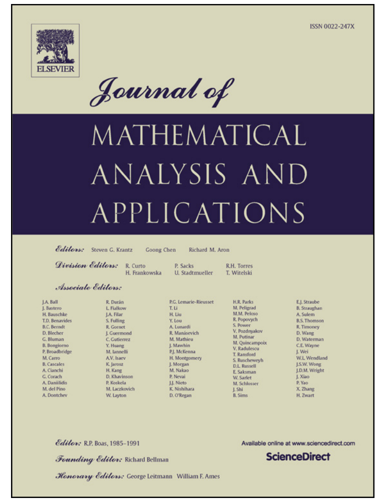
Accepted Manuscript



Summation Formulae for A Class of Terminating Balanced q-Series

Xiaojing Chen, Wenchang Chu

PII: S0022-247X(17)30185-3
DOI: http://dx.doi.org/10.1016/j.jmaa.2017.02.035
Reference: YJMAA 21160

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 10 October 2016

SUMMATION FORMULAE FOR A CLASS OF TERMINATING BALANCED q-SERIES

XIAOJING CHEN AND WENCHANG CHU

Abstract. A class of terminating balanced q-series are investigated. Thirty summation formulae are established by employing Carlitz' inversions and the polynomial argument, as well as contiguous relations.

1. Introduction and Motivation

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of natural numbers and nonnegative integers, respectively. For an indeterminate x, the shifted factorial with the base q is defined by $(x;q)_0 = 1$ and

$$(x;q)_n \;=\; (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n \in \mathbb{N}.$$

Then the Gaussian binomial coefficient reads as

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} = \frac{(q^{m-n+1};q)_n}{(q;q)_n} \quad \text{where} \quad m,\ n \in \mathbb{N}.$$

In 1973, Carlitz [4] found a well–known pair of inverse series relations, which can be reproduced as follows. Let $\{a_k, b_k\}_{k\geq 0}$ be two sequences such that the p-polynomials defined by

$$p(x;0) \equiv 1 \quad \text{and} \quad p(x;n) = \prod_{k=0}^{n-1}(a_k + xb_k) \quad \text{for} \quad n = 1, 2, \cdots$$

differ from zero at $x = q^{-m}$ for $m \in \mathbb{N}_0$. Then the following inverse relations hold

$$\left.\begin{aligned} f(n) &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} p(q^{-k};n)g(k), \\ g(n) &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{a_k + q^{-k}b_k}{p(q^{-n};k+1)} f(k). \end{aligned}\right\} \tag{1}$$

Alternatively, if the p-polynomials differ from zero at $x = q^m$ for $m \in \mathbb{N}_0$, Carlitz deduced, under the base change $q \to q^{-1}$, another pair of inversions:

$$\left.\begin{aligned} f(n) &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} p(q^k;n)g(k), \\ g(n) &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_k + q^k b_k}{p(q^n;k+1)} f(k). \end{aligned}\right\} \tag{2}$$

2000 *Mathematics Subject Classification.* Primary 33D15, Secondary 05A30.
Key words and phrases. Basic hypergeometric series; Contiguous relations; Carlitz inversions; Polynomial argument; q-Pfaff–Saalschütz theorem; Bailey's well-poised ${}_6\psi_6$ series identity.
Email addresses: upcxjchen@163.com and chu.wenchang@unisalento.it.

These inverse pairs have been shown by Chu [5, 6] to be very useful in proving terminating q-series identities. As further applications, this paper will investigate the following balanced series

$$\Omega(w,x,y)=\sum_{k\ge0}\left[\begin{matrix}x,y\\qw\end{matrix}\middle|\,q\right]_k\frac{(wxy;q^3)_k}{(xy;q)_{2k}}q^k. \tag{3}$$

When the series is nonterminating, it seems that there does not exist closed expression. However, when the Ω-series is terminating, we do have the following summation formula due to Andrews [1, Eq. 4.7] (see also Gessel–Stanton [8, Eq. 4.32]):

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^ny\\q\end{matrix}\middle|\,q\right]_k\frac{(y;q^3)_k}{(y;q)_{2k}}q^k=\chi(n\equiv_3 0)\left[\begin{matrix}q,q^2\\qy,q^2y\end{matrix}\middle|\,q^3\right]_{\lfloor\frac{n}{3}\rfloor}y^{\lfloor\frac{n}{3}\rfloor}. \tag{4}$$

Here and forth, we fix $\delta=0,\,1$ and utilize, for brevity, three notations: $\lfloor x\rfloor$ for the integer part of a real number x, $i\equiv_m j$ for the congruence of i and j modulo m, and χ for the logical function with $\chi(\text{true})=1$ and $\chi(\text{false})=0$ otherwise.

Observe that Andrews' identity (4) corresponds to only one of the four terminating cases of the Ω-series listed below:

- $w=1$ and $x=q^{-n}$.
- $xy=q^{1+\delta}$ and $wxy=q^{-3n}$.
- $x=q^{-n}$ and $y=q^{1+\delta+n}$.
- $w=1$ and $xy=q^{-3n}$.

We shall also consider the following terminating Ω-series perturbed by two integer parameters λ and μ

$$\Omega_{\lambda,\mu}(w,x,y)=\sum_{k\ge0}\left[\begin{matrix}x,q^\lambda y\\qw\end{matrix}\middle|\,q\right]_k\frac{(q^\mu wxy;q^3)_k}{(xy;q)_{2k}}q^k. \tag{5}$$

By employing contiguous relations, thirty summation formulae will be proved with most of them having not appeared previously. In order to ensure the accuracy, all the formulae presented in this paper are checked with *Mathematica* commands.

The rest of the paper will be organized as follows. In the next section, three contiguous relations for the above $\Omega_{\lambda,\mu}(w,x,y)$-series will be established for subsequent applications. From Section 3 to Section 5, we shall utilize the Carlitz inversions to derive summation formulae for the $\Omega_{\lambda,\mu}(w,x,y)$-sums corresponding to the first three terminating cases. Then the polynomial argument will be employed in Section 6 to prove summation formulae for the fourth terminating case. Finally, the reversal series of $\Omega_{\lambda,\mu}(w,x,y)$ will be examined in Section 7 and four summation formulae will be exemplified.

Throughout the paper, the product and quotient of shifted factorials will be abbreviated respectively to

$$\begin{aligned}[\alpha,\,\beta,\,\cdots,\,\gamma;q]_n&=(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n,\\ \left[\begin{matrix}\alpha,\,\beta,\,\cdots,\,\gamma\\A,\,B,\,\cdots,\,C\end{matrix}\middle|\,q\right]_n&=\frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n}.\end{aligned}$$

The q-series is defined, according to Bailey [2] and Gasper–Rahman [9], by

$${}_{1+\ell}\phi_\ell\left[\begin{matrix}a_0,\,a_1,\,\cdots,\,a_\ell\\b_1,\,\cdots,\,b_\ell\end{matrix}\middle|\,q;z\right]=\sum_{n=0}^{\infty}\left[\begin{matrix}a_0,\,a_1,\,\cdots,\,a_\ell\\q,\,\,b_1,\,\cdots,\,b_\ell\end{matrix}\middle|\,q\right]_n z^n.$$

If one of its numerator parameters is of the form q^{-m} with $m \in \mathbb{N}_0$, then we say that the ${}_{1+\ell}\phi_\ell$-series terminates. In particular, there are two important classes of q-series. One class is well–poised when their parameters can be paired off in columns such that two entries in each column have always the same product, i.e., $qa_0 = a_1b_1 = \cdots = a_\ell b_\ell$. Another class is called balanced or Saalschützian if the parameters satisfy the condition $qa_0a_1\cdots a_\ell = b_1b_2\cdots b_\ell$. It is routine to check that when $\lambda+\mu = 0$, the $\Omega_{\lambda,\mu}(w, x, y)$–series defined in (5) is balanced for the four terminating cases.

