



Longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on \mathbb{R}^N ☆



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ABSTRACT

The paper studies the longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on \mathbb{R}^N : $u_{tt} - \Delta u_t - M(\|\nabla u\|^2)\Delta u + u_t + g(x, u) = f(x)$. We establish the well-posedness, the existence of the global and exponential attractors in natural energy space $\mathcal{H} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ in critical nonlinearity case. These results improve not only the recent ones achieved by Yang (2007) [32], but also ones achieved by Conti, Pata and Squassina (2005) [9] to some extent.

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1. Introduction

In this paper, we are concerned with the well-posedness and the existence of global and exponential attractors to the Cauchy problem of the Kirchhoff equation with strong dissipation and critical nonlinearity

$$u_{tt} - \Delta u_t - M(\|\nabla u\|^2)\Delta u + u_t + g(x, u) = f(x) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $N \geq 3$, $M(s)$, $g(x, u)$ are nonlinear functions specified later, and $f(x)$ is an external force term.

When the space dimension $N = 1$, Eq. (1.1), with $M(s) = a + bs$ ($a \geq 0$, $b > 0$) and without dissipative terms $-\Delta u_t$ and u_t , was firstly introduced by Kirchhoff [18] to describe small vibrations of an elastic stretched string. The term $-\Delta u_t$ occurs in the study of the motion of viscoelastic materials, for instance, the string is made up of the viscoelastic material of rate-type [10], and it indicates that the stress is proportional not only to the strain, as with the Hooke law, but also to the strain rate as in a linearized Kelvin–Voigt material.

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There have been many researches on the global existence and decay properties of solutions to the Kirchhoff equation with dissipation $-\Delta u_t$ or u_t or $h(u_t)$, with $h(s)s \geq 0$, $s \in \mathbb{R}$. For the IBVP of the type of Eq. (1.1) on a bounded domain $\Omega \subset \mathbb{R}^N$, there have been a lot of well-posedness results in the literature (see [4,19,23–25] and references therein). There are also a few recent results on the global attractor (see [12,22,36,35]), but all these results require the sub-criticality of the source term $g(x, u)$.

Chueshov [7] first studied the well-posedness and the global attractor for the IBVP of the Kirchhoff equation with strong nonlinear damping

$$u_{tt} - \sigma(\|\nabla u\|^2)\Delta u_t - \phi(\|\nabla u\|^2)\Delta u + g(u) = h(x) \quad (1.3)$$

in natural energy space $\mathcal{H}(\Omega) = H_0^1(\Omega) \cap L^{p+1}(\Omega) \times L^2(\Omega)$. His results allow that the growth exponent p of the nonlinearity $g(u)$ is supercritical, that is, $p^* < p < p^{**}$, with $p^* \equiv \frac{N+2}{(N-2)^+}$, $p^{**} \equiv \frac{N+4}{(N-4)^+}$. Here the growth exponent p^* is called critical for $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ as $p \leq p^*$. He established a finite-dimensional global attractor in the sense of partially strong topology in $\mathcal{H}(\Omega)$. In particular, in nonsupercritical case: (i) the partially strong topology becomes strong; (ii) an exponential attractor is obtained in $\mathcal{H}(\Omega)$ by virtue of the strong quasi-stability estimates. Moreover, Chueshov [6] also studied the global well-posedness and the longtime dynamics for the Kirchhoff equations with a structural damping of the form $\sigma(\|\nabla u\|^2)(-\Delta)^\theta u_t$, with $1/2 \leq \theta < 1$, at an abstract level. For the related works on the quasilinear wave equations (rather than the semilinear ones) with strong damping, one can see [5,15,22].

Recently, Yang, Ding and Liu [34] put forward a functional analysis method and used it to construct a bounded absorbing set in $\mathcal{H}(\Omega)$, which is of higher global regularity. They removed the restriction of “partially strong topology” for $\mathcal{H}(\Omega)$ in [7] and established a strong global attractor in supercritical nonlinearity case.

But for the Kirchhoff equation (1.1) on an unbounded domain, there are only a few recent results (see [32]) on the existence of global attractor, and it seems that little is known on that of exponential attractor. The reason is that when Ω is unbounded, the compactness of the Sobolev embedding which is indispensable for constructing the global attractor is lost. Several remedies for the evolution equation on an unbounded domain have been found to overcome this difficulty. One of them consists in working in weighted Sobolev spaces as some authors done (see [1,2,11,16,21,37]). But unfortunately, this attempt is no use for Eq. (1.1) because of the appearance of the Kirchhoff nonlinearity $M(\|\nabla u\|^2)\Delta u$. Other approaches developed for unbounded domain are the usual Sobolev spaces (see [29]). One of them consists in using a suitable semigroup decomposition as done by Feireisl in [13,14] for the damped wave equation, but the method ceases to be effective for the strongly damped wave equation because the finite propagation speed of initial disturbances, which plays a key role in [13,14], is lost.

Wang [30] presented a new method of investigating the existence of global attractor for reaction–diffusion equations in unbounded domains. By approaching \mathbb{R}^N by a bounded domain Ω_k , and combining the compactness of Sobolev embedding in bounded domain Ω_k with the tail estimates with respect to spatial variables, he proved the asymptotic compactness of the solution semigroup and then established the global attractor in $L^2(\mathbb{R}^N)$. This method has been also used in other reaction–diffusion equations and systems (see [17,28]).

For the strongly damped semilinear wave equation on \mathbb{R}^3 :

$$u_{tt} - \Delta u_t - \Delta u + g(x, u) + \phi(x)u_t = f(x, t), \quad (1.4)$$

Belleri and Pata [3] presented a technique based on a decomposition of the solution by means of suitable cut-off functions to investigate its longtime dynamics. In particular, when $f(x, t) \equiv f(x)$ and the nonlinearity $g(x, u)$ is of subcritical growth 3, they showed that Eq. (1.4) possesses a global attractor.

Conti, Pata and Squassina [9] further studied the existence of global attractor for the strongly damped semilinear wave equation on \mathbb{R}^3 with critical nonlinearity:

$$u_{tt} - \Delta u_t - \Delta u + g(x, u) + \phi(x, u_t) = f(x). \quad (1.5)$$

Based on the technique introduced in [3], they improved the main result of [3], namely, they allowed the nonlinear term $g(x, u)$ to reach the critical power 5. But what about the fractal dimension of the global attractor? What about the existence of the exponential attractor? These questions remain unanswered.

For the investigation of the longtime dynamics of the Kirchhoff equation on \mathbb{R}^N , recently, by combining the decomposition of the solution semigroup with the tail estimates, both on time and spatial variables, Yang [32] has proved the existence of global attractor of Eq. (1.1) in phase space $X_2 = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ provided that $M(s) = 1 + s^{m/2}$, $m \geq 1$, $g \in C^1(\mathbb{R}^N \times \mathbb{R})$ and

$$|\frac{\partial g}{\partial s}| \leq \beta(|s|^{p-1} + L_1(x)), \quad |\nabla_x g| \leq \gamma(|s|^p + L_2(x)),$$

with $1 \leq p < p^*$ and $p \leq \tilde{p}$, where $p^* = \frac{N+2}{(N-2)_+}$, $\tilde{p} = \frac{4}{(N-4)_+}$, $L_1 \in L^{\frac{2(p+1)}{p-1}}(\mathbb{R}^N)$, $L_2 \in L^2(\mathbb{R}^N)$.

Obviously, there are still two unsolved questions in [32]:

- (i) $g(x, u)$ in [32] excludes $g(u)$ as its special case because of the assumptions on L_1, L_2 ;
- (ii) the growth exponent p^* of the nonlinearity is critical for the natural energy space $\mathcal{H} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, but is not critical for the phase space X_2 .

Does Eq. (1.1) possess global and exponential attractors in natural energy space \mathcal{H} in critical nonlinearity case: $p = p^*$? To the best of our knowledge, the questions are still open.

The purpose of the present paper is to solve these questions. First, we use the energy method combined with the monotone technology and a new limiting process to get the well-posedness of solutions in natural energy space \mathcal{H} . Second, we use the direct tail cut-off method in \mathcal{H} , which is different from those used in [3,9,32], and the recently developed quasi-stable estimate to establish the existence of the global and exponential attractors in \mathcal{H} for $1 \leq p \leq p^*$. These results improve the recent ones archived by the first author in [32].

The novelty of this paper is that it overcomes the essential difficulties: “both the Sobolev embedding on \mathbb{R}^N and the critical growth of g cause the lack of compactness” and establishes the well-posedness, the existence of the global and exponential attractors for the Kirchhoff equation with critical nonlinearity. Especially, our results cover the case $M(s) \equiv 1$, and at this time Eq. (1.1) becomes Eq. (1.5), with $\phi(x, u_t) = u_t$, so the results of the present paper also improve the results in [9] to some extent.

The paper is organized as follows. In Section 2, we discuss the well-posedness of the Cauchy problem (1.1)–(1.2). In Section 3, we establish the existence of the global attractor. In Section 4, we study the exponential attractor.

2. Well-posedness

We first introduce the following abbreviations:

$$L^p = L^p(\mathbb{R}^N), \quad W^{s,p} = W^{s,p}(\mathbb{R}^N), \quad H^s = W^{s,2}, \quad \int = \int_{\mathbb{R}^N}, \quad \|\cdot\| = \|\cdot\|_{L^2}, \quad \|\cdot\|_p = \|\cdot\|_{L^p},$$

with $p \geq 1$. The notation (\cdot, \cdot) for L^2 -inner product will also be used for the notation of duality pairing between the dual spaces. We use the same letter C to denote different positive constants, $C(\cdot, \cdot)$ to denote

positive constants depending on the quantities appearing in parenthesis. The sign $H_1 \hookrightarrow H_2$ denotes that the functional space H_1 continuously embeds into H_2 and $H_1 \hookrightarrow\hookrightarrow H_2$ denotes that H_1 compactly embeds into H_2 . Let

$$\mathcal{H} = H^1 \times L^2,$$

which is equipped with the usual graph norm.

Assumption 1.

(i) $g = g(x, s) \in C(\mathbb{R}^N \times \mathbb{R})$, with $g(x, \cdot) \in C^2(\mathbb{R})$ for almost every $x \in \mathbb{R}^N$, and

$$(G_1) \quad g(\cdot, 0) \in L^2;$$

$$(G_2) \quad |g'(x, 0)| \leq C, \quad |g''(x, s)| \leq c_1(1 + |s|^{p-2}), \quad x \in \mathbb{R}^N, \quad s \in \mathbb{R}, \quad 2 \leq p \leq p^*;$$

$$(G_3) \quad \liminf_{|s| \rightarrow \infty} \frac{g(x, s)}{s} \geq 0 \text{ uniformly as } |x| \leq r_0;$$

$$(G_4) \quad (g(x, s) - g(x, 0))s \geq c_2 s^2, \quad \forall s \in \mathbb{R}, \quad |x| > r_0,$$

where and in the following $g'(x, s)$ stands for the derivative for the second variable, $p^* = \frac{N+2}{N-2}$ ($N \geq 3$), c_i and r_0 are positive constants.

(ii) $M \in C^1(\mathbb{R}^+)$, $M'(s) \geq 0$, $M(0) \triangleq M_0 > 0$.

(iii) $f \in L^2$, $\xi_u(0) = (u_0, u_1) \in \mathcal{H}$, $\|(u_0, u_1)\|_{\mathcal{H}} \leq R_0$.

Without loss of generality, we assume $g(x, 0) = 0$. Or else, let $g_1(x, s) = g(x, s) - g(x, 0)$, $f_1(x) = f(x) - g(x, 0)$, and replace g and f in Eq. (1.1) by g_1 and f_1 , respectively. Obviously, $g_1(x, 0) = 0$, $f_1 \in L^2$, and $g_1(x, s)$ satisfies Assumption 1: (i).

