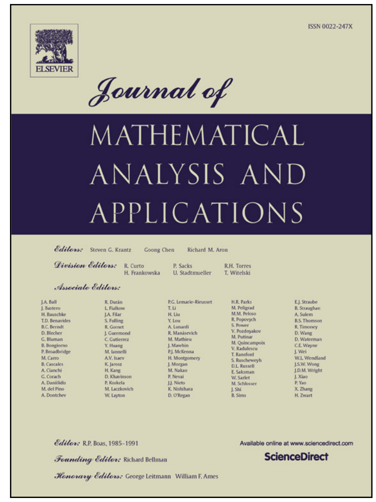
Accepted Manuscript



Existence and uniqueness for a concrete carbonation process with hysteresis

Toyohiko Aiki, Sergey A. Timoshin

PII: S0022-247X(17)30011-2
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.12.086
Reference: YJMAA 21026

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 14 October 2016

Existence and uniqueness for a concrete carbonation process with hysteresis

Toyohiko Aiki* and Sergey A. Timoshin†

January 4, 2017

Abstract

A partial differential system arising in the moisture transport in concrete carbonation modeling is considered. The system consists of a diffusion equation for moisture and an ordinary differential equation accounting for the hysteresis effect appearing in the process. The existence of solutions for this system supplied with suitable initial boundary conditions and having fairly general nonlinear right-hand sides is established. When the dimension of the space domain is one, the uniqueness of a solution is also provided.

Keywords: evolution system, hysteresis, concrete carbonation.

1 Introduction

In the space-time cylinder $Q(T) := [0,T] \times \Omega$, where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $T > 0$ is a fixed final time, consider the system

$$\rho u_t - \operatorname{div}(g(u)\nabla u) = h(u,w) \quad \text{in } Q(T), \tag{1.1}$$

$$w_t + \partial I(u;w) \ni F(u,w) \quad \text{in } Q(T), \tag{1.2}$$

$$u = u_b \quad \text{on } (0,T) \times \partial\Omega, \tag{1.3}$$

*Department of Mathematical and Physical Sciences, Faculty of Science, Japan Women's University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo, 112-8681, Japan, E-mail: aikit@fc.jwu.ac.jp.

†Institute for System Dynamics and Control Theory, Siberian Branch, Russian Academy of Sciences, Lermontov str. 134, 664033 Irkutsk, Russia, E-mail: sergey.timoshin@gmail.com.

$$u(0) = u_0, \quad w(0) = w_0 \quad \text{on } \Omega. \tag{1.4}$$

Here, ρ is a given positive constant, $I(u;\cdot)$ is the indicator function of the interval $[f_*(u), f^*(u)]$, $\partial I(u;\cdot)$ is its subdifferential, g, h, F, f_*, f^* are given functions with the properties specified in the next section, u_b, u_0, w_0 are given boundary and initial conditions.

For convenience, denote system (1.1)–(1.4) by (P). System (P) in case when $F \equiv 0$ and $h = wf(t,x)$ with $f \in L^\infty(Q(T))$, $f \geq 0$ was studied in [1,2] as a simplified model for moisture transport appearing in concrete carbonation process when hysteresis effects are taken into account. The unknowns $u = u(t,x)$ and $w = w(t,x)$ represent then the relative humidity and the degree of saturation, respectively, ρ is the density of saturated vapor and f is the quantity of water produced as a result of chemical reactions during concrete carbonation. Eq. (1.1) is the diffusion equation for moisture and relation (1.2) models the so-called generalized play operator generated by the curves $w = f_*(u)$ and $w = f^*(u)$, see [3–5] for details. The introduction of the latter operator to the model accounts for hysteretic relationship between u and w, playing in this case the roles of the input and output functions, respectively. We refer the reader to [6] for more physical background on the model. We mention also the articles [7–9], where systems related to (P) with $g(u) \equiv 1$ were considered.

The purpose of the present paper is to prove the existence of a solution to system (P) in case of sufficiently general F and h and, when $N = 1$, to establish the uniqueness of such a solution. The reason for considering general nonlinearities on the right-hand sides of (1.1), (1.2) is two-fold. First, in this form the proposed system might be a better approximation of a rather complicated from the mathematical investigation point of view model of the moisture transport in cementitious materials proposed originally in [10] and then simplified in [1,2]. Second, such a generalization often represents a challenging and interesting task from the mathematical point of view.

Our approach to establish the existence for problem (P) is in some aspects somewhat close to that used in [1]. In particular, first we construct a family of suitable approximate problems based on the Yosida regularization $\partial I_\lambda(u;\cdot)$, $\lambda > 0$ of the subdifferential $\partial I(u;\cdot)$ and establish the existence of solutions for these approximate problems for each $\lambda > 0$. Then, we derive a priori estimates independent of λ for solutions of approximate problems. Finally, we prove existence of a solution to problem (P) through the passage-to-the-limit procedure. We note that in contrast to [1] we do not singularly perturb Eq. (1.2) to get suitable compactness properties. Instead, in order to legitimate the passage-to-the-limit we exploit essentially the properties derived from the specific structure of the approximate equations for (1.2). This also allowed us, inter alia, to treat general nonlinearities in Eqs. (1.1), (1.2).

At the end of the introduction we make some remarks on the uniqueness proof for problem (P). In case when $F \equiv 0$ and $h = wf(t,x)$, the uniqueness of a solution to the problem is proved for $N = 3$ in [11]. We note that the corresponding proof is rather long and complicated due to the facts that the continuous dependence of the solution w of (1.2) on u is only valid in the topology of L^∞ (see Lemma 6.1 below) and (1.1) is quasi-linear. Thus, in the present paper we prove the uniqueness only for the case of $N = 1$. However, since in our problem $F \not\equiv 0$ the difficulty for the proof of uniqueness persists. To circumvent such a difficulty, in [9] Kenmochi, Minchev and Okazaki added u_t to (1.1), applied the Hilpert inequality, and then proved the uniqueness. The idea to overcome this difficulty in our paper is to impose some monotonicity or smallness assumptions on F. Making use of these assumptions we obtain the continuous dependence for (1.2) as stated in Lemma 6.1. Then, in order to apply Lemma 6.1 to prove the uniqueness we need an estimate for the $L^\infty-$norm of the difference of two solutions u_1 and u_2. Accordingly, we multiply the corresponding difference of the equations (1.1) for u_1 and u_2 by $-(u_1-u_2)_{xx}$ and get the required estimate. A similar argument to treat the quasi-linear equation (1.1) is outlined in [2].

2 Preliminaries and hypotheses on the data

In this section, we recall some notions which we use in the paper, state several known auxiliary results, and posit assumptions on the coefficients and functions describing problem (P).

Throughout the paper we denote by H the Hilbert space $L^2(\Omega)$ with the standard inner product $(\cdot,\cdot)_H$, and by V the Sobolev space $H^1(\Omega)$.

Let f_*, f^* be two Lipschitz continuous functions defined on $\mathbb{R}$. Recall that the subdifferential of the indicator function $I(u;\cdot)$, $u \in H$,

$$I(u;w) := \begin{cases} 0 & \text{if } \; w \in K(u), \\ +\infty & \text{otherwise,} \end{cases}$$

of the set $K(u) = \{w \in H;\ f_*(u) \le w \le f^*(u) \ \text{ a.e. on } \Omega\}$ has the form:

$$\partial I(u;w) = \begin{cases} \emptyset & \text{if } \; w \notin K(u), \\ [0,+\infty) & \text{if } \; w = f^*(u) > f_*(u) \ \text{ a.e. on } \Omega, \\ \{0\} & \text{if } \; f_*(u) < w < f^*(u) \ \text{ a.e. on } \Omega, \\ (-\infty,0] & \text{if } \; w = f_*(u) < f^*(u) \ \text{ a.e. on } \Omega, \\ (-\infty,+\infty) & \text{if } \; w = f_*(u) = f^*(u) \ \text{ a.e. on } \Omega. \end{cases} \tag{2.1}$$

For $\lambda > 0$ the Yosida regularization of $\partial I(u;w)$ is the function

$$\partial I_\lambda(u;w) = \frac{1}{\lambda}[w - f^*(u)]^+ - \frac{1}{\lambda}[f_*(u) - w]^+, \quad u, w \in H. \tag{2.2}$$

It is the subdifferential of the convex function

$$I_\lambda(u;w) = \frac{1}{2\lambda}\left|[w - f^*(u)]^+\right|_H^2 - \frac{1}{2\lambda}\left|[f_*(u) - w]^+\right|_H^2, \quad u, w \in H.$$

Lemma 2.1. (cf. [8, Lemma 4.1]) *For $u, w \in W^{1,2}(0,T;H)$ we have*

$$\frac{d}{dt} I_\lambda(u;w) \le (\partial I_\lambda(u;w), w_t)_H + C_0 |\partial I_\lambda(u;w)|_H |u_t|_H \quad \text{a.e. on } [0,T], \lambda > 0,$$

where $C_0 := \max\{|f_*'|_{L^\infty(\mathbb{R})}, |f^{*\prime}|_{L^\infty(\mathbb{R})}\}$.