2. Contiguous Relations for the Ω-Series

In this section, we shall prove three contiguous relations for the $\Omega_{\lambda,\mu}$-series, that will be utilized to derive summation formulae in next four sections.

According to the linear equation

$$1 - q^{\lambda+k-1}y = \frac{w - q^{\lambda-1}y}{w - x}(1 - q^k x) - \frac{x - q^{\lambda-1}y}{w - x}(1 - q^k w)$$

we can manipulate the $\Omega_{\lambda,\mu}$-series defined in (5) as follows

$$\begin{aligned}
\Omega_{\lambda,\mu} =& \sum_{k\geq 0} \frac{1 - q^{\lambda+k-1}y}{1 - q^{\lambda-1}y} \left[\begin{matrix} x, q^{\lambda-1}y \\ qw \end{matrix}\middle| q\right]_k \frac{(q^\mu wxy; q^3)_k}{(xy; q)_{2k}} q^k \\
=& \sum_{k\geq 0} \frac{(1 - q^k x)(1 - q^{\lambda-1}y/w)}{(1 - x/w)(1 - q^{\lambda-1}y)} \left[\begin{matrix} x, q^{\lambda-1}y \\ qw \end{matrix}\middle| q\right]_k \frac{(q^\mu wxy; q^3)_k}{(xy; q)_{2k}} q^k \\
&+ \sum_{k\geq 0} \frac{(1 - q^k w)(1 - q^{\lambda-1}y/x)}{(1 - w/x)(1 - q^{\lambda-1}y)} \left[\begin{matrix} x, q^{\lambda-1}y \\ qw \end{matrix}\middle| q\right]_k \frac{(q^\mu wxy; q^3)_k}{(xy; q)_{2k}} q^k \\
=& \frac{(1 - x)(1 - q^{\lambda-1}y/w)}{(1 - x/w)(1 - q^{\lambda-1}y)} \Omega_{\lambda,\mu}(w, qx, y/q) \\
&+ \frac{(1 - w)(1 - q^{\lambda-1}y/x)}{(1 - w/x)(1 - q^{\lambda-1}y)} \Omega_{\lambda-1,\mu+1}(w/q, x, y).
\end{aligned}$$

This can be equivalently restated in the following lemma.

Lemma 1 (Contiguous relation)**.**

$$\begin{aligned}
\frac{1 - qw}{1 - x}\Omega_{\lambda-1,\mu+1}(w, x, y) \quad =& \quad \frac{(1 - qw/x)(1 - q^{\lambda-1}y)}{(1 - x)(1 - q^{\lambda-1}y/x)} \Omega_{\lambda,\mu}(qw, x, y) \\
&+ \quad \frac{qw}{x} \frac{(1 - q^{\lambda-2}y/w)}{(1 - q^{\lambda-1}y/x)} \Omega_{\lambda,\mu}(qw, qx, y/q).
\end{aligned}$$

By means of this lemma, we shall be able to evaluate the $\Omega_{\lambda-1,\mu+1}$-sums from $\Omega_{\lambda,\mu}$-sums as long as the latter one has closed expression.

Similarly, applying the linear equation

$$\begin{aligned}1-q^{2k}xy &= \frac{w^2-xy}{(w-x)(w-q^\lambda y)}(1-q^kx)(1-q^{\lambda+k}y)\\ &+ \frac{xy-q^{2\lambda}y^2}{(x-q^\lambda y)(-w+q^\lambda y)}(1-q^kx)(1-q^kw)\\ &+ \frac{xy-x^2}{(w-x)(x-q^\lambda y)}(1-q^{\lambda+k}y)(1-q^kw)\end{aligned}$$

we can establish another contiguous relation

$$\begin{aligned}\Omega_{\lambda,\mu}(w,x,y) =&\frac{(w^2-xy)(1-x)(1-q^\lambda y)}{(1-xy)(w-x)(w-q^\lambda y)}\Omega_{\lambda+1,\mu-1}(w,qx,y)\\ &+\frac{(xy-q^{2\lambda}y^2)(1-x)(1-w)}{(1-xy)(x-q^\lambda y)(-w+q^\lambda y)}\Omega_{\lambda,\mu}(w/q,qx,y)\\ &+\frac{(xy-x^2)(1-q^\lambda y)(1-w)}{(1-xy)(w-x)(x-q^\lambda y)}\Omega_{\lambda,\mu}(w/q,x,qy).\end{aligned}$$

After some simplification, we state it as the following lemma.

Lemma 2 (Contiguous relation)**.**

$$\begin{aligned}\Omega_{\lambda+1,\mu-1}(w,x,y) =&\frac{(1-xy/q)(w-x/q)(w-q^\lambda y)}{(w^2-xy/q)(1-x/q)(1-q^\lambda y)}\Omega_{\lambda,\mu}(w,x/q,y)\\ &+\frac{(xy/q-q^{2\lambda}y^2)(w-x/q)(1-w)}{(w^2-xy/q)(x/q-q^\lambda y)(1-q^\lambda y)}\Omega_{\lambda,\mu}(w/q,x,y)\\ &+\frac{(x^2/q^2-xy/q)(w-q^\lambda y)(1-w)}{(w^2-xy/q)(x/q-q^\lambda y)(1-x/q)}\Omega_{\lambda,\mu}(w/q,x/q,qy).\end{aligned}$$

This lemma will be used to evaluate the $\Omega_{\lambda+1,\mu-1}$-sums from $\Omega_{\lambda,\mu}$-sums.

Finally, in accordance with the expression

$$1=\frac{q^\lambda y}{q^\lambda y-x}(1-q^kx)+\frac{x}{x-q^\lambda y}(1-q^{\lambda+k}y)$$

we can verify, without difficulty, the following third recurrence.

Lemma 3 (Contiguous relation)**.**

$$\Omega_{\lambda,\mu}(w,x,y)=\frac{q^\lambda y(1-x)}{q^\lambda y-x}\Omega_{\lambda+1,\mu}(w,qx,y/q)+\frac{x(1-q^\lambda y)}{x-q^\lambda y}\Omega_{\lambda+1,\mu}(w,x,y).$$

It will be employed to find closed formulae for $\Omega_{\lambda,\mu}$-sums from the $\Omega_{\lambda+1,\mu}$-sums.

3. Evaluation of the $\Omega_{\lambda,\mu}(1,q^{-n},y)$-Sums

By applying the recurrence relations established in the last section to Andrews' identity (4), we shall prove more closed formulae for the $\Omega_{\lambda,\mu}(1,q^{-n},y)$-sums. Our starting point is the crucial formula (4). To maintain the integrity, we present another proof of it by means of inverse series relations.