Example. We now give some examples for the functions g , M , f satisfying Assumption 1. Let

$$g(x, s) = a_0(x)|s|^{p-1}s + a_1(x)s,$$

where $a_0, a_1 \in C(\mathbb{R}^N)$, $0 \leq a_0(x) \leq C$, $0 < c_2 \leq a_1(x) \leq C$. Obviously, assumptions (G_1) and (G_3) hold, and

$$\begin{aligned} g'(x, s) &= p a_0(x)|s|^{p-1} + a_1(x), \quad |g''(x, s)| = |p(p-1)a_0(x)|s|^{p-3}s| \leq c_1(1 + |s|^{p-2}), \\ g(x, s)s &\geq c_2 s^2, \quad |g'(x, 0)| = |a_1(x)| \leq C. \end{aligned}$$

That is, g satisfies Assumption 1: (i).

Let

$$M(s) = a + b s^\alpha, \quad f(x) = \frac{1}{1 + |x|^\beta},$$

where $a > 0$, $b \geq 0$, $\alpha \geq 1$, $\beta > N/2$. Obviously, M and f satisfy Assumption 1: (ii) and (iii), respectively.

Theorem 2.1. Let Assumption 1 be in force. Then problem (1.1)–(1.2) admits a unique solution $u \in C(\mathbb{R}^+; H^1)$, with $u_t \in C(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; H^1)$. This solution possesses the following properties:

(1)

$$\|u(t)\|_{H^1}^2 + \|u_t(t)\|^2 + \int_0^t \|u_t(\tau)\|_{H^1}^2 d\tau \leq C(R_0, \|f\|), \quad t > 0. \quad (2.1)$$

(2) For any $a > 0$,

$$u_t \in L^\infty(a, T; H^1), \quad u_{tt} \in L^\infty(a, T; L^2),$$

and there exists a small constant $\kappa > 0$ such that

$$\|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 \leq \frac{t^2 + 1}{t^2} C(R_0) e^{-\kappa t} + C(\|f\|), \quad t > 0. \quad (2.2)$$

(3) The following Lipschitz continuity holds:

$$\|z(t)\|_{H^1}^2 + \|z_t(t)\|^2 + \int_0^t \|z_t(\tau)\|_{H^1}^2 d\tau \leq C e^{kt} (\|z(0)\|_{H^1}^2 + \|z_t(0)\|^2), \quad t \geq 0, \quad (2.3)$$

for some $k > 0$, where $z = u - v$, u, v are the solutions of problem (1.1)–(1.2) corresponding to initial data (u_0, u_1) and (v_0, v_1) , respectively.

We first state some lemmas, which are indispensable for our proof.

Lemma 2.1. (See [27].) Let X, B and Y be the Banach spaces, $X \hookrightarrow B \hookrightarrow Y$,

$$W = \{u \in L^p(0, T; X) | u_t \in L^1(0, T; Y)\}, \quad \text{with } 1 \leq p < \infty,$$

$$W_1 = \{u \in L^\infty(0, T; X) | u_t \in L^r(0, T; Y)\}, \quad \text{with } r > 1.$$

Then,

$$W \hookrightarrow L^p(0, T; B), \quad W_1 \hookrightarrow C([0, T]; B).$$

Lemma 2.2. (See [29].) Let X, Y be two Banach spaces such that $X \hookrightarrow Y$. If $\phi \in L^\infty(0, T; X) \cap C_w([0, T]; Y)$, then $\phi \in C_w([0, T]; X)$.

Lemma 2.3. (See [20, 31].) Let Ω be an open set in \mathbb{R}^N . Then $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, with $q = \frac{2N}{N-2}$ ($N \geq 3$), that is,

$$\|u\|_{L^q(\Omega)} \leq c_0 \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega),$$

where $c_0 = c_0(N)$ is a positive constant independent of Ω .

Lemma 2.4. Let Ω be an open set in \mathbb{R}^N . Then

$$\|u\|_{L^{p+1}(\Omega)} \leq C \|u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega)$$

for $1 \leq p \leq p^* \equiv \frac{N+2}{N-2}$ ($N \geq 3$), where C is a positive constant independent of Ω .

Proof. Lemma 2.3 implies that the conclusion of Lemma 2.4 holds for $p = p^*$. When $1 \leq p < p^*$, by the interpolation theorem and Lemma 2.3,

$$\|u\|_{L^{p+1}(\Omega)} \leq C(\theta) \|u\|_{L^{p^*+1}(\Omega)}^\theta \|u\|_{L^2(\Omega)}^{1-\theta} \leq C(\theta) \|u\|_{H^1(\Omega)},$$

with $\theta = \frac{N(p-1)}{2(p+1)}$.

Lemma 2.5. Let $y : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an absolutely continuous function satisfying

$$\frac{d}{dt}y(t) + 2\epsilon y(t) \leq h(t)y(t) + z(t), \quad t > 0,$$

where $\epsilon > 0$, $z \in L^1_{loc}(\mathbb{R}^+)$, $\int_s^t h(\tau)d\tau \leq \epsilon(t-s) + m$ for $t \geq s \geq 0$ and some $m > 0$. Then

$$y(t) \leq e^m \left(y(0)e^{-\epsilon t} + \int_0^t |z(\tau)|e^{-\epsilon(t-\tau)}d\tau \right), \quad t > 0.$$

Lemma 2.6 (Gronwall-type lemma). (See [3].) Let X be a Banach space, and let $\mathcal{Z} \subset C(\mathbb{R}^+, X)$. Let $\Phi : X \rightarrow \mathbb{R}$ be a continuous function such that

$$\sup_{t \in \mathbb{R}^+} \Phi(z(t)) \geq -\eta, \quad \Phi(z(0)) \leq K$$

for some $\eta, K \geq 0$ and every $z \in \mathcal{Z}$. In addition, assume that for every $z \in \mathcal{Z}$ the function $t \mapsto \Phi(z(t))$ is continuously differentiable, and satisfies the differential inequality

$$\frac{d}{dt}\Phi(z(t)) + \delta \|z(t)\|_X^2 \leq k$$

for some $\delta > 0$ and $k \geq 0$ independent of $z \in \mathcal{Z}$. Then, for every $\gamma > 0$ there is $t_0 = \frac{\eta+K}{\gamma} > 0$ such that

$$\Phi(z(t)) \leq \sup_{\zeta \in X} \{\Phi(\zeta) : \delta \|\zeta\|_X^2 \leq k + \gamma\}, \quad t \geq t_0.$$

Proof of Theorem 2.1. Let $\Omega = \Omega_R$ be a ball in \mathbb{R}^N with radius R . We first consider the auxiliary IBVP of Eq. (1.1) on Ω :

$$\begin{cases} u_{tt} - \Delta u_t - M(\|\nabla u\|_{L^2(\Omega)}^2)\Delta u + u_t + g(x, u) = f(x), & (x, t) \in \Omega \times \mathbb{R}^+, \\ u|_{\partial\Omega} = 0, \\ u(x, 0) = \tilde{u}_0^R(x), \quad u_t(x, 0) = \tilde{u}_1^R(x), & x \in \Omega, \end{cases} \quad (2.4)$$

where the functions \tilde{u}_i^R ($i = 0, 1$) are of the forms: $\tilde{u}_i^R(x) = \theta(|x|)u_i(x)$. Here $\theta(x)$ is a smooth function,

$$\theta(x) = 1 \quad \text{as } |x| \leq R-1; \quad \theta(x) = 0 \quad \text{as } |x| \geq R \text{ and } 0 \leq \theta(x) \leq 1, \quad |\nabla\theta(x)| \leq C, \quad x \in \mathbb{R}^N.$$

Let

$$\mathcal{H}(\Omega) = H_0^1(\Omega) \times L^2(\Omega).$$

For brevity, for problem (2.4), we let the notation (\cdot, \cdot) stand for either the $L^2(\Omega)$ -inner product or the duality pairing between dual spaces.

Lemma 2.7. Let [Assumption 1](#): (i) be in force, $\Omega = \Omega_R$, with $R > r_0$. Then for any $\eta > 0$, there exist $\rho(\eta) > 0$ and $c(\eta) \geq 0$ such that the following inequalities hold for every $u \in H_0^1(\Omega)$:

$$\begin{aligned}\mathcal{G}(u) &\geq -\eta\|u\|_{L^2(\Omega)} - c(\eta), \\ (g(\cdot, u), u) + \eta\|\nabla u\|_{L^2(\Omega)}^2 &\geq \rho(\eta)\|u\|_{L^2(\Omega)}^2 - c(\eta),\end{aligned}\tag{2.5}$$

where $\mathcal{G}(u) = \int_{\Omega} G(x, u)dx$, $G(x, u) = \int_0^u g(x, \tau)d\tau$.

Proof. For any $\theta \in H_0^1(\Omega_R)$, by virtue of [Lemma 2.3](#) and the fact $L^q(\Omega_{r_0}) \hookrightarrow L^2(\Omega_{r_0})$, with $q = \frac{2N}{N-2}$,

$$\begin{aligned}c_3\|\theta\|_{L^2(\Omega_R)}^2 - \|\nabla\theta\|_{L^2(\Omega_R)}^2 &\leq c_3 \int_{\Omega_R \setminus \Omega_{r_0}} |\theta|^2 dx + c_3\|\theta\|_{L^2(\Omega_{r_0})}^2 - c_0^{-2}\|\theta\|_{L^q(\Omega_R)}^2 \\ &\leq c_3 \int_{\Omega_R \setminus \Omega_{r_0}} |\theta|^2 dx + (c_3|\Omega_{r_0}|^{\frac{2}{N}} - c_0^{-2})\|\theta\|_{L^q(\Omega_{r_0})}^2 \\ &\leq \int_{\Omega_R \setminus \Omega_{r_0}} |\theta|^2 dx,\end{aligned}\tag{2.6}$$

with $0 < c_3 \leq \min\{1, c_0^{-2}|\Omega_{r_0}|^{-\frac{2}{N}}\}$. By virtue of (2.6), similar to the proof in [26] for $\Omega = \mathbb{R}^N$, one can easily get the conclusion of [Lemma 2.7](#). We omit the process here. The proof is complete.

Now, we formally give some a priori estimates to the solutions of problem (2.4). Using the multiplier $u_t + \epsilon u$ in (2.4), we get

$$\frac{d}{dt}\Phi(\xi_u(t)) + K(\xi_u(t)) = 0,\tag{2.7}$$

where $\xi_u(t) = (u(t), u_t(t))$,

$$\begin{aligned}\Phi(\xi_u(t)) &= \frac{1}{2}\left(\|u_t\|_{L^2(\Omega)}^2 + \int_0^{\|\nabla u\|_{L^2(\Omega)}^2} M(s)ds + 2\mathcal{G}(u) - 2(u, f)\right) \\ &\quad + \epsilon\left((u, u_t) + \frac{1}{2}\|u\|_{H^1(\Omega)}^2\right), \\ K(\xi_u(t)) &= \|u_t\|_{H^1(\Omega)}^2 - \epsilon\|u_t\|_{L^2(\Omega)}^2 + \epsilon M(\|\nabla u\|_{L^2(\Omega)}^2)\|\nabla u\|_{L^2(\Omega)}^2 + \epsilon(g(\cdot, u), u) - \epsilon(u, f).\end{aligned}\tag{2.8}$$

