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Geodesic mapping onto Kählerian space of the third kind ☆

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A R T I C L E   I N F O

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A B S T R A C T

In the present paper we study geodesic mappings between generalized Riemannian $\mathbb{GR}_N$ and generalized Kählerian spaces of the third type $\mathbb{G}\underset{3}{\mathbb{K}}_N$, and specially the case when these spaces have the same torsions at corresponding points. Using the non-symmetric metric tensor we find necessary and sufficient conditions for the existence of geodesic mapping $f : \mathbb{GR}_N \to \mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ with respect to the four kinds of covariant derivatives.

1. Introduction

In 1922 Cartan put forward a modification of General Relativity Theory (GRT), by relaxing the assumption that the affine connection has vanishing antisymmetric part (torsion tensor), and relating the torsion to the density of the intrinsic angular momentum. Also, the torsion is implicit in the 1928 Einstein theory of gravitation with teleparallelism. Afterwards, several mathematicians dealt with non-symmetric affine connection, for example, Eisenhart [2], Prvanović, Minčić [13–16] etc. Sinyukov [18] introduced the concept of almost geodesic mappings between affine connected spaces without torsion. Mikeš and coauthors [8–12] gave some significant contributions to the study of geodesic and almost geodesic mappings of affine connected, Riemannian and Einstein spaces.

The fundamental (0,2) tensor g_{ij} in a non-symmetric (generalized) Riemannian space is in general non-symmetric. It is decomposed in two parts, the symmetric part $g_{\underline{ij}}$ and the skewsymmetric part $g_{\underset{\vee}{ij}}$, where

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}) = \frac{1}{2}g_{(ij)}, \quad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}) = \frac{1}{2}g_{[ij]}. \tag{1.1}$$

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The Levi-Civita connection corresponding to the symmetric non-degenerate (0,2) tensor $g_{\underline{ij}}$ we denote by $\Gamma^i_{\underline{jk}}$. The lowering and the raising of indices is defined via the tensors $g_{\underline{ij}}$ and $g^{\underline{ij}}$ respectively, where $g^{\underline{ij}}$ is defined by the equation

$$g_{\underline{ij}} g^{\underline{jk}} = \delta_i^k \tag{1.2}$$

and δ_i^k is the Kronecker symbol. The Koszul formula reads

$$\Gamma_{i.\underline{jk}} = \frac{1}{2}(g_{\underline{ji},k} - g_{\underline{jk},i} + g_{\underline{ik},j}), \qquad g_{\underline{ij},k} = \frac{\partial g_{\underline{ij}}}{\partial x^k}. \tag{1.3}$$

We denote the (0,3) torsion tensor with respect to $g_{\underline{ij}}$ by the same letter, $T_{k.ij} := g_{\underline{kp}} T^p_{ij}$.

Based on non-symmetry of the connection in a generalized Riemannian space one can define four kinds of covariant derivatives. For example, for a tensor a^i_j in $\mathbb{GR}_N$ we have

$$\begin{aligned} a^i_{j\underset{1}{|}m} &= a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{jm} a^i_p, \\ a^i_{j\underset{2}{|}m} &= a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{mj} a^i_p, \\ a^i_{j\underset{3}{|}m} &= a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{mj} a^i_p, \\ a^i_{j\underset{4}{|}m} &= a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{jm} a^i_p. \end{aligned} \tag{1.4}$$

In the paper [6], a generalized Riemannian space is considered and the connection coefficients are explicitly given, namely

Theorem 1.1. *Let* $(\mathbb{GR}_N, g = g_{\underline{ij}} + g_{\underset{\vee}{ij}})$ *be a generalized Riemannian space and* $\Gamma^i_{\underline{jk}}$ *be the Levi-Civita connection of* $g_{\underline{ij}}$. *Let* Γ^i_{jk} *be a linear connection with torsion* T^i_{jk}. *Then* Γ^i_{jk} *is unique determined by the following formula*

$$\Gamma_{i.jk} = \Gamma_{i.\underline{jk}} + \frac{1}{4}\Big[T_{i.jk} + T_{k.ij} - T_{j.ki}\Big] - \frac{1}{2}\Big[g_{\underline{ik}\underset{1}{|}j} + g_{\underline{ij}\underset{1}{|}k} - g_{\underline{kj}\underset{1}{|}i}\Big]. \tag{1.5}$$