Recall the q-Pfaff–Saalschütz summation theorem (cf. Bailey [2, §8.4] and Gasper–Rahman [9, II-12]) about the terminating balanced series:

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, \; a, \quad b \\ c, \; q^{1-n}ab/c \end{matrix}\Big|\; q;q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix}\Big|\; q\right]_n. \tag{6} $$

Its particular case

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, \; q^{1-n}, \quad q^{2-n} \\ q^3x, \; q^{3-3n}/x \end{matrix}\Big|\; q^3;q^3\right] = \frac{(x;q)_{2n}}{(x;q^3)_n(qx;q)_n} $$

can be reformulated as the following q-binomial sum

$$ \sum_{k\geq 0}(-1)^k \begin{bmatrix} n \\ 3k \end{bmatrix} (q^{-3k}x;q^3)_n \left[\begin{matrix} q, \quad q^2 \\ q^3x, \; q^3/x \end{matrix}\Big|\; q^3\right]_k q^{\binom{1+3k}{2}} = \frac{(x;q)_{2n}}{(qx;q)_n}. $$

Observing that this equality matches the first equation of (1) under the specifications

$$ f(n) \;\to\; \frac{(x;q)_{2n}}{(qx;q)_n}, \qquad p(y;n) \;\to\; (xy;q^3)_n; $$

$$ g(k) \;\to\; \chi(k \equiv_3 0) \left[\begin{matrix} q, \quad q^2 \\ q^3x, \; q^3/x \end{matrix}\Big|\; q^3\right]_{\frac{k}{3}} q^{\binom{k+1}{2}}; $$

we obtain the following dual relation from the second equation of (1)

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-q^{2k}x}{(q^{-n}x;q^3)_{k+1}} \frac{(x;q)_{2k}}{(qx;q)_k} = \chi(n \equiv_3 0) \left[\begin{matrix} q, \quad q^2 \\ q^3x, \; q^3/x \end{matrix}\Big|\; q^3\right]_{\lfloor \frac{n}{3} \rfloor} q^{\binom{n+1}{2}} $$

which can be restated equivalently as

$$ \sum_{k=0}^{n} \frac{(q^{-n};q)_k(qx;q)_{2k}}{(q;q)_k(qx;q)_k(q^{3-n}x;q^3)_k} q^k = \chi(n \equiv_3 0) \left[\begin{matrix} q, \quad q^2 \\ q^3x, \; 1/x \end{matrix}\Big|\; q^3\right]_{\lfloor \frac{n}{3} \rfloor}. \tag{7} $$

Its reversal under the replacement $x \to q^{-2n}/y$ confirms the identity (4).

Theorem 4 ($\Omega_{0,0}(1,q^{-n},q^ny)$: Andrews [1])**.**

$$ \sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^ny \\ q \end{matrix}\Big|\; q\right]_k \frac{(y;q^3)_k}{(y;q)_{2k}} q^k = \chi(n \equiv_3 0) \left[\begin{matrix} q, q^2 \\ qy, q^2y \end{matrix}\Big|\; q^3\right]_{\lfloor \frac{n}{3} \rfloor} (y)^{\lfloor \frac{n}{3} \rfloor}. $$

There is also an alternative proof by applying Carlitz' inversions to the q-Dougall sum. The interested reader can find it in Chu [5, Eq. 4.4d].

Now we are going to derive summation formulae for the $\Omega_{\lambda,\mu}(1,q^{-n},y)$-sums by utilizing the contiguous relations proved in the last section.

First, letting $\lambda = \mu = 0$ and $w \to q^{-1}$, $x \to q^{-n-1}$, $y \to q^{n-1}y$ in Lemma 1 and then applying Theorem 4, we derive the following summation formula.

Corollary 5 ($\Omega_{-1,1}(1,q^{-n},q^ny)$)**.**

$$ \sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n-1}y \\ q \end{matrix}\Big|\; q\right]_k \frac{(qy;q^3)_k}{(y;q)_{2k}} q^k = \frac{(qy)^{\lfloor \frac{n}{3} \rfloor}}{1-q^{2n-1}y} \left[\begin{matrix} q, q^2 \\ y, y/q \end{matrix}\Big|\; q^3\right]_{\lfloor \frac{n+1}{3} \rfloor} \times \begin{cases} 1-y/q, & n \equiv_3 0; \\ 0, & n \equiv_3 1; \\ y^2/q - y, & n \equiv_3 2. \end{cases} $$

By combining the last formula with Lemma 1 specified by $\lambda = -1$, $\mu = 1$ and $w \to q^{-1}$, $x \to q^{-n-1}$, $y \to q^{n-1}y$, we obtain another balanced series identity.

Corollary 6 ($\Omega_{-2,2}(1, q^{-n}, q^n y)$)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n-2}y \\ q \end{matrix}\middle| q\right]_k \frac{(q^2y;q^3)_k}{(y;q)_{2k}} q^k = (q^2y)^{\lfloor \frac{n}{3} \rfloor} \left[\begin{matrix} q, q^2 \\ y, qy \end{matrix}\middle| q^3\right]_{\lfloor \frac{n}{3} \rfloor} \times \begin{cases} \dfrac{(1-q^{n-2}y)(1-q^{n-3}y)}{(1-q^{2n-2}y)(1-q^{2n-3}y)}, & n \equiv_3 0; \\ \dfrac{y(1-q^n)(1-q^{n+1})}{q(1-q^{2n-1}y)(1-q^{2n-2}y)}, & n \equiv_3 1; \\ \dfrac{-y(1+q)(1-q^n)(1-q^{n-1})}{(1-q^{2n-1}y)(1-q^{2n-3}y)}, & n \equiv_3 2. \end{cases}$$

Similarly, letting $\lambda = 0$, $\mu = 0$ and $w \to 1$, $x \to q^{-n}$, $y \to q^n y$ in Lemma 2 and then invoking Theorem 4, we derive the following closed formula.

Corollary 7 ($\Omega_{1,-1}(1, q^{-n}, q^n y)$)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n+1}y \\ q \end{matrix}\middle| q\right]_k \frac{(y/q;q^3)_k}{(y;q)_{2k}} q^k = \left(\frac{y}{q}\right)^{\lfloor \frac{n+2}{3} \rfloor} \left[\begin{matrix} q, q^2 \\ y, qy \end{matrix}\middle| q^3\right]_{\lfloor \frac{n+2}{3} \rfloor}.$$

By applying Lemma 2 to the last formula, we can further evaluate $\Omega_{2,-2}(1, q^{-n}, q^n y)$. We limit ourselves to record the following formula corresponding to $n \equiv_3 1$ because the resulting expressions for the remaining two cases are too complicated.

Corollary 8 ($\Omega_{2,-2}(1, q^{-3m-1}, q^{3m+1}y)$)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-1-3m}, q^{3+3m}y \\ q \end{matrix}\middle| q\right]_k \frac{(y/q^2;q^3)_k}{(y;q)_{2k}} q^k = \frac{y^{m+1}(1-q^2)(1-q^3)}{q^{2m+2}(1-y)(1-qy)} \left[\begin{matrix} q^4, q^5 \\ q^2y, q^3y \end{matrix}\middle| q^3\right]_m.$$

Finally, by means of the Lemma 3, we can show further five summation formulae for $\Omega_{\lambda,\mu}(1, q^{-n}, q^n y)$, where the formulae to be employed will be indicated in the headers of corollaries.