Obviously, $\Phi : \mathcal{H}(\Omega) \mapsto \mathbb{R}$ is a continuous function. Making use of (2.5) and the fact: $M(s) \geq M_0 > 0$, we infer from (2.8) that

$$\begin{aligned}\Phi(\xi_u(t)) &\geq \kappa(\|u_t\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2) - C(\|f\|_{L^2(\Omega)}), \\ K(\xi_u(t)) &\geq \frac{1}{2}\|u_t\|_{H^1(\Omega)}^2 + \kappa(\|u_t\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2) - C(\|f\|_{L^2(\Omega)})\end{aligned}\tag{2.9}$$

for $\epsilon > 0$ suitably small, where and in the following κ stands for a small positive constant. Obviously,

$$\begin{aligned}\Phi(\xi_u(0)) &\leq C\left(\|\tilde{u}_1^R\|_{L^2(\Omega)}^2 + \|\tilde{u}_0^R\|_{H^1(\Omega)}^2 + M(\|\nabla \tilde{u}_0^R\|_{L^2(\Omega)}^2)\|\nabla \tilde{u}_0^R\|_{L^2(\Omega)}^2 + \|\tilde{u}_0^R\|_{L^{p+1}(\Omega)}^{p+1}\right) + C(\|f\|_{L^2(\Omega)}) \\ &\leq C(R_0, \|f\|_{L^2(\Omega)}).\end{aligned}$$

Inserting (2.9) into (2.7), we have

$$\frac{d}{dt}\Phi(\xi_u(t)) + \kappa\|\xi_u(t)\|_{\mathcal{H}(\Omega)}^2 \leq C(\|f\|_{L^2(\Omega)}). \quad (2.10)$$

Applying Lemma 2.6 to (2.10) (with $\eta = C(\|f\|_{L^2(\Omega)})$, $k = C(\|f\|_{L^2(\Omega)})$, $K = C(R_0, \|f\|_{L^2(\Omega)})$, $\delta = \kappa$, $\gamma = 1$) we get

$$\Phi(\xi_u(t)) \leq \sup_{\zeta \in \mathcal{H}(\Omega)} \{\Phi(\zeta) \mid \|\zeta\|_{\mathcal{H}(\Omega)}^2 \leq \frac{C(\|f\|_{L^2(\Omega)}) + 1}{\kappa}\}, \quad t \geq t_0 = C(R_0, \|f\|_{L^2(\Omega)}).$$

Therefore,

$$\|u(t)\|_{H^1(\Omega)}^2 + \|u_t(t)\|_{L^2(\Omega)}^2 \leq C(R_0, \|f\|_{L^2(\Omega)}), \quad t \geq t_0 = C(R_0, \|f\|_{L^2(\Omega)}). \quad (2.11)$$

Integrating (2.10) on $(0, t)$, with $t \leq t_0$, we get

$$\|u(t)\|_{H^1(\Omega)}^2 + \|u_t(t)\|_{L^2(\Omega)}^2 \leq C(R_0, \|f\|_{L^2(\Omega)}), \quad \forall t \in [0, t_0]. \quad (2.12)$$

Letting $\epsilon = 0$ in (2.7), integrating the resulting expression over $(0, t)$ and making use of (2.11)–(2.12) we get

$$\int_0^t \|u_t(\tau)\|_{H^1(\Omega)}^2 d\tau \leq C(R_0, \|f\|_{L^2(\Omega)}). \quad (2.13)$$

The combination of (2.11)–(2.12) with (2.13) means that (2.1) holds on Ω . On account of $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, $L^{1+\frac{1}{p}}(\Omega) \hookrightarrow H^{-1}(\Omega)$, we infer from Eq. (2.4) and (2.11)–(2.13) that

$$\int_0^T \|u_{tt}(t)\|_{H^{-1}(\Omega)}^2 dt \leq C(R_0, \|f\|_{L^2(\Omega)}, T). \quad (2.14)$$

Differentiating Eq. (2.4) with respect to t and letting $v = u_t$, we get

$$v_{tt} - \Delta v_t - M(\|\nabla u\|_{L^2(\Omega)}^2)\Delta v - 2M'(\|\nabla u\|_{L^2(\Omega)}^2)(\nabla u, \nabla u_t)\Delta u + v_t + g'(\cdot, u)v = 0. \quad (2.15)$$

Using the multiplier $v_t + \epsilon v$ in (2.15), we have

$$\begin{aligned} & \frac{d}{dt}\Psi(t) + \|v_t\|_{H^1(\Omega)}^2 + \kappa\|v\|_{L^2(\Omega)}^2 + (g'(\cdot, u)v, v_t + \epsilon v) \\ &= \epsilon\|v_t\|_{L^2(\Omega)}^2 + \kappa\|v\|_{L^2(\Omega)}^2 + M'(\|\nabla u\|_{L^2(\Omega)}^2)(\nabla u, \nabla u_t)\|\nabla v\|_{L^2(\Omega)}^2 \\ & \quad - 2M'(\|\nabla u\|_{L^2(\Omega)}^2)(\nabla u, \nabla u_t)(\nabla u, \nabla v_t) \\ & \quad - \epsilon\left(M(\|\nabla u\|_{L^2(\Omega)}^2)\|\nabla v\|_{L^2(\Omega)}^2 + 2M'(\|\nabla u\|_{L^2(\Omega)}^2)(\nabla u, \nabla v)^2\right) \\ & \leq \epsilon\|v_t\|_{L^2(\Omega)}^2 + \kappa\|v\|_{L^2(\Omega)}^2 + C\|\nabla u_t\|_{L^2(\Omega)}(\|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla v_t\|_{L^2(\Omega)}) + C\|\nabla v\|_{L^2(\Omega)}^2 \\ & \leq \epsilon\|v_t\|_{L^2(\Omega)}^2 + \kappa\|v\|_{L^2(\Omega)}^2 + C\|\nabla u_t\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}^2 + \epsilon\|v_t\|_{H^1(\Omega)}^2 + C\|v\|_{H^1(\Omega)}^2, \end{aligned} \quad (2.16)$$

where

$$\begin{aligned}\Psi(t) &= \frac{1}{2} \left(\|v_t\|_{L^2(\Omega)}^2 + M(\|\nabla u\|_{L^2(\Omega)}^2) \|\nabla v\|_{L^2(\Omega)}^2 \right) + \epsilon \left((v, v_t) + \frac{1}{2} \|v\|_{H^1(\Omega)}^2 \right) \\ &\sim \|v\|_{H^1(\Omega)}^2 + \|v_t\|_{L^2(\Omega)}^2\end{aligned}\quad (2.17)$$

for $\epsilon > 0$ suitably small, and where we have used (2.11)–(2.12). By Lemma 2.4,

$$\begin{aligned}& |(g'(\cdot, u)v, v_t + \epsilon v)| \\ & \leq C \left(\|v\|_{L^2(\Omega)} \|v_t\|_{L^2(\Omega)} + \epsilon \|v\|_{L^2(\Omega)}^2 + \|u\|_{L^{p+1}(\Omega)}^{p-1} (\epsilon \|v\|_{L^{p+1}(\Omega)}^2 + \|v\|_{L^{p+1}(\Omega)} \|v_t\|_{L^{p+1}(\Omega)}) \right) \\ & \leq \epsilon \|v_t\|_{L^2(\Omega)}^2 + C \|v\|_{L^2(\Omega)}^2 + C \|u\|_{H^1(\Omega)}^{p-1} (\epsilon \|v\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)} \|v_t\|_{H^1(\Omega)}) \\ & \leq 2\epsilon \|v_t\|_{H^1(\Omega)}^2 + C \|v\|_{H^1(\Omega)}^2,\end{aligned}\quad (2.18)$$

where and in the context C is a positive constant independent of Ω . Inserting (2.18) into (2.16), we get

$$\frac{d}{dt} \Psi(t) + \kappa \Psi(t) + \frac{1}{2} \|v_t\|_{H^1(\Omega)}^2 \leq C \|\nabla u_t\|_{L^2(\Omega)} \Psi(t) + C \|v\|_{H^1(\Omega)}^2. \quad (2.19)$$

When $0 < t \leq 1$, multiplying (2.19) by t^2 , we have

$$\begin{aligned}& \frac{d}{dt} \left(t^2 \Psi(t) \right) + \kappa t^2 \Psi(t) + \frac{1}{2} t^2 \|v_t\|_{H^1(\Omega)}^2 \\ & \leq C \|\nabla u_t\|_{L^2(\Omega)} t^2 \Psi(t) + C \|v\|_{H^1(\Omega)}^2 + C t (\|v_t\|_{L^2(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2) \\ & \leq \frac{1}{2} t^2 \|v_t\|_{H^1(\Omega)}^2 + C \|\nabla u_t\|_{L^2(\Omega)} t^2 \Psi(t) + C (\|v\|_{H^1(\Omega)}^2 + \|v_t\|_{H^{-1}(\Omega)}^2),\end{aligned}\quad (2.20)$$

where we have used the interpolation inequality

$$C t \|v_t\|_{L^2(\Omega)}^2 \leq C t \|v_t\|_{H^1(\Omega)} \|v_t\|_{H^{-1}(\Omega)} \leq \frac{1}{2} t^2 \|v_t\|_{H^1(\Omega)}^2 + C \|v_t\|_{H^{-1}(\Omega)}^2.$$

On account of

$$C \int_s^t \|\nabla u_t(\tau)\|_{L^2(\Omega)} d\tau \leq C \left(\int_s^t \|\nabla u_t(\tau)\|_{L^2(\Omega)}^2 d\tau \right)^{1/2} (t-s)^{1/2} \leq \frac{\kappa}{2} (t-s) + m$$

for $t > s \geq 0$ and some $m > 0$, applying Lemma 2.5 to (2.20), we have

$$t^2 \Psi(t) \leq C_1, \quad \|u_t(t)\|_{H^1(\Omega)}^2 + \|u_{tt}(t)\|_{L^2(\Omega)}^2 \leq \frac{C_1}{t^2}, \quad 0 < t \leq 1, \quad (2.21)$$

where and in the following $C_1 = C(R_0, \|f\|_{L^2(\Omega)})$.

When $t \geq 1$, applying Lemma 2.5 to (2.19) on $(1, t)$, we get

$$\|u_t(t)\|_{H^1(\Omega)}^2 + \|u_{tt}(t)\|_{L^2(\Omega)}^2 \leq C_1 e^{-\kappa t} + C_1 < C_1. \quad (2.22)$$

The combination of (2.21) with (2.22) gives

$$\|u_t(t)\|_{H^1(\Omega)}^2 + \|u_{tt}(t)\|_{L^2(\Omega)}^2 \leq \frac{t^2 + 1}{t^2} C_1 e^{-\kappa t} + C(\|f\|_{L^2(\Omega)}), \quad t > 0. \quad (2.23)$$

Now, we look for the approximate solutions of Eq. (2.4) of the form

$$u^n(t) = \sum_{j=1}^n T_{jn}(t)w_j,$$

where $-\Delta w_j = \lambda_j w_j$, $j = 1, 2, \dots$, $w_j|_{\partial\Omega} = 0$, with

$$\begin{aligned} & (u_{tt}^n, w_j) + (\nabla u_t^n, \nabla w_j) + M(\|\nabla u^n\|_{L^2(\Omega)}^2)(\nabla u^n, \nabla w_j) + (u_t^n, w_j) \\ & + (g(\cdot, u^n), w_j) = (f, w_j), \quad t > 0, \quad j = 1, 2, \dots, n, \\ & u^n(0) = \tilde{u}_{0n}, \quad u_t^n(0) = \tilde{u}_{1n}, \end{aligned} \quad (2.24)$$

and where $(\tilde{u}_{0n}, \tilde{u}_{1n}) \rightarrow (\tilde{u}_0^R, \tilde{u}_1^R)$ in $\mathcal{H}(\Omega)$. Obviously, the estimates (2.1) (on Ω), (2.14) and (2.23) hold for u^n . So we can extract a subsequence, still denoted by $\{u^n\}$, such that

$$\begin{aligned} u^n &\rightarrow u \quad \text{weakly}^* \text{ in } L^\infty(0, T; H_0^1(\Omega)); \\ u_t^n &\rightarrow u_t \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)); \\ \tilde{u}_{tt}^n &\rightarrow u_{tt} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)); \\ -M(\|\nabla u^n\|_{L^2(\Omega)}^2)\Delta u^n &\rightarrow \xi \quad \text{weakly}^* \text{ in } L^\infty(0, T; H^{-1}(\Omega)). \end{aligned} \quad (2.25)$$

It follows from (2.25) and Lemma 2.1 that

$$\begin{aligned} u_t^n &\rightarrow u_t \quad \text{in } L^2(0, T; L^2(\Omega)); \\ u^n &\rightarrow u \quad \text{in } C([0, T]; L^2(\Omega)) \text{ and a.e. in } \Omega \text{ for any } t \in [0, T]; \\ g(\cdot, u^n) &\rightarrow g(\cdot, u) \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^{1+\frac{1}{p}}(\Omega)). \end{aligned} \quad (2.26)$$

Letting $n \rightarrow \infty$ in (2.24), we get that the limiting function $u \in L^\infty(0, T; H_0^1(\Omega))$ solves

$$\begin{aligned} u_{tt} - \Delta u_t + \xi + u_t + g(\cdot, u) &= f, \\ u(0) &= \tilde{u}_0^R, \quad u_t(0) = \tilde{u}_1^R. \end{aligned}$$

Now we show $\xi = -M(\|\nabla u\|_{L^2(\Omega)}^2)\Delta u$. Define the operator $B : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$,

$$(Bu, v) = M(\|\nabla u\|_{L^2(\Omega)}^2)(\nabla u, \nabla v), \quad \forall u, v \in H_0^1(\Omega).$$

Lemma 2.8.

