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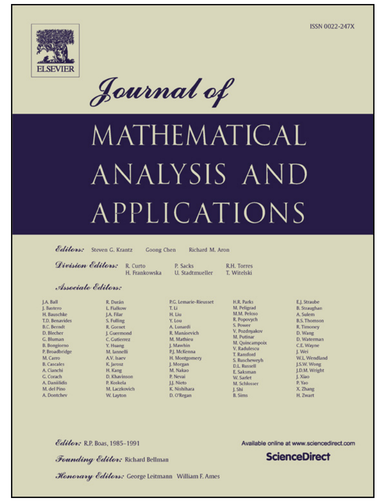

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EXISTENCE OF STRONG SOLUTIONS AND DECAY OF TURBULENT SOLUTIONS OF NAVIER-STOKES FLOW WITH NONZERO DIRICHLET BOUNDARY DATA

REINHARD FARWIG, HIDEO KOZONO, AND DAVID WEGMANN

Abstract. Recently, Leray's problem of the L^2-decay of a special weak solution to the Navier-Stokes equations with nonhomogeneous boundary values was studied by the authors, exploiting properties of the approximate solutions converging to this solution. In this paper this result is generalized to the case of an arbitrary weak solution satisfying the strong energy inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain or an exterior domain with $\partial\Omega \in C^{1,1}$ and let us consider the non-stationary Navier-Stokes equations with viscosity $\nu = 1$ and data f, β, u_0 in the form

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \\ u_{|\partial\Omega} = \beta, \quad u(0) = u_0 \end{aligned} \tag{1.1}$$

in the time interval $[0,T)$, $0 < T \le \infty$. To prove the existence of a weak solution to (1.1) we assume that

$$\begin{aligned} & f = \operatorname{div} F,\ F \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)),\ u_0 \in \mathrm{L}^2_\sigma(\Omega), \\ & \beta \in \mathrm{L}^4(0,T;W^{-\frac14,4}(\partial\Omega)) \cap \mathrm{L}^{s_0}(0,T;W^{-\frac{1}{q_0},q_0}(\partial\Omega)), \\ & \frac{2}{s_0} + \frac{3}{q_0} = 1,\ 2 < s_0 < \infty,\ 3 < q_0 < \infty, \int_{\partial\Omega} n\cdot\beta \, \mathrm{d}\sigma = 0, \end{aligned} \tag{1.2}$$

where $n = n(x)$ denotes the exterior normal on the boundary at $x \in \partial\Omega$. The existence of weak solutions was studied by several authors since the early 2000s, see [1, 4–14, 20]. The most common way to deal with non-homogeneous boundary data is to split the problem into two parts. The first one is to construct a solution b to the non-stationary Stokes equations

$$\begin{aligned} b_t - \Delta b + \nabla \bar{p} = 0, \quad \operatorname{div} b = 0 \\ b_{|\partial\Omega} = \beta, \quad b(0) = 0. \end{aligned} \tag{1.3}$$

Then a function u is a solution to (1.1) if and only if the function $v := u - b$ is a solution to

$$\begin{aligned} v_t - \Delta v + (v+b)\cdot\nabla(v+b) + \tilde{p} = f, \quad \operatorname{div} v = 0 \\ v_{|\partial\Omega} = 0, \quad v(0) = u_0 \end{aligned} \tag{1.4}$$

and we will call the equations (1.4) *perturbed Navier-Stokes equations.* To consider (1.4) instead of (1.1) leads to the advantage of not dealing with non-homogeneous boundary data, at the cost of a

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more difficult non-linear term. Nevertheless, we will construct a solution to (1.1) via constructing a solution v to (1.4). Since $\operatorname{div} v = \operatorname{div} b = 0$ the non-linear term in (1.4) can be rewritten as

$$(v+b)\cdot\nabla(v+b) = \operatorname{div}\left((v+b)\otimes(v+b)\right) = \operatorname{div}\left(v\otimes v + v\otimes b + b\otimes v + b\otimes b\right).$$

Here $\otimes$ denotes the *dyadic product*, *i.e.*, $v\otimes b := (v_i b_j)_{1\le i,j\le 3}$, and the divergence is taken column-wise. For simplicity we will omit the symbol $\otimes$ and write $vb := v\otimes b$. Hence, with the $\mathrm{L}^2(\Omega)$-scalar product $\langle\cdot,\cdot\rangle$, we have $\langle bv, \nabla w\rangle = -\langle bw, \nabla v\rangle$ since $v = 0$ on $\partial\Omega$. To deal with the non-linear term during the process of construction of a weak solution one needs to assume that

$$vb + bv + bb \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$$

or equivalently

$$b \in \mathrm{L}^{s_0}(0,T;\mathrm{L}^{q_0}(\Omega)) \cap \mathrm{L}^4(0,T;\mathrm{L}^4(\Omega)). \tag{1.5}$$

Since we do not need any integrability conditions for ∇b, it usually suffices to deal with a *very weak* solution b to (1.3) and so we just need to assume that β takes values in a dual space of a trace space, *cf.* (1.2). Let us define the notion of a weak solution, a turbulent solution, and a strong solution to (1.4).

Definition 1.1. Let b satisfy (1.5) and assume $u_0 \in \mathrm{L}^2_\sigma(\Omega)$ and $f = \operatorname{div} F$, $F \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. Then a vector field v on $(0,T)\times\Omega$ is called a Leray-Hopf type *weak solution* to the perturbed Navier-Stokes system (1.4) if the following conditions are satisfied:

(1) $v \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, $\nabla v \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$,

(2) for each test function $\varphi \in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$ the equality

$$-\langle v,\varphi_t\rangle_{\Omega,T} + \langle\nabla v,\nabla\varphi\rangle_{\Omega,T} - \langle(v+b)(v+b),\nabla\varphi\rangle_{\Omega,T} = \langle u_0,\varphi(0)\rangle_\Omega - \langle F,\nabla\varphi\rangle_{\Omega,T} \tag{1.6}$$

is fulfilled,

(3) the energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F-(v+b)b,\nabla v\rangle\,\mathrm{d}\tau \tag{1.7}$$

holds for all $t\in(0,T)$.

After a redefinition on a null set we may assume that the weak solution is weakly continuous in time with values in $\mathrm{L}^2(\Omega)$, *cf.* [32, Ch. V, 1.3.1 Theorem]. As already mentioned, (1.7) is called energy inequality. Furthermore, we will say that a weak solution v fulfills the *strong energy inequality*, if in addition to (1.7)

$$\frac{1}{2}\|v(t)\|_2^2 + \int_s^t \|\nabla v\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|v(s)\|_2^2 - \int_s^t \langle F-(v+b)b,\nabla v\rangle\,\mathrm{d}\tau \tag{1.8}$$

holds for almost all $s\in(0,T)$ and all $t\in(s,T)$, and finally, a solution v fulfills the *energy equality*, if

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F-(v+b)b,\nabla v\rangle\,\mathrm{d}\tau \tag{1.9}$$

holds for all $t\in(0,T)$.

Definition 1.2. Let v be a weak solution to (1.4) in the sense of Definition 1.1.