Corollary 9 ($\Omega_{0,-1}(1, q^{-n}, q^n y)$ by Corollary 7)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n}y \\ q \end{matrix}\middle| q\right]_k \frac{(y/q;q^3)_k}{(y;q)_{2k}} q^k = \left(\frac{y}{q}\right)^{\lfloor \frac{n+2}{3} \rfloor} \left[\begin{matrix} q, q^2 \\ y, qy \end{matrix}\middle| q^3\right]_{\lfloor \frac{n+2}{3} \rfloor} \times \begin{cases} 1, & n \equiv_3 0; \\ \dfrac{1-q^n y}{1-q^{n+1}}, & n \equiv_3 1; \\ 1, & n \equiv_3 2. \end{cases}$$

Corollary 10 ($\Omega_{-1,-1}(1, q^{-n}, q^n y)$ by Corollary 9)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n-1}y \\ q \end{matrix}\middle| q\right]_k \frac{(y/q;q^3)_k}{(y;q)_{2k}} q^k = \left(\frac{y}{q}\right)^{\lfloor \frac{n+2}{3} \rfloor} \left[\begin{matrix} q, q^2 \\ y, qy \end{matrix}\middle| q^3\right]_{\lfloor \frac{n+1}{3} \rfloor} \times \begin{cases} 1, & n \equiv_3 0; \\ \dfrac{1-q^{2n}}{1-q^{2n-1}y}, & n \equiv_3 1; \\ \dfrac{1-q^{2n-2}y^2}{1-q^{2n-1}y}, & n \equiv_3 2. \end{cases}$$

Corollary 11 ($\Omega_{-1,0}(1, q^{-n}, q^n y)$ by Theorem 4)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n-1}y \\ q \end{matrix}\middle| q\right]_k \frac{(y;q^3)_k}{(y;q)_{2k}} q^k = y^{\lfloor \frac{n+2}{3} \rfloor} \left[\begin{matrix} q, q^2 \\ qy, q^2y \end{matrix}\middle| q^3\right]_{\lfloor \frac{n}{3} \rfloor} \times \begin{cases} \dfrac{1-q^{n-1}y}{1-q^{2n-1}y}, & n \equiv_3 0; \\ \dfrac{q^{n-1}(1-q^n)}{1-q^{2n-1}y}, & n \equiv_3 1; \\ 0, & n \equiv_3 2. \end{cases}$$

Corollary 12 ($\Omega_{-2,0}(1, q^{-n}, q^n y)$ by Corollary 11)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n-2}y \\ q \end{matrix}\middle| q\right]_k \frac{(y;q^3)_k}{(y;q)_{2k}} q^k = y^{\lfloor \frac{n+2}{3} \rfloor} \left[\begin{matrix} q, q^2 \\ qy, q^2y \end{matrix}\middle| q^3\right]_{\lfloor \frac{n+1}{3} \rfloor} \times \begin{cases} \dfrac{(1-q^{n-1}y)(1-q^{n-2}y)}{(1-q^{2n-1}y)(1-q^{2n-2}y)}, & n \equiv_3 0; \\ \dfrac{(1+q)q^{n-2}(1-q^n)(1-q^{n-2}y)}{(1-q^{2n-1}y)(1-q^{2n-3}y)}, & n \equiv_3 1; \\ \dfrac{yq^{2n-4}(1-q^{n-1}y)(1-q^n y)}{(1-q^{2n-2}y)(1-q^{2n-3}y)}, & n \equiv_3 2. \end{cases}$$

Corollary 13 ($\Omega_{-2,1}(1, q^{-n}, q^n y)$ by Corollary 5)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{n-2}y \\ q \end{matrix}\middle| q\right]_k \frac{(qy;q^3)_k}{(y;q)_{2k}} q^k = \frac{q-y}{q} \left[\begin{matrix} q, q^2 \\ y, y/q \end{matrix}\middle| q^3\right]_{\lfloor \frac{n+2}{3} \rfloor} \times \begin{cases} \dfrac{(qy)^{\lfloor \frac{n}{3} \rfloor}(1-q^{n-3}y-q^{n-2}y+q^{2n-2}y)}{(1-q^{2n-1}y)(1-q^{2n-3}y)}, & n \equiv_3 0; \\ \dfrac{q^{n-3}(qy)^{\lfloor \frac{n+2}{3} \rfloor}(1-q^{n-1}y)(1-q^{n-2}y)}{(1-q^{n+1})(1-q^{2n-2}y)(1-q^{2n-3}y)}, & n \equiv_3 1; \\ \dfrac{-(q)^{\lfloor \frac{n}{3} \rfloor}(y)^{\lfloor \frac{n+2}{3} \rfloor}(1-q^{n-2}y)}{(1-q^{2n-1}y)(1-q^{2n-2}y)}, & n \equiv_3 2. \end{cases}$$

Following the same procedure, it is possible to evaluate the $\Omega_{\lambda,\mu}(1, q^{-n}, q^n y)$-sums in closed form for any couple of integer parameters λ and μ satisfying the condition $\lambda+\mu \le 0$. However we shall not enlarge the list of identities due to their complexity and space limitations.

4. Evaluation of the $\Omega_{\lambda,\mu}(q^{-1-\delta-\mu-3n}, x, q^{1+\delta}/x)$-Sums

As a particular case of (6), it is routine to verify the following equality

$$ {}_3\phi_2\left[\begin{matrix} q^{1-2n}, & q^{-2n}, & q^{-2n-1} \\ & q^{1-3n}x, & q^{2-3n}/x \end{matrix}\Big| \, q^3; q^3\right] = \frac{q^{n^2}(x;q)_n(q/x;q)_n}{(qx;q^3)_n(q^2/x;q^3)_n}. $$

Writing the ${}_3\phi_2$-series as a finite sum $\sum_{k=0}^{n}$ and then reversing the summation order by $k \to n-k$, we can reformulate it, after some simplifications, as follows:

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^3} (q^{-3k-1};q)_n \frac{(qx;q^3)_k(q^2/x;q^3)_k}{(q;q^3)_{k+1}(q^2;q^3)_k} q^{3\binom{k+1}{2}} = \frac{(q^3;q^3)_n(x;q)_n(q/x;q)_n}{(q;q)_{2n+1}}, $$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{q^3}$ denotes the Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ with the base q being replaced by q^3. According to the inverse pair (1), we get the dual relation below

$$ \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^3} q^{3\binom{n-k}{2}} \frac{1-q^{-2k-1}}{(q^{-3n-1};q)_{k+1}} \frac{(q^3;q^3)_k(x;q)_k(q/x;q)_k}{(q;q)_{2k+1}} = \frac{(qx;q^3)_n(q^2/x;q^3)_n}{(q;q^3)_{n+1}(q^2;q^3)_n} q^{3\binom{n+1}{2}}. $$

This is equivalent to the following summation formula.