Problem (1.1)–(1.4) is considered under the following hypotheses:

(A1) ρ is a positive constant;

(A2) $g \in C^2((0,\infty))$ and $g(r) \geq g_0$ for $r > 0$, where g_0 is a positive constant. In addition, $g(r) = G'(r)$ for a continuous function $G : (0,\infty) \to \mathbb{R}$;

(A3) $h, F : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz continuous and h is nonnegative and bounded on $\mathbb{R}^2$;

(A4) $f_*, f^* \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ with $0 \leq f_* \leq f^* \leq w_*$ on $\mathbb{R}$, where w_* is a positive constant;

(A5) $u_b \in W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega))$ with $u_b \geq \kappa_0$ a.e. on $Q(T)$ for some positive constant κ_0;

(A6) $u_0 \in L^\infty(\Omega) \cap V$, $w_0 \in L^\infty(\Omega)$ with $u_0 \geq \kappa_0$, $w_0 \geq 0$ a.e. on Ω, $u_0 = u_b(0)$ a.e. on $\partial\Omega$ and $f_*(u_0) \leq w_0 \leq f^*(u_0)$.

Note that these hypotheses are consistent with physically justified assumptions on the concrete carbonation model considered in [1, 2] for the case of $F \equiv 0$ and $h = wf(t,x)$ with a nonnegative function $f \in L^\infty(Q(T))$.

Next, we define a notion of solution to our problem (P).

Definition 2.1. *A pair* $\{u, w\}$ *is called a solution of system* (1.1)–(1.4) *if*

$$\begin{aligned} &u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T];H^2(\Omega)), \quad u > 0,\\ &w \in W^{1,2}(0,T;H), \end{aligned}$$

and there exists a function $\xi \in L^2(0,T;H)$ *such that*

$$w_t + \xi = F(u,w) \quad \text{a.e. in } Q(T),$$
$$\xi(t) \in \partial I(u(t); w(t)) \quad \text{for a.e. } t \in [0,T],$$

$$\rho u_t - \operatorname{div}(\nabla G(u)) = h(v,w) \quad \text{a.e. in } Q(T),$$
$$u = u_b \quad \text{a.e. on } (0,T) \times \partial\Omega,$$
$$u(0) = u_0, \quad w(0) = w_0 \quad \text{a.e. on } \Omega.$$

3 L^∞-boundedness of solutions

In this section, we obtain a priori L^∞-bounds for all possible solutions of system (1.1)–(1.4). To this end, let $\{u, w\}$ be an arbitrary solution to problem (P). First, we note that in view of (2.1) Eq. (1.2) requires the following constraint:

$$f_*(u) \leq w \leq f^*(u) \quad \text{a.e. on } Q(T). \tag{3.1}$$

Hence, taking account of $(A4)$ we immediately see that

$$0 \le w \le w_* \quad \text{a.e. on } Q(T).$$

Next, testing (1.1) by $-[\hat{u}+\kappa_0]^+$, where $\hat{u} = -u$, κ_0 is the same constant as in $(A5)$, $(A6)$, from $(A3)$ we obtain

$$\frac{\rho}{2}\frac{d}{dt}\left|[\hat{u}+\kappa_0]^+\right|_H^2 + \int_\Omega \Delta G(u(t))[\hat{u}+\kappa_0]^+ dx \le 0 \quad \text{for a.e. } t \in [0,T]. \tag{3.2}$$

We note that by virtue of $(A5)$, $(A2)$

$$\begin{aligned}\int_\Omega \Delta G(u(t))[\hat{u}+\kappa_0]^+ dx &= -\int_\Omega \nabla G(u(t))\nabla[\hat{u}+\kappa_0]^+ dx \\ &= -\int_\Omega g(u(t))\nabla u(t)\nabla[\hat{u}+\kappa_0]^+ dx \\ &= \int_\Omega g(u(t))\left|\nabla[\hat{u}+\kappa_0]^+\right|_H^2 dx \ge 0 \quad \text{for a.e. } t \in [0,T].\end{aligned}$$

From (3.2) we then infer that

$$u \ge \kappa_0 \quad \text{a.e. on } Q(T).$$

Let now

$$\begin{aligned} M_1 &:= \max\{|u_0|_{L^\infty(\Omega)}, |u_b|_{L^\infty(\Omega)}\}, \\ M_2 &:= \rho^{-1}|h|_{L^\infty(\mathbb{R}^2)}, \\ p(t) &:= M_1 + M_2 t. \end{aligned}$$

Testing (1.1) by $[u-p]^+$ we deduce that

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}\left|[u-p]^+\right|_H^2 + \int_\Omega \nabla G(u)\nabla[u-p]^+ dx \\ &\qquad = \int_\Omega (h(u,w) - \rho M_2)[u-p]^+ dx \le 0 \quad \text{a.e. on } [0,T].\end{aligned}$$

Hence,

$$u \le p \le M_1 + M_2 T := M_0 \quad \text{a.e. on } Q(T).$$

Therefore, for any solution $\{u, w\}$ of problem (P) we have

$$\kappa_0 \le u \le M_0, \quad 0 \le w \le w_*. \tag{3.3}$$

Now that every solution of (1.1)–(1.4) is bounded, without loss of generality, we may assume (cutting off outside the set $\{\kappa_0 \le u \le M_0, 0 \le w \le w_*\}$, if necessary) that the functions h, F are both bounded and Lipschitz continuous on $\mathbb{R}^2$. We denote by L a common Lipschitz constant of h and F.

4 Approximation problem

In order to prove the existence of a solution to our problem (P), in this section, we approximate the latter by a family of suitable problems depending on two approximation parameters which we introduce next.

For $\lambda > 0$ and $u \in H$, let $\partial I_\lambda(u;\cdot)$ be the Yosida regularization of $\partial I(u;\cdot)$. Furthermore, for a given $m > 0$ let g_m be a function on $\mathbb{R}$ such that $g_m \in C^1(\mathbb{R})$, $g_m(r) = g(r)$ for $\frac{1}{m} \le r \le m$ and $g_m \ge g_0$ on $\mathbb{R}$, where g_0 is the constant from $(A2)$. Setting $G_m(r) = \int_1^r g_m(s)\,ds$ for $r \in \mathbb{R}$, we consider the following approximate problem denoted by $(P)_{\lambda,m}$:

$$\rho u_t - \operatorname{div}(\nabla G_m(u)) = h(u,w) \quad \text{in } Q(T), \tag{4.1}$$

$$w_t + \partial I_\lambda(u;w) = F(u,w) \quad \text{in } Q(T), \tag{4.2}$$

$$u = u_b \quad \text{on } (0,T)\times\partial\Omega, \tag{4.3}$$

$$u(0) = u_0, \quad w(0) = w_0 \quad \text{on } \Omega. \tag{4.4}$$

A pair of functions $\{u,w\}$ is called a solution to $(P)_{\lambda,m}$ if $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T];H^2(\Omega))$, $u > 0$ a.e. on $Q(T)$, $w \in W^{1,2}(0,T;H)$ and (4.1)–(4.4) hold.