Many different types of generalized Riemannian spaces are found in literature, one of which is the generalized Riemannian space in the sense of Eisenhart's definition [2], i.e. it is a differentiable N-dimensional manifold, equipped with a non-symmetric basic tensor g_{ij} (i.e. $g_{ij} \neq g_{ji}$). The connection coefficients of this space are explicitly given by

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad \Gamma^i_{jk} = g^{\underline{ip}} \Gamma_{p.jk}. \tag{1.6}$$

Generally $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. Therefore, one can define the anti-symmetric part of Γ^i_{jk}

$$\Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) = \frac{1}{2}\Gamma^i_{[jk]} \tag{1.7}$$

The quantity $\Gamma^i_{\underset{\vee}{jk}} = T^i_{jk}$ is the *torsion tensor* of the space $\mathbb{GR}_N$.

In the rest of the paper we will consider a generalized Riemannian space in the sense of Eisenhart's definition.

It is easy to see that the equation (1.6) can be written in the form

$$\begin{aligned}\Gamma_{i.jk} &= \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j})\\ &= \frac{1}{2}(g_{\underline{ji},k} - g_{\underline{jk},i} + g_{\underline{ik},j}) + \frac{1}{2}(g_{\underset{\vee}{ji},k} - g_{\underset{\vee}{jk},i} + g_{\underset{\vee}{ik},j})\\ &= \Gamma_{i.\underline{jk}} + \frac{1}{2}dg_{ijk},\end{aligned}$$

where $\Gamma_{i.\underline{jk}}$ is the Levi-Civita connection corresponding to the symmetric tensor $g_{\underline{ij}}$ and dg_{ijk} is the exterior derivative of the anti-symmetric part $g_{\underset{\vee}{ij}}$. From the previous equation, we see that the symmetric part $g_{\underline{ij}}$ of g_{ij} is covariantly constant, and the torsion $T_{i.jk}$ is totally skew-symmetric determined by the equation $T_{i.jk} = \frac{1}{2}dg_{ijk}$.

Cappozziello–Lambiase–Stornaiolo classification for the torsion in $\mathbb{GR}_4$. A property of torsion is that it can be decomposed with respect to the Lorentz group into three irreducible tensors (for more details see [1]), i.e. it can be written as

$$T^i_{jk} = {}^tT^i_{jk} + {}^aT^i_{jk} + {}^vT^i_{jk}. \tag{1.8}$$

In the previous equation, we have

$${}^vT^i_{jk} = \frac{1}{3}(T_j\delta^i_k - T_k\delta^i_j) \quad ({}^vT_j = {}^vT^k_{jk}), \tag{1.9}$$

$${}^aT^i_{jk} = g^{\overline{ip}}T_{[p.jk]} = T^i_{jk}, \tag{1.10}$$

which is called the *axial torsion*, and

$${}^tT^i_{jk} = T^i_{jk} - {}^aT^i_{jk} - {}^vT^i_{jk}, \tag{1.11}$$

which is the *traceless* part of torsion. In this case, ${}^tT^i_{jk} = -{}^vT^i_{jk}$.

Among all forms of generalized Riemannian spaces, specially are interesting quarter-symmetric spaces and their particular form, the semi-symmetric spaces. Further considerations of these component and spaces we leave for future work.

1.1. Generalized Kählerian spaces of the third kind

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [20], M. Prvanović [17], J. Mikeš [5,8–12], Domašev, N. Pušić, T. Otsuki and Y. Tasiro, S.S. Pujar, N.S. Sinjukov, U.C. De [7] and many others.

An N-dimensional Riemannian space with the metric tensor g_{ij} is a *Kählerian space* $\mathbb{K}_N$ if there exists an almost complex structure F^i_j, such that

$$\begin{gathered} F^h_pF^p_i = -\delta^h_i\,,\\ g_{pq}F^p_iF^q_j = g_{ij}, \quad g^{ij} = g^{pq}F^i_pF^j_q,\\ F^h_{i;j} = 0, \end{gathered} \tag{1.12}$$

where $(;)$ denotes the covariant derivative with respect to the metric tensor g_{ij}.