Theorem 14 ($\Omega_{0,0}(q^{-3n-1}, x, q/x)$: Chu [5, Eq. 3.9a])**.**

$$ \sum_{k=0}^{n}\left[\begin{matrix} x, q/x \\ q^{-3n} \end{matrix}\Big| \, q\right]_k \frac{(q^{-3n};q^3)_k}{(q;q)_{2k}} q^k = \left[\begin{matrix} qx, q^2/x \\ q, q^2 \end{matrix}\Big| \, q^3\right]_n. $$

Alternatively, by examining another particular case of (6)

$$ {}_3\phi_2\left[\begin{matrix} q^{-2n}, & q^{-2n-1}, & q^{-2n-2} \\ & q^{-3n-1}x, & q^{1-3n}/x \end{matrix}\Big| \, q^3; q^3\right] = \frac{q^{n^2+n}(x;q)_n(q^2/x;q)_n}{(q^2x;q^3)_n(q^4/x;q^3)_n} $$

and carrying out the same procedure, we would find the identity below.

Theorem 15 ($\Omega_{0,0}(q^{-3n-2}, x, q^2/x)$: Chu [5, Eq. 3.9b])**.**

$$ \sum_{k=0}^{n}\left[\begin{matrix} x, q^2/x \\ q^{-1-3n} \end{matrix}\Big| \, q\right]_k \frac{(q^{-3n};q^3)_k}{(q^2;q)_{2k}} q^k = \left[\begin{matrix} q^2x, q^4/x \\ q^2, q^4 \end{matrix}\Big| \, q^3\right]_n. $$

Specifying $\lambda = 0, \mu = 0$ and $w \to q^{-2-\delta-3n}, y \to q^{1+\delta}/x$ in Lemma 1 and then applying Theorem 14 and Theorem 15, we find, respectively, the following two summation formulae.

Corollary 16 ($\Omega_{-1,1}(q^{-3n-2}, x, q/x)$)**.**

$$ \sum_{k=0}^{n}\left[\begin{matrix} x, 1/x \\ q^{-3n-1} \end{matrix}\Big| \, q\right]_k \frac{(q^{-3n};q^3)_k}{(q;q)_{2k}} q^k = \frac{1-q^{3n+1}x}{(1+x)(1-q)}\left\{\left[\begin{matrix} qx, q^2/x \\ q^2, q^4 \end{matrix}\Big| \, q^3\right]_n - \frac{q(1-x/q)}{1-q^{3n+1}x}\left[\begin{matrix} q^2x, q^4/x \\ q^2, q^4 \end{matrix}\Big| \, q^3\right]_n\right\}. $$

Corollary 17 ($\Omega_{-1,1}(q^{-3n-3},x,q^2/x)$).

$$\sum_{k=0}^{n}\begin{bmatrix}x,q/x\\q^{-3n-2}\end{bmatrix}q\Big]_k\frac{(q^{-3n};q^3)_k}{(q^2;q)_{2k}}q^k = \frac{x(1-q)}{q-x^2}\left\{\begin{bmatrix}x,q^3/x\\q,\ q^2\end{bmatrix}q^3\Big]_{n+1}-\frac{1-q^{3n+2}x}{1-x/q}\begin{bmatrix}x/q,q/x\\q,\ q^2\end{bmatrix}q^3\Big]_{n+1}\right\}.$$

Similarly, letting $\lambda=-1$ and $\mu=0$ in Lemma 3 and then appealing to Theorem 14 and Theorem 15, we obtain two further identities.

Corollary 18 ($\Omega_{-1,0}(q^{-3n-1},x,q/x)$).

$$\sum_{k=0}^{n}\left[\begin{matrix}x,1/x\\q^{-3n}\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q;q)_{2k}}q^k = \frac{1}{1+x}\left[\begin{matrix}q^2x,q/x\\q,\ q^2\end{matrix}\middle|\,q^3\right]_n+\frac{x}{1+x}\left[\begin{matrix}qx,q^2/x\\q,\ q^2\end{matrix}\middle|\,q^3\right]_n.$$

Corollary 19 ($\Omega_{-1,0}(q^{-3n-2},x,q^2/x)$).

$$\sum_{k=0}^{n}\left[\begin{matrix}x,q/x\\q^{-3n-1}\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^2;q)_{2k}}q^k = \frac{1-x}{1-x^2/q}\left[\begin{matrix}q^3x,q^3/x\\q^2,\ q^4\end{matrix}\middle|\,q^3\right]_n+\frac{x-x^2/q}{1-x^2/q}\left[\begin{matrix}q^2x,q^4/x\\q^2,\ q^4\end{matrix}\middle|\,q^3\right]_n.$$

5. Evaluation of the $\Omega_{\lambda,\mu}(w,q^{-n},q^{1+n+\delta})$-Sums

For the bilateral q-series, Bailey [3] discovered the following very well-poised ${}_6\psi_6$-series identity (see also Chu [7] and Gasper–Rahman [9, II-33]):

$$\sum_{k=-\infty}^{\infty}\frac{1-q^{2k}a}{1-a}\left[\begin{matrix}b, & c, & d, & e\\qa/b, & qa/c, & qa/d, & qa/e\end{matrix}\middle|\,q\right]_k\left(\frac{qa^2}{bcde}\right)^k = \left[\begin{matrix}q,\ qa,\ q/a,\ qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de\\qa/b,qa/c,qa/d,qa/e,\ q/b,q/c,q/d,q/e,qa^2/bcde\end{matrix}\middle|\,q\right]_\infty \tag{8}$$

provided that $|qa^2/bcde|<1$ for convergence.

First, it is not hard to check, under $q\to q^3$, the following special case

$$\frac{(q;q)_n(q^2;q)_n(qw;q^3)_n}{(q;q)_{2n}(qw;q)_n}=\sum_{k=-\infty}^{\infty}\frac{1-q^{6k+1}}{1-q}\left[\begin{matrix}q/w,\ q^{-n},\ q^{1-n},\ q^{2-n}\\q^3w,\ q^{4+n},\ q^{3+n},\ q^{2+n}\end{matrix}\middle|\,q^3\right]_k q^{k(1+3n)}w^k.$$

For the last bilateral series, splitting it into two sums $\sum_{k\geq 0}$ and $\sum_{k<0}$, and then replacing the summation index k by $-1-k$ for the second sum, we can reformulate the resulting equality as

$$\begin{aligned}\frac{(q;q)_n(q;q)_n(qw;q^3)_n}{(qw;q)_n(q;q)_{2n}}&=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\3k\end{bmatrix}q^{\binom{3k}{2}}\frac{1-q^{6k+1}}{(q^{n+1};q)_{3k+1}}\frac{(qw)^k(q/w;q^3)_k(q;q)_{3k}}{(q^3w;q^3)_k}\\&+\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\3k+2\end{bmatrix}q^{\binom{3k+2}{2}}\frac{1-q^{6k+5}}{(q^{n+1};q)_{3k+3}}\frac{(qw)^{k+1}(1/w;q^3)_{k+1}(q;q)_{3k+2}}{(q^2w;q^3)_{k+1}}.\end{aligned}$$

This equality fits perfectly into the second equation displayed in (2) under the parameter specifications

$$\begin{aligned} g(n) &= \frac{(q;q)_n(q;q)_n(qw;q^3)_n}{(qw;q)_n(q;q)_{2n}}, \qquad p(y;n)=(qy;q)_n;\\ f(k) &= q^{\binom{k}{2}}\begin{cases} \dfrac{(q;q)_{3i}(q/w;q^3)_i}{(q^3w;q^3)_i}(qw)^i, & k=3i;\\ 0, & k=3i+1;\\ \dfrac{(1/w;q^3)_{i+1}(q;q)_{3i+2}}{(q^2w;q^3)_{i+1}}(qw)^{i+1}, & k=3i+2. \end{cases} \end{aligned}$$

The dual relation corresponding to the first equation of (2) reads as

$$\begin{aligned} &\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}(q^{1+k};q)_n\frac{(q;q)_k(q;q)_k(qw;q^3)_k}{(qw;q)_k(q;q)_{2k}}\\ &=q^{\binom{n}{2}}(q;q)_n\begin{cases} (qw)^m\dfrac{(q/w;q^3)_m}{(q^3w;q^3)_m}, & n=3m;\\ 0, & n=3m+1;\\ (qw)^{m+1}\dfrac{(1/w;q^3)_{m+1}}{(q^2w;q^3)_{m+1}}, & n=3m+2. \end{cases} \end{aligned}$$

This can be restated as the following summation formula.