(1) The solution v is called a *turbulent solution* to (1.4), if v fulfills the strong energy inequality (1.8).

(2) The solution v is called a *strong solution in the sense of Serrin* to (1.4), if

$$v \in \mathrm{L}^s(0,T;\mathrm{L}^q(\Omega)) \text{ for some } 2<s<\infty \text{ and } 3<q<\infty, \text{ where } \frac{2}{s}+\frac{3}{q}=1.$$

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Note that if v is a strong solution to (1.4), using a density argument one can prove that v can be used as a test function in (1.6). Therefore, this solution fulfills the energy equality (1.9). Furthermore, following the proof of [32, Ch. V, 1.4.1 Theorem] one can prove that after redefinition on a null set $v \in C^0([0,T);\mathrm{L}^2(\Omega))$.

To state our first main theorem, let A denote the Stokes operator in a bounded or exterior domain Ω and let $e^{-\tau A}$, $\tau \geq 0$, denote the semigroup of the Stokes operator. Let us define

$$\|u_0\|_{\mathcal{B}_T^{q,s}} := \Big(\int_0^T \|e^{-\tau A}u_0\|_q^s \mathrm{d}\tau\Big)^{\frac{1}{s}}.$$

For more details to this Besov-type norm see [15–18]. Furthermore, in the Theorems 1.3, 1.5-1.7 we need to assume that b fulfills an additional integrability condition. Let us assume that

$$b \in \mathrm{L}^{s_1}(0,\infty;\mathrm{L}^{q_1}(\Omega)), \quad \text{where } \frac{1}{s_0}+\frac{1}{s_1}=\frac{1}{q_0}+\frac{1}{q_1}=\frac{1}{2}. \tag{1.10}$$

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{1,1}$ be a domain with compact boundary. Let $u_0 \in L^2_\sigma(\Omega)$ such that $\|u_0\|_{\mathcal{B}_T^{q_0,s_0}} < \infty$ and $F \in L^{\frac{s_0}{2}}(0,T;L^{\frac{q_0}{2}}(\Omega))$, $2 < s_0 < \infty$, $3 < q_0 < \infty$, $2/s_0 + 3/q_0 = 1$ and let b satisfy (1.5) and (1.10). Then there exists a constant $\varepsilon_\star > 0$ with the following property: For any time $0 < T' \leq T$ such that*

$$\|F\|_{L^{\frac{s_0}{2}}(0,T';L^{\frac{q_0}{2}}(\Omega))} + \|b\|_{L^{s_0}(0,T';L^{q_0}(\Omega))} + \|u_0\|_{\mathcal{B}_T^{q_0,s_0}} < \varepsilon_\star,$$

there exists a unique strong solution u to (1.4) with

$$u \in L^{s_0}(0,T';L^{q_0}(\Omega)).$$

In a second step, we will deal with a generalization of Serrin's Uniqueness Theorem.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{1,1}$ be a domain with compact boundary. Furthermore, let b fulfill (1.5) and let $f = \mathrm{div} F$, $F \in L^2(0,T;L^2(\Omega))$ and $u_0 \in L^2_\sigma(\Omega)$. Finally, let v be a weak solution to (1.4) and let w be a strong solution to (1.4) in $\Omega \times (0,T)$. Then $v = w$.*

Note that in our Definition 1.1 it is already assumed that the weak solution v fulfills the energy inequality. Hence, in the case $b = 0$ this result reproves Serrin's Uniqueness Theorem.

In Theorems 1.5, 1.6 and 1.7 below we will consider Leray's problem of the L^2-decay of a weak solution to the perturbed Navier-Stokes equations (1.4). Let us remark that so far only the existence of a *special solution* to (1.4) which tends to 0 is known, see [13]. This solution was constructed using Yosida approximation and this generalized the result of Borchers and Miyakawa [3]. Theorems 1.5-1.7 will extend the results of [13] to the case of *arbitrary turbulent solutions*.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{1,1}$, let $F \in L^2(0,\infty;L^2(\Omega))$, $f = \mathrm{div}\,\mathrm{F}$, $u_0 \in L^2_\sigma(\Omega)$, and let b satisfy (1.5) and (1.10). Furthermore, assume that there exists a $T > 0$ such that $F \in L^{\frac{s_0}{2}}(T,\infty;L^{\frac{q_0}{2}}(\Omega))$. Then every turbulent solution v to (1.4) in the sense of Definition 1.1 fulfills*

$$\lim_{t\to\infty} \|v(t)\|_2 = 0.$$

Let $\alpha \in (0,1)$, $\beta > 0$, and assume in addition that

$$\|F\|^2_{L^2(\alpha t,t;L^2(\Omega))} + \|b\|^4_{L^4(\alpha t,t;L^4(\Omega))} + \|b\|^{s_0}_{L^{s_0}(\alpha t,t;L^{q_0}(\Omega))} = O(\exp(-\beta t)) \quad \text{as } t \to \infty. \tag{1.11}$$

Then it holds

$$\|v(t)\|_2 = O(\exp(-t\gamma)) \quad \text{as } t \to \infty$$

for every turbulent solution v, where $\gamma := \min\{(1-\alpha)\rho, \beta\}$ and $\sqrt{\rho}$ denotes the largest constant for which Poincaré's inequality holds, i.e., *$\rho\|u\|_2^2 \leq \|\nabla u\|_2^2$ for all $u \in W_0^{1,2}(\Omega)$.*

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Theorem 1.6. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{1,1}$. In addition to the assumptions $F \in L^2(0,\infty; L^2(\Omega))$, $f = \operatorname{div} \mathrm{F}$, $u_0 \in L^2_\sigma(\Omega)$ and (1.5) and (1.10) for b suppose that there exist $\frac{6}{5} < r < 2$ and $2 < s_2 < s_0$ such that $F \in L^2(0,\infty; L^r(\Omega))$ and $b \in L^4(0,\infty; L^{2r}(\Omega)) \cap L^{s_2}(0,\infty; L^{q_0}(\Omega))$. Furthermore, assume that there exists a $T > 0$ such that $F \in L^{\frac{s_0}{2}}(T,\infty; L^{\frac{q_0}{2}}(\Omega))$. Then every turbulent solution v to (1.4) in the sense of Definition 1.1 fulfills*

$$\lim_{t\to\infty} \|v(t)\|_2 = 0.$$

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{1,1}$, assume $F \in L^2(0,\infty; L^2(\Omega))$, $f = \operatorname{div} \mathrm{F}$, $u_0 \in L^2_\sigma(\Omega)$, and let b satisfy (1.5) and (1.10). Furthermore, assume that there exists a $T > 0$ such that $F \in L^{\frac{s_0}{2}}(T,\infty; L^{\frac{q_0}{2}}(\Omega))$. Assume that*