In the papers [16,19] different kinds of Kählerian spaces with torsion are considered. Taking into account non-symmetry of the connection and there being defined four kinds of covariant derivatives of tensors, the almost complex structure can be covariantly constant with respect to for kinds of differentiation.

Kählerian spaces of the first type are characterized by the equation $F^i_{j\underset{1}{|}k} = 0$. In [19] we proved that Kählerian spaces of the first type are equivalent to the Kählerian spaces of the second type ($F^i_{j\underset{2}{|}k} = 0$). Now, we define a new class of Kählerian spaces, more precisely

Definition 1.1. A generalized N-dimensional Riemannian space with a non-symmetric metric tensor g_{ij}, is a **generalized Kählerian space of the third kind** $\underset{3}{\mathbb{GK}}_N$ if there exists an almost complex structure F^i_j, so that

$$F^h_p F^p_i = -\delta^h_i\,, \tag{1.13}$$

$$g_{\underline{pq}}\, F^p_i F^q_j = g_{\underline{ij}}, \quad g^{\underline{ij}} = g^{\underline{pq}} F^i_p F^j_q, \tag{1.14}$$

$$F^h_{i\underset{3}{|}j} = 0, \tag{1.15}$$

$$F^h_{i;j} = 0, \tag{1.16}$$

where $\underset{3}{|}$ denotes the covariant derivative of the third kind with respect to the connection Γ^i_{jk} ($\Gamma^i_{jk} \neq \Gamma^i_{kj}$) and ($;$) denotes the covariant derivative with respect to the symmetric part of the metric tensor $\Gamma^i_{\underline{jk}}$.

From (1.14), using (1.13), we get $F_{ij} = -F_{ji}$, $F^{ij} = -F^{ji}$, where we denote $F_{ji} = F^p_j g^{\underline{pi}}$, $F^{ji} = F^j_p g^{\underline{pi}}$.
From these considerations we immediately have the following theorem:

Theorem 1.2. *For the almost complex structure F^i_j of $\underset{3}{\mathbb{GK}}_N$ the relations*

$$F^h_{i\underset{1}{|}j} = 2F^h_p \Gamma^p_{\underset{\vee}{ji}}, \quad F^h_{i\underset{2}{|}j} = 2F^p_i \Gamma^h_{\underset{\vee}{jp}}, \quad F^h_{i\underset{4}{|}j} = 0, \tag{1.17}$$

are valid, where $\Gamma^h_{\underset{\vee}{ij}}$ is the torsion tensor.

2. Geodesic mapping

The study of the theory of geodesic mappings of Riemannian spaces, affine connection spaces and their generalizations has been an active field over the past several decades. Many new and interesting results appeared in the papers of N.S. Sinyukov [18], J. Mikeš [8], G.S. Hall [4,3], etc. The investigation of geodesic mappings for special spaces is an important and active research topic.

In this section we consider a geodesic mapping $f : \mathbb{GR}_N \to \underset{3}{\mathbb{G\overline{K}}}_N$.

Definition 2.1. A diffeomorphism $f : \mathbb{GR}_N \to \underset{3}{\mathbb{G\overline{K}}}_N$ is **geodesic**, if geodesics of the space $\mathbb{GR}_N$ are mapped to geodesics of the space $\underset{3}{\mathbb{G\overline{K}}}_N$.

At the corresponding points M and $\overline{M}$ we can put

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk}, \quad (i, j, k = 1, ..., N), \tag{2.1}$$

where P^i_{jk} is *the deformation tensor* of the connection Γ of $\mathbb{GR}_N$ according to the mapping $f : \mathbb{GR}_N \to \underset{3}{\mathbb{G\overline{K}}}_N$.

