J. Math. Anal. Appl. 458 (2018) 174–192

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global well-posedness for a L^2-critical nonlinear higher-order Schrödinger equation

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A R T I C L E   I N F O

Article history:
Received 3 March 2017
Available online 6 September 2017
Submitted by A. Mazzucato

Keywords:
Cubic higher-order Schrödinger equation
Global well-posedness
Almost conservation law

A B S T R A C T

We prove the global well-posedness for a L^2-critical defocusing cubic higher-order Schrödinger equation, namely

$$i\partial_t u + \Lambda^k u = -|u|^2 u,$$

where $\Lambda = \sqrt{-\Delta}$ and $k \geq 3$, $k \in \mathbb{Z}$ in $\mathbb{R}^k$ with initial data $u_0 \in H^\gamma$, $\gamma > \gamma(k) := \frac{k(4k-1)}{14k-3}$.

1. Introduction and main results

Let $k \geq 2$, $k \in \mathbb{Z}$. We consider the Cauchy problem for the defocusing cubic nonlinear higher-order Schrödinger equation posed on $\mathbb{R}^k$, namely

$$\begin{cases} i\partial_t u(t,x) + \Lambda^k u(t,x) &= -|u(t,x)|^2 u(t,x), \quad t \geq 0, x \in \mathbb{R}^k, \\ u(0,x) &= u_0(x) \in H^\gamma(\mathbb{R}^k), \end{cases} \tag{NLS$_k$}$$

where $\Lambda = \sqrt{-\Delta}$ is the Fourier multiplier by $|\xi|$. When $k = 2$, (NLS$_k$) corresponds to the well-known Schrödinger equation (see e.g. [11], [19], [18], [20], [21], [13], [16], [15], [12], [1], [2] and references therein). When $k = 4$, it is the fourth-order Schrödinger equation take into consideration the role of small fourth-order dispersion in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity (see e.g. [7], [8], [3], [4]).

It is worth noticing that the (NLS$_k$) is L^2-critical in the sense that if u is a solution to (NLS$_k$) on $(-T, T)$ with initial data u_0, then

$$u_\lambda(t,x) = \lambda^{-k/2} u(\lambda^{-k} t, \lambda^{-1} x), \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jmaa.2017.09.004

is also a solution of (NLS$_k$) on $(-\lambda^k T, \lambda^k T)$ with initial data $u_\lambda(0)$ and

$$\|u_\lambda(0)\|_{L^2(\mathbb{R}^k)} = \|u_0\|_{L^2(\mathbb{R}^k)}.$$

It is known (see e.g. [18], [22], [23]) that (NLS$_k$) is locally well-posed in $H^\gamma(\mathbb{R}^k)$ when $\gamma > 0$. Moreover, these local solutions enjoy mass conservation, i.e.

$$\|u(t)\|_{L^2(\mathbb{R}^k)} = \|u_0\|_{L^2(\mathbb{R}^k)}, \tag{1.2}$$

and $H^{k/2}$ solutions have the conserved energy, i.e.

$$E(u(t)) := \int\limits_{\mathbb{R}^k} \frac{1}{2}|\Lambda^{k/2}u(t,x)|^2 + \frac{1}{4}|u(t,x)|^4 dx = E(u_0). \tag{1.3}$$

The conservations of mass and energy combine with the persistence of regularity (see e.g. [23]) immediately yield the global well-posedness for (NLS$_k$) with initial data in $H^\gamma(\mathbb{R}^k)$ when $\gamma \geq k/2$. Note also (see [22]) that one has the local well-posedness for (NLS$_k$) when initial data $u_0 \in L^2(\mathbb{R}^k)$ but the time of existence depends not only on the size but also on the profile of the initial data. In addition, if $\|u_0\|_{L^2(\mathbb{R}^k)}$ is small enough, then (NLS$_k$) is global well-posed and scattering in $L^2(\mathbb{R}^k)$. It is conjectured that (NLS$_k$) is in fact globally well-posed for initial data in $H^\gamma(\mathbb{R}^k)$ with $\gamma \geq 0$. This paper concerns with the global well-posedness of (NLS$_k$) in $H^\gamma(\mathbb{R}^k)$ when $0 < \gamma < k/2$. Let us recall known results for the defocusing cubic Schrödinger equation in $\mathbb{R}^2$, i.e. (NLS$_2$). The first attempt to this problem due to Bourgain in [11] where he used a "Fourier truncation" approach to prove the global existence for $\gamma > 3/5$. It was then improved for $\gamma > 4/7$ by I-team in [13]. The proof is based on the almost conservation of a modified energy functional. The idea is to replace the conserved energy $E(u)$, which is not available when $\gamma < 1$, by an "almost conserved" quantity $E(I_N u)$ with $N \gg 1$ where I_N is a smoothing operator which behaves like the identity for low frequencies $|\xi| \leq N$ and like a fractional integral operator of order $1-\gamma$ for high frequencies $|\xi| \geq 2N$. Since $I_N u$ is not a solution to (NLS$_2$), we may expect an energy increment. The key idea is to show that on the time interval of local existence, the increment of the modified energy $E(I_N u)$ decays with respect to a large parameter N. This allows to control $E(I_N u)$ on time interval where the local solution exists, and we can iterate this estimate to obtain a global in time control of the solution by means of the bootstrap argument. Fang–Grillakis then upgraded this result to $\gamma \geq 1/2$ in [24]. Later, Colliander–Grillakis–Tzirakis improved for $\gamma > 2/5$ in [15] using an almost interaction Morawetz inequality. Subsequent paper [12] has decreased the necessary regularity to $\gamma > 1/3$. Afterwards, Dodson established in [1] the global existence for (NLS$_2$) when $\gamma > 1/4$. The proof combines the almost conservation law and an improved interaction Morawetz estimate. Recently, Dodson in [2] proved the global well-posedness and scattering for (NLS$_2$) for initial data $u_0 \in L^2(\mathbb{R}^2)$ using the bilinear estimate and a frequency localized interaction Morawetz estimate. We next recall some known results about the global well-posedness below energy space for the fourth-order Schrödinger equation. In [6], the author considered the more general fourth-order Schrödinger equation, namely