$$\begin{aligned}
\nabla b &\in L^{s_2}(0,\infty; L^{q_2}(\Omega)), \quad \frac{2}{s_2} + \frac{3}{q_2} = 2, \quad q_2 > 2,\\
b &\in L^{s_3}(0,\infty; L^{q_3}(\Omega)), \quad \frac{2}{s_3} + \frac{3}{q_3} =: S(b), \quad \frac{3}{2} < q_3 \le 3,\\
b \cdot \nabla b &\in L^1(0,\infty; L^2(\Omega)),\\
\nabla F &\in L^1(0,\infty; L^2(\Omega)),\\
F &\in L^{s_4}(0,\infty; L^{q_4}(\Omega)), \quad \frac{2}{s_4} + \frac{3}{q_4} =: S(F), \quad \frac{3}{2} \le q_4 \le 2,
\end{aligned} \tag{1.12}$$

and that

$$\|\nabla F\|_{L^1(t/2,t;L^2(\Omega))} + \|\nabla b\|_{L^{s_2}(t/2,t,L^{q_2}(\Omega))} + \|b\cdot\nabla b\|_{L^1(t/2,t,L^2(\Omega))} = O(t^{-\alpha_2}) \quad \text{as } t \to \infty \tag{1.13}$$

for some $\alpha_2 > 0$. Moreover, suppose that

$$\|e^{-tA}u_0\|_2 \in O(t^{-\alpha_1}) \tag{1.14}$$

for some $\alpha_1 > 0$. Then every turbulent solution v to (1.4) fulfills

$$\|v(t)\|_2 = O(t^{-\alpha}) \quad \text{as } t \to \infty \tag{1.15}$$

where

$$\alpha := \min\left\{\frac{3}{4}, S(b) - \frac{5}{4}, S(F) - \frac{5}{4}, \alpha_1, \alpha_2\right\}. \tag{1.16}$$

Remark 1.8.
1. Leray's question of the L^2-decay of a weak solution was answered first by Kato [27] and Masuda [28].
2. Borchers and Miyakawa in [3, Theorem 1] proved a polynomial decay rate for weak solutions in the exterior domain case. Their result was generalized to the case of $f \neq 0 \neq b$ in [13].
3. In this paper, Theorem 2 in [3] is generalized to the case of $f \neq 0 \neq b$.
4. Our result is closely related to the stability of solutions because the velocity fields b and v in (1.4) may be regarded as the basic flow and its perturbation, respectively. Similarly to the class (1.12), there are a number of results to deal with the optimal class of b which exhibits stability. See for instance, Hishida-Schonbek [24] and Karch-Pilarczyk-Schonbek [26].

The outline of this paper is as follows. After some preliminaries and a summary of well known results to the Stokes equations we will prove the existence of a strong solution in Section 4 and the uniqueness of a strong solution in Section 5. Finally, we will prove our decay results in the last Section 6.

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2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a domain with compact boundary and $\partial\Omega \in C^{1,1}$. As usual, we will use the Lebesgue spaces $\mathrm{L}^s(\Omega)$ with norm $\|\cdot\|_s$, $1 \le s \le \infty$ and Lebesgue-Bochner spaces $\mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))$, $1 \le s, q \le \infty$, equipped with the norm $\|\cdot\|_{\mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))} =: \|\cdot\|_{q,s,T}$; the index T will be omitted frequently. The pairing of functions will be denoted by $\langle\cdot,\cdot\rangle$ or $\langle\cdot,\cdot\rangle_{\Omega,T}$ respectively. Furthermore, we will use standard Sobolev spaces $W^{s,q}(\Omega)$ with corresponding trace spaces or dual of trace spaces $W^{s-1/q,q}(\partial\Omega)$, respectively, $s \in \mathbb{R}$, $1 \le q \le \infty$.

To define weak and very weak solutions to (1.3) and (1.4) we need to introduce the space of test functions $C_0^\infty(\Omega)$, of solenoidal test functions $C_{0,\sigma}^\infty(\Omega) := \{\varphi \in C_0^\infty(\Omega) \mid \operatorname{div}\varphi = 0\}$ and the set

$$C_{0,\sigma}^2(\overline{\Omega}) := \{\varphi \in C^2(\overline{\Omega}) \mid \operatorname{div}\varphi = 0,\ \operatorname{supp}(\varphi) \subset \overline{\Omega} \text{ is compact and } \varphi = 0 \text{ on } \partial\Omega\}.$$

Finally, let $\mathrm{L}^q_\sigma(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$.

The existence of a projection $\mathrm{P}_q\colon \mathrm{L}^q(\Omega) \to \mathrm{L}^q_\sigma(\Omega)$, the so called Helmholtz projection, is well known for all $1 < q < \infty$. Since $\mathrm{P}_q(f) = \mathrm{P}_r(f)$ for all $1 < q < r < \infty$ and $f \in \mathrm{L}^r(\Omega) \cap \mathrm{L}^q(\Omega)$ we will omit the index q. Using the Helmholtz projection let us define the Stokes operator

$$A := A_q := -\mathrm{P}_q\Delta\colon \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap \mathrm{L}^q_\sigma(\Omega) \subset \mathrm{L}^q_\sigma(\Omega) \to \mathrm{L}^q_\sigma(\Omega).$$

For $-1 \le \alpha \le 1$ the fractional powers $A_q^\alpha\colon \mathcal{D}(A_q^\alpha) \to \mathrm{L}^q_\sigma(\Omega)$ are well defined, injective, densely defined with dense range and it holds that $(A_q^\alpha)^{-1} = A_q^{-\alpha}$. It is well known that

$$\|A_q^{\frac12} u\|_q \le c\|\nabla u\|_q, 1 < q < \infty. \tag{2.1}$$

A converse estimate holds if Ω is bounded or, if Ω is an exterior domain, for $1 < q < 3$. Moreover, it holds that $\|A_2^{\frac12} u\|_2 = \|\nabla u\|_2$ for all $u \in \mathcal{D}(A_2^{\frac12})$. Note that we also have

$$\|u\|_q \le c\|A^\alpha u\|_\gamma,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 1 < \gamma < 3. \tag{2.2}$$

The Stokes operator $-A_q$ generates a bounded analytic semigroup $\{e^{-tA_q} \mid t \ge 0\}$ on $\mathrm{L}^q_\sigma(\Omega)$ satisfying

$$\begin{aligned} &\|A_q^\alpha e^{-tA_q} v\|_q \le ct^{-\alpha}\|v\|_q,\ 0 \le \alpha \le 1,\ t > 0,\\ &\|e^{-tA_q} v\|_p \le ct^{-\frac32(\frac1q-\frac1p)}\|v\|_q,\ 1 < q \le p < \infty\\ &\|\nabla e^{-tA_q} v\|_p \le ct^{-\frac12-\frac32(\frac1q-\frac1p)}\|v\|_q,\ 1 < q \le p \le 3 \end{aligned} \tag{2.3}$$

for all $v \in \mathrm{L}^q_\sigma(\Omega)$, see [3, 23]. Moreover, the Stokes operator admits the property of maximal regularity, *i.e.*, for all $f \in \mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))$, $1 < s, q < \infty$ the instationary Stokes system

$$v_t + A_q v = P_q f, \quad v(0) = 0$$

has a unique solution $v \in C([0,T];\mathrm{L}^q(\Omega))$ such that $v_t, A_q v \in \mathrm{L}^s(0,T;\mathrm{L}^q_\sigma(\Omega))$ satisfying the *a priori* estimate

$$\|v_t\|_{q,s,T} + \|A_q\|_{q,s,T} \le c\|f\|_{q,s,T} \tag{2.4}$$

with a constant c independent of T. If $f = \operatorname{div} F$, this solution is given by

$$v(t) = e^{-tA}u_0 + A^{\frac12}\int_0^t e^{-(t-s)A}A^{-\frac12}\mathrm{P}\operatorname{div} F(s)\,\mathrm{d}s. \tag{2.5}$$

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We note that the formal operator $A^{-\frac{1}{2}}\mathrm{P}\mathrm{div}$ is well-defined as a bounded operator from $\mathrm{L}^q(\Omega)$ to $\mathrm{L}^q_\sigma(\Omega)$ for each $1 < q < \infty$. For these and further properties of the Stokes operator we refer to [21, 22, 29, 32].