Theorem 2.1. [15] *A necessary and sufficient condition that the mapping* $f : \mathbb{GR}_N \to \mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ *to be geodesic is that the deformation tensor* P^i_{jk} *from* (2.1) *has the form*

$$P^i_{jk} = \delta^i_j\, \psi_k + \delta^i_k\, \psi_j + \xi^i_{jk}, \tag{2.2}$$

where

$$\psi_i = \frac{1}{N+1}(\overline{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha}), \quad \xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}). \tag{2.3}$$

We remark that in $\mathbb{G}\underset{3}{\mathbb{K}}_N$ the following equations are valid:

$$\Gamma^\alpha_{\underset{\vee}{i\alpha}} = 0, \quad \xi^\alpha_{i\alpha} = 0, \quad F^\alpha_\alpha = 0. \tag{2.4}$$

Let us construct the geodesic mapping $f : \mathbb{GR}_4 \to \mathbb{G}\underset{3}{\overline{\mathbb{K}}}_4$.

Example 2.1. Let the generalized Riemannian space $\mathbb{GR}_4$ be given by basic metric tensor $g_{ij} = g_{\underline{ij}} + g_{\underset{\vee}{ij}}$ where

$$(g_{\underline{ij}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{x^1}x^3 & e^{x^1}x^2 \\ 0 & 0 & e^{x^1}x^2 & e^{x^1}(x^2+x^3) \end{bmatrix} \quad (g_{\underset{\vee}{ij}}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & (x^1)^2 & (x^2)^2 \\ 0 & -(x^1)^2 & 0 & 1 \\ 0 & -(x^2)^2 & -1 & 0 \end{bmatrix} \tag{2.5}$$

Also, let

$$(\overline{g}_{\underline{ij}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^3 & x^2 \\ 0 & 0 & x^2 & 1 \end{bmatrix} \quad \text{and} \quad (\overline{F}^h_i) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{2.6}$$

are symmetric part of basic metric tensor and almost complex structure of the space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_4$. Then the conditions (1.13), (1.14) and (1.16) are satisfied. If $\overline{\Gamma}^h_{\underset{\vee}{ij}}$ we choose so that the conditions $\overline{\Gamma}^h_{\underset{\vee}{pj}}\overline{F}^p_i - \overline{\Gamma}^p_{\underset{\vee}{ji}}\overline{F}^h_p = 0$ are valid then the condition (1.15) is satisfied i.e. the space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_4$ is the fourth dimensional generalized Kählerian space of the third kind. If the mapping $f : \mathbb{GR}_4 \to \mathbb{G}\underset{3}{\overline{\mathbb{K}}}_4$ is geodesic, then

$$\begin{aligned} &\psi_k = \frac{1}{5}(\overline{\Gamma}^p_{kp} - \Gamma^p_{kp}), \quad (k = 1, \cdots, 4) \quad \text{where} \\ &\overline{\Gamma}^p_{kp} = \frac{\partial}{\partial x^k} \ln \sqrt{det(\overline{g}_{\underline{ij}})} = \frac{1}{2}\frac{\partial}{\partial x^k} \ln\,(x^3 - (x^2)^2) \\ &\Gamma^p_{kp} = \frac{\partial}{\partial x^k} \ln \sqrt{det(g_{\underline{ij}})} = \frac{1}{2}\frac{\partial}{\partial x^k}[x^1 + \frac{1}{2}\ln\,(x^2x^3 - (x^2)^2) + (x^3)^2]. \qquad \square \end{aligned} \tag{2.7}$$

In [11] Mikeš et al. have provided necessary and sufficient conditions for the existence of a geodesic mapping of a Riemannian space onto a Kählerian space.

Theorem 2.2. *The Riemannian space* $\mathbb{R}_N$ *admits a nontrivial geodesic mapping onto the Kählerian space* $\overline{\mathbb{K}}_N$ *if and only if, in the common coordinate system* x *with respect to the mapping, the conditions*

$$
\begin{aligned}
a)\quad & \overline{g}_{ij;k} = 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik};\\
b)\quad & \overline{F}^h_{i;k} = \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^\alpha_i\psi_\alpha;
\end{aligned}
\tag{2.8}
$$

hold, where $\psi_i \neq 0$ and the tensors $\overline{g}_{ij}$ and $\overline{F}^h_i$ satisfy the following conditions:

$$
det(\overline{g}_{ij}) \neq 0, \quad \overline{F}^h_\alpha\overline{g}_{\alpha j} + \overline{F}^\alpha_j\overline{g}_{\alpha i} = 0, \quad \overline{F}^h_\alpha\overline{F}^\alpha_i = -\delta^h_i. \tag{2.9}
$$

Then $\overline{g}_{ij}$ and $\overline{F}^h_i$ are the metric tensor and the structure of $\overline{\mathbb{K}}_N$, respectively.