Below, we prove the existence of solutions for problems $(P)_{\lambda,m}$, $\lambda, m > 0$. We split the proof into three steps. In the first one, we establish the continuity of the solution operator that with a function u from an appropriate class associates the solution w of the ODE (4.2) and derive a priori estimates for w. Then, we obtain a similar result for the solution operator which with a function w associates the solution u of the PDE (4.1). Finally, we use these continuity properties and estimates to construct a solution to $(P)_{\lambda,m}$ by the Schauder fixed point theorem.

4.1 Step 1: ODE mapping

We fix $\lambda, m > 0$ and let

$$\mathcal{K}_C := \{z \in L^2(0,T;H);\ |z|_{W^{1,2}(0,T;H)} + |z|_{L^\infty(0,T;V)} \le C\},$$

for a positive constant C to be determined later. For a given $\tilde{u} \in \mathcal{K}_C$ consider the following problem:

$$w_t + \partial I_\lambda(\tilde{u};w) = F(\tilde{u},w) \quad \text{in } Q(T), \tag{4.5}$$

$$w(0) = w_0 \quad \text{on } \Omega. \tag{4.6}$$

Since $\partial I_\lambda(\tilde{u};w) - F(\tilde{u},w)$ is Lipschitz continuous, by general existence-uniqueness theorem for ODEs system (4.5), (4.6) admits a unique solution $w \in W^{1,2}(0,T;H)$.

Lemma 4.1. *If w is a solution to (4.5), (4.6), then*

$$|w| \le R \tag{4.7}$$

for $R = w_* + |F|_{L^\infty(\mathbb{R}^2)}T$, where w_* is the constant from $(A4)$.

Proof. Let $p(t) := w_* + |F|_{L^\infty(\mathbb{R}^2)}t$. Then, testing (4.5) by $[w-p]^+$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left|[w-p]^+\right|_H^2 + \int_\Omega \partial I_\lambda(\tilde{u};w)[w-p]^+dx$$
$$= \int_\Omega \left(F(\tilde{u},w) - |F|_{L^\infty(\mathbb{R}^2)}\right)[w-p]^+dx \le 0 \quad \text{a.e. on } [0,T].$$

Since the second term on the left-hand side of this inequality is non-negative (cf. (2.2)) we have

$$w \le p \le w_* + |F|_{L^\infty(\mathbb{R}^2)}T =: R.$$

Testing next (4.5) by $-[\hat{w}-p]^+$ with $\hat{w} = -w$ we see that

$$\frac{1}{2}\frac{d}{dt}\left|[\hat{w}-p]^+\right|_H^2 = -\int_\Omega \left(|F|_{L^\infty(\mathbb{R}^2)} + F(\tilde{u},w)\right)[\hat{w}-p]^+dx$$
$$+\int_\Omega \partial I_\lambda(\tilde{u};w)[\hat{w}-p]^+dx \quad \text{a.e. on } [0,T].$$

Since $\partial I_\lambda(\tilde{u};w)[\hat{w}-p]^+ \le 0$ (cf. (2.2)), from the above inequality we infer that

$$w \ge -p \ge -R,$$

so that the claim of lemma follows. □

For $C > 0$, let $S_1 : \mathcal{K}_C \to L^2(0,T;H)$ be given by

$$S_1\tilde{u} = w, \quad \tilde{u} \in \mathcal{K}_C,$$

where w is the solution of (4.5), (4.6) for $\tilde{u} \in \mathcal{K}_C$.

Lemma 4.2. *The operator S_1 is continuous.*

Proof. Let $\tilde{u}_i, \tilde{u} \in \mathcal{K}_C$, $i \ge 1$, $\tilde{u}_i \to \tilde{u}$ in $L^2(0,T;H)$ as $i \to \infty$, and $w_i := S_1\tilde{u}_i$, $i \ge 1$, $w := S_1\tilde{u}$. Then, by virtue of Eq. (4.5) we have

$$(w_i - w)_t + \partial I_\lambda(\tilde{u}_i;w_i) - \partial I_\lambda(\tilde{u};w) = F(\tilde{u}_i,w_i) - F(\tilde{u},w) \quad \text{in } Q(T).$$

Multiplying this equality by $w_i - w$, using the Lipschitz continuity of ∂I_λ, F and Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}|w_i(t)-w(t)|_H^2 \le C_1|\tilde{u}_i(t)-\tilde{u}(t)|_H^2 + C_2|w_i(t)-w(t)|_H^2 \quad \text{for a.e. } t \in [0,T],$$

where C_1, C_2 are some positive constants depending on λ and the Lipschitz constants of f_*, f^* and F. The Gronwall inequality allows us to conclude now that $w_i \to w$ in $L^2(0,T;H)$. Hence, S_1 is continuous as claimed. □

4.2 Step 2: PDE mapping

Let

$$W_R := \{z \in L^2(0,T;H);\ |z| \leq R \quad \text{a.e. on } Q(T)\},$$

where R is the constant from (4.7). For a given $\tilde{w} \in W_R$ consider the following problem:

$$\rho u_t - \operatorname{div}(\nabla G_m(u)) = h(u, \tilde{w}) \quad \text{in } Q(T), \tag{4.8}$$

$$u = u_b \quad \text{on } (0,T) \times \partial\Omega, \tag{4.9}$$

$$u(0) = u_0 \quad \text{on } \Omega. \tag{4.10}$$

Define the function

$$\varphi^t(z) := \begin{cases} |\nabla z|_H^2 & \text{if} \quad z \in V \text{ with } z = G_m(u_b) \text{ on } \partial\Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.11}$$

and consider the following problem:

$$\rho u_t + \partial\varphi^t(G_m(u)) \ni h(u, \tilde{w}) \quad \text{in } H,\ t \in [0,T], \tag{4.12}$$

$$u(0) = u_0 \quad \text{in } H. \tag{4.13}$$

Since G_m is bi-Lipschitz continuous and h is continuous and bounded invoking the abstract theory as developed by Kenmochi (cf. [12, Theorem 2.8.1 and Proposition 3.2.2]) we conclude that problem (4.12), (4.13) has a solution $u \in W^{1,2}(0,T;H)$ such that $\varphi^t(G_m(u))$ is bounded on $[0,T]$. From the bi-Lipschitz continuity of G_m we then infer that $u(t) \in H^1(\Omega)$ for every $t \in [0,T]$. Furthermore, from the fact that $G_m(u) = G_m(u_b)$ a.e. on $(0,T) \times \partial\Omega$ we obtain that $u = u_b$ a.e. on $(0,T)\times\partial\Omega$. Therefore, u is also a solution to (4.8)–(4.10) with $u \in L^\infty(0,T;H^1(\Omega))$. In addition, [13, Lemma 3.7.1] implies that $u \in L^2(0,T;H^2(\Omega))$.

Next, we establish the uniqueness result for problem (4.8)–(4.10).

Lemma 4.3. *For a given $\tilde{w} \in W_R$ problem* (4.8)–(4.10) *has a unique solution.*

Proof. Let u_1, u_2 be two solutions of (4.8)–(4.10) for a given $\tilde{w} \in W_R$. Then, we have

$$\rho(u_1 - u_2)_t - \Delta(G_m(u_1) - G_m(u_2)) = h(u_1, \tilde{w}) - h(u_2, \tilde{w}). \tag{4.14}$$

We denote by sign : $\mathbb{R} \to \{-1, 0, 1\}$ the sign function, that is, $\operatorname{sign}(z) = 1$ if $z > 0$, $\operatorname{sign}(z) = 0$ if $z = 0$, $\operatorname{sign}(z) = -1$ if $z < 0$, and by $\operatorname{sign}_\delta$ its regularization with a positive parameter δ, which is defined as

$$\operatorname{sign}_\delta(z) = \begin{cases} 1 & \text{for} \quad z \geq \delta, \\ z/\delta & \text{for} \quad z \in (-\delta, \delta), \\ -1 & \text{for} \quad z \leq -\delta. \end{cases}$$