$$i\partial_t u + \lambda\Delta u + \mu\Delta^2 u + \nu|u|^{2m}u = 0,$$

and established the global well-posedness in $H^\gamma(\mathbb{R}^n)$ for $\gamma > 1 + \frac{mn-9+\sqrt{(4m-mn+7)^2+16}}{4m}$ under the assumption $4 < mn < 4m+2$ and of course some conditions on λ, μ and ν. For the mass-critical fourth-order Schrödinger equation in high dimensions $n \geq 5$, Pausader–Shao proved in [5] that the L^2-solution is global and scattering under some conditions. Recently, Miao–Wu–Zhang in [9] showed the global existence and scattering below energy space for the defocusing cubic fourth-order Schrödinger equation in $\mathbb{R}^n$ with $n = 5, 6, 7$. To our knowledge, there is no result concerning the global existence (possibly scattering) for (NLS$_4$).

The purpose of this paper is to prove the global existence of (NLS$_k$) with $k \geq 3$, $k \in \mathbb{Z}$ below the energy space $H^{k/2}(\mathbb{R}^k)$.

Theorem 1.1. *Let $k \geq 3$, $k \in \mathbb{Z}$. The initial value problem (NLS$_k$) is globally well-posed in $H^\gamma(\mathbb{R}^k)$ for any $k/2 > \gamma > \gamma(k) := \frac{k(4k-1)}{14k-3}$. Moreover, the solution satisfies*

$$\|u(T)\|_{H^\gamma(\mathbb{R}^k)} \leq C(1+T)^{\frac{(4k-1)(k-2\gamma)}{2((14k-3)\gamma-k(4k-1))}+},$$

for $|T| \to \infty$, where the constant C depends only on $\|u_0\|_{H^\gamma(\mathbb{R}^k)}$.

The proof of this theorem is based on the I-method similar to [13] (see also [6]). We shall consider a modified I-operator and show a suitable "almost conservation law" for the higher-order Schrödinger equation. The global well-posedness then follows by a usual scheme as in [13].

This paper is organized as follows. In Section 2, we recall some linear and bilinear estimates for the higher-order Schrödinger equation, and also a modified I-operator together with its basic properties. We will show in Section 3 an almost conservation law and a modified local well-posed result. The proof of Theorem 1.1 is proved in Section 4. Throughout this paper, we shall use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C. The notation $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$ to denote $A \leq cB$ for some small constant $c > 0$. We also use the Japanese bracket $\langle a \rangle := \sqrt{1+|a|^2} \sim 1 + |a|$ and $a\pm := a \pm \epsilon$ with some universal constant $0 < \epsilon \ll 1$.

2. Preliminaries

2.1. Littlewood–Paley decomposition

Let φ be a smooth, real-valued, radial function in $\mathbb{R}^k$ such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| \geq 2$. Let $M = 2^k$, $k \in \mathbb{Z}$. We denote the Littlewood–Paley operators by

$$\begin{aligned} \widehat{P_{\leq M} f}(\xi) &:= \varphi(M^{-1}\xi)\hat{f}(\xi), \\ \widehat{P_{>M} f}(\xi) &:= (1-\varphi(M^{-1}\xi))\hat{f}(\xi), \\ \widehat{P_M f}(\xi) &:= (\varphi(M^{-1}\xi) - \varphi(2M^{-1}\xi))\hat{f}(\xi), \end{aligned}$$

where $\hat{\cdot}$ is the spatial Fourier transform. We similarly define

$$P_{<M} := P_{\leq M} - P_M, \quad P_{\geq M} := P_{>M} + P_M,$$

and for $M_1 \leq M_2$,

$$P_{M_1 < \cdot \leq M_2} := P_{\leq M_2} - P_{\leq M_1} = \sum_{M_1 < M \leq M_2} P_M.$$

We have the following so called Bernstein's inequalities (see e.g. [10, Chapter 2] or [21, Appendix]).