(i) The operator $B : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ is monotone, that is, for any $u, v \in H_0^1(\Omega)$,

$$(Bu - Bv, u - v) \geq 0.$$

(ii) The operator B is semicontinuous, that is, for any $u, v, w \in H_0^1(\Omega)$,

$$(B(u + \lambda v) - Bu, w) \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Proof. (i) For any $u, v \in H_0^1(\Omega)$,

$$(Bu - Bv, u - v) = M_{12}(t) \|\nabla(u - v)\|_{L^2(\Omega)}^2 + \tilde{M}_{12}(t) \left(\nabla(u + v), \nabla(u - v) \right)^2,$$

where

$$\begin{aligned} M_{12}(t) &= \frac{1}{2} \left(M(\|\nabla u\|_{L^2(\Omega)}^2) + M(\|\nabla v\|_{L^2(\Omega)}^2) \right) > 0, \\ \tilde{M}_{12}(t) &= \frac{1}{2} \int_0^1 M'(\lambda \|\nabla u\|_{L^2(\Omega)}^2 + (1 - \lambda) \|\nabla v\|_{L^2(\Omega)}^2) d\lambda > 0. \end{aligned} \quad (2.27)$$

So the operator B is monotone.

(ii) When $\lambda \rightarrow 0$, for any $u, v, w \in H_0^1(\Omega)$,

$$\begin{aligned} \|\nabla u + \lambda \nabla v\|_{L^2(\Omega)}^2 &= \|\nabla u\|_{L^2(\Omega)}^2 + 2\lambda(\nabla u, \nabla v) + \lambda^2 \|\nabla v\|_{L^2(\Omega)}^2 \rightarrow \|\nabla u\|_{L^2(\Omega)}^2, \\ M(\|\nabla u + \lambda \nabla v\|_{L^2(\Omega)}^2) &\rightarrow M(\|\nabla u\|_{L^2(\Omega)}^2), \\ (\nabla(u + \lambda v), \nabla w) &\rightarrow (\nabla u, \nabla w), \quad (B(u + \lambda v), w) \rightarrow (Bu, w), \end{aligned}$$

that is, the operator B is semicontinuous.

Let

$$\xi_n(t) = \int_0^t (Bu^n - Bv, u^n - v) d\tau \quad (\geq 0),$$

where u^n as shown in (2.24) and $v \in H_0^1(\Omega)$. Obviously,

$$\begin{aligned} \int_0^t (Bu^n, u^n) d\tau &= \int_0^t -(u_{tt}^n - \Delta u_t^n + u_t^n + g(\cdot, u^n) - f, u^n) d\tau \\ &= -(u_t^n, u^n) + (\tilde{u}_{1n}, \tilde{u}_{0n}) + \int_0^t \|u_t^n(\tau)\|_{L^2(\Omega)}^2 d\tau - \frac{1}{2} \left(\|u^n(t)\|_{H^1(\Omega)}^2 - \|\tilde{u}_{0n}\|_{H^1(\Omega)}^2 \right) \\ &\quad + \int_0^t (f, u^n) d\tau - \int_0^t (g(\cdot, u^n), u^n) d\tau. \end{aligned}$$

It follows from (2.26) that

$$g(x, u^n)u^n \rightarrow g(x, u)u \quad \text{a.e. in } Q_t = \Omega \times [0, t].$$

We infer from assumptions (G_3) and (G_4) that

$$g(x, s)s + s^2 \geq -C(r_0), \quad |x| \leq r_0; \quad g(x, s)s \geq c_2 s^2 \geq 0, \quad |x| > r_0.$$

So

$$g(x, s)s + s^2 \geq -C(r_0), \quad x \in \Omega, \quad s \in \mathbb{R}.$$

By the Fatou lemma and (2.26),

$$\begin{aligned} \int_0^t \int_{\Omega} (g(x, u)u + u^2) dx d\tau &\leq \liminf_{n \rightarrow \infty} \int_0^t \int_{\Omega} (g(x, u^n)u^n + (u^n)^2) dx d\tau \\ &\leq \liminf_{n \rightarrow \infty} \int_0^t \int_{\Omega} (g(x, u^n)u^n dx d\tau + \int_0^t \int_{\Omega} u^2 dx d\tau, \\ \int_0^t (g(\cdot, u), u) d\tau &\leq \liminf_{n \rightarrow \infty} \int_0^t (g(\cdot, u^n), u^n) d\tau. \end{aligned} \quad (2.28)$$

By virtue of (2.25)–(2.26) and (2.28),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_0^t (Bu^n, u^n) d\tau &\leq -(u_t, u) + (\tilde{u}_0^R, \tilde{u}_1^R) + \int_0^t \|u_t(\tau)\|_{L^2(\Omega)}^2 d\tau - \frac{1}{2} \|u(t)\|_{H^1(\Omega)}^2 \\ &\quad + \frac{1}{2} \|\tilde{u}_0^R\|_{H^1(\Omega)}^2 + \int_0^t (f, u) d\tau - \int_0^t (g(\cdot, u), u) d\tau. \end{aligned}$$

Obviously,

$$\int_0^t (Bu^n, v) d\tau \rightarrow \int_0^t (\xi, v) d\tau, \quad \int_0^t (Bv, u^n) d\tau \rightarrow \int_0^t (Bv, u) d\tau$$

as $n \rightarrow \infty$. Therefore,

$$0 \leq \limsup_{n \rightarrow \infty} \xi_n(t) \leq \int_0^t (\xi - Bv, u - v) d\tau.$$

Taking $v = u - \lambda w$, with $\lambda > 0$, $w \in H_0^1(\Omega)$, we get

$$0 \leq \int_0^t (\xi - B(u - \lambda w), w) d\tau.$$

Letting $\lambda \rightarrow 0$, we infer from Lemma 2.8 that

$$0 \leq \int_0^t (\xi - Bu, w) d\tau.$$

By the arbitrariness of w , $\xi = Bu$, u is a solution of the problem (2.4). By the lower semi-continuity of the norm of the weak* limit, the estimates (2.1)–(2.2) (on Ω) hold for u .

Now, we show the existence of solutions of the Cauchy problem (1.1)–(1.2). For brevity, in the following, we use the abbreviations as is shown in the beginning of this section.

Let $u^R \in L^\infty(0, T; H_0^1(\Omega_R))$ be the solution of the auxiliary problem (2.4). Define the natural extension of u^R on \mathbb{R}^N :

$$\tilde{u}^R = \begin{cases} u^R, & |x| \leq R, \\ 0, & |x| > R, \end{cases} \quad f_R = \begin{cases} f, & |x| \leq R, \\ 0, & |x| > R. \end{cases}$$

A simple calculation shows that

$$\nabla \tilde{u}^R = \begin{cases} \nabla u^R, & |x| \leq R, \\ 0, & |x| > R. \end{cases}$$

Indeed, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, noticing that $u^R|_{\partial\Omega_R} = 0$, we have

$$\int \tilde{u}^R \nabla \phi dx = \int_{\Omega_R} u^R \nabla \phi dx = - \int_{\Omega_R} \nabla u^R \phi dx = - \int \nabla \tilde{u}^R \phi dx.$$

Obviously, $\tilde{u}^R \in L^\infty(0, T; H^1(\mathbb{R}^N))$ solves the Cauchy problem

$$\begin{cases} \tilde{u}_{tt}^R - \Delta \tilde{u}_t^R - M(\|\nabla \tilde{u}^R\|^2) \Delta \tilde{u}^R + \tilde{u}_t^R + g(\cdot, \tilde{u}^R) = f_R & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ \tilde{u}^R(0) = \tilde{u}_0^R, \quad \tilde{u}_t^R(0) = \tilde{u}_1^R, \end{cases} \quad (2.29)$$

and the estimates (2.1)–(2.2) and (2.14) hold for \tilde{u}^R . Since

$$\begin{aligned} g(x, \tilde{u}^R) &= \left(g'(x, \mu \tilde{u}^R) - g'(x, 0) \right) \tilde{u}^R + g'(x, 0) \tilde{u}^R \\ &= g''(x, \mu \theta \tilde{u}^R) \mu |\tilde{u}^R|^2 + g'(x, 0) \tilde{u}^R, \end{aligned} \quad (2.30)$$

where $0 < \mu, \theta < 1$, by assumption (G_2) and the Sobolev embedding: $L^{1+\frac{1}{p}} \hookrightarrow H^{-1}$, $L^2 \hookrightarrow H^{-1}$,

$$\begin{aligned} |g(x, \tilde{u}^R) - g'(x, 0) \tilde{u}^R| &= |g''(x, \mu \theta \tilde{u}^R) \mu |\tilde{u}^R|^2| \leq C(1 + |\tilde{u}^R|^{p-2}) |\tilde{u}^R|^2, \\ \|g(\cdot, \tilde{u}^R) - g'(\cdot, 0) \tilde{u}^R\|_{H^{-1}} &\leq C \|g''(x, \mu \theta \tilde{u}^R) \mu |\tilde{u}^R|^2\|_{1+\frac{1}{p}} \leq C(\|\tilde{u}^R\|_{\frac{2(p+1)}{p}}^2 + \|\tilde{u}^R\|_{p+1}^p) \leq C, \\ \|g(\cdot, \tilde{u}^R)\|_{H^{-1}} &\leq \|g'(\cdot, 0) \tilde{u}^R\|_{H^{-1}} + C \leq C(\|\tilde{u}^R\| + 1) \leq C, \quad t \geq 0. \end{aligned} \quad (2.31)$$

So there exists a limiting function defined on \mathbb{R}^N , still denoted by u , such that (subsequence if necessary)

$$\begin{aligned} \tilde{u}^R &\rightarrow u \quad \text{weakly}^* \text{ in } L^\infty(0, T; H^1); \\ \tilde{u}_t^R &\rightarrow u_t \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2) \cap L^2(0, T; H^1); \\ \tilde{u}_{tt}^R &\rightarrow u_{tt} \quad \text{weakly in } L^2(0, T; H^{-1}); \\ g(\cdot, \tilde{u}^R) &\rightarrow \vartheta \quad \text{weakly}^* \text{ in } L^\infty(0, T; H^{-1}); \\ M(\|\nabla \tilde{u}^R\|^2) \Delta \tilde{u}^R &\rightarrow \zeta \quad \text{weakly}^* \text{ in } L^\infty(0, T; H^{-1}). \end{aligned}$$

Lemma 2.9.

$$\|\tilde{u}_0^R - u_0\|_{H^1} + \|\tilde{u}_1^R - u_1\| + \|f_R - f\| \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