Our idea is to establish appropriate necessary and sufficient conditions for the existence of geodesic mappings with respect to the four kinds of covariant derivatives. The details follow.

Theorem 2.3. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$
\begin{aligned}
a)\quad & \overline{g}_{ij\underset{1}{|}k} = \overline{g}_{i\underset{\vee}{j}\underset{1}{\overline{|}}k} + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi^p_{ik}\overline{g}_{pj} + \xi^p_{jk}\overline{g}_{ip};\\
b)\quad & \overline{F}^h_{i\underset{1}{|}k} = \overline{F}^h_{i\underset{1}{\overline{|}}k} + \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^p_i\psi_p - \xi^h_{pk}\overline{F}^p_i + \xi^p_{ik}\overline{F}^h_p;
\end{aligned}
\tag{2.10}
$$

hold with respect to the first kind of covariant derivatives, where $\psi_i \neq 0$ and the tensors $\overline{g}_{ij}$ and $\overline{F}^h_i$ satisfy the following conditions:

$$
det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}^\alpha_i\overline{g}_{\underline{\alpha j}} + \overline{F}^\alpha_j\overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}^h_\alpha\overline{F}^\alpha_i = -\delta^h_i. \tag{2.11}
$$

Then $\overline{g}_{ij}$ and $\overline{F}^h_i$ are the metric tensor and the almost complex structure of $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$, respectively.

Proof. The equation (2.10)(a) guarantees the existence of a geodesic mapping from the generalized Riemannian space $\mathbb{GR}_N$ onto the generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_N$ with the metric tensor $\overline{g}_{ij}$ with respect to the first kind of covariant derivatives.

The formula (2.10)(b) implies that the structure $\overline{F}^h_i$ in $\mathbb{G}\overline{\mathbb{R}}_N$ is covariantly constant with respect to the third kind of covariant derivative. The algebraic conditions (2.11) guarantee that $\overline{g}_{ij}$ and $\overline{F}^h_i$ are the metric tensor and the structure of $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$, respectively.

The deformation tensor is determined by equation (2.2), i.e.

$$
\overline{\Gamma}^h_{ij} - \Gamma^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \xi^h_{ij}. \tag{2.12}
$$

For the structure $\overline{F}$, we have the following equations:

$$
\overline{F}^h_{i\underset{1}{|}k} = \overline{F}^h_{i,k} + \Gamma^h_{pk}\overline{F}^p_i - \Gamma^p_{ik}\overline{F}^h_p, \quad \overline{F}^h_{i\underset{2}{|}k} = \overline{F}^h_{i,k} + \Gamma^h_{kp}\overline{F}^p_i - \Gamma^p_{ki}\overline{F}^h_p. \tag{2.13}
$$

Substituting Γ^h_{ij} from (2.12) in (2.13), we get

$$
\begin{aligned}
\overline{F}^h_{i\underset{1}{|}k} &= \overline{F}^h_{i,k} + (\overline{\Gamma}^h_{pk} - \psi_p\delta^h_k - \psi_k\delta^h_p - \xi^h_{pk})\overline{F}^p_i - (\overline{\Gamma}^p_{ik} - \psi_i\delta^p_k - \psi_k\delta^p_i - \xi^p_{ik})\overline{F}^h_p\\
&= \overline{F}^h_{i,k} + \overline{\Gamma}^h_{pk}\overline{F}^p_i - \psi_p\delta^h_k\overline{F}^p_i - \psi_k\delta^h_p\overline{F}^p_i - \xi^h_{pk}\overline{F}^p_i - \overline{\Gamma}^p_{ik}\overline{F}^h_p + \psi_i\delta^p_k\overline{F}^h_p + \psi_k\delta^p_i\overline{F}^h_p + \xi^p_{ik}\overline{F}^h_p\\
&= \overline{F}^h_{i\underset{1}{\overline{|}}k} - \psi_p\delta^h_k\overline{F}^p_i - \psi_k\delta^h_p\overline{F}^p_i - \xi^h_{pk}\overline{F}^p_i + \psi_i\delta^p_k\overline{F}^h_p + \psi_k\delta^p_i\overline{F}^h_p + \xi^p_{ik}\overline{F}^h_p
\end{aligned}
\tag{2.14}
$$