Theorem 20 ($\Omega_{0,0}(w,q^{-n},q^{1+n})$).

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{1+n}\\ qw\end{matrix}\,\middle|\,q\right]_k\frac{(qw;q^3)_k}{(q;q)_{2k}}q^k=\begin{cases} \dfrac{(q/w;q^3)_m}{(q^3w;q^3)_m}(qw)^m, & n=3m;\\ 0, & n=3m+1;\\ \dfrac{(1/w;q^3)_{m+1}}{(q^2w;q^3)_{m+1}}(qw)^{m+1}, & n=3m+2. \end{cases}$$

Letting $w\to 0$ and $w\to\infty$, we find the following interesting identities

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+k\\2k\end{bmatrix}q^{\binom{n-k}{2}} = (-1)^{\lfloor\frac{n+1}{3}\rfloor}q^{\frac{n(2n-1)}{3}}\chi(n\not\equiv_3 1), \tag{9a}$$

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+k\\2k\end{bmatrix}q^{\binom{n-k}{2}+k(k-1)} = (-1)^{\lfloor\frac{n+1}{3}\rfloor}q^{\frac{n(n-2)}{3}}\chi(n\not\equiv_3 1). \tag{9b}$$

These two identities are reciprocal in the sense that one is equivalent to another under the base replacement $q\to q^{-1}$. In addition, we point out that (9a) is equivalent to the case $m=2n$ of the following identity due to Warnaar [10]:

$$\sum_{k=0}^{m}(-1)^k\begin{bmatrix}m-k\\k\end{bmatrix}q^{\binom{k}{2}} = \chi(m\not\equiv_3 2)(-1)^{\lfloor\frac{m}{3}\rfloor}q^{\frac{m(m-1)}{6}}. \tag{10}$$

Analogously, Bailey's identity (8) can also be specialized, under the base change $q\to q^3$, to another equality

$$\frac{(q;q)_n(q^3;q)_n(q^2w;q^3)_n}{(q^2;q)_{2n}(qw;q)_n}=\sum_{k=-\infty}^{\infty}\frac{1-q^{6k+2}}{1-q^2}\left[\begin{matrix}q^2/w,\ q^{-n},\ q^{1-n},\ q^{2-n}\\ q^3w,\ q^{5+n},\ q^{4+n},\ q^{3+n}\end{matrix}\,\middle|\,q^3\right]_k q^{k(2+3n)}w^k$$

which can be reformulated further as

$$\frac{(q;q)_n(q^2;q)_n(q^2w;q^3)_n}{(qw;q)_n(q^2;q)_{2n}} = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ 3k \end{bmatrix} q^{\binom{3k}{2}} \frac{1-q^{6k+2}}{(q^{n+2};q)_{3k+1}} \frac{(q^2w)^k(q^2/w;q^3)_k(q;q)_{3k}}{(q^3w;q^3)_k}$$
$$+\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ 3k+1 \end{bmatrix} q^{\binom{3k+1}{2}} \frac{1-q^{6k+4}}{(q^{n+2};q)_{3k+2}} \frac{(q^2w)^{k+1}(1/w;q^3)_{k+1}(q;q)_{3k+1}}{q(qw;q^3)_{k+1}}.$$

By comparing this last relation with the second equation displayed in (2) under the parameter specifications

$$g(n) = \frac{(q;q)_n(q^2;q)_n(q^2w;q^3)_n}{(qw;q)_n(q^2;q)_{2n}}, \qquad p(y;n) = (q^2y;q)_n;$$
$$f(k) = q^{\binom{k}{2}}(q^2w)^{\lfloor \frac{k}{3} \rfloor} \begin{cases} \dfrac{(q;q)_{3i}(q^2/w;q^3)_i}{(q^3w;q^3)_i}, & k=3i; \\ -qw\dfrac{(1/w;q^3)_{i+1}(q;q)_{3i+1}}{(qw;q^3)_{i+1}}, & k=3i+1; \\ 0, & k=3i+2; \end{cases}$$

we get correspondingly the dual relation from the first equation of (2):

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(q^{2+k};q)_n \frac{(q;q)_k(q^2;q)_k(q^2w;q^3)_k}{(qw;q)_k(q^2;q)_{2k}}$$
$$= q^{\binom{n}{2}}(q;q)_n(q^2w)^{\lfloor \frac{n}{3} \rfloor} \begin{cases} \dfrac{(q^2/w;q^3)_m}{(q^3w;q^3)_m}, & n=3m; \\ -qw\dfrac{(1/w;q^3)_{m+1}}{(qw;q^3)_{m+1}}, & n=3m+1; \\ 0, & n=3m+2. \end{cases}$$

This can be restated as another summation theorem.

Theorem 21 ($\Omega_{0,0}(w,q^{-n},q^{2+n})$)**.**

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, q^{2+n} \\ qw \end{matrix} \middle| q \right]_k \frac{(q^2w;q^3)_k}{(q;q)_{2k+1}} q^k = \begin{cases} \dfrac{(q^2w)^m}{1-q^{3m+1}} \left[\begin{matrix} q^2/w \\ q^3w \end{matrix} \middle| q^3 \right]_m, & n=3m; \\ \dfrac{-(q^2w)^{m+1}}{q(1-q^{3m+2})} \left[\begin{matrix} 1/w \\ qw \end{matrix} \middle| q^3 \right]_{m+1}, & n=3m+1; \\ 0, & n=3m+2. \end{cases}$$

Letting $w \to 0$ and $w \to \infty$ respectively, we obtain the following two reciprocal sums, with the first one corresponding to the case $m = 2n+1$ of (10).

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n+k+1 \\ 1+2k \end{bmatrix} q^{\binom{n-k}{2}} = (-1)^{\lfloor \frac{n}{3} \rfloor} q^{\frac{n(2n+1)}{3}} \chi(n \not\equiv_3 2), \tag{11a}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n+k+1 \\ 1+2k \end{bmatrix} q^{\binom{n-k}{2}+k^2} = (-1)^{\lfloor \frac{n}{3} \rfloor} q^{\frac{n(n-1)}{3}} \chi(n \not\equiv_3 2). \tag{11b}$$

Based on these two $\Omega_{0,0}(w,q^{-n},q^{1+n+\delta})$ sums with $\delta = 0,\ 1$ displayed in Theorems 20 and 21, we may derive further formulae via contiguous relations. Four of them are listed as follows.