Proof. Obviously,

$$\begin{aligned} &\|\tilde{u}_0^R - u_0\|_{H^1}^2 + \|\tilde{u}_1^R - u_1\|^2 + \|f_R - f\|^2 \\ &\leq 2(C^2 + 1) \int_{|x| \geq R-1} (|u_0|^2 + |\nabla u_0|^2) dx + \int_{|x| \geq R-1} (|u_1|^2 + |f|^2) dx \rightarrow 0. \end{aligned}$$

Let $\tilde{u}^{R_i}, \tilde{u}^{R_j}$ be the solutions of the Cauchy problem (2.29) corresponding to initial data $(\tilde{u}_0^{R_i}, \tilde{u}_1^{R_i})$ and $(\tilde{u}_0^{R_j}, \tilde{u}_1^{R_j}) \in \mathcal{H}$. Then $z = \tilde{u}^{R_i} - \tilde{u}^{R_j}$ solves

$$\begin{aligned} z_{tt} - \Delta z_t - M_{ij}(t)\Delta z + z_t - \tilde{M}_{ij}(t)(\nabla(\tilde{u}^{R_i} + \tilde{u}^{R_j}), \nabla z)\Delta(\tilde{u}^{R_i} + \tilde{u}^{R_j}) \\ = -g(\cdot, \tilde{u}^{R_i}) + g(\cdot, \tilde{u}^{R_j}) + f_{R_i} - f_{R_j}, \\ z(0) = \tilde{u}_0^{R_i} - \tilde{u}_0^{R_j} \equiv z_0, \quad z_t(0) = \tilde{u}_1^{R_i} - \tilde{u}_1^{R_j} \equiv z_1, \end{aligned} \quad (2.32)$$

where

$$M_{ij}(t) = \frac{1}{2} \left(M(\|\nabla \tilde{u}^{R_i}\|^2) + M(\|\nabla \tilde{u}^{R_j}\|^2) \right), \quad \tilde{M}_{ij}(t) = \frac{1}{2} \int_0^1 M'(\lambda \|\nabla \tilde{u}^{R_i}\|^2 + (1-\lambda) \|\nabla \tilde{u}^{R_j}\|^2) d\lambda.$$

Using the multiplier $z_t + \epsilon z$ in (2.32), we have

$$\frac{d}{dt} H_1(t) + \|z_t\|_{H^1}^2 = K_1(t) - \left(g(\cdot, \tilde{u}^{R_i}) - g(\cdot, \tilde{u}^{R_j}) - (f_{R_i} - f_{R_j}), z_t + \epsilon z \right), \quad (2.33)$$

where

$$H_1(t) = \frac{1}{2} \left(\|z_t\|^2 + \epsilon \|z\|_{H^1}^2 \right) + \epsilon(z, z_t) \sim \|z\|_{H^1}^2 + \|z_t\|^2$$

for $\epsilon > 0$ suitably small, and

$$\begin{aligned} K_1(t) &= -M_{ij}(t)(\nabla z, \nabla z_t) - \tilde{M}_{ij}(t)(\nabla(\tilde{u}^{R_i} + \tilde{u}^{R_j}), \nabla z_t)(\nabla(\tilde{u}^{R_i} + \tilde{u}^{R_j}), \nabla z) \\ &\quad - \epsilon \left(M_{ij}(t) \|\nabla z\|^2 + \tilde{M}_{ij}(t) (\nabla(\tilde{u}^{R_i} + \tilde{u}^{R_j}), \nabla z)^2 \right) + \epsilon \|z_t\|^2 \\ &\leq \frac{1}{8} \|z_t\|_{H^1}^2 + C \|z\|_{H^1}^2. \end{aligned} \quad (2.34)$$

Obviously,

$$\begin{aligned} &\left| - \left(g(\cdot, \tilde{u}^{R_i}) - g(\cdot, \tilde{u}^{R_j}) - (f_{R_i} - f_{R_j}), z_t + \epsilon z \right) \right| \\ &\leq C \int (|\tilde{u}^{R_i}|^{p-1} + |\tilde{u}^{R_j}|^{p-1} + 1) |z| (|z_t| + \epsilon |z|) dx + \frac{1}{2} (\|z_t\|^2 + \|z\|^2) + \frac{1}{2} \|f_{R_i} - f_{R_j}\|^2 \\ &\leq C \|z\| (\|z_t\| + \epsilon \|z\|) + C (\|\tilde{u}^{R_i}\|_{p+1}^{p-1} + \|\tilde{u}^{R_j}\|_{p+1}^{p-1}) \|z\|_{p+1} (\|z_t\|_{p+1} + \epsilon \|z\|_{p+1}) \\ &\quad + \frac{1}{2} (\|z_t\|^2 + \|z\|^2) + \frac{1}{2} \|f_{R_i} - f_{R_j}\|^2 \\ &\leq \frac{1}{8} \|z_t\|_{H^1}^2 + C (\|z_t\|^2 + \|z\|_{H^1}^2) + \frac{1}{2} \|f_{R_i} - f_{R_j}\|^2. \end{aligned} \quad (2.35)$$

Inserting (2.34)–(2.35) into (2.33), we get

$$\begin{aligned} \frac{d}{dt} H_1(t) + \frac{1}{8} \|z_t\|_{H^1}^2 &\leq C H_1(t) + \|f_{R_i} - f_{R_j}\|^2, \\ \|z(t)\|_{H^1}^2 + \|z_t(t)\|^2 + \int_0^t \|z_t(\tau)\|_{H^1}^2 d\tau &\leq C e^{kt} (\|z_0\|_{H^1}^2 + \|z_1\|^2) + C(T) \|f_{R_i} - f_{R_j}\|^2. \end{aligned} \quad (2.36)$$

Estimate (2.36) and Lemma 2.9 imply that $\{(\tilde{u}^{R_i}, \tilde{u}_t^{R_i})\}$ is a Cauchy sequence in \mathcal{H} , that is,

$$\begin{aligned}\tilde{u}^{R_i} &\rightarrow u \quad \text{in } L^\infty(0, T; H^1), \quad \tilde{u}_t^{R_i} \rightarrow u_t \quad \text{in } L^\infty(0, T; L^2); \\ \tilde{u}^{R_i} &\rightarrow u, \quad g(x, \tilde{u}^{R_i}) \rightarrow g(x, u) \quad \text{a.e. in } \mathbb{R}^N \times [0, T].\end{aligned}$$

We claim that

$$\vartheta = g(\cdot, u).$$

Indeed, it follows from (2.30) that

$$g(x, \tilde{u}^{R_i}) = g''(x, \mu_i \theta_i \tilde{u}^{R_i}) \mu_i (\tilde{u}^{R_i})^2 + g'(x, 0) \tilde{u}^{R_i},$$

with $0 < \mu_i, \theta_i < 1$. By the compactness of the number sequences $\{\mu_i\}$, $\{\theta_i\}$, we have (subsequence if necessary)

$$\begin{aligned}\mu_i &\rightarrow \mu, \quad \theta_i \rightarrow \theta, \quad \mu_i \theta_i \tilde{u}^{R_i} \rightarrow \mu \theta u \quad \text{a.e. in } \mathbb{R}^N \times [0, T], \\ g''(x, \mu_i \theta_i \tilde{u}^{R_i}) \mu_i (\tilde{u}^{R_i})^2 &\rightarrow g''(x, \mu \theta u) \mu u^2, \quad g'(x, 0) \tilde{u}^{R_i} \rightarrow g'(x, 0) u \quad \text{a.e. in } \mathbb{R}^N \times [0, T].\end{aligned}$$

By the uniqueness of the limit

$$g(x, u) = g''(x, \mu \theta u) \mu u^2 + g'(x, 0) u \quad \text{a.e. in } \mathbb{R}^N \times [0, T].$$

Obviously (see (2.31)),

$$\|g''(x, \mu_i \theta_i \tilde{u}^{R_i}) \mu_i (\tilde{u}^{R_i})^2\|_{1+\frac{1}{p}} + \|g'(x, 0) \tilde{u}^{R_i}\| \leq C.$$

So,

$$\begin{aligned}g''(x, \mu_i \theta_i \tilde{u}^{R_i}) \mu_i (\tilde{u}^{R_i})^2 &\rightarrow g''(x, \mu \theta u) \mu u^2 \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^{1+\frac{1}{p}}); \\ g'(x, 0) \tilde{u}^{R_i} &\rightarrow g'(x, 0) u \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2).\end{aligned}$$

Therefore, for any $\phi \in H^1(\hookrightarrow L^{p+1})$

$$\begin{aligned}(g(\cdot, \tilde{u}^{R_i}), \phi) &= (g''(x, \mu_i \theta_i \tilde{u}^{R_i}) \mu_i (\tilde{u}^{R_i})^2 + g'(x, 0) \tilde{u}^{R_i}, \phi) \\ &\rightarrow (g''(x, \mu \theta u) \mu u^2 + g'(x, 0) u, \phi) = (g(\cdot, u), \phi),\end{aligned}$$

that is, $\vartheta = g(\cdot, u)$.

For any $v \in H^1$,

$$\begin{aligned}&\left(M(\|\nabla \tilde{u}^{R_i}\|^2) \Delta \tilde{u}^{R_i} - M(\|\nabla u\|^2) \Delta u, v \right) \\ &= - \left(M(\|\nabla \tilde{u}^{R_i}\|^2) - M(\|\nabla u\|^2) \right) (\nabla \tilde{u}^{R_i}, \nabla v) - M(\|\nabla u\|^2) (\nabla \tilde{u}^{R_i} - \nabla u, \nabla v) \rightarrow 0.\end{aligned}$$

By the uniqueness of the limit, $M(\|\nabla u\|^2) \Delta u = \zeta$.

Therefore, $u \in L^\infty(0, T; H^1)$, with $u_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, is a solution of the Cauchy problem (1.1)–(1.2). By the lower semi-continuity of the norm of the weak* limit, the estimates (2.1)–(2.2) hold for u .

Let u, v be two solutions of the Cauchy problem (1.1)–(1.2) corresponding to initial data (u_0, u_1) and $(v_0, v_1) \in \mathcal{H}$, respectively, $z = u - v$. Similar to the proof of (2.36) (replacing $f_{R_i} - f_{R_j}$ by 0 there), we get (2.3).