Testing (4.14) by $\mathrm{sign}_\delta(G_m(u_1) - G_m(u_2))$ and using the Lipschitz continuity of h we obtain

$$\begin{aligned}&\rho \int_\Omega (u_{1t}(t) - u_{2t}(t))\,\mathrm{sign}_\delta(G_m(u_1)(t) - G_m(u_2)(t))\,dx\\&+ \int_\Omega \nabla(G_m(u_1(t)) - G_m(u_2(t)))\nabla\,\mathrm{sign}_\delta(G_m(u_1(t)) - G_m(u_2(t)))\,dx\\&\le L|u_1(t) - u_2(t)|_{L^1(\Omega)} \quad \text{for a.e. } t \in [0,T]. \qquad (4.15)\end{aligned}$$

Since $\mathrm{sign}'_\delta \ge 0$, the second term on the left-hand side of this inequality is non-negative. Letting δ tend to zero and using the fact that $\mathrm{sign}(u_1 - u_2) = \mathrm{sign}(G_m(u_1) - G_m(u_2))$ a.e. in $Q(T)$, from (4.15) we infer that

$$\rho \frac{d}{dt}|u_1(t) - u_2(t)|_{L^1(\Omega)} \le L|u_1(t) - u_2(t)|_{L^1(\Omega)} \quad \text{for a.e. } t \in [0,T].$$

The Gronwall argument now implies the claim of lemma. □

Lemma 4.4. *If u is a solution to problem* (4.8)–(4.10) *for $\tilde{w} \in W_R$, then*

$$\kappa_0 \le u \le M_0 \quad \textit{a.e. in } Q(T),$$

where κ_0, M_0 are the same constants as in Section 3.

The proof of this lemma follows the lines of the proof of a priori estimates for u given in Section 3.

Now, let $S_2 : W_R \to L^2(0,T;H)$ be given by the rule

$$S_2\tilde{w} = u, \quad \tilde{w} \in W_R,$$

where u is the solution of (4.8)–(4.10) for $\tilde{w} \in W_R$.

Lemma 4.5. *The operator S_2 is continuous.*

Proof. Let $\tilde{w}_i, \tilde{w} \in W_R$, $i \ge 1$, $\tilde{w}_i \to \tilde{w}$ in $L^2(0,T;H)$ as $i \to \infty$, and $u_i := S_2\tilde{w}_i$, $i \ge 1$, $u := S_2\tilde{w}$. Then, from Eq. (4.8) it follows that

$$\rho(u_i - u)_t - \Delta(G_m(u_i) - G_m(u)) = h(u_i, \tilde{w}_i) - h(u, \tilde{w}) \quad \text{in } Q(T).$$

Multiplying this equality by $\mathrm{sign}_\delta(G_m(u_i) - G_m(u))$, in view of the Lipschitz continuity of h we obtain

$$\begin{aligned}&\rho \int_\Omega (u_{it}(t) - u_t(t))\,\mathrm{sign}_\delta(G_m(u_i)(t) - G_m(u)(t))\,dx\\&\le \int_\Omega (|u_i(t) - u(t)| + |\tilde{w}_i(t) - \tilde{w}(t)|)\,dx \quad \text{for a.e. } t \in [0,T].\end{aligned}$$

Letting $\delta \to 0$ we conclude as in Lemma 4.4 that

$$\rho \frac{d}{dt}|u_i(t) - u(t)|_{L^1(\Omega)} \le L\left(|u_i(t) - u(t)|_{L^1(\Omega)} + |\tilde{w}_i(t) - \tilde{w}(t)|_{L^1(\Omega)}\right)$$

for a.e. $t \in [0,T]$. The Gronwall argument then implies that

$$u_i \to u \quad \text{in } L^1(0,T;L^1(\Omega)) \quad \text{as } i \to \infty.$$

The claim of lemma finally follows from the inequality

$$|u_i - u|_{L^2(0,T;H)} \leq \sqrt{2M_0}|u_i - u|^{1/2}_{L^1(0,T;L^1(\Omega))}.$$

□

4.3 Step 3: Fixed point argument

Consider the superposition of the operators S_1 and S_2 defined in two previous steps. Lemmas 4.1, 4.2 and 4.5 imply that this superposition $S := S_2 \circ S_1 : \mathcal{K}_C \to L^2(0,T;H)$ is continuous in $L^2(0,T;H)$. Since the set $\mathcal{K}_C$ is non-empty, convex and compact in $L^2(0,T;H)$, in order to apply the Schauder fixed point theorem to establish the existence of a solution to problem $(P)_{\lambda,m}$ it remains to show that S takes its values in the set $\mathcal{K}_C$ for an appropriate choice of the constant C. The following lemma serves this purpose.

Lemma 4.6. *If u is a solution to problem (4.8)–(4.10) for a given $\tilde{w} \in W_R$, then*

$$|u|_{L^\infty(0,T;V)} + |u|_{L^2(0,T;H^2(\Omega))} \leq M, \tag{4.16}$$

for some M independent of $\tilde{w}$.

Proof. Testing (4.8) by $G_m(u)_t$ we obtain

$$\begin{aligned}\rho(u_t(t), G_m(u(t))_t)_H &+ (-\Delta G_m(u(t)), G_m(u(t))_t)_H \\ &= (h(u(t),\tilde{w}(t)), G_m(u(t))_t)_H \quad \text{for a.e. } t \in [0,T].\end{aligned}$$

Then, [14, Proposition 3.2] implies that

$$\begin{aligned}\frac{d}{dt}\varphi^t(G_m(u(t))) &- (h(u(t),\tilde{w}(t)) - \rho u_t(t), G_m(u(t))_t)_H \\ &\leq \delta|h(u(t),\tilde{w}(t)) - \rho u_t(t)|^2_H \\ &+ \Big(\frac{1}{2\delta}|a'(t)|^2 + |a'(t)|\Big)(1 + \varphi^t(G_m(u(t))))\end{aligned} \tag{4.17}$$

for a.e. $t \in [0,T]$ for some $\delta > 0$, where φ^t is defined by (4.11) and $a(t) = \text{const} \cdot \int_0^t |G_m(u_b(\tau))_\tau|_H d\tau$ (cf. [12, Proposition 3.2.2]). From (4.17) we further obtain

$$\begin{aligned}\Big(\frac{\rho g_0}{2} - 2\delta\rho^2\Big)&|u_t(t)|^2_H + \frac{d}{dt}\varphi^t(G_m(u(t))) \\ &\leq M_1 + 2\delta M_2 + \Big(\frac{1}{2\delta}|a'(t)|^2 + |a'(t)|\Big)(1 + \varphi^t(G_m(u(t))))\end{aligned} \tag{4.18}$$

for a.e. $t \in [0,T]$, where $M_1 = (|h|_{L^\infty(\mathbb{R}^2)} g_* |\Omega|^{1/2})^2/(2\rho g_0)$ with $g_* = \sup_{\kappa_0 \le r \le M_0} g_m(r)$, $M_2 = |h|^2_{L^\infty(\mathbb{R}^2)} |\Omega|$. Taking δ such that $\rho g_0/2 - 2\delta\rho^2 > 0$ and applying Gronwall's lemma to (4.18) we have

$$|u|_{W^{1,2}(0,T;H)} + |u|_{L^\infty(0,T;V)} \le M_3, \tag{4.19}$$

for some constant $M_3 > 0$.

Finally, proceeding exactly as in [1, Lemma 7] we obtain the assertion of lemma. $\square$

Now, setting $C = M + M_3$, where M and M_3 are the constants from (4.16) and (4.19), respectively, we see that the operator S acts from $\mathcal{K}_C$ into $\mathcal{K}_C$. Therefore, from the Schauder fixed point theorem we conclude that S has a fixed point, which we denote by $u_{\lambda,m} \in \mathcal{K}_C$, $\lambda > 0$, $m > 0$. This, in particular, means that $\{u_{\lambda,m}, w_{\lambda,m}\}$, where $w_{\lambda,m} := S_1 u_{\lambda,m}$, is a solution to problem $(P)_{\lambda,m}$, $\lambda > 0$, $m > 0$.