$$= \overline{F}^h_{i\overline{|}_1 k} - \psi_p \delta^h_k \overline{F}^p_i - \psi_k \overline{F}^h_i - \xi^h_{pk} \overline{F}^p_i + \psi_i \overline{F}^h_k + \psi_k \overline{F}^h_i + \xi^p_{ik} \overline{F}^h_p$$
$$= \overline{F}^h_{i\overline{|}_1 k} - \psi_p \delta^h_k \overline{F}^p_i + \psi_i \overline{F}^h_k - \xi^h_{pk} \overline{F}^p_i + \xi^p_{ik} \overline{F}^h_p,$$

where $|$, and $\overline{|}$ are respectively the covariant derivatives in $\mathbb{GR}_N$ and $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$. □

Theorem 2.4. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} a)\quad & \overline{g}_{ij\underset{2}{|}k} = \overline{g}_{ij\underset{\vee 2}{\overline{|}}k} + 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \xi^p_{ki} \overline{g}_{pj} + \xi^p_{kj} \overline{g}_{ip}; \\ b)\quad & \overline{F}^h_{i\underset{2}{|}k} = \overline{F}^h_{i\underset{2}{\overline{|}}k} + \overline{F}^h_k \psi_i - \delta^h_k \overline{F}^p_i \psi_p - \xi^h_{kp} \overline{F}^p_i + \xi^p_{ki} \overline{F}^h_p; \end{aligned} \tag{2.15}$$

hold with respect to the second kind of covariant derivatives, where $\psi_i \neq 0$ and the tensors $\overline{g}_{ij}$ and $\overline{F}^h_i$ satisfy the following conditions:

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}^\alpha_i \overline{g}_{\underline{\alpha j}} + \overline{F}^\alpha_j \overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}^h_\alpha \overline{F}^\alpha_i = -\delta^h_i. \tag{2.16}$$

Then $\overline{g}_{ij}$ and $\overline{F}^h_i$ are the metric tensor and the almost complex structure of $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$, respectively.

Proof. In $\mathbb{GR}_N$ for the second kind of covariant derivatives, we have

$$\begin{aligned} \overline{F}^h_{i\underset{2}{|}k} &= \overline{F}^h_{i,k} + (\overline{\Gamma}^h_{kp} - \psi_k \delta^h_p - \psi_p \delta^h_k - \xi^h_{kp}) \overline{F}^p_i - (\overline{\Gamma}^p_{ki} - \psi_k \delta^p_i - \psi_i \delta^p_k - \xi^p_{ki}) \overline{F}^h_p \\ &= \overline{F}^h_{i,k} + \overline{\Gamma}^h_{kp} \overline{F}^p_i - \psi_k \delta^h_p \overline{F}^p_i - \psi_p \delta^h_k \overline{F}^p_i - \xi^h_{kp} \overline{F}^p_i - \overline{\Gamma}^p_{ki} \overline{F}^h_p + \psi_k \delta^p_i \overline{F}^h_p + \psi_i \delta^p_k \overline{F}^h_p + \xi^p_{ki} \overline{F}^h_p \\ &= \overline{F}^h_{i\underset{2}{\overline{|}}k} - \psi_k \delta^h_p \overline{F}^p_i - \psi_p \delta^h_k \overline{F}^p_i - \xi^h_{kp} \overline{F}^p_i + \psi_k \delta^p_i \overline{F}^h_p + \psi_i \delta^p_k \overline{F}^h_p + \xi^p_{ki} \overline{F}^h_p \\ &= \overline{F}^h_{i\underset{2}{\overline{|}}k} - \psi_k \overline{F}^h_i - \psi_p \delta^h_k \overline{F}^p_i - \xi^h_{kp} \overline{F}^p_i + \psi_k \overline{F}^h_i + \psi_i \overline{F}^h_k + \xi^p_{ki} \overline{F}^h_p \\ &= \overline{F}^h_{i\underset{2}{\overline{|}}k} + \psi_i \overline{F}^h_k - \psi_p \delta^h_k \overline{F}^p_i - \xi^h_{kp} \overline{F}^p_i + \xi^p_{ki} \overline{F}^h_p. \qquad \square \end{aligned} \tag{2.17}$$