3. Stokes Equations and Weak Solutions

As already discussed a weak solution to (1.1) will be constructed as a sum of a weak solution to (1.4) and a very weak solution to (1.3). Therefore, we recall the notion of a very weak solution to the stationary Stokes equations

$$-\Delta b + \nabla p = 0, \quad \operatorname{div} b = 0, \quad b_{|\partial\Omega} = \beta \tag{3.1}$$

and the instationary Stokes equations

$$\begin{aligned} b_t - \Delta b + \nabla p = 0, \quad \operatorname{div} b = 0 \\ b(0) = 0, \quad b_{|\partial\Omega} = \beta \end{aligned} \tag{3.2}$$

for the special case of homogeneous data except for nonzero boundary values.

Definition 3.1. Let $\Omega \subset \mathbb{R}^3$ be a domain with compact $C^{1,1}$-boundary. Further, let $1 < q < \infty$ and let $\beta \in W^{-\frac{1}{q},q}(\partial\Omega)$ such that $\int_{\partial\Omega} n \cdot \beta \,\mathrm{d}\sigma = 0$ in an appropriate weak sense. A vector field $b \in \mathrm{L}^q(\Omega)$ is called a very weak solution to (3.1) if

$$\begin{aligned} -\langle b, \Delta\varphi\rangle = -\langle \beta, n\cdot\nabla\varphi\rangle_{\partial\Omega} \\ \operatorname{div}\, b = 0, \quad b_{|\partial\Omega}\cdot n = \beta\cdot n \end{aligned} \tag{3.3}$$

for all $\varphi \in C^2_{0,\sigma}(\overline{\Omega})$. Here and hereafter n denotes the outer unit normal vector of Ω.

Note that the term $\langle \beta, n\cdot\nabla\varphi\rangle_{\partial\Omega}$ only prescribes the tangential part of b on $\partial\Omega$ since div $\varphi = 0$ and hence $n\cdot\nabla\varphi$ is tangential. A similar remark holds for the time-dependent case to be defined in Definition 3.2 below. The existence and uniqueness of very weak solutions to the stationary Stokes equations for arbitrary data $\beta \in W^{-\frac{1}{q},q}(\partial\Omega)$ is well known, see for instance [9, Theorem 1.6]. Note that

$$\|b\|_{\mathrm{L}^q(\Omega)} \le c\|\beta\|_{W^{-\frac{1}{q},q}(\partial\Omega)} \tag{3.4}$$

with a constant c independent of β.

Definition 3.2. Let $\Omega \subset \mathbb{R}^3$ be a domain with compact $C^{1,1}$-boundary. Let $1 < q < \infty$ and $\beta \in \mathrm{L}^s(0,\infty; W^{-\frac{1}{q},q}(\partial\Omega))$ such that $\int_{\partial\Omega} n\cdot\beta(t)\,\mathrm{d}\sigma = 0$ for *a.a.* t in an appropriate weak sense. A function $b \in \mathrm{L}^s(0,\infty;\mathrm{L}^q(\Omega))$ is called a very weak solution to (3.2) if

$$\begin{aligned} -\langle b, \varphi_t\rangle_{\Omega,\infty} - \langle b, \Delta\varphi\rangle_{\Omega,\infty} = -\langle \beta, n\cdot\nabla\varphi\rangle_{\partial\Omega,\infty} \\ \operatorname{div}\, b = 0,\ b(0) = 0,\ b_{|\partial\Omega}\cdot n = \beta\cdot n \end{aligned} \tag{3.5}$$

is satisfied for every $\varphi \in C^1_0([0,\infty); C^2_{0,\sigma}(\overline{\Omega}))$.

Next, we will briefly discuss the construction of very weak solutions to (3.2). Let $\gamma = \gamma(t)$ denote the solution to

$$-\Delta\gamma(t) + \nabla p = 0, \quad \operatorname{div}\gamma(t) = 0, \quad \gamma(t)_{|\partial\Omega} = \beta(t).$$

In [9, Lemma 4.1] it is shown that the unique very weak solution to (3.2) is given by

$$b(t) = \int_0^t A_q e^{-(t-\tau)A_q}\mathrm{P}_q\gamma(\tau)\,\mathrm{d}\tau. \tag{3.6}$$

Combining (3.4) and (3.6) we conclude from the maximal regularity estimate that

$$\|b\|_{q,s} \le c\|\beta\|_{\mathrm{L}^s(0,\infty;W^{-\frac{1}{q},q}(\partial\Omega))}. \tag{3.7}$$

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4. Existence of Strong Solutions

In this section, we are going to prove the existence of a strong solution to (1.4) in the sense of Definition 1.2. Note that the existence of a strong solution in the bounded domain case is proved in [19]. Hence, let us assume that Ω is an exterior domain such that $\partial\Omega \in C^{1,1}$. For the proof we will follow the ideas of [19]. Let us start with an elementary lemma, see [19, 32, 33].

Lemma 4.1. *Let X be a Banach space and $\mathcal{F}\colon X \to X$ be a continuous operator such that $\|\mathcal{F}(x)\| \le \alpha(\|x\| + \beta)^2$ for all $x \in X$ and some $\alpha, \beta > 0$, $4\alpha\beta < 1$, and let us assume that*

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \le \alpha(\|x\| + \|y\| + 2\beta)\|x - y\| \tag{4.1}$$

for all $x, y \in X$. Then there exists an $r > 0$ such that $\mathcal{F}(\overline{B_r}) \subset \overline{B_r}$; moreover, in $\overline{B_r}$ there exists a unique fixed point of $\mathcal{F}$. Here $\overline{B_r}$ denotes the closed ball $\overline{B_r} := \{x \in X \mid \|x\| \le r\}$.