5 Existence for problem (P)

Reasoning as in Lemmas 4.1, 4.4 we see that for a solution $\{u_{\lambda,m}, w_{\lambda,m}\}$ to problem $(P)_{\lambda,m}$, $\lambda > 0$, $m > 0$ obtained at the end of previous section the following estimates hold

$$\kappa_0 \le u_{\lambda,m} \le M_0, \qquad |w_{\lambda,m}| \le R \quad \text{a.e. on } Q(T).$$

Therefore, fixing m such that $\frac{1}{m} \le \kappa_0$ and $M_0 \le m$, we deduce that $\{u_{\lambda,m}, w_{\lambda,m}\}$ is also a solution to the problem consisting of Eqs. (4.1), (4.2)–(4.4) and denoted by $(P)_\lambda$, $\lambda \in (0,1]$. Below, we derive a priori uniform estimates independent of λ for solutions $\{u_\lambda, w_\lambda\}$ to $(P)_\lambda$, $\lambda \in (0,1]$ and use them to establish the existence of a solution to problem (P) through the passage-to-the-limit procedure.

Lemma 5.1. *The set $\{u_\lambda\}_{\lambda\in(0,1]}$ is bounded in $W^{1,2}(0,T;H)$, $L^\infty(0,T;V)$ and $L^2(0,T;H^2(\Omega))$. In addition, $\{G_m(u_\lambda)\}_{\lambda\in(0,1]}$ is bounded in $L^2(0,T;H^2(\Omega))$.*

Proof. First, as shown above, we have

$$\kappa_0 \le u_\lambda \le M_0 \quad \text{a.e. on } Q(T),\ \lambda \in (0,1].$$

Next, arguing as in Lemma 4.6 we can prove that $\{u_\lambda\}_{\lambda\in(0,1]}$ is bounded in $W^{1,2}(0,T;H)$ and $L^\infty(0,T;V)$. Then, since $\rho u_{\lambda t} - \Delta G_m(u_\lambda) = h(u_\lambda, w_\lambda)$ on $Q(T)$, we see that $\{\Delta G_m(u_\lambda)\}_{\lambda\in(0,1]}$ is bounded in $L^2(0,T;H)$. Hence, [13, Lemma 3.7.1] implies that $\{G_m(u_\lambda)\}_{\lambda\in(0,1]}$ is bounded in $L^2(0,T;H^2(\Omega))$. Finally, from the equality

$$\Delta G_m(u_\lambda) = g_m(u_\lambda)\Delta u_\lambda + g_m'(u_\lambda)|\nabla u_\lambda|^2$$

and the fact that $g_m \ge g_0$ we infer that $\{\Delta u_\lambda\}_{\lambda\in(0,1]}$ is bounded in $L^2(0,T;H)$ and thus $\{u_\lambda\}_{\lambda\in(0,1]}$ is bounded in $L^2(0,T;H^2(\Omega))$. $\square$

Lemma 5.2. *The set* $\{w_\lambda\}_{\lambda\in(0,1]}$ *is bounded in* $W^{1,2}(0,T;H)$, $\{\partial I_\lambda(u_\lambda;w_\lambda)\}_{\lambda\in(0,1]}$ *is bounded in* $L^2(0,T;H)$, *and* $\{I_\lambda(u_\lambda;w_\lambda)\}_{\lambda\in(0,1]}$ *is bounded in* $L^\infty(0,T)$.

Proof. First, we see again that

$$|w_\lambda| \le R \quad \text{a.e. on } Q(T),\ \lambda\in(0,1].$$

Next, let $\lambda\in(0,1]$. Testing (4.2) by $w_{\lambda t}$ we obtain

$$|w_{\lambda t}|_H^2 + (\partial I_\lambda(u_\lambda;w_\lambda), w_{\lambda t})_H \le |F(u_\lambda,w_\lambda)|_H |w_{\lambda t}|_H \quad \text{a.e. on } [0,T].$$

From Lemma 2.1 it then follows that

$$|w_{\lambda t}|_H^2 + \frac{d}{dt} I_\lambda(u_\lambda;w_\lambda) \le C_0 |\partial I_\lambda(u_\lambda;w_\lambda)|_H |u_{\lambda t}|_H + |F(u_\lambda,w_\lambda)|_H |w_{\lambda t}|_H$$

a.e. on $[0,T]$. In view of Eq. (4.2) using Young's inequality we further obtain

$$|w_{\lambda t}|_H^2 + 2\frac{d}{dt} I_\lambda(u_\lambda;w_\lambda) \le C_1\left(1+|u_{\lambda t}|_H^2\right) \quad \text{a.e. on } [0,T]$$

for $C_1 = \max\{2(C_0+C_0^2), (\frac{C_0}{2}+2)|F|^2_{L^\infty(\mathbb{R}^2)}|\Omega|\}$. Integrating this inequality and taking account of the facts that $I_\lambda(u_\lambda;w_\lambda)\ge 0$, $I_\lambda(u_0;w_0)=0$ (cf. $(A6)$) from Lemma 5.1 we infer that $\{w_\lambda\}_{\lambda\in(0,1]}$ is bounded in $W^{1,2}(0,T;H)$

Now, testing (4.2) by $\partial I_\lambda(u_\lambda;w_\lambda)$ and using Lemma 2.1 again we have

$$\begin{aligned}\frac{d}{dt} I_\lambda(u_\lambda;w_\lambda) + |\partial I_\lambda(u_\lambda;w_\lambda)|_H^2 &\le |F(u_\lambda,w_\lambda)|_H |\partial I_\lambda(u_\lambda;w_\lambda)|_H \\ &\quad + C_0 |\partial I_\lambda(u_\lambda;w_\lambda)|_H |u_{\lambda t}|_H \quad \text{a.e. on } [0,T].\end{aligned}$$

Applying Young's inequality to the terms on the right-hand side of this inequality we deduce that

$$\frac{d}{dt} I_\lambda(u_\lambda;w_\lambda) + \frac{1}{2}|\partial I_\lambda(u_\lambda;w_\lambda)|_H^2 \le C_2\left(1+|u_{\lambda t}|_H^2\right) \quad \text{a.e. on } [0,T]$$

for some $C_2>0$ depending on C_0, $|F|^2_{L^\infty(\mathbb{R}^2)}$ and $|\Omega|$. Then, integrating the last inequality we see as above that $\{\partial I_\lambda(u_\lambda;w_\lambda)\}_{\lambda\in(0,1]}$ is bounded in $L^2(0,T;H)$ and $\{I_\lambda(u_\lambda;w_\lambda)\}_{\lambda\in(0,1]}$ is bounded in $L^\infty(0,T)$. □

On account of Lemmas 5.1 and 5.2, by weak and weak-star compactness results, there exists a null sequence λ_j, $j\ge 1$, in $(0,1]$ and functions $u,w\in W^{1,2}(0,T;H)$, $\xi\in L^2(0,T;H)$ such that

$$\begin{aligned} u_j := u_{\lambda_j} \to u \quad &\text{weakly in } W^{1,2}(0,T;H)\cap L^2(0,T;H^2(\Omega)), \\ &\text{weakly-star in } L^\infty(0,T;V) \text{ and in } C([0,T];H), \end{aligned} \tag{5.1}$$

$$w_j := w_{\lambda_j} \to w \quad \text{weakly in } W^{1,2}(0,T;H), \tag{5.2}$$

$$\partial I_{\lambda_j}(u_j;w_j) \to \xi \quad \text{weakly in } L^2(0,T;H). \tag{5.3}$$

We will show below that also

$$w_j \to w \quad \text{in } C([0,T];H). \tag{5.4}$$

In fact, from (4.2) for any $i, j \geq 1$, $i \neq j$, we have

$$w_{jt} - w_{it} + \partial I_{\lambda_j}(u_j; w_j) - \partial I_{\lambda_i}(u_i; w_i) = F(u_j, w_j) - F(u_i, w_i).$$

Testing this equality by $w_j - w_i$, using the Lipschitz continuity of F and Young's inequality we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|w_j - w_i|_H^2 + (\partial I_{\lambda_j}(u_j; w_j) - \partial I_{\lambda_i}(u_i; w_i), w_j - w_i)_H \\ &\qquad \leq 2L(|w_j - w_i|_H^2 + |u_j - u_i|_H^2). \end{aligned} \tag{5.5}$$