First, letting $\lambda = 0$ and $\mu = 0$ in Lemma 1 yields the following two identities.

Corollary 22 $(\Omega_{-1,1}(w,q^{-n},q^{1+n}))$**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{n}\\ qw\end{matrix}\Big|\,q\right]_k\frac{(q^2w;q^3)_k}{(q;q)_{2k}}q^k
=\begin{cases}\dfrac{(q^2w)^m}{1+q^{3m}}\left\{\dfrac{(1/w;q^3)_m}{(qw;q^3)_m}+\dfrac{(q^2/w;q^3)_m}{(q^3w;q^3)_m}\right\}, & n=3m;\\[2ex] \dfrac{-q^{2m+1}w^{m+1}}{1+q^{3m+1}}\dfrac{(1/w;q^3)_{m+1}}{(qw;q^3)_{m+1}}, & n=3m+1;\\[2ex] \dfrac{-q^{2m+1}w^{m}}{1+q^{3m+2}}\dfrac{(q^2/w;q^3)_{m}}{(q^3w;q^3)_{m}}, & n=3m+2.\end{cases}$$

Corollary 23 $(\Omega_{-1,1}(w,q^{-n},q^{2+n}))$**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{1+n}\\ qw\end{matrix}\Big|\,q\right]_k\frac{(q^3w;q^3)_k}{(q;q)_{2k+1}}q^k
=\begin{cases}\dfrac{q^{3m}w^m}{1-q^{6m+1}}\dfrac{(q/w;q^3)_m}{(qw;q^3)_m}, & n=3m;\\[2ex] \dfrac{q^{3m+1}w^{m}}{1-q^{6m+3}}\left\{\dfrac{(q^2/w;q^3)_m}{(q^2w;q^3)_m}-w\dfrac{(q/w;q^3)_{m+1}}{(qw;q^3)_{m+1}}\right\}, & n=3m+1;\\[2ex] \dfrac{-q^{3m+2}w^{m+1}}{1-q^{6m+5}}\dfrac{(q^2/w;q^3)_{m+1}}{(q^2w;q^3)_{m+1}}, & n=3m+2.\end{cases}$$

Then letting $\lambda=-1$ and $\mu=0$ in Lemma 3, we deduce two further formulae.

Corollary 24 $(\Omega_{-1,0}(w,q^{-n},q^{1+n}))$**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{n}\\ qw\end{matrix}\Big|\,q\right]_k\frac{(qw;q^3)_k}{(q;q)_{2k}}q^k
=\begin{cases}\dfrac{w^m}{1+q^{3m}}\left\{q^{4m}\dfrac{(1/w;q^3)_m}{(q^2w;q^3)_m}+q^m\dfrac{(q/w;q^3)_m}{(q^3w;q^3)_m}\right\}, & n=3m;\\[2ex] \dfrac{q^{4m+1}w^{m}}{1+q^{3m+1}}\dfrac{(q/w;q^3)_{m}}{(q^3w;q^3)_{m}}, & n=3m+1;\\[2ex] \dfrac{(qw)^{m+1}}{1+q^{3m+2}}\dfrac{(1/w;q^3)_{m+1}}{(q^2w;q^3)_{m+1}}, & n=3m+2.\end{cases}$$

Corollary 25 $(\Omega_{-1,0}(w,q^{-n},q^{2+n}))$**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},q^{1+n}\\ qw\end{matrix}\Big|\,q\right]_k\frac{(q^2w;q^3)_k}{(q;q)_{2k+1}}q^k
=\begin{cases}\dfrac{(q^2w)^m}{1-q^{6m+1}}\dfrac{(q^2/w;q^3)_m}{(q^3w;q^3)_m}, & n=3m;\\[2ex] \dfrac{q^{2m+1}w^{m}}{1-q^{6m+3}}\left\{q^{3m+1}\dfrac{(q^2/w;q^3)_m}{(q^3w;q^3)_m}-w\dfrac{(1/w;q^3)_{m+1}}{(qw;q^3)_{m+1}}\right\}, & n=3m+1;\\[2ex] \dfrac{-q^{5m+4}w^{m+1}}{1-q^{6m+5}}\dfrac{(1/w;q^3)_{m+1}}{(qw;q^3)_{m+1}}, & n=3m+2.\end{cases}$$

6. Evaluation of the $\Omega_{\lambda,\mu}(1,x,q^{-3n}/x)$-Sums

For the fourth terminating case, we begin with the following formula.

Theorem 26 ($\Omega_{0,0}(1,x,q^{-3n}/x)$).

$$\sum_{k=0}^{n}\left[\begin{matrix}x,q^{-3n}/x\\ q\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^{-3n};q)_{2k}}q^k=\frac{1}{x^n}\left[\begin{matrix}qx,q^2x\\ q,q^2\end{matrix}\middle|\,q^3\right]_n.$$

Proof. By multiplying both sides by x^n, the last equality becomes a polynomial identity of degree $2n$. Therefore we need only to verify the last equality for $2n+1$ distinct values of x. First, it is obvious that the equality holds for $x=1$. Then for $x=q^{1+\delta-3\ell}$ with $\delta=0,\ 1$ and $\ell=1,2,\cdots,n$, the right member is equal to zero. The corresponding left member reads as

$$\begin{aligned}&\sum_{k=0}^{n}\left[\begin{matrix}q^{1+\delta-3\ell},q^{3\ell-3n-1-\delta}\\ q\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^{-3n};q)_{2k}}q^k\\ =&\sum_{k=0}^{\lfloor 3n/2\rfloor}\left[\begin{matrix}q^{1+\delta-3\ell},q^{3\ell-3n-1-\delta}\\ q\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^{-3n};q)_{2k}}q^k,\end{aligned}$$

where the upper limit of the sum is justified by

$$k\le\min\big\{3\ell-1-\delta,3n-3\ell+1+\delta\big\}\le\lfloor 3n/2\rfloor\quad\text{for}\quad 1\le\ell\le n.$$

According to (4), the last sum is annihilated for $1+\delta\not\equiv_3 0$. Consequently, we have validated the equality in question for $2n+1$ distinct values of x and completed the proof of theorem. □

Letting $\lambda=0,\mu=0$ and $w\to q^{-1},x\to x/q,y\to q^{-3n-2}/x$ in Lemma 1 and then applying Theorem 26, we derive the following identity.

Corollary 27 ($\Omega_{-1,1}(1,x,q^{-1-3n}/x)$).

$$\begin{aligned}&\sum_{k=0}^{n}\left[\begin{matrix}x,q^{-2-3n}/x\\ q\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^{-1-3n};q)_{2k}}q^k\\ =&\frac{x^{-n}(1-q^{2+3n})}{1-q^{2+3n}x^2}\left\{\left[\begin{matrix}qx,q^2x\\ q,q^2\end{matrix}\middle|\,q^3\right]_{n+1}-q^{n+1}\left[\begin{matrix}x,qx\\ q,q^2\end{matrix}\middle|\,q^3\right]_{n+1}\right\}.\end{aligned}$$

Alternatively, letting $\lambda=-1$ and $\mu=0$ in Lemma 3 and then invoking Theorem 26, we get another summation formula.