Proof. Let $y_1 := (2\alpha)^{-1}(1 - \sqrt{1 - 4\alpha\beta}) = 2\beta(1 + \sqrt{1 - 4\alpha\beta})^{-1} \in (\beta, 2\beta)$ denote the smallest root of $y - \alpha y^2 - \beta$ and let $r := y_1 - \beta$. Let $x \in \overline{B_r}$. Then

$$\|\mathcal{F}(x)\| \le \alpha(\|x\| + \beta)^2 \le \alpha y_1^2 = y_1 - \beta = r$$

and hence $\mathcal{F}(\overline{B_r}) \subset \overline{B_r}$. Furthermore, estimate (4.1) implies that $\mathcal{F}$ is a strict contraction and the Banach Fixed Point Principle completes the proof. □

The main part of the proof of Theorem 1.3 will be given in the next Proposition. To fix some notation let $X = X_T = \mathrm{L}^s(0, T; \mathrm{L}^q_\sigma(\Omega))$ and define

$$\begin{aligned} v_0(t) &:= e^{-tA}u_0 + \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A}(A^{-\frac{1}{2}}\mathrm{P}\mathrm{div}\,)F(\tau)\,\mathrm{d}\tau \\ &=: V_0(t) + V_1(t). \end{aligned}$$

Then we consider on X the nonlinear operator

$$\mathcal{F} : w \mapsto \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A}(A^{-\frac{1}{2}}\mathrm{P}\mathrm{div}\,)((w + v_0 + b)(w + v_0 + b))(\tau)\,\mathrm{d}\tau.$$

Proposition 4.2. *Let Ω be an exterior domain with $\partial\Omega \in C^{1,1}$, let $u_0 \in \mathcal{B}_T^{q_0,s_0}$. Furthermore, let $F \in L^{\frac{s_0}{2}}(0, T; L^{\frac{q_0}{2}}(\Omega))$, $2 < s_0 < \infty$, $3 < q_0 < \infty$, $2/s_0 + 3/q_0 = 1$, and let b satisfy (1.5). Then there exists a constant $\varepsilon_\star > 0$ with the following property: For any $0 < T' \le T$ such that*

$$\|F\|_{L^{\frac{s_0}{2}}(0,T';L^{\frac{q_0}{2}}(\Omega))} + \|b\|_{L^{s_0}(0,T',L^{q_0}(\Omega))} + \|u_0\|_{\mathcal{B}_{T'}^{q_0,s_0}} < \varepsilon_\star,$$

the operator $\mathcal{F}\colon X_{T'} \to X_{T'}$ fulfills the assumptions of Lemma 4.1.

Proof. First we show that $v_0 \in X = X_{T'}$. Actually, $V_0 \in X$ by definition since $u_0 \in \mathcal{B}_T^{q_0,s_0}$, and $\|V_0\|_X \le \|u_0\|_{\mathcal{B}_T^{q_0,s_0}}$. Concerning V_1 we use (2.3) and the boundedness of $A^{-\frac{1}{2}}\mathrm{P}\mathrm{div}$ to get for $t \in (0, T)$ that

$$\begin{aligned} \|V_1(t)\|_{q_0} &\le \int_0^t \|e^{-(t-\tau)A/2} A^{\frac{1}{2}} e^{-(t-\tau)A/2}(A^{-\frac{1}{2}}\mathrm{P}\mathrm{div}\,)F(\tau)\|_{q_0}\,\mathrm{d}\tau \\ &\le c\int_0^t (t-\tau)^{-\frac{3}{2q_0}}\|A^{\frac{1}{2}} e^{-(t-\tau)A/2}(A^{-\frac{1}{2}}\mathrm{P}\mathrm{div}\,)F(\tau)\|_{q_0/2}\,\mathrm{d}\tau \\ &\le c\int_0^t (t-\tau)^{-\frac{3}{2q_0}-\frac{1}{2}}\|F(\tau)\|_{q_0/2}\,\mathrm{d}\tau. \end{aligned}$$

Then the Hardy-Littlewood-Sobolev inequality implies that

$$\|V_1\|_{q_0,s_0} \le c\|F\|_{q_0/2,s_0/2}.$$

Hence $v_0 \in X$ and $\|v_0\|_X \le c(\|u_0\|_{\mathcal{B}_T^{q_0,s_0}} + \|F\|_{q_0/2,s_0/2})$.

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For an arbitrary $w \in X$ we apply the estimates above to $\mathcal{F}w(t)$ to get that

$$\begin{aligned}\|\mathcal{F}w\|_{q_0,s_0} &\le c\|(w+v_0+b)(w+v_0+b)\|_{q_0/2,s_0/2}\\ &\le c\|w+v_0+b\|^2_{q_0,s_0}.\end{aligned}$$

Thus $\mathcal{F}$ maps $X_{T'}$ into $X_{T'}$ and

$$\|\mathcal{F}(w)\|_X \le c(\|w\|_X + \beta(T'))^2$$

with $\beta(T') = \|v_0\|_{q_0,s_0,T'} + \|b\|_{q_0,s_0,T'}$. By analogy, for given $w_1, w_2 \in X$, we obtain that

$$\begin{aligned}\|\mathcal{F}(w_1)-\mathcal{F}(w_2)\|_X &\le c\|w_1-w_2\|_{q_0,s_0}\big(\|w_1+v_0+b_0\|_{q_0,s_0} + \|w_2+v_0+b_0\|_{q_0,s_0}\big)\\ &\le c\|w_1-w_2\|_X(\|w_1\|_X + \|w_2\|_X + 2\beta(T')).\end{aligned}$$

Choosing $T' \in (0,T]$ so small such that $4c\beta(T') < 1$ we complete the proof of Proposition 4.2. □

Using Proposition 4.2 we will easily prove Theorem 1.3.

Proof of Theorem 1.3. Note that $v := v_0 + w$ fulfills

$$v(t) = e^{-tA}u_0 + \int_0^t A^{\frac12}e^{-(t-\tau)A}(A^{-\frac12}\mathrm{P}\mathrm{div}\,)(F-(v+b)(v+b))(\tau)\,\mathrm{d}\tau \tag{4.2}$$

and hence v is a weak solution to (1.4) if v is in the Leray-Hopf class. Let us prove first that $\nabla v \in \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))$. The assumptions on F, b in (1.2), (1.5), and (1.10) implies that $\tilde F := F - vb - bv - bb \in \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))$. Thus, we obtain that

$$\tilde v = e^{-tA}u_0 + \int_0^t A^{\frac12}e^{-(t-\tau)A}A^{-\frac12}\mathrm{P}\mathrm{div}\,\tilde F(\tau)\,\mathrm{d}\tau$$

is a weak solution to the nonstationary Stokes equations with initial value u_0 and right hand side $\mathrm{div}\,\tilde F$, see [32, Ch. V, Theorem 2.4.1]; in particular, $\nabla\tilde v \in \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))$. Thus $\tilde v$ is in the Leray-Hopf class with

$$\|\tilde v\|_{q_1,s_1,T'} \le c(\|\tilde v\|_{2,\infty,T'} + \|\nabla\tilde v\|_{2,2,T'})$$

where s_1, q_1 is defined as in (1.10) and consequently satisfies $\frac{2}{s_1} + \frac{3}{q_1} = \frac32$.