Set

$$S_{ij} = (\partial I_{\lambda_j}(u_j; w_j) - \partial I_{\lambda_i}(u_i; w_i), w_j - w_i)_H. \tag{5.6}$$

Then, from (2.2) it follows that

$$\begin{aligned} S_{ij} = \Bigg(&\frac{1}{\lambda_j}[w_j - f^*(u_j)]^+ - \frac{1}{\lambda_j}[f_*(u_j) - w_j]^+ \\ &- \frac{1}{\lambda_i}[w_i - f^*(u_i)]^+ + \frac{1}{\lambda_i}[f_*(u_i) - w_i]^+, w_j - w_i \Bigg)_H. \end{aligned}$$

Next, we estimate the value of S_{ij} from below. There are nine possible cases to consider. First, we assume that $w_j \geq f^*(u_j)$, $w_i \geq f^*(u_i)$. Then, we have

$$\begin{aligned} S_{ij} = \Bigg(&\frac{1}{\lambda_j}(w_j - f^*(u_j)) - \frac{1}{\lambda_i}(w_i - f^*(u_i)), \\ &\lambda_j \frac{1}{\lambda_j}(w_j - f^*(u_j)) - \lambda_i \frac{1}{\lambda_i}(w_i - f^*(u_i)) + f^*(u_j) - f^*(u_i) \Bigg)_H \\ \geq\ & \lambda_j |\partial I_{\lambda_j}(u_j; w_j)|_H^2 + \lambda_i |\partial I_{\lambda_i}(u_i; w_i)|_H^2 - (\lambda_j + \lambda_i)|\partial I_{\lambda_j}(u_j; w_j)|_H |\partial I_{\lambda_i}(u_i; w_i)|_H \\ &- (|\partial I_{\lambda_j}(u_j; w_j)|_H + |\partial I_{\lambda_i}(u_i; w_i)|_H)|f^*(u_j) - f^*(u_i)|_H. \end{aligned}$$

In the case when $w_j \geq f^*(u_j)$, $f_*(u_i) \leq w_i < f^*(u_i)$ we have

$$S_{ij} = \left(\frac{1}{\lambda_j}(w_j - f^*(u_j)), w_j - w_i\right)_H \geq -|\partial I_{\lambda_j}(u_j; w_j)|_H |f^*(u_j) - f^*(u_i)|_H.$$

If $w_j \geq f^*(u_j)$, $w_i < f_*(u_i)$, then

$$\begin{aligned} S_{ij} &= \left(\frac{1}{\lambda_j}(w_j - f^*(u_j)) + \frac{1}{\lambda_i}(f_*(u_i) - w_i), w_j - w_i\right)_H \\ &\geq -\left(|\partial I_{\lambda_j}(u_j; w_j)|_H + |\partial I_{\lambda_i}(u_i; w_i)|_H\right)|f^*(u_j) - f^*(u_i)|_H. \end{aligned}$$

The remaining cases:

$$\begin{aligned}
f_*(u_j) < w_j < f^*(u_j),\ & w_i \geq f^*(u_i)\\
f_*(u_j) < w_j < f^*(u_j),\ & f_*(u_i) \leq w_i < f^*(u_i)\\
f_*(u_j) < w_j < f^*(u_j),\ & w_i < f_*(u_i)\\
w_j \leq f_*(u_j),\ & w_i \geq f^*(u_i)\\
w_j \leq f_*(u_j),\ & f_*(u_i) \leq w_i < f^*(u_i)\\
w_j \leq f_*(u_j),\ & w_i < f_*(u_i)
\end{aligned}$$

are fully symmetric and are thus treated likewise. Therefore, in all possible cases we see that

$$\begin{aligned}
S_{ij} &\geq -(\lambda_j + \lambda_i)|\partial I_{\lambda_j}(u_j; w_j)|_H |\partial I_{\lambda_i}(u_i; w_i)|_H\\
&\quad - (|\partial I_{\lambda_j}(u_j; w_j)|_H + |\partial I_{\lambda_i}(u_i; w_i)|_H)\,(|f^*(u_j) - f^*(u_i)|_H + |f_*(u_j) - f_*(u_i)|_H)\\
&\quad =: \delta_{ij}.
\end{aligned}$$

Consequently, from (5.5), (5.6) we infer that

$$|w_j - w_i|_H^2(t) \leq 4L \int_0^t |w_j - w_i|_H^2(\tau)\, d\tau + 4L \int_0^t |u_j - u_i|_H^2(\tau)\, d\tau + 2 \int_0^t \delta_{ij}(\tau)\, d\tau.$$

Since $u_j \to u$ in $C([0,T]; H)$, invoking the Gronwall argument and Lemma 5.2 we conclude from the last inequality that w_j, $j \geq 1$, is a Cauchy sequence in the space $C([0,T]; H)$. Hence, according to (5.2) we obtain the convergence (5.4).

Now, from the monotonicity of $\partial\varphi^t$ and Eq. (4.1) via the representation (4.11) it follows that for any $v \in L^2(0,T;V)$ with $v = G_m(u_b)$ a.e. on $(0,T) \times \partial\Omega$ we have

$$\int_0^T (h(u_j, w_j) - \rho u_{jt}, v - G_m(u_j))_H dt \leq \int_0^T \varphi^t(v)\, dt - \int_0^T \varphi^t(G_m(u_j))\, dt. \quad (5.7)$$

Let $z \in L^2(0,T;H)$ and $z_k \in C_0^\infty(Q(T))$, $k \geq 1$ be such that $z_k \to z$ in $L^2(0,T;H)$ as $k \to \infty$. Then, we have

$$\begin{aligned}
\int_0^T (\nabla G_m(u_j) - \nabla G_m(u), z)_H dt &= \int_0^T (\nabla G_m(u_j) - \nabla G_m(u), z - z_k)_H dt\\
&+ \int_0^T (g_m(u_j) - g_m(u), z_k \nabla u_j)_H dt + \int_0^T (\nabla u_j - \nabla u, g_m(u) z_k)_H dt. \quad (5.8)
\end{aligned}$$

We note that $G_m(u_j)$, $G_m(u)$ are bounded in $L^\infty(0,T;V)$ and $z_k \nabla u_j$, $g_m(u) z_k$ are bounded in $L^2(0,T;H)$. Letting $j \to \infty$ and fixing a suitable number k we infer from (5.1) and (5.8) that $\nabla G_m(u_j) \to \nabla G_m(u)$ weakly in $L^2(0,T;H)$. Hence,

$$\frac{1}{2} \int_0^T |\nabla G_m(u)|_H^2 dt \leq \liminf_{j \to \infty} \frac{1}{2} \int_0^T |\nabla G_m(u_j)|_H^2 dt.$$

Therefore, passing to the limit as $j \to \infty$ in (5.7) we conclude in view of (5.1), (5.4) that $h(u,w) - \rho u_t \in \partial\varphi^t(G_m(u))$ a.e. on $[0,T]$, which in turn implies (1.1) as well as the initial and boundary conditions for u.

Finally, given the convergences (5.1)–(5.4) we see that the pair $\{u,w\}$ is a solution to problem (P) provided

$$\xi \in \partial I(u;w) \quad \text{a.e. on } [0,T]. \tag{5.9}$$

In order to establish this latter fact, let z be an arbitrary function from $L^2(0,T;H)$ such that $f_*(u) \le z \le f^*(u)$ a.e. on $Q(T)$ and for each $j \ge 1$ let

$$z_j := \max\{\min\{z, f^*(u_j)\}, f_*(u_j)\} \quad \text{on } Q(T).$$

Clearly, $f_*(u_j) \le z_j \le f^*(u_j)$ on $Q(T)$, $j \ge 1$, and $z_j \to z$ in $L^2(0,T;H)$ as $j \to \infty$. Hence,

$$\int_0^T (\partial I_{\lambda_j}(u_j;w_j), z_j - w_j)_H dt \le \int_0^T (I_{\lambda_j}(u_j;z_j) - I_{\lambda_j}(u_j;w_j))dt = 0, \tag{5.10}$$

$j \ge 1$. On the other hand, from (2.2) we see that

$$[w_j - f^*(u_j)]^+ - [f_*(u_j) - w_j]^+ = \lambda_j \partial I_{\lambda_j}(u_j;w_j) \to 0$$

in $L^2(0,T;H)$ as $j \to \infty$. Consequently, we have $f_*(u) \le w \le f^*(u)$ a.e. on $Q(T)$. Passing now to the limit as $j \to \infty$ in (5.10) we conclude that (5.9) holds and $\{u,w\}$ is thus a solution to problem (P).