Corollary 28 ($\Omega_{-1,0}(1,x,q^{-3n}/x)$).

$$\begin{aligned}&\sum_{k=0}^{n}\left[\begin{matrix}x,q^{-3n-1}/x\\ q\end{matrix}\middle|\,q\right]_k\frac{(q^{-3n};q^3)_k}{(q^{-3n};q)_{2k}}q^k\\ =&\frac{1}{x^n(1-q^{1+3n}x^2)}\left\{\frac{1-x}{q^n}\left[\begin{matrix}q^2x,q^3x\\ q,q^2\end{matrix}\middle|\,q^3\right]_n+x(1-qx)\left[\begin{matrix}q^2x,q^4x\\ q,q^2\end{matrix}\middle|\,q^3\right]_n\right\}.\end{aligned}$$

Finally, by combining Lemma 3 with Corollary 27, we obtain the identity below.

Corollary 29 ($\Omega_{-2,1}(1,x,q^{-1-3n}/x)$)**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}x,q^{-3-3n}/x\\q\end{matrix}\Big|\,q\right]_k\frac{q^k(q^{-3n};q^3)_k}{(q^{-1-3n};q)_{2k}}=\frac{(qx)^{-n}(1-x)(1-q^{2+3n})}{(1-q^{3+3n}x^2)(1-q^{4+3n}x^2)}\left[\begin{matrix}q^2x,q^3x\\q,q^2\end{matrix}\Big|\,q^3\right]_{n+1}$$
$$-\frac{qx(1-q^{3+3n}x)(1-q^{2+3n})}{(x/q)^n(1-q^{2+3n}x^2)(1-q^{3+3n}x^2)}\left[\begin{matrix}x,qx\\q,q^2\end{matrix}\Big|\,q^3\right]_{n+1}$$
$$+\frac{(qx-q+x-q^{4+3n}x^2)(1-q^{2+3n})}{x^n(1-q^{2+3n}x^2)(1-q^{4+3n}x^2)}\left[\begin{matrix}qx,q^2x\\q,q^2\end{matrix}\Big|\,q^3\right]_{n+1}.$$

7. Formulae for the Reversal $\Lambda(w,x,y)$-Series

Recall the $\Omega(w,x,y)$-series defined by (3). By considering the "negative part" of the sum with summation indices $k<0$ and then making the replacement $k\to -1-k$, we can express the resulting series equivalently as

$$\Lambda(w,x,y):=\sum_{k\geq 0}\left[\begin{matrix}w\\qx,qy\end{matrix}\Big|\,q\right]_k\frac{(qxy;q)_{2k}}{(q^3wxy;q^3)_k}q^k.$$

When both $\Omega(w,x,y)$-series and $\Lambda(w,x,y)$-series are terminating, it is not hard to check that they are reversal each other. Therefore, by examining reversals, we may translate the summation formulae for $\Omega(w,x,y)$-sums obtained in the preceding sections into $\Lambda(w,x,y)$-sums. Four of them are highlighted as examples.

Example 30 ($\Lambda(q^{-n},x,1)$: Reversal of Theorem 4)**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n}\\q,qx\end{matrix}\Big|\,q\right]_k\frac{(qx;q)_{2k}}{(q^{3-n}x;q^3)_k}q^k=\chi(n\equiv_3 0)\left[\begin{matrix}q,\quad q^2\\q^3x,\,1/x\end{matrix}\Big|\,q^3\right]_{\lfloor\frac{n}{3}\rfloor}.$$

Example 31 ($\Lambda(q^{n+1},x,q^{-1-n}/x)$: Reversal of Theorems 14 and 15)**.**

$$\sum_{k=0}^{\lfloor n/2\rfloor}\left[\begin{matrix}q^{1+n}\\qx,q^{-n}/x\end{matrix}\Big|\,q\right]_k\frac{(q^{-n};q)_{2k}}{(q^3;q^3)_k}q^k=\frac{(q^{2-n}x;q^3)_n}{(qx;q)_n}.$$

Example 32 ($\Lambda(w,1,q^{-1-n})$: Reversal of Theorems 20 and 21)**.**

$$\sum_{k=0}^{\lfloor n/2\rfloor}\left[\begin{matrix}w\\q,q^{-n}\end{matrix}\Big|\,q\right]_k\frac{(q^{-n};q)_{2k}}{(q^{2-n}w;q^3)_k}q^k=\chi(n\not\equiv_3 2)\left[\begin{matrix}q^{2-n}/w\\q^{2-n}w\end{matrix}\Big|\,q^3\right]_{\lfloor\frac{n}{3}\rfloor}w^{\lfloor\frac{n}{3}\rfloor}.$$

Example 33 ($\Lambda(q^{-n},x,q^n/x)$: Reversal of Theorem 26)**.**

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n}\\qx,q^{1+n}/x\end{matrix}\Big|\,q\right]_k\frac{(q^{1+n};q)_{2k}}{(q^3;q^3)_k}q^k=\frac{(q^{1-n}/x;q^3)_n(q^{2-n}/x;q^3)_n}{(q^{-n}/x;q)_n(q^{1+n}/x;q)_n}.$$

Acknowledgement: *This work is supported for the first author by the Natural Sciences Foundation of China under Grant No. 11201484 and the Fundamental Research Funds for the Central Universities (14CX02016A).*

References

[1] G. E. Andrews, *Connection coefficient problems and partitions,* Relations between combinatorics and other parts of mathematics (Ohio State Univ., Columbus, Ohio, 1978), pp. 1–24; Proc. Sympos. Pure Math. 34: Amer. Math. Soc., Providence, R.I., 1979.

[2] W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge University Press, Cambridge, 1935.

[3] W. N. Bailey, *Series of hypergeometric type which are infinite in both directions,* Quart. J. Math. (Oxford) 7 (1936), 105–115.
[4] L. Carlitz, *Some inverse series relations,* Duke Math. J. 40 (1973), 893-901.
[5] W. Chu, *Inversion Techniques and Combinatorial Identities: Strange evaluations of basic hypergeometric series,* Compositio Mathematica 91 (1994), 121-144; **MR** 95h:33011 & **Zbl** 807:33014.
[6] W. Chu, *Basic Hypergeometric Identities: An introductory revisiting through the Carlitz inversions,* Forum Mathematicum 7 (1995), 117-129; MR95m:33015 & Zbl.815:05009.
[7] W. Chu, *Bailey's very well–poised* ${}_6\psi_6$*-series identity,* Journal of Combinatorial Theory (Series A) 113:6 (2006), 966–979.
[8] I. Gessel & D. Stanton, *Applications of q-Lagrange inversion to basic hypergeometric series,* Trans. Amer. Math. Soc. 277 (1983), 173–203.
[9] G. Gasper & M. Rahman, *Basic Hypergeometric Series* (2nd ed.), Cambridge University Press, Cambridge, 2004.
[10] S. O. Warnaar, *q-Hypergeometric proofs of polynomial analogues of the triple product identity, Lebesgue's identity and Euler's pentagonal number theorem,* The Ramanujan Journal 8:4 (2005), 467–474.

School of Statistics, Qufu Normal University
Qufu 273165 (Shandong), P. R. China

E-mail address: upcxjchen@163.com

Dipartimento di Matematica e Fisica "Ennio De Giorgi"
Università del Salento, Lecce–Arnesano P. O. Box 193
73100 Lecce, Italy

E-mail address: chu.wenchang@unisalento.it