Noticing that $g(\cdot, u)u_t \in L^1([0, T] \times \mathbb{R}^N)$, using the multiplier u_t in (1.1) and integrating the resulting expression over (t_0, t) , we get

$$E(\xi_u(t)) + \int_{t_0}^t \|u_t(\tau)\|_{H^1}^2 d\tau = E(\xi_u(t_0)) \quad (2.37)$$

for $t \geq t_0 \geq 0$, where

$$E(\xi_u) = \frac{1}{2} \left(\|u_t\|^2 + \int_0^{\|\nabla u\|^2} M(s) ds \right) + \int G(x, u) dx - (f, u).$$

Letting $t \rightarrow t_0$ in (2.37), we arrive at

$$E(\xi_u(t_0)) = \lim_{t \rightarrow t_0} E(\xi_u(t)). \quad (2.38)$$

Since $u \in H^1(0, T; H^1) \hookrightarrow C([0, T]; H^1)$, $u \in C([0, T]; H^1)$,

$$\begin{aligned} u(t) &\rightarrow u(t_0), \quad \nabla u(t) \rightarrow \nabla u(t_0) \quad \text{in } L^2 \text{ and a.e. in } \mathbb{R}^N \text{ as } t \rightarrow t_0, \\ \int_0^{\|\nabla u(t_0)\|^2} M(s) ds &= \lim_{t \rightarrow t_0} \int_0^{\|\nabla u(t)\|^2} M(s) ds. \end{aligned}$$

By (2.14), $u_t \in L^\infty(0, T; L^2) \cap H^1(0, T; H^{-1}) \hookrightarrow C([0, T]; H^{-1})$, $L^2 \hookrightarrow H^{-1}$, so $u_t \in C_w([0, T]; L^2)$ (see Lemma 2.2),

$$u_t(t) \rightarrow u_t(t_0) \quad \text{weakly in } L^2, \quad \|u_t(t_0)\| \leq \liminf_{t \rightarrow t_0} \|u_t(t)\|.$$

By assumptions (G_3) – (G_4) ,

$$G(x, s) + s^2 \geq -C(r_0), \quad |x| \leq r_0; \quad G(x, s) = \int_0^s g(x, \tau) d\tau \geq 0, \quad |x| > r_0.$$

By the Fatou lemma,

$$\begin{aligned} \int_{\Omega_{r_0}} G(x, u(t_0)) dx + \|u(t_0)\|_{L^2(\Omega_{r_0})}^2 &\leq \liminf_{t \rightarrow t_0} \int_{\Omega_{r_0}} (G(x, u(t)) + u^2(t)) dx \\ &\leq \liminf_{t \rightarrow t_0} \int_{\Omega_{r_0}} G(x, u(t)) dx + \|u(t_0)\|_{L^2(\Omega_{r_0})}^2, \\ \int_{\mathbb{R}^N \setminus \Omega_{r_0}} G(x, u(t_0)) dx &\leq \liminf_{t \rightarrow t_0} \int_{\mathbb{R}^N \setminus \Omega_{r_0}} G(x, u(t)) dx, \\ \int G(x, u(t_0)) dx &= \left(\int_{\Omega_{r_0}} + \int_{\mathbb{R}^N \setminus \Omega_{r_0}} \right) G(x, u(t_0)) dx \leq \liminf_{t \rightarrow t_0} \int G(x, u(t)) dx. \end{aligned}$$

Therefore, it follows from (2.38) that

$$\begin{aligned}
& \limsup_{t \rightarrow t_0} \frac{1}{2} \|u_t(t)\|^2 + \liminf_{t \rightarrow t_0} \int G(x, u(t)) dx \\
& \leq \lim_{t \rightarrow t_0} \left(\frac{1}{2} \|u_t(t)\|^2 + \int G(x, u(t)) dx \right) = \frac{1}{2} \|u_t(t_0)\|^2 + \int G(x, u(t_0)) dx \\
& \leq \liminf_{t \rightarrow t_0} \frac{1}{2} \|u_t(t)\|^2 + \liminf_{t \rightarrow t_0} \int G(x, u(t)) dx,
\end{aligned}$$

which implies

$$\lim_{t \rightarrow t_0} \|u_t(t)\|^2 = \|u_t(t_0)\|^2, \quad \lim_{t \rightarrow t_0} \int G(x, u(t)) dx = \int G(x, u(t_0)) dx.$$

So $\|u_t(t)\| \in C[0, T]$. By the uniform convexity of L^2 , $u_t \in C([0, T]; L^2)$.

Therefore, problem (1.1)–(1.2) possesses a unique solution u , with $(u, u_t) \in C([0, T], H^1 \times L^2)$, and the estimates (2.1)–(2.3) hold. Theorem 2.1 is proved.

3. Global attractor

We denote the solution in Theorem 2.1 by $S(t)(u_0, u_1) = (u(t), u_t(t))$. Theorem 2.1 shows that the solution operators $S(t)$ compose a continuous semigroup in \mathcal{H} .

Theorem 3.1. *In addition to Assumption 1, if also $M \in C^2(\mathbb{R}^+)$, $g'(x, s) > -l$ for some constant $l > 0$. Then the dynamical system $(S(t), \mathcal{H})$ possesses a compact global attractor \mathcal{A} .*

Proof. Estimate (2.1) shows that the dynamical system $(S(t), \mathcal{H})$ is dissipative and

$$B_0 = \{(u, v) \in \mathcal{H} \mid \|u\|_{H^1}^2 + \|v\|^2 \leq R_0^2\}$$

is an absorbing set for R_0 suitably large, so there exists a $t_0 > 0$ such that $S(t)B_0 \subset B_0$ for $t > t_0$. Let

$$\mathcal{B} = [\cup_{t \geq t_0+1} S(t)B_0]_{\mathcal{H}},$$

where $[\cdot]_{\mathcal{H}}$ denotes the closure in \mathcal{H} . Obviously, \mathcal{B} is a forward invariant and bounded closed absorbing set, and it is complete with respect to the norm of \mathcal{H} . So it is enough to show that the dynamical system $(S(t), \mathcal{B})$ has a global attractor.

We construct the functions

$$\begin{aligned}
K_0(s) &= \begin{cases} 0, & 0 \leq s \leq 1, \\ s-1, & 1 < s \leq 2, \\ 1, & s > 2, \end{cases} \\
K_\delta(s) &= (\rho_\delta * K_0)(s) = \int_{\mathbb{R}} \rho_\delta(s-y) K_0(y) dy,
\end{aligned}$$

where $\rho_\delta(s)$ is the standard mollifier on \mathbb{R} with $\text{supp } \rho_\delta \subset [-\delta, \delta]$. Obviously,

$$K_\delta \in C^\infty(\mathbb{R}), \quad 0 \leq K_\delta(s) \leq 1, \quad K_\delta(s) = 0 \quad \text{as } 0 \leq s < 1; \quad K_\delta(s) = 1 \quad \text{as } s > 2,$$

with $0 < \delta \ll 1$. Let $\varphi(x) = K_\delta(\frac{|x|}{R})$, with $R > r_0$. A simple calculation shows that

$$\varphi(x) = 0 \quad \text{as } |x| < R, \quad 0 \leq \varphi(x) \leq 1 \quad \text{and} \quad |\nabla \varphi(x)|^2 \leq \frac{C}{R^2} \varphi(x), \quad x \in \mathbb{R}^N. \quad (3.1)$$

Lemma 3.1. Let *Assumption 1*: (i) be in force and $g'(x, s) > -l$. Then

$$\begin{aligned} \int \varphi^2 G(x, u) dx &\geq \frac{c_2}{2} \|\varphi u\|^2, \\ \int \varphi^2 (g(x, u)u - \eta G(x, u)) dx &\geq \eta \|\varphi u\|^2 \end{aligned}$$

for $\eta : 0 < \eta \leq \frac{2c_2}{2c_2+l+2} (< 1)$.

Proof. When $|x| > r_0$, taking account of $g(x, 0) = 0$, $g'(x, s) > -l$ and making use of assumption (G_4) , we have

$$\begin{aligned} G(x, u) &\geq \frac{c_2}{2} u^2, \\ g(x, u)u - \eta G(x, u) &= \eta \int_0^u (g(x, u) - g(x, \xi)) d\xi + (1 - \eta)g(x, u)u \\ &\geq \left(c_2(1 - \eta) - \frac{\eta l}{2}\right) u^2 \geq \eta u^2. \end{aligned}$$

Since $\varphi(x) = 0$ as $|x| \leq r_0$ ($< R$), we have

$$\begin{aligned} \int \varphi^2 G(x, u) dx &= \int_{|x| > r_0} \varphi^2 G(x, u) dx \geq \frac{c_2}{2} \|\varphi u\|^2, \\ \int \varphi^2 (g(x, u)u - \eta G(x, u)) dx &= \int_{|x| > r_0} \varphi^2 (g(x, u)u - \eta G(x, u)) dx \\ &\geq \eta \int_{|x| > r_0} |\varphi u|^2 dx = \eta \|\varphi u\|^2. \end{aligned}$$

Lemma 3.2 (Tail estimate). Let $S(t)(u_0, u_1) = (u(t), u_t(t))$, with $(u_0, u_1) \in \mathcal{B}$. Then for any $\epsilon > 0$, there exist positive constants $R_1 = R_1(R_0)$ and $T_0 = T_0(R_0)$ such that

$$\|(u(t), u_t(t))\|_{\mathcal{H}(\Omega_{2R}^C)} < \epsilon \quad \text{as } R > R_1, \quad t > T_0,$$

where and in the following Ω_{2R} is the ball in \mathbb{R}^N with radius $2R$, $\Omega_{2R}^C = \mathbb{R}^N \setminus \Omega_{2R}$.

Proof. Using the multiplier $\varphi^2(u_t + \epsilon u)$ in (1.1), taking account of the boundedness of (u, u_t) in \mathcal{H} and (3.1), we have

$$\begin{aligned} &\frac{d}{dt} H_2(t) + K_2(t) \\ &= -2 \int \varphi \nabla \varphi \left((u_t + \epsilon u) \nabla u_t + M(\|\nabla u\|^2)(u_t + \epsilon u) \nabla u \right) dx + M'(\|\nabla u\|^2)(\nabla u, \nabla u_t) \|\varphi \nabla u\|^2 \\ &\leq \frac{1}{2} (\|\varphi u_t\|^2 + \|\varphi \nabla u_t\|^2) + \epsilon^2 \|\varphi \nabla u\|^2 + C(\|u_t \nabla \varphi\|^2 + \|\nabla \varphi \nabla u\|^2 + \|u \nabla \varphi\|^2) + C \|\nabla u_t\| \|\varphi \nabla u\|^2 \\ &\leq \frac{1}{2} (\|\varphi u_t\|^2 + \|\varphi \nabla u_t\|^2) + \epsilon^2 \|\varphi \nabla u\|^2 + \frac{C}{R^2} + C \|\nabla u_t\| \|\varphi \nabla u\|^2, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
H_2(t) &= \frac{1}{2} \left(\|\varphi u_t\|^2 + M(\|\nabla u\|^2) \|\varphi \nabla u\|^2 + 2 \int \varphi^2 (G(x, u) - fu) dx \right) \\
&\quad + \epsilon \left((\varphi^2 u, u_t) + \frac{1}{2} (\|\varphi u\|^2 + \|\varphi \nabla u\|^2) \right), \\
K_2(t) &= (1 - \epsilon) \|\varphi u_t\|^2 + \|\varphi \nabla u_t\|^2 + \epsilon M(\|\nabla u\|^2) \|\varphi \nabla u\|^2 + \epsilon (\varphi^2 g(\cdot, u), u) - \epsilon (\varphi^2 u, f).
\end{aligned}$$

Taking account of $M(s) \geq M_0 > 0$ and exploiting [Lemma 3.1](#), we get

$$H_2(t) \geq \kappa (\|\varphi u_t\|^2 + \|\varphi u\|^2 + \|\varphi \nabla u\|^2) - C \|\varphi f\|^2, \quad (3.3)$$

$$\begin{aligned}
K_2(t) &- \frac{1}{2} (\|\varphi u_t\|^2 + \|\varphi \nabla u_t\|^2) - \epsilon^2 \|\varphi \nabla u\|^2 - \eta \epsilon H_2(t) \\
&= \left(\frac{1}{2} - \epsilon - \frac{\eta \epsilon}{2} \right) \|\varphi u_t\|^2 + \frac{1}{2} \|\varphi \nabla u_t\|^2 + \epsilon \left(\left(1 - \frac{\eta}{2} \right) M(\|\nabla u\|^2) - \epsilon \left(1 + \frac{\eta}{2} \right) \right) \|\varphi \nabla u\|^2 \\
&\quad + \epsilon \int \varphi^2 (g(x, u)u - \eta G(x, u)) dx - \epsilon (1 - \eta) \int \varphi^2 u f dx - \epsilon^2 \eta \left((\varphi^2 u, u_t) + \frac{1}{2} \|\varphi u\|^2 \right) \\
&\geq \kappa (\|\varphi u_t\|^2 + \|\varphi \nabla u_t\|^2 + \|\varphi \nabla u\|^2 + \|\varphi u\|^2) - C \|\varphi f\|^2
\end{aligned} \quad (3.4)$$

for $\epsilon > 0$ suitably small. Inserting (3.4) into (3.2), we have

$$\frac{d}{dt} H_2(t) + \eta \epsilon H_2(t) \leq C \|\nabla u_t\| H_2(t) + C \|\varphi f\|^2 (\|\nabla u_t\| + 1) + \frac{C}{R^2}. \quad (3.5)$$

Applying [Lemma 2.5](#) to (3.5), we get

$$\begin{aligned}
H_2(t) &\leq C H_2(0) e^{-\kappa t} + C \int_0^t e^{-\kappa(t-\tau)} \left(\|\varphi f\|^2 (\|\nabla u_t(\tau)\| + 1) + \frac{1}{R^2} \right) d\tau \\
&\leq C H_2(0) e^{-\kappa t} + C \left(\frac{1}{R^2} + \|\varphi f\|^2 \right), \\
\|(u(t), u_t(t))\|_{\mathcal{H}(\Omega_{2R}^C)}^2 &\leq C e^{-\kappa t} + C \left(\frac{1}{R^2} + \|f\|_{L^2(\Omega_R^C)}^2 \right).
\end{aligned} \quad (3.6)$$

(3.6) implies the conclusion of [Lemma 3.2](#).