6 Uniqueness for problem (P)

In this section, we consider the uniqueness for problem (P) in case $N = 1$. Then, (P) is given by

$$\rho u_t - (g(u)u_x)_x = h(u,w) \quad \text{in } Q(T), \tag{6.1}$$

$$w_t + \partial I(u;w) \ni F(u,w) \quad \text{in } Q(T), \tag{6.2}$$

$$u(\cdot,0) = b_0, u(\cdot,1) = b_1 \quad \text{on } (0,T), \tag{6.3}$$

$$u(0) = u_0, \quad w(0) = w_0 \quad \text{on} (0,1), \tag{6.4}$$

where b_0 and b_1 are given functions on $[0,T]$.
Here, we assume the following conditions for F and b_i, $i = 0,1$:

(A3-1) For any $u \in \mathbb{R}$ the function $F(u,w)$ is nonincreasing with respect to $w \in \mathbb{R}$.

(A5-1) For $i = 0,1$ we have $b_i \in W^{1,2}(0,T)$ and $b_i \ge \kappa_0$ on $[0,T]$.

The next theorem guarantees the uniqueness of a solution of (P).

Theorem 6.1. *Under* (A1), (A2), (A3), (A3-1), (A4), (A5-1), (A6) *let* $\{u_1,w_1\}$ *and* $\{u_2,w_2\}$ *be solutions of* (P). *If* $\delta \le u_i \le M$ *on* $Q(T)$ *for* $i = 1,2$, *where* δ *and* M *are positive constants, then* $u_1 = u_2$ *and* $w_1 = w_2$ *on* $Q(T)$.

For the proof of Theorem 6.1 we prepare the following two lemmas.

Lemma 6.1. *Under the same assumptions as in Theorem* 6.1, *there exist positive constants* C_1 *and* $0 < T_1 \le T$ *such that*

$$|w_1(t) - w_2(t)|_{L^\infty(0,1)} \le C_1 |u_1 - u_2|_{L^\infty(0,t_1;L^\infty(0,1))} \text{ for } 0 \le t \le t_1 \le T_1. \tag{6.5}$$

Proof. We put $u = u_1 - u_2$ and $w = w_1 - w_2$. By Definition 2.1 for $i = 1, 2$ there exists $\xi_i \in L^2(0, T; H)$ such that

$$\xi_i(t) \in \partial I(u_i(t), w_i(t)) \text{ for a.e. } t \in [0, T],$$

$$w_{it} + \xi_i = F(u_i, w_i)(=: F_i) \text{ a.e. on } Q(T).$$

Let $t_1 \in (0, T]$ be fixed, and $\ell(t_1) = |u|_{L^\infty(0,t_1;L^\infty(0,1))}$

$$v_1(t) = w_1(t) - [w(t) - C(t+1)\ell(t_1)]^+ \quad \text{for } 0 \le t \le t_1,$$

where $C \ge C_*$ is a positive constant, $C_* = \max\{C_1, C_2\}$, and C_1 and C_2 are the Lipschitz constants of f_* and f^*, respectively. Then it holds that

$$f_*(u_1) \le v_1 \le f^*(u_1) \quad \text{a.e. on } Q(t_1). \tag{6.6}$$

In fact, it is clear that $v_1 \le f^*(u_1)$ a.e. on $Q(t_1)$. If $w \le C(t+1)\ell(t_1)$, then $v_1 \ge f_*(u_1)$. Otherwise, we observe that

$$\begin{aligned} v_1 &= w_2 + C(t+1)\ell(t_1) \\ &\ge f_*(u_2) + C(t+1)\ell(t_1) \\ &\ge -C_*\ell(t_1) + f_*(u_1) + C(t+1)\ell(t_1) \\ &\ge f_*(u_1). \end{aligned}$$

Thus we get (6.6). Similarly, we put

$$v_2(t) = w_2(t) + [w(t) - C(t+1)\ell(t_1)]^+ \quad \text{for } 0 \le t \le t_1,$$

and obtain

$$f_*(u_2) \le v_2 \le f^*(u_2) \quad \text{a.e. on } Q(t_1).$$

Immediately, we see that

$$\xi_i(w_i - v_i) \ge 0 \quad \text{a.e. on } Q(t_1) \text{ for } i = 1, 2.$$

Accordingly, we have

$$\left.\begin{aligned} w_{1t}[w - C(t+1)\ell(t_1)]^+ &\le F_1[w - C(t+1)\ell(t_1)]^+, \\ -w_{2t}[w - C(t+1)\ell(t_1)]^+ &\le -F_2[w - C(t+1)\ell(t_1)]^+ \end{aligned}\right\} \text{ a.e. on } Q(t_1),$$

so that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|[w-C(t+1)\ell(t_1)]^+|^2\\ \le&(F_1-F_2-C\ell(t_1))[w-C(t+1)\ell(t_1)]^+\\ \le&(L|u|+F(u_1,w_1)-F(u_1,w_2)-C\ell(t_1))[w-C(t+1)\ell(t_1)]^+ \text{ a.e. on } Q(t_1),\end{aligned}$$

where recall that L is the Lipschitz constant of F. Here, we take C with $C \ge \max\{L, C_*\}$. Since F is a nonincreasing function with respect to w, we have

$$(F(u_1,w_1)-F(u_1,w_2))[w-C(t+1)\ell(t_1)]^+ \le 0,$$

so that

$$\frac{1}{2}\frac{d}{dt}|[w-C(t+1)\ell(t_1)]^+|^2 \le 0 \text{ a.e. on } Q(t_1). \tag{6.7}$$

Then, (6.5) is a direct consequence of (6.7). □

Lemma 6.2. *For $i = 1,2$, $u_{ix} \in L^4(Q(T)) \cap L^2(0,T;L^\infty(0,1))$.*

Proof. For $i = 1,2$ by the Gagliardo-Nirenberg inequality there exists a positive constant M_* such that

$$|u_{ix}(t)|_{L^4(0,1)} \le M_*(|u_{ixx}(t)|_H^{1/4}|u_{ix}(t)|_H^{3/4} + |u_{ix}(t)|_H) \text{ for a.e. } t \in [0,T].$$

Since $u_i \in L^\infty(0,T;H^1(0,1))$, it is easy to see that

$$\begin{aligned}&\int_0^T |u_{ix}|^4_{L^4(0,1)}dt\\ \le&M_*^4\int_0^T(|u_{ixx}|_H|u_{ix}|_H^3+|u_{ix}|_H^4)dt\\ \le&M_*^4(|u_{ix}|^3_{L^\infty(0,T;H)}\int_0^T|u_{ixx}|_Hdt+|u_{ix}|^4_{L^\infty(0,T;H)}T),\end{aligned}$$

and $u_{ix} \in L^4(Q(T))$.
It is clear that $|u_{ix}(t)|^2_{L^\infty(0,1)} \le 2(|u_{ixx}(t)|_H^2 + |u_{ix}(t)|_H^2)$ for $0 \le t \le T$. This implies that $u_{ix} \in L^2(0,T;L^\infty(0,1))$. □

In the rest of this section, we prove Theorem 6.1.