Let $J_n := (1 + \frac1n A^{\frac12})^{-1}$ denote the Yosida operator and let $v_n = J_n v$. We have

$$J_n\mathrm{P}\mathrm{div}\,(vv) = J_n\mathrm{P}\mathrm{div}\,(v(J_n^{-1}v_n)) = J_n\mathrm{P}(v\cdot\nabla v_n) + (\frac1n J_nA^{\frac12})(A^{-\frac12}\mathrm{P}\mathrm{div}\,)(v(A^{\frac12}v_n)).$$

Using the boundedness of $A^{-\frac12}\mathrm{P}\mathrm{div}$ and of J_n, $\frac1n A^{\frac12}J_n$ uniformly in $n \in \mathbb{N}$ as well as the converse estimate to (2.1) we obtain with $\frac1\gamma = \frac12 + \frac1{q_0}$ that

$$\|J_n\mathrm{P}\mathrm{div}\,(vv)(t)\|_\gamma \le c\|v(t)\|_{q_0}\|A^{\frac12}v_n(t)\|_2.$$

Due to (2.2) with $\alpha = \frac{3}{2\gamma} - \frac34$ and (2.3) we get for v, see (4.2), the estimate

$$\begin{aligned}\|A^{\frac12}v_n(t)\|_2 &\le \|A^{\frac12}J_n\tilde v(t)\|_2 + c\int_0^t \|A^{\frac12+\alpha}e^{-(t-\tau)A}\|\|J_n\mathrm{P}\mathrm{div}\,(vv)(\tau)\|_\gamma\,\mathrm{d}\tau\\ &\le \|A^{\frac12}J_n\tilde v(t)\|_2 + c\int_0^t (t-\tau)^{-1+\frac1s}(\|v(\tau)\|_{q_0}\|A^{\frac12}v_n(\tau)\|_2)\,\mathrm{d}\tau.\end{aligned}$$

Thus, the Hardy-Littlewood-Sobolev inequality yields the estimate

$$\|A^{\frac12}v_n\|_{2,2,T'} \le \|A^{\frac12}J_n\tilde v\|_{2,2,T'} + c\|v\|_{q_0,s_0,T'}\|A^{\frac12}v_n\|_{2,2,T'}.$$

By choosing $\varepsilon_\star$ small enough we can assume that $c\|v\|_{q_0,s_0,T'} \le \frac12$ and hence by an absorption argument we obtain that

$$\|A^{\frac12}v_n\|_{2,2,T'} \le 2\|A^{\frac12}J_n\tilde v\|_{2,2,T'} \le c\|\nabla\tilde v\|_{2,2,T'}.$$

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Since the last estimate is independent of n, a reflexivity argument implies $A^{\frac{1}{2}}v \in \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))$.

As a second step let us prove that $vv \in \mathrm{L}^2(0,T';\mathrm{L}^2(\Omega))$. Therefore, let

$$\bar{v}(t) := -\int_0^t e^{-(t-\tau)A}\mathrm{P}\mathrm{div}\,(vv)\,\mathrm{d}\tau$$

and choose s_2, q_2 such that

$$\frac{1}{q_2} = \frac{1}{2} + \frac{1}{q_0}, \quad \frac{1}{s_2} = \frac{1}{2} + \frac{1}{s_0}.$$

Then we may estimate, using also q_1, s_1 as in (1.10),

$$\begin{aligned}\|\bar{v}(t)\|_{q_1} &\le c\int_0^t \|A^{\frac{3}{q_0}}e^{-(t-\tau)A}\mathrm{P}(v\cdot\nabla v)(\tau)\|_{q_2}\,\mathrm{d}\tau\\ &\le c\int_0^t (t-\tau)^{-\frac{3}{q_0}}\|v\|_{q_0}\|\nabla v\|_2\,\mathrm{d}\tau.\end{aligned}$$

Therefore, the Hardy-Littlewood-Sobolev inequality implies that

$$\|\bar{v}\|_{q_1,s_1,T'} \le c\|v\|_{q_0,s_0,T'}\|\nabla v\|_{2,2,T'},$$

and we conclude that $v = \tilde{v} + \bar{v} \in \mathrm{L}^{s_1}(0,T';\mathrm{L}^{q_1}(\Omega))$. Thus,

$$\|vv\|_{2,2,T'} \le \|v\|_{q_1,s_1,T'}\|v\|_{q_0,s_0,T'} < \infty.$$

As for $\tilde{v}$ above we obtain that v is a weak solution to the Stokes equations with right hand side $F - vv - bv - vb - bb \in L^2(0,T';L^2(\Omega))$ and thus v fulfills the energy equality (1.9). Especially, $v \in \mathrm{L}^\infty(0,T';\mathrm{L}^2(\Omega))$ and hence v is in the Leray-Hopf class. In summary, we have proved Theorem 1.3 except for the uniqueness of the strong solution. The uniqueness will be shown in the next section. □

5. On Serrin's Uniqueness Condition

In this section we are going to prove Theorem 1.4, *i.e.*, a version of Serrin's Uniqueness Theorem [31] for the equation (1.4). To prove this theorem we will follow the ideas of [32, Ch. V, 1.5]. Throughout this section let u denote a weak solution to (1.4) and let $w \in \mathrm{L}^{s_0}(0,T;\mathrm{L}^{q_0}(\Omega))$, $0 < T \le \infty$, denote a strong solution to (1.4). Furthermore, let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a mollifier, *i.e.*, $0 \le \rho \in C_0^\infty(\mathbb{R})$ is even and $\int \rho(t)\,\mathrm{d}t = 1$. Then we define the convolution operator in time by

$$v_\varepsilon(t) := \int_0^T \frac{1}{\varepsilon}\rho\Big(\frac{t-\tau}{\varepsilon}\Big)v(\tau)\,\mathrm{d}\tau, \quad \varepsilon > 0,$$

for a given $v \in \mathrm{L}^s(0,T;\mathrm{L}^q(\Omega))$, $1 \le s, q \le \infty$. All convergence results needed in the proof are stated in the following Lemma 5.1.

Lemma 5.1. *Choose $s_1, q_1 \in (1,\infty)$ such that*

$$\frac{1}{2} = \frac{1}{s_1} + \frac{1}{s_0} = \frac{1}{q_1} + \frac{1}{q_0} \tag{5.1}$$

and let q_0', s_0' denote the conjugate exponents of q_0, s_0. Then

$$\begin{aligned}\|\nabla u_\varepsilon - \nabla u\|_{2,2} &+ \|\nabla w_\varepsilon - \nabla w\|_{2,2} + \|w_\varepsilon - w\|_{q_0,s_0} + \|F_\varepsilon - F\|_{2,2} + \|(bu)_\varepsilon - bu\|_{2,2}\\ &+ \|(bw)_\varepsilon - bw\|_{2,2} + +\|(ww)_\varepsilon - ww\|_{2,2} + \|\mathrm{div}\,(uu)_\varepsilon - \mathrm{div}\,uu\|_{q_0',s_0'} \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

Furthermore, for all $t \in (0,T)$, it holds that

$$w_\varepsilon(t) \to w(t) \text{ in } L^2(\Omega), \quad u_\varepsilon(t) \rightharpoonup u(t) \text{ in } L^2(\Omega).$$

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Proof. Most of these results on convergence can be proved easily with a standard mollifier argument. Let us just mention that $\frac{2}{s_1}+\frac{3}{q_1}=\frac{3}{2}$ and $\frac{1}{2}+\frac{1}{s_1}=\frac{1}{s_0'}$, $\frac{1}{2}+\frac{1}{q_1}=\frac{1}{q_0'}$. Hence

$$\|\mathrm{div}\,(uu)\|_{q_0',s_0'} \le \|u\|_{q_1,s_1}\|\nabla u\|_{2,2} \le c(\|u\|_{2,\infty}+\|\nabla u\|_{2,2})\|\nabla u\|_{2,2} < \infty$$

and

$$\begin{aligned}\|ww\|_{2,2} &\le \|w\|_{q_0,s_0}\|w\|_{q_1,s_1} < \infty\\ \|bu\|_{2,2} &\le \|b\|_{q_0,s_0}\|u\|_{q_1,s_1} < \infty.\end{aligned}$$

The statement of the pointwise convergence follows from the (weak) L^2-continuity in time of u or w, respectively. ☐

Let us start to prove Theorem 1.4.