In a similar way, we can obtain the following corresponding results for the third and the fourth kind of covariant derivatives:

Theorem 2.5. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} a)\quad & \overline{g}_{ij\underset{3}{|}k} = \overline{g}_{ij\underset{\vee 3}{\overline{|}}k} + 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \xi^p_{ik} \overline{g}_{pj} + \xi^p_{kj} \overline{g}_{ip}; \\ b)\quad & \overline{F}^h_{i\underset{3}{|}k} = \psi_i \overline{F}^h_k - \psi_p \delta^h_k \overline{F}^p_i - \xi^h_{pk} \overline{F}^p_i + \xi^p_{ki} \overline{F}^h_p, \end{aligned} \tag{2.18}$$

hold with respect to the third kind of covariant derivatives, where $\psi_i \neq 0$ and the tensors $\overline{g}_{ij}$ and $\overline{F}^h_i$ satisfy the following conditions:

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}^\alpha_i \overline{g}_{\underline{\alpha j}} + \overline{F}^\alpha_j \overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}^h_\alpha \overline{F}^\alpha_i = -\delta^h_i. \tag{2.19}$$

Then $\overline{g}_{ij}$ and $\overline{F}^h_i$ are the metric tensor and the almost complex structure of $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$, respectively.

Theorem 2.6. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} a)\quad & \overline{g}_{ij\underset{4}{|}k} = \overline{g}_{\underset{\vee}{ij}\underset{4}{\overline{|}}k} + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi^p_{ki}\overline{g}_{pj} + \xi^p_{jk}\overline{g}_{ip}; \\ b)\quad & \overline{F}^h_{i\underset{4}{|}k} = \psi_i\overline{F}^h_k - \psi_p\delta^h_k\overline{F}^p_i - \xi^h_{kp}\overline{F}^p_i + \xi^p_{ik}\overline{F}^h_p, \end{aligned} \tag{2.20}$$

hold with respect to the fourth kind of covariant derivatives, where $\psi_i \neq 0$ and the tensors $\overline{g}_{ij}$ and $\overline{F}^h_i$ satisfy the following conditions:

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}^\alpha_i\overline{g}_{\underline{\alpha j}} + \overline{F}^\alpha_j\overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}^h_\alpha\overline{F}^\alpha_i = -\delta^h_i. \tag{2.21}$$

Then $\overline{g}_{ij}$ and $\overline{F}^h_i$ are the metric tensor and the almost complex structure of $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$, respectively.

2.1. Equitorsion geodesic mappings

Equitorsion mappings play an important role in the theories of geodesic, conformal and holomorphically projective transformations between two spaces of non-symmetric affine connection.

Definition 2.2. [15] A mapping $f : \mathbb{GR}_N \to \mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ is an **equitorsion geodesic mapping** if the torsion tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ are equal. Then from (2.1), (2.2) and (2.12) it follows

$$\overline{\Gamma}^h_{\underset{\vee}{ij}} - \Gamma^h_{\underset{\vee}{ij}} = \xi^h_{ij} = 0, \tag{2.22}$$

where $\underset{\vee}{ij}$ denotes an antisymmetrization with respect to i, j.