Proof of Theorem 6.1. We put $u = u_1 - u_2$, $w = w_1 - w_2$, $h_1 = h(u_1,w_1)$ and $h_2 = h(u_2,w_2)$. Easily, we get

$$\rho u_t - (g(u_1)u_{1x} - g(u_2)u_{2x})_x = h_1 - h_2 \quad \text{on } Q(T). \tag{6.8}$$

We multiply (6.8) by $-u_{xx}$ and integrate it. Then, we observe that

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}\int_0^1|u_x|^2dx+\int_0^1(g(u_1)u_{1x}-g(u_2)u_{2x})_xu_{xx}dx\\ =&-\int_0^1(h_1-h_2)u_{xx}dx \text{ a.e. on } [0,T],\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^1 (g(u_1)u_{1x} - g(u_2)u_{2x})_x u_{xx} dx \\
=&\int_0^1 g(u_1)|u_{xx}|^2 dx + \int_0^1 (g(u_1) - g(u_2))u_{2xx}u_{xx} dx \\
&+ \int_0^1 g'(u_1)(|u_{1x}|^2 - |u_{2x}|^2)u_{xx} dx + \int_0^1 (g'(u_1) - g'(u_2))|u_{2x}|^2 u_{xx} dx \\
=:&I_1 + I_2 + I_3 + I_4 \text{ a.e. on } [0,T].
\end{aligned}
$$

First, by (A2) we note that

$$
I_1 \geq g_0 \int_0^1 |u_{xx}|^2 dx \text{ a.e. on } [0,T].
$$

Here, since $\delta \leq u_i \leq M$ on $Q(T)$, there exists a positive constant C_g such that $|g'(u_i)| \leq C_g$, $|g(u_1) - g(u_2)| \leq C_g|u|$, $|g'(u_1) - g'(u_2)| \leq C_g|u|$ a.e. on $[0,T]$ for $i = 1,2$. By elementary calculations, for instance, $|u|_{L^\infty(0,1)} \leq |u_x|_H$, we observe that

$$
\begin{aligned}
|I_2| \leq& C_g \int_0^1 |u||u_{2xx}||u_{xx}| dx \\
\leq& \frac{g_0}{8}|u_{xx}|_H^2 + \frac{2C_g^2}{g_0}|u_x|_H^2 |u_{2xx}|_H^2,
\end{aligned}
$$

$$
\begin{aligned}
|I_3| \leq& C_g \int_0^1 (|u_{1x}| + |u_{2x}|)|u_x||u_{xx}| dx \\
\leq& \frac{g_0}{8}|u_{xx}|_H^2 + \frac{4C_g^2}{g_0}(|u_{1x}|_{L^\infty(0,1)}^2 + |u_{2x}|_{L^\infty(0,1)}^2)|u_x|_H^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_4| \leq& C_g \int_0^1 |u||u_{2x}|^2 |u_{xx}| dx \\
\leq& \frac{g_0}{8}|u_{xx}|_H^2 + \frac{2C_g^2}{g_0}|u_{2x}|_{L^4(0,1)}^4 |u_x|_H^2 \text{ a.e. on } [0,T].
\end{aligned}
$$

Next, we obtain

$$
\begin{aligned}
&-\int_0^1 (h_1 - h_2)u_{xx} dx \\
\leq& L\int_0^1 (|u| + |w|)|u_{xx}| dx \\
\leq& \frac{g_0}{4}|u_{xx}|_H^2 + \frac{2L^2}{g_0}(|u_x|_H^2 + |w|_H^2) \text{ a.e. on } [0,T],
\end{aligned}
$$

where recall that L is the Lipschitz constant of h. By adding these inequalities we have

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}|u_x(t)|_H^2 + \frac{g_0}{4}|u_{xx}(t)|_H^2 \\ \le &\frac{\rho}{2}E(t)|u_x(t)|_H^2 + \frac{\rho}{2}C_2|w(t)|_H^2 \text{ for a.e. } t \in [0,T],\end{aligned}$$

where C_2 is a positive constant and

$$\begin{aligned}E(t) =&\frac{2}{\rho}\left(\frac{4C_g^2}{g_0}\left(|u_{1x}(t)|^2_{L^\infty(0,1)} + |u_{1x}(t)|^2_{L^\infty(0,1)}\right)\right. \\ &\left. + \frac{2C_g^2}{g_0}|u_{2x}(t)|^4_{L^4(0,1)} + \frac{2C_g^2}{g_0}|u_{2xx}(t)|_H^2 + \frac{2L^2}{g_0}\right) \text{ for } t \in [0,T].\end{aligned}$$

Then, Lemma 6.2 guarantees that $E \in L^1(0,T)$.
Thanks to the Gronwall inequality, we infer that

$$|u_x(t)|_H^2 \le C_2 \exp\left(\int_0^t E(\tau)d\tau\right)\int_0^t |w(\tau)|_H^2 d\tau \text{ for } t \in [0,T]. \tag{6.9}$$

Moreover, on account of Lemma 6.1 there exist positive constants C_1 and T_1 satisfying (6.5). Next, we substitute (6.5) into (6.9) and see that

$$|u_x(t)|_H^2 \le C_1C_3\int_0^{t_1} |u|^2_{L^\infty(0,t_1;L^\infty(0,1))}d\tau \text{ for } 0 \le t_1 \le T_1,$$

where $C_3 = C_2\exp(\int_0^T E(\tau)d\tau)$, and

$$|u(t)|^2_{L^\infty(0,1)} \le C_1C_3t_1|u|^2_{L^\infty(0,t_1;L^\infty(0,1))} \text{ for } 0 \le t_1 \le T_1.$$

Hence, for t_1 with $C_1C_3t_1 < 1$ it holds that $|u|_{L^\infty(0,t_1;L^\infty(0,1))} = 0$. This shows that $u = 0$ and $w = 0$ on $Q(t_1)$. Since the choice of t_1 is independent of the initial values, we can say that $u = 0$ and $w = 0$ on $Q(T)$. Thus, we have proved the uniqueness. □

References

[1] T. Aiki and K. Kumazaki, *Well-posedness of a mathematical model for moisture transport appearing in concrete carbonation process*, Adv. Math. Sci. Appl. 21 (2011), no. 2, 361–381.

[2] T. Aiki and K. Kumazaki, *Mathematical model for hysteresis phenomenon in moisture transport of concrete carbonation process*, Phys. B 407 (2012), 1424–1426.

[3] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions.* Appl. Math. Sci., **121**, Springer-Verlag, New York, 1996.

[4] P. Krejčí, *Hysteresis, Convexity and Dissipation in Hyperbolic Equations*. Gakuto Int. Ser. Math. Sci. Appl., Vol. 8, Gakkōtosho, Tokyo, 1996.

[5] A. Visintin, *Differential Models of Hysteresis*. Appl. Math. Sci. 111, Springer-Verlag, Berlin, 1994.

[6] T. Aiki and K. Kumazaki, *Mathematical modelling of concrete carbonation process with hysteresis effect*, RIMS, Kyoto Univ., sūrikaisekikenkyūsho, kōkyūuroku, 1792 (2012), 99–107.

[7] N. Kenmochi, T. Koyama and G.H. Meyer, *Parabolic PDEs with hysteresis and quasivariational inequalities*. Nonlinear Anal. **34** (1998), 665–686.

[8] P. Colli, N. Kenmochi and M. Kubo, *A phase field model with temperature dependent constraint*. J. Math. Anal. Appl. **256** (2001), 668–685.

[9] N. Kenmochi, E. Minchev and T. Okazaki, *On a system of nonlinear PDE's with diffusion and hysteresis effects*. Adv. Math. Sci. Appl. **14** (2004), 633–664.

[10] K. Maekawa, T. Ishida and T. Kishi, *Multi-scale modeling of concrete performance*. J. Adv. Concr. Technol. **1** (2003), 91–126.

[11] A. Aiki and K. Kumazaki, *Uniqueness of solutions to a mathematical model describing moisture transport in concrete materials*. Netw. Heterog. Media **9** (2014), 683—707.

[12] N. Kenmochi, *Solvability of nonlinear evolution equations with time-dependent constraints and applications*, Bull. Fac. Educ., Chiba Univ. **30** (1981) 1–87.

[13] O.A. Ladyženskaja and N.N. Ural'ceva, *Équations aux dérivées partielles de type elliptique*. Dunod, Paris, 1968.

[14] Y. Yamada, *On evolution equations generated by subdifferential operators*, J. Fac. Sci. Univ. Tokyo, Sect. I A 23 (1976), 491–515.