Proof of Theorem 1.4. Let $0<t_0<t_1<T$ and let $\varphi \in C_0^\infty((t_0,t_1))$. Using equation (1.6) for u with the test function $(\varphi w_\varepsilon)_\varepsilon$ we get an integral identity of the form

$$-\int_0^T \langle u(\tau), ((\varphi w_\varepsilon)_\varepsilon)_t(\tau)\rangle\,\mathrm{d}\tau = \int_0^T \langle \ldots, (\varphi \nabla w_\varepsilon)_\varepsilon(\tau)\rangle\,\mathrm{d}\tau$$

where the left-hand term may be rewritten as $\int_{t_0}^{t_1}\langle \partial_t u_\varepsilon, w_\varepsilon\rangle \varphi\,\mathrm{d}\tau$ since supp $\varphi \subset (t_0,t_1)$ and ρ is even. Since this identity holds for arbitrary $\varphi \in C_0^\infty((t_0,t_1))$, we conclude that the equality

$$\langle \partial_t u_\varepsilon(t), w_\varepsilon(t)\rangle + \langle \nabla u_\varepsilon(t), \nabla w_\varepsilon(t)\rangle - \langle ((u+b)(u+b))_\varepsilon(t), \nabla w_\varepsilon(t)\rangle = -\langle F_\varepsilon(t), \nabla w_\varepsilon(t)\rangle \tag{5.2}$$

holds for all $t\in(t_0,t_1)$. Adding equation (5.2) and the corresponding equation (1.6) for w tested with $(\varphi u_\varepsilon)_\varepsilon$, integrating from t_0 to t_1, and using the simple identity

$$\int_{t_0}^{t_1} \langle \partial_t u_\varepsilon(\tau), w_\varepsilon(\tau)\rangle + \langle u_\varepsilon(\tau), \partial_t w_\varepsilon(\tau)\rangle\,\mathrm{d}\tau = \langle u_\varepsilon(t_1), w_\varepsilon(t_1)\rangle - \langle u_\varepsilon(t_0), w_\varepsilon(t_0)\rangle,$$

we obtain

$$\begin{aligned}&\langle u_\varepsilon(t_1), w_\varepsilon(t_1)\rangle - \langle u_\varepsilon(t_0), w_\varepsilon(t_0)\rangle + 2\int_{t_0}^{t_1}\langle \nabla u_\varepsilon(\tau), \nabla w_\varepsilon(\tau)\rangle\,\mathrm{d}\tau\\ &\quad -\int_{t_0}^{t_1}\langle ((u+b)(u+b))_\varepsilon(\tau), \nabla w_\varepsilon(\tau)\rangle + \langle ((w+b)(w+b))_\varepsilon(\tau), \nabla u_\varepsilon(\tau)\rangle\,\mathrm{d}\tau\\ &\quad = -\int_{t_0}^{t_1}\langle F_\varepsilon(\tau), \nabla u_\varepsilon(\tau)+\nabla w_\varepsilon(\tau)\rangle\,\mathrm{d}\tau.\end{aligned}$$

Taking the limit $\varepsilon\to 0$ as well as $t_0 \to 0$ we get

$$\begin{aligned}&\langle u(t_1), w(t_1)\rangle + 2\int_0^{t_1}\langle \nabla u(\tau), \nabla w\rangle\,\mathrm{d}\tau\\ &\quad -\int_{t_0}^{t_1}\langle (u+b)(u+b), \nabla w(\tau)\rangle + \langle (w+b)(w+b), \nabla u\rangle\,\mathrm{d}\tau\\ &\quad = -\int_0^{t_1}\langle F, \nabla u+\nabla w\rangle\,\mathrm{d}\tau + \|u_0\|_2^2.\end{aligned} \tag{5.3}$$

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Note that in (5.3) the term $\langle bu, \nabla w\rangle + \langle bw, \nabla u\rangle = 0$. Adding the energy inequality (1.7) for u, the energy equality (1.9) for w and subtracting (5.3), we see that

$$\begin{aligned}
0 \geq{}& \frac{1}{2}\|u(t_1)\|_2^2 + \frac{1}{2}\|w(t_1)\|_2^2 - \langle u(t_1), w(t_1)\rangle + \int_0^{t_1} \|\nabla u\|_2^2 + \|\nabla w\|_2^2 - 2\langle \nabla u, \nabla w\rangle \,\mathrm{d}\tau \\
&- \int_0^{t_1} \langle (u+b)b, \nabla u\rangle + \langle (w+b)b, \nabla w\rangle \,\mathrm{d}\tau \\
&+ \int_0^{t_1} \langle (u+b)(u+b), \nabla w\rangle + \langle (w+b)(w+b), \nabla u\rangle \,\mathrm{d}\tau \\
={}& \frac{1}{2}\|u(t_1) - w(t_1)\|_2^2 + \int_0^{t_1} \|\nabla (u-w)\|_2^2 \,\mathrm{d}\tau \\
&- \int_0^{t_1} \langle (u-w)b, \nabla(u-w)\rangle - \langle uu, \nabla w\rangle - \langle ww, \nabla u\rangle \,\mathrm{d}\tau \\
={}& \frac{1}{2}\|W(t_1)\|_2^2 + \int_0^{t_1} \|\nabla W\|_2^2 \,\mathrm{d}\tau - \int_0^{t_1} \langle Wb, \nabla W\rangle - \langle uu, \nabla w\rangle - \langle ww, \nabla u\rangle \,\mathrm{d}\tau
\end{aligned}$$

with $W := u - w$. Note that

$$\langle uu, \nabla w\rangle + \langle ww, \nabla u\rangle = -\langle W\cdot\nabla W, w\rangle$$

and hence

$$\frac{1}{2}\|W(t)\|_2^2 + \int_0^{t_1} \|\nabla W(\tau)\|_2^2 \,\mathrm{d}\tau \leq \int_0^{t_1} \langle Wb, \nabla W\rangle + \langle W\cdot\nabla W, w\rangle \,\mathrm{d}\tau.$$

Let us define the energy norm $\|\cdot\|_{t_1}$ by

$$\|W\|_{t_1}^2 := \frac{1}{2}\sup_{0<t<t_1} \|W(t)\|_2^2 + \int_0^{t_1} \|\nabla W(\tau)\|_2^2 \,\mathrm{d}\tau.$$