Remark 3.2. For any bounded set $B \subset \mathcal{H}$, there exists a $t_B > 0$ such that $S(t)B \subset \mathcal{B}$ as $t \geq t_B$. So we infer from [Lemma 3.2](#) that for any $\epsilon > 0$, there exist positive constants $R_1 = R_1(R_0)$ and $T_1 = t_B + T_0$ such that

$$\|(u(t), u_t(t))\|_{\mathcal{H}(\Omega_{2R}^C)} < \epsilon \quad \text{as } R > R_1, \quad t > T_1,$$

where $(u(t), u_t(t)) = S(t)(u_0, u_1)$, $(u_0, u_1) \in B$.

Lemma 3.3. Let the assumptions of [Theorem 3.1](#) be in force, and u, v be two solutions of the Cauchy problem (1.1)–(1.2) corresponding to initial data $(u_0, u_1), (v_0, v_1) \in \mathcal{B}$, $z = u - v$. Then

$$\|(z, z_t)(t)\|_{\mathcal{H}}^2 \leq C \|(z_0, z_1)\|_{\mathcal{H}}^2 e^{-\kappa t} + C \int_0^t e^{-\kappa(t-\tau)} \|z(\tau)\|^2 d\tau. \quad (3.7)$$

Proof. Obviously, $z = u - v$ solves

$$\begin{aligned}
z_{tt} - \Delta z_t - M_{12}(t) \Delta z + z_t - \tilde{M}_{12}(t) (\nabla(u + v), \nabla z) \Delta(u + v) + g(\cdot, u) - g(\cdot, v) &= 0, \\
z(0) = u_0 - v_0 \equiv z_0, \quad z_t(0) = u_1 - v_1 \equiv z_1,
\end{aligned} \quad (3.8)$$

where $M_{12}(t)$ and $\tilde{M}_{12}(t)$ are as shown in (2.27). Using the multiplier z_t in (3.8), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|z_t\|^2 + M_{12}(t) \|\nabla z\|^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \right) + \|z_t\|_{H^1}^2 \\ &= \frac{1}{2} \left(M'(t) (\nabla u, \nabla u_t) + M'(t) (\nabla v, \nabla v_t) \right) \|\nabla z\|^2 + \tilde{M}_{12}(t) (\nabla(u_t + v_t), \nabla z) (\nabla(u+v), \nabla z) \\ &+ \frac{1}{2} \int_0^1 M''(\lambda \|\nabla u\|^2 + (1-\lambda) \|\nabla v\|^2) \left(\lambda (\nabla u, \nabla u_t) + (1-\lambda) (\nabla v, \nabla v_t) \right) d\lambda (\nabla(u+v), \nabla z)^2 \\ &- (g(\cdot, u) - g(\cdot, v), z_t) \\ &\leq C(\|\nabla u_t\| + \|\nabla v_t\|) \|\nabla z\|^2 - (g(\cdot, u) - g(\cdot, v), z_t). \end{aligned} \quad (3.9)$$

(i) When $1 \leq p < p^*$, there exists a $\delta : 0 < \delta < 1$ such that $H^{1-\delta} \hookrightarrow L^{p+1}$. By the interpolation theorem,

$$\begin{aligned} |(g(\cdot, u) - g(\cdot, v), z_t)| &\leq C \int (|u|^{p-1} + |v|^{p-1} + 1) |z| |z_t| dx \\ &\leq C \|z\| \|z_t\| + C(\|u\|_{p+1}^{p-1} + \|v\|_{p+1}^{p-1}) \|z\|_{p+1} \|z_t\|_{p+1} \\ &\leq C \|z\| \|z_t\| + C \|z\| \|z_t\|_{H^{1-\delta}} \|z_t\|_{H^1} \leq C \|z\| \|z_t\| + C \|z\|^\delta \|z\|_{H^1}^{1-\delta} \|z_t\|_{H^1} \\ &\leq \epsilon \|z_t\|_{H^1}^2 + \epsilon^2 \|z\|_{H^1}^2 + C \|z\|^2. \end{aligned} \quad (3.10)$$

Inserting (3.10) into (3.9), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|z_t\|^2 + M_{12}(t) \|\nabla z\|^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \right) + (1-\epsilon) \|z_t\|_{H^1}^2 \\ &\leq \epsilon^2 \|z\|_{H^1}^2 + C(\|\nabla u_t\| + \|\nabla v_t\|) \|\nabla z\|^2 + C \|z\|^2. \end{aligned} \quad (3.11)$$

Similarly, using the multiplier z in (3.8) and adding $\|z\|^2$ to both sides, we have

$$\begin{aligned} & \frac{d}{dt} \left((z_t, z) + \frac{1}{2} \|z\|_{H^1}^2 \right) + \|z\|^2 + M_{12}(t) \|\nabla z\|^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \\ &\leq \|z_t\|^2 + \|z\|^2 - (g(\cdot, u) - g(\cdot, v), z) \leq \|z_t\|^2 + \epsilon \|z\|_{H^1}^2 + C \|z\|^2. \end{aligned} \quad (3.12)$$

(3.11) + $\epsilon \times$ (3.12) gives

$$\frac{d}{dt} H_3(t) + K_3(t) \leq C(\|\nabla u_t\| + \|\nabla v_t\|) \|\nabla z\|^2 + C \|z\|^2, \quad (3.13)$$

where

$$\begin{aligned} H_3(t) &= \frac{1}{2} \left(\|z_t\|^2 + M_{12}(t) \|\nabla z\|^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \right) + \epsilon \left((z_t, z) + \frac{1}{2} \|z\|_{H^1}^2 \right) \\ &\sim \|z_t\|^2 + \|z\|_{H^1}^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2, \\ K_3(t) &= (1-2\epsilon) \|z_t\|_{H^1}^2 + \epsilon \|z\|^2 - 2\epsilon^2 \|z\|_{H^1}^2 + \epsilon M_{12}(t) \|\nabla z\|^2 + \epsilon \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \\ &\sim \|z_t\|_{H^1}^2 + \|z\|_{H^1}^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \end{aligned}$$

for $\epsilon > 0$ suitably small. Therefore, there exists a $\kappa > 0$ such that

$$K_3(t) - \kappa H_3(t) \geq 0. \quad (3.14)$$

Inserting (3.14) into (3.13), we have

$$\begin{aligned} \frac{d}{dt}H_3(t) + \kappa H_3(t) &\leq C(\|\nabla u_t\| + \|\nabla v_t\|)H_3(t) + C\|z\|^2, \\ \|(z, z_t)(t)\|_{\mathcal{H}}^2 &\leq C\|(z_0, z_1)\|_{\mathcal{H}}^2 e^{-\kappa t} + C \int_0^t e^{-\kappa(t-\tau)} \|z(\tau)\|^2 d\tau. \end{aligned} \quad (3.15)$$

(ii) When $p = p^*$, noticing that $p^* \geq 2$ implies $N \leq 6$ and $H^1 \hookrightarrow L^3$, we have

$$\begin{aligned} (g(\cdot, u) - g(\cdot, v), z_t) &= \frac{1}{2} \frac{d}{dt} \int_0^1 \int \left(g'(x, \lambda u + (1-\lambda)v) + l \right) z^2 d\lambda dx + \tilde{G}(t), \\ |\tilde{G}(t)| &= \left| \frac{1}{2} \int_0^1 \int g''(x, \lambda u + (1-\lambda)v) \left(\lambda u_t + (1-\lambda)v_t \right) z^2 d\lambda dx + l(z, z_t) \right| \\ &\leq C(\|u_t\|_3 + \|v_t\|_3) \|z\|_3^2 + C(\|u\|_{p^*+1}^{p^*-2} + \|v\|_{p^*+1}^{p^*-2}) (\|u_t\|_{p^*+1} + \|v_t\|_{p^*+1}) \|z\|_{p^*+1}^2 + l(z, z_t) \\ &\leq C(\|u_t\|_{H^1} + \|v_t\|_{H^1}) \|z\|_{H^1}^2 + \epsilon \|z_t\|^2 + C\|z\|^2, \end{aligned}$$

and

$$(g(\cdot, u) - g(\cdot, v), z) \geq -l\|z\|^2.$$

Hence, we infer from (3.9) and (3.12) that

$$\frac{d}{dt} \tilde{H}_3(t) + \tilde{K}_3(t) \leq C(\|u_t\|_{H^1} + \|v_t\|_{H^1}) \|z\|_{H^1}^2 + C\|z\|^2, \quad (3.16)$$

where

$$\begin{aligned} \tilde{H}_3(t) &= \frac{1}{2} \left[\|z_t\|^2 + M_{12}(t) \|\nabla z\|^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \right. \\ &\quad \left. + \int_0^1 \int \left(g'(x, \lambda u + (1-\lambda)v) + l \right) z^2 dx d\lambda \right] + \epsilon \left((z_t, z) + \frac{1}{2} \|z\|_{H^1}^2 \right) \\ &\sim \|z_t\|^2 + \|z\|_{H^1}^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2, \\ \tilde{K}_3(t) &= \|z_t\|_{H^1}^2 - 2\epsilon \|z_t\|^2 + \epsilon \|z\|^2 + \epsilon M_{12}(t) \|\nabla z\|^2 + \epsilon \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 - \epsilon^2 \|z\|_{H^1}^2 \\ &\sim \|z_t\|_{H^1}^2 + \|z\|_{H^1}^2 + \tilde{M}_{12}(t) (\nabla(u+v), \nabla z)^2 \end{aligned}$$

for $\epsilon > 0$ suitably small. Repeating the proof of (3.14)–(3.15), we still get (3.7). Lemma 3.3 is proved.

