leq C\|\mathbf{u}\|. \tag{B.3}$$

Using the embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$ we establish (i).

Let $\mathcal{W} : L^2(\Omega) \longrightarrow H^2(\Omega) \cap X^s$, $\mathcal{W}(\mathbf{u}) := \mathbf{u}^s$, denote the filter mapping. Then from (B.3) $\mathcal{W}$ is a bounded (linear) transformation from $L^2(\Omega) \longrightarrow H^2(\Omega) \cap X^s$.

Let $\mathcal{W}^* : \left(H^2(\Omega) \cap X^s\right)^* \longrightarrow L^2(\Omega)$ denote the adjoint operator of $\mathcal{W}$. (The existence of $\mathcal{W}^*$ follows immediately from the Riesz Representation Theorem.)

Now, for $\boldsymbol{\eta} \in \left(H^2(\Omega) \cap X^s\right)^*$

$$\begin{aligned}\langle \mathbf{u}_n^s - \mathbf{u}^s,\ \boldsymbol{\eta}\rangle_{H^2,(H^2)^*} &= \langle \mathcal{W}(\mathbf{u}_n) - \mathcal{W}(\mathbf{u})\ ,\ \boldsymbol{\eta}\rangle_{H^2,(H^2)^*} = \langle \mathcal{W}(\mathbf{u}_n - \mathbf{u})\ ,\ \boldsymbol{\eta}\rangle_{H^2,(H^2)^*} \\ &= (\mathbf{u}_n - \mathbf{u}\ ,\ \mathcal{W}^*(\boldsymbol{\eta})) \\ &\longrightarrow 0 \text{ as } n \to \infty,\end{aligned}$$

as $\mathbf{u}_n$ converges weakly in $L^2(\Omega)$ to $\mathbf{u}$. Hence as $H^2(\Omega) \cap X^s$ is compactly embedded in $L^\infty(\Omega) \cap X^s$, then $\mathbf{u}_n^s$ converges to $\mathbf{u}^s$ strongly in $L^\infty(\Omega)$. □

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