Using (5.1) and the estimate $\|W\|_{q_1,s_1,t_1} \leq c\|W\|_{t_1}$ we get that

$$\begin{aligned}
\|W\|_{t_1}^2 &\leq \int_0^{t_1} \langle Wb, \nabla W\rangle + \langle w, W\cdot\nabla W\rangle \,\mathrm{d}\tau \\
&\leq (\|b\|_{q_0,s_0,t_1} + \|w\|_{q_0,s_0,t_1})\|\nabla W\|_{2,2,t_1}\|W\|_{q_1,s_1,t_1} \\
&\leq c(\|b\|_{q_0,s_0,t_1} + \|w\|_{q_0,s_0,t_1})\|W\|_{t_1}^2.
\end{aligned}$$

Choosing t_1 such that $c(\|b\|_{q_0,s_0,t_1} + \|w\|_{q_0,s_0,t_1}) < 1$ we see that $W_{|(0,t_1)} = 0$. Considering the shifted function $w(\cdot - t_1)$, $u(\cdot - t_1)$ and using an induction argument one proves easily that $W = 0$ and hence $u = w$. □

6. Decay of Turbulent Solutions

In this final section we will prove the decay results for turbulent solutions. The main idea for the proofs of Theorems 1.5 - 1.7 is to construct a strong solution in some time interval (T,∞) which coincides with the turbulent solution and to use the results in [13]. How to choose the time T and a suitable initial value for the strong solution requires some considerations and will be presented in Lemma 6.1. The proof of this lemma is based on ideas from interpolation theory, but elementary.

Lemma 6.1. *Let $v \in L^\infty(0,\infty; L^2(\Omega))$ such that $\nabla v \in L^2(0,\infty; L^2(\Omega))$, let $T > 0$ and let $\varepsilon > 0$. Then there exists a $t > T$ such that*

$$\|v(t)\|_{\mathcal{B}_\infty^{q,s}(\Omega)} < \varepsilon,\ \frac{2}{s} + \frac{3}{q} = 1,\ 3 < q < \infty$$

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Proof. For every $\delta > 0$ and $T > 0$ there exists a $t \geq T$ such that $\|v(t)\|_2 \leq \|v\|_{L^\infty(0,\infty;L^2(\Omega))}$ and $\|v(t)\|_6 \leq c\|v(t)\|_{\dot W^{1,2}} < \delta$. To control, with $x = v(t)$, the term $\|x\|_{\mathcal{B}^{q,s}_\infty(\Omega)}$, fix $0 < R < \infty$. In the first case let $q \geq 6$. Using the L^p-L^q-estimates of the Stokes semigroup (2.3) we obtain the estimate

$$\begin{aligned}\int_0^\infty \|e^{-\tau A}x\|_q^s \,\mathrm{d}\tau &\leq \int_0^R \|e^{-\tau A}x\|_q^s \,\mathrm{d}\tau + \int_R^\infty \|e^{-\tau A}x\|_q^s \,\mathrm{d}\tau \\ &\leq c\int_0^R \tau^{-\frac{3}{2}(\frac{1}{6}-\frac{1}{q})s}\|x\|_6^s \,\mathrm{d}\tau + c\int_R^\infty \tau^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})s}\|x\|_2^s \,\mathrm{d}\tau \\ &\leq c\big(R^{\frac{s}{4}}\|x\|_6^s + R^{-\frac{s}{4}}\|x\|_2^s\big)\end{aligned}$$

since $-\frac{3}{2}(\frac{1}{6}-\frac{1}{q})s = \frac{s}{4}-1$ and $-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})s = -\frac{s}{4}-1$. Choosing $R = \big(\|x\|_2/\|x\|_6\big)^2$, we arrive at the estimate

$$\|x\|_{\mathcal{B}^{q,s}_\infty(\Omega)} \leq c\|x\|_2^{\frac{1}{2}}\|x\|_6^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}, \quad x = v(t).$$

In the second case let $3 < q < 6$. Thus there exists $\alpha \in (0,1)$ such that $\frac{\alpha}{6} + \frac{1-\alpha}{2} = \frac{1}{q}$. Then

$$\begin{aligned}\int_0^\infty \|e^{-\tau A}x\|_q^s \,\mathrm{d}\tau &\leq c\int_0^R \|e^{-\tau A}x\|_2^{(1-\alpha)s}\|e^{-\tau A}x\|_6^{\alpha s} \,\mathrm{d}\tau + c\int_R^\infty \tau^{-\frac{s}{4}-1}\|x\|_2^s \,\mathrm{d}\tau \\ &\leq c\|x\|_2^{(1-\alpha)s}\big(R\|x\|_6^{\alpha s} + R^{-\frac{s}{4}}\|x\|_2^{\alpha s}\big)\end{aligned}$$

The choice $R^{1+\frac{1}{4}} = \big(\|x\|_2/\|x\|_6\big)^{\alpha s}$ yields the estimate

$$\int_0^\infty \|e^{-\tau A}x\|_q^s \,\mathrm{d}\tau \leq c\|x\|_2^{(1-\alpha)s}\big(\|x\|_2\|x\|_6^{\frac{s}{4}}\big)^{\frac{\alpha s}{1+s/4}} \leq C\delta^\beta, \quad x = v(t),$$

where $\beta = \frac{\alpha s^2}{4+s}$. This completes the proof of the lemma. □

Finally, we prove the main Theorems 1.5-1.7. The ideas of these proofs will be the same in all three cases, so let us present the proof of Theorem 1.6 only.

Proof of Theorem 1.6. Let $\varepsilon_\star$ denote the constant in Theorem 1.3. Let us choose $T \geq 0$ such that

$$\|F\|_{\mathrm{L}^{\frac{s_0}{2}}(T,\infty;L^{\frac{q_0}{2}}(\Omega))} + \|b\|_{\mathrm{L}^{s_0}(T,\infty;L^{q_0}(\Omega))} \leq \frac{\varepsilon_\star}{2}.$$

Furthermore, using Lemma 6.1 we obtain a $t \geq T$ such that

$$\|u(t)\|_{\mathcal{B}^{q,s}_\infty(\Omega)} < \frac{\varepsilon_\star}{2}.$$

and such that u fulfills the energy inequality on $[t,\infty)$. Theorem 1.3 implies the existence of a strong solution $\tilde u$ in $[t,\infty)$ such that $u(t) = \tilde u(t)$ and hence $u_{|[t,\infty)} = \tilde u$ by Theorem 1.4. Furthermore, using [13, Theorem 1.3] we obtain the existence of a weak solution $\bar u$ to (1.4) in $[t,\infty)$ with $u(t) = \bar u(t)$ such that $u(\tau) \to 0$ as $\tau \to \infty$. Using Theorem 1.4 again we see that $u_{[t,\infty)} = \bar u$. Hence Theorem 1.6 is proved. □

Note that Theorem 1.5 and Theorem 1.7 can be proved in the same way. In the case of a bounded domain the existence of a strong solution is proved in [19] and the corresponding decay results can be found in [13, Theorem 1.2, Theorem 1.4].

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