Asymptotic compactness. Let $\{(u_0^n, u_1^n)\}$ be a bounded sequence in \mathcal{B} , $S(t)(u_0^n, u_1^n) = (u^n(t), u_t^n(t))$. Applying Lemma 3.3 to $w^{m,n}(t) = u^n(t+t_n-T) - u^m(t+t_m-T)$, with $t_n > t_m > T > 0$, $t \geq 0$, we obtain

$$\|(w^{m,n}, w_t^{m,n})(t)\|_{\mathcal{H}}^2 \leq C e^{-\kappa t} + C \sup_{0 \leq s \leq t} \|u^n(t_n - T + s) - u^m(t_m - T + s)\|^2.$$

By taking $t = T$ we get

$$\begin{aligned}
& \| (u^n(t_n) - u^m(t_m), u_t^n(t_n) - u_t^m(t_m)) \|_{\mathcal{H}}^2 \\
& \leq C e^{-\kappa T} + C \sup_{0 \leq s \leq T} \| u^n(t_n - s) - u^m(t_m - s) \|_{L^2(\Omega_{2R}^C)}^2 \\
& \quad + C \sup_{0 \leq s \leq T} \| u^n(t_n - s) - u^m(t_m - s) \|_{L^2(\Omega_{2R})}^2.
\end{aligned} \tag{3.17}$$

Since

$$C([0, T], H^1(\Omega_{2R})) \cap C^1([0, T], L^2(\Omega_{2R})) \hookrightarrow C([0, T], L^2(\Omega_{2R})), \tag{3.18}$$

we can extract a subsequence, still denoted by $\{u^n\}$, such that

$$u^n \rightarrow u \quad \text{in } C([0, T], L^2(\Omega_{2R}))$$

for any fixed $T > 0$, $R > 0$. For any $\epsilon > 0$, by fixing $R : R > R_1$, $T : C e^{-\kappa T} < \epsilon/4$ and taking n, m large enough such that $t_n - T > T_0$, $t_m - T > T_0$, we infer from [Lemma 3.2](#) and [\(3.17\)](#) that

$$C e^{-\kappa T} + C \sup_{0 \leq s \leq T} \| u^n(t_n - s) - u^m(t_m - s) \|_{L^2(\Omega_{2R}^C)}^2 < \frac{\epsilon}{2}, \tag{3.19}$$

$$C \sup_{0 \leq s \leq T} \| u^n(t_n - s) - u^m(t_m - s) \|_{L^2(\Omega_{2R})}^2 < \frac{\epsilon}{2}. \tag{3.20}$$

The combination of [\(3.17\)](#) with [\(3.19\)–\(3.20\)](#) yields

$$\| (u^n(t_n) - u^m(t_m), u_t^n(t_n) - u_t^m(t_m)) \|_{\mathcal{H}}^2 < \epsilon,$$

that is, $S(t)$ is asymptotically compact on \mathcal{B} . Therefore, the dynamical system $(S(t), \mathcal{B})$, and hence the dynamical system $(S(t), \mathcal{H})$, has a global attractor \mathcal{A} . [Theorem 3.1](#) is proved.

4. Exponential attractor

Definition. Let X be a complete metric space. A set $\mathcal{A}_{exp} \subset X$ is said to be an exponential attractor of the dynamical system $(S(t), X)$ if

- (1) it is a compact set in X ;
- (2) it has finite fractal dimension in X , i.e. $\dim_f \{\mathcal{A}_{exp}, X\} < +\infty$;
- (3) it is a forward invariant set, i.e. $S(t)\mathcal{A}_{exp} \subset \mathcal{A}_{exp}$, $t \geq 0$;
- (4) it attracts exponentially the bounded sets in X , that is, for any bounded set $B \subset X$, there exists a positive constant γ such that

$$\text{dist}_X \{S(t)B, \mathcal{A}_{exp}\} \leq C(\|B\|_X) e^{-\gamma t}, \quad t \geq 0,$$

where $\|B\|_X = \sup_{\zeta \in B} \|\zeta\|_X$.

Theorem 4.1. Under the assumptions of [Theorem 3.1](#), the dynamical system $(S(t), \mathcal{H})$ possesses an exponential attractor \mathcal{A}_{exp} .

In order to prove [Theorem 4.1](#), we need the following lemma.

Lemma 4.2. (See [\[8\]](#).) Let X be a Banach space and M be a bounded closed set in X . Assume that the mapping $V : M \mapsto M$ possesses the properties:

(i) V is Lipschitz on M , i.e., there exists an $L > 0$ such that

$$\|Vv_1 - Vv_2\| \leq L\|v_1 - v_2\|, \quad \forall v_1, v_2 \in M;$$

(ii) there exist the compact seminorms $n_1(x)$, $n_2(x)$ on X such that

$$\|Vv_1 - Vv_2\| \leq \eta\|v_1 - v_2\| + K\left(n_1(v_1 - v_2) + n_2(Vv_1 - Vv_2)\right)$$

for any $v_1, v_2 \in M$, where $0 < \eta < 1$ and $K > 0$ are constants.

Then for any $\kappa > 0$ and $\delta \in (0, 1 - \eta)$ there exists a forward invariant compact set $A_{\kappa, \delta} \subset M$ of finite fractal dimension such that

$$\text{dist}(V^k M, A_{\kappa, \delta}) \leq q^k, \quad k = 1, 2, \dots,$$

where $q = \eta + \delta < 1$, and

$$\dim_f A_{\kappa, \delta} \leq \left[\ln \frac{1}{\delta + \eta} \right]^{-1} \cdot \left[\ln m_0 \left(\frac{2K(1 + L^2)^{1/2}}{1 - \eta} \right) + \kappa \right],$$

where $m_0(R)$ is the maximal number of pairs (x_i, y_i) in $X \times X$ possessing the properties

$$\|x_i\|^2 + \|y_i\|^2 \leq R^2, \quad n_1(x_i - x_j) + n_2(y_i - y_j) > 1, \quad i \neq j.$$

That is, the discrete dynamical system (V^k, M) possesses an exponential attractor $A_{\kappa, \delta}$.

Proof of Theorem 4.1. We have known that the dynamical system $(S(t), \mathcal{H})$ has a forward invariant bounded absorbing set \mathcal{B} , which is closed in \mathcal{H} . Especially, we see from the estimate (2.2) that \mathcal{B} is bounded in $H^1 \times H^1$, and for any $\xi_u = (u_0, u_1) \in \mathcal{B}$, $\xi_u(t) = S(t)\xi_u = (u(t), u_t(t)) \in \mathcal{B}$, and

$$\|u(t)\|_{H^1} + \|u_t(t)\|_{H^1} + \|u_{tt}(t)\| \leq C, \quad t \geq 0. \quad (4.1)$$

Define the operator

$$V = S(T) : \mathcal{B} \mapsto \mathcal{B}.$$

Obviously, $V\mathcal{B} \subset \mathcal{B}$ and V is Lipschitz on \mathcal{B} . For any $\xi_u, \xi_v \in \mathcal{B}$, we infer from Lemma 3.3 and (2.3) that

$$\begin{aligned} \|V\xi_u - V\xi_v\|_{\mathcal{H}}^2 &\leq Ce^{-\kappa(T-t_0)}\|\xi_u(t_0) - \xi_v(t_0)\|_{\mathcal{H}}^2 + C \int_{t_0}^T e^{-\kappa(T-s)}\|u(s) - v(s)\|^2 ds \\ &\leq \eta_T^2 \|\xi_u - \xi_v\|_{\mathcal{H}}^2 + C \max_{t_0 \leq s \leq T} \|u(s) - v(s)\|^2, \end{aligned}$$

that is,

$$\|V\xi_u - V\xi_v\|_{\mathcal{H}} \leq \eta_T \|\xi_u - \xi_v\|_{\mathcal{H}} + Cn_1(\xi_u - \xi_v),$$

where

$$\eta_T^2 = Ce^{(\kappa+k)t_0}e^{-\kappa T}, \quad n_1(\xi_u) = \max_{t_0 \leq s \leq T} \|u(s)\|.$$

We claim that $n_1(\xi_u)$ is a compact seminorm on \mathcal{H} . Indeed, for any bounded sequence $\{\xi_{u^n}\} \subset \mathcal{H}$, $\|\xi_{u^n}\|_{\mathcal{H}} \leq R^*$, on account of

$$\begin{aligned} n_1(\xi_{u^n} - \xi_{u^m}) &= \max_{t_0 \leq s \leq T} \|u^n(s) - u^m(s)\| \\ &\leq \max_{t_0 \leq s \leq T} \|u^n(s) - u^m(s)\|_{L^2(\Omega_{2R})} + \max_{t_0 \leq s \leq T} \|u^n(s) - u^m(s)\|_{L^2(\Omega_{2R}^C)}, \end{aligned}$$

for any $\epsilon > 0$, taking $R > R_1$, $t_0 > T_1$ ($= t_{R^*} + T_0$), we infer from Remark 3.2 that

$$\max_{t_0 \leq s \leq T} \|u^n(s) - u^m(s)\|_{L^2(\Omega_{2R}^C)} < \epsilon/2.$$

For fixed R and t_0 , taking T such that $0 < \eta_T < 1$, by (3.18) (replacing 0 there by t_0), one can extract a subsequence, still denoted by $\{u^n\}$, such that

$$u^n \rightarrow u \quad \text{in } C([t_0, T]; L^2(\Omega_{2R})).$$

That is, there exists an $N > 0$ such that

$$\max_{t_0 \leq s \leq T} \|u^n(s) - u^m(s)\|_{L^2(\Omega_{2R})} < \epsilon/2 \quad \text{as } n, m > N.$$

Therefore,

$$n_1(\xi_{u^n} - \xi_{u^m}) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

The claim is valid.

By Lemma 4.2, the discrete dynamical system (V^k, \mathcal{B}) has an exponential attractor \mathbb{A} , where $V^k = S(kT)$. Let

$$\mathcal{A}_{exp} = \bigcup_{0 \leq t \leq T} S(t)\mathbb{A}.$$

By the standard method (cf. [33,35]), one easily knows that \mathcal{A}_{exp} is an exponential attractor of the dynamical system $(S(t), \mathcal{B})$. So there exists a $\gamma > 0$ such that

$$dist_{\mathcal{H}}\{S(t)\mathcal{B}, \mathcal{A}_{exp}\} \leq Ce^{-\gamma t}, \quad t \geq 0.$$

We claim that \mathcal{A}_{exp} is an exponential attractor of the dynamical system $(S(t), \mathcal{H})$. Indeed,

- (i) Obviously, \mathcal{A}_{exp} is a forward invariant set.
- (ii) For every bounded set $B \subset \mathcal{H}$, there exists a $t_B > 0$ such that $S(t)B \subset \mathcal{B}$ as $t \geq t_B$, so

$$\begin{aligned} dist_{\mathcal{H}}\{S(t)B, \mathcal{A}_{exp}\} &= dist_{\mathcal{H}}\{S(t - t_B)S(t_B)B, \mathcal{A}_{exp}\} \\ &\leq dist_{\mathcal{H}}\{S(t - t_B)\mathcal{B}, \mathcal{A}_{exp}\} \leq Ce^{-\gamma(t - t_B)} \leq C(\|B\|_{\mathcal{H}})e^{-\gamma t}. \end{aligned}$$

When $t < t_B$,

$$dist_{\mathcal{H}}\{S(t)B, \mathcal{A}_{exp}\} \leq Ce^{\gamma t}e^{-\gamma t} \leq C(\|B\|_{\mathcal{H}})e^{-\gamma t}.$$

(iii) Define the operator

$$F : [0, T] \times \mathbb{A} \rightarrow \mathcal{B}, \quad F(t, \xi_u) = \xi_u(t) = S(t)\xi_u, \quad \xi_u \in \mathbb{A}.$$

It follows from (4.1) and (2.3) that

$$\begin{aligned} \|F(t_1, \xi_u) - F(t_2, \xi_u)\|_{\mathcal{H}} &= \|S(t_1)\xi_u - S(t_2)\xi_u\|_{\mathcal{H}} \leq \left| \int_{t_1}^{t_2} \|\xi'_u(\tau)\|_{\mathcal{H}} d\tau \right| \leq C|t_1 - t_2|, \\ \|F(t, \xi_{u_1}) - F(t, \xi_{u_2})\|_{\mathcal{H}} &= \|S(t)\xi_{u_1} - S(t)\xi_{u_2}\|_{\mathcal{H}} \leq C\|\xi_{u_1} - \xi_{u_2}\|_{\mathcal{H}} \end{aligned} \quad (4.2)$$

for all $\xi_u, \xi_{u_1}, \xi_{u_2} \in \mathcal{B}, t, t_1, t_2 \in [0, T]$, that is, the mapping F is Lipschitz continuous, so $\mathcal{A}_{exp} = F\{[0, T] \times \mathbb{A}\}$ (the image of the set $[0, T] \times \mathbb{A}$) is a compact set in \mathcal{H} , and

(iv)

$$\dim_f(\mathcal{A}_{exp}, \mathcal{H}) \leq 1 + \dim_f\{\mathbb{A}, \mathcal{H}\} < +\infty.$$

So, \mathcal{A}_{exp} is the desired exponential attractor. Theorem 4.1 is proved.

Remark 4.1. Theorem 4.1 implies that the global attractor \mathcal{A} in Theorem 3.1 has finite fractal dimension.

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