In the special case of this kind of mappings, the previous Theorems 2.3–2.6 become:

Theorem 2.7. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} a)\quad & \overline{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{jk}} + \psi_j\overline{g}_{\underline{ik}}; \\ b)\quad & \overline{F}^h_{i\underset{1}{|}k} = \overline{F}^h_{i\underset{1}{\overline{|}}k} + \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^p_i\psi_p; \end{aligned} \tag{2.23}$$

hold with respect to the first kind of covariant derivatives, where $\psi_i \neq 0$ and the tensors $\overline{g}_{ij}$ and $\overline{F}^h_i$ satisfy the following conditions:

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}^\alpha_i\overline{g}_{\underline{\alpha j}} + \overline{F}^\alpha_j\overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}^h_\alpha\overline{F}^\alpha_i = -\delta^h_i. \tag{2.24}$$

Theorem 2.8. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} a)\quad & \overline{g}_{\underline{ij}\underset{2}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{jk}} + \psi_j\overline{g}_{\underline{ik}}; \\ b)\quad & \overline{F}^h_{i\underset{2}{|}k} = \overline{F}^h_{i\underset{2}{\overline{|}}k} + \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^p_i\psi_p; \end{aligned} \tag{2.25}$$

hold with respect to the second kind of covariant derivatives, where $\psi_i \neq 0$ *and the tensors* $\overline{g}_{ij}$ *and* $\overline{F}_i^h$ *satisfy the following conditions:*

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}_i^\alpha \overline{g}_{\underline{\alpha j}} + \overline{F}_j^\alpha \overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}_\alpha^h \overline{F}_i^\alpha = -\delta_i^h. \tag{2.26}$$

Theorem 2.9. *The generalized Riemannian space* $\mathbb{GR}_N$ *admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space* $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ *if and only if, in the common coordinate system* x *with respect to the mapping, the conditions*

$$\begin{aligned} a) \quad & \overline{g}_{\underline{ij}\underset{3}{|}k} = 2\psi_k \overline{g}_{\underline{ij}} + \psi_i \overline{g}_{\underline{jk}} + \psi_j \overline{g}_{\underline{ik}}; \\ b) \quad & \overline{F}^h_{i\underset{3}{|}k} = \psi_i \overline{F}_k^h - \psi_p \delta_k^h \overline{F}_i^p \end{aligned} \tag{2.27}$$

hold with respect to the third kind of covariant derivatives, where $\psi_i \neq 0$ *and the tensors* $\overline{g}_{ij}$ *and* $\overline{F}_i^h$ *satisfy the following conditions:*

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}_i^\alpha \overline{g}_{\underline{\alpha j}} + \overline{F}_j^\alpha \overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}_\alpha^h \overline{F}_i^\alpha = -\delta_i^h. \tag{2.28}$$

Theorem 2.10. *The generalized Riemannian space* $\mathbb{GR}_N$ *admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space* $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ *if and only if, in the common coordinate system* x *with respect to the mapping, the conditions*

$$\begin{aligned} a) \quad & \overline{g}_{\underline{ij}\underset{4}{|}k} = 2\psi_k \overline{g}_{\underline{ij}} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{\underline{ik}}; \\ b) \quad & \overline{F}^h_{i\underset{4}{|}k} = \psi_i \overline{F}_k^h - \psi_p \delta_k^h \overline{F}_i^p, \end{aligned} \tag{2.29}$$

hold with respect to the fourth kind of covariant derivatives, where $\psi_i \neq 0$ *and the tensors* $\overline{g}_{ij}$ *and* $\overline{F}_i^h$ *satisfy the following conditions:*

$$det(\overline{g}_{\underline{ij}}) \neq 0, \quad \overline{F}_i^\alpha \overline{g}_{\underline{\alpha j}} + \overline{F}_j^\alpha \overline{g}_{\underline{\alpha i}} = 0, \quad \overline{F}_\alpha^h \overline{F}_i^\alpha = -\delta_i^h. \tag{2.30}$$

3. Conclusion

The present article presents a continuation of the work on developing the general ideas outlined and suggested in [16,19,21] by introducing the notion of generalized Kähler spaces of the third kind.

The following two results constitute the main contribution of the paper:

1. New explicit necessary and sufficient conditions for the existence of geodesic mappings onto $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ are given in Section 2.
2. New explicit necessary and sufficient conditions for the existence of equitorsion geodesic mappings onto $\mathbb{G}\underset{3}{\overline{\mathbb{K}}}_N$ are given in Section 2.1.

In this way we hope to have given useful contribution to the development of the theory of geodesic mappings and its applications.

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