Lemma 2.1. *Let $\gamma \geq 0$ and $1 \leq p \leq q \leq \infty$.*

$$\|P_{\geq M} f\|_{L^p_x} \lesssim M^{-\gamma}\|\Lambda^\gamma P_{\geq M} f\|_{L^p_x},$$
$$\|P_{\leq M}\Lambda^\gamma f\|_{L^p_x} \lesssim M^{\gamma}\|P_{\leq M} f\|_{L^p_x},$$

$$\|P_M \Lambda^{\pm\gamma} f\|_{L^p_x} \sim M^{\pm\gamma} \|P_M f\|_{L^p_x},$$
$$\|P_{\leq M} f\|_{L^q_x} \lesssim M^{k/p-k/q} \|P_{\leq M} f\|_{L^p_x},$$
$$\|P_M f\|_{L^q_x} \lesssim M^{k/p-k/q} \|P_M f\|_{L^p_x}.$$

2.2. Norms and Strichartz estimates

Let $\gamma, b \in \mathbb{R}$. The Bourgain space $X^{\gamma,b}_{\tau=|\xi|^k}$ is the closure of space-time Schwartz space $\mathscr{S}_{t,x}$ under the norm

$$\|u\|_{X^{\gamma,b}_{\tau=|\xi|^k}} := \| \langle\xi\rangle^\gamma \left\langle \tau - |\xi|^k \right\rangle^b \tilde{u}\|_{L^2_\tau L^2_\xi},$$

where $\tilde{\cdot}$ is the space-time Fourier transform, i.e.

$$\tilde{u}(\tau,\xi) := \iint e^{-i(t\tau+x\xi)} u(t,x) dt dx.$$

We shall use $X^{\gamma,b}$ instead of $X^{\gamma,b}_{\tau=|\xi|^k}$ when there is no confusion. We recall a following special property of $X^{\gamma,b}$ space (see e.g. [21, Lemma 2.9]).

Lemma 2.2. *Let $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ and Y be a Banach space of functions on $\mathbb{R} \times \mathbb{R}^k$. If*

$$\|e^{it\tau} e^{it\Lambda^k} f\|_Y \lesssim \|f\|_{H^\gamma_x},$$

for all $f \in H^\gamma_x$ and all $\tau \in \mathbb{R}$, then

$$\|u\|_Y \lesssim \|u\|_{X^{\gamma,1/2+}},$$

for all $u \in \mathscr{S}_{t,x}$. Moreover, if

$$\|[e^{it\tau} e^{it\Lambda^k} f_1][e^{it\zeta} e^{it\Lambda^k} f_2]\|_Y \lesssim \|f_1\|_{H^{\gamma_1}} \|f_2\|_{H^{\gamma_2}_x},$$

for all $f_1 \in H^{\gamma_1}_x$, $f_2 \in H^{\gamma_2}_x$ and all $\tau, \zeta \in \mathbb{R}$, then

$$\|u_1 u_2\|_Y \lesssim \|u_1\|_{X^{\gamma_1,1/2+}} \|u_2\|_{X^{\gamma_2,1/2+}},$$

for all $u_1, u_2 \in \mathscr{S}_{t,x}$.

Throughout this paper, a pair (p,q) is called admissible in $\mathbb{R}^k$ if

$$(p,q) \in [2,\infty]^2, \quad (q,p) \neq (2,\infty), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{2.4}$$

We recall the following Strichartz estimate (see e.g. [22], [3]).

Proposition 2.3. *Let $k \geq 2$, $k \in \mathbb{Z}$. Suppose that u is a solution to*

$$i\partial_t u(t,x) + \Lambda^k u(t,x) = F(t,x), \quad u(0,x) = u_0(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^k.$$

Then for all (p,q) and (a,b) admissible pairs,

$$\|u\|_{L^p_t L^q_x} \lesssim \|u_0\|_{L^2_x} + \|F\|_{L^{a'}_t L^{b'}_x}.$$

Here (a, a') and (b, b') are Hölder exponents.

A direct consequence of Lemma 2.2 and Proposition 2.3 is the following linear estimate in $X^{\gamma,b}$ space.

Corollary 2.4. *Let (p, q) be an admissible pair. Then*

$$\|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{0,1/2+}}, \tag{2.5}$$

for all $u \in \mathscr{S}_{t,x}$.

We also have the following bilinear estimate in $\mathbb{R}^k$.

Proposition 2.5. *Let $k \geq 2$, $k \in \mathbb{Z}$ and $M_1, M_2 \in 2^{\mathbb{Z}}$ be such that $M_1 \leq M_2$. Then*

$$\|[e^{it\Lambda^k} P_{M_1} u_0][e^{it\Lambda^k} P_{M_2} v_0]\|_{L^2_t L^2_x} \lesssim (M_1/M_2)^{(k-1)/2} \|u_0\|_{L^2_x} \|v_0\|_{L^2_x}.$$

Proof. We refer the reader to [11] for the standard case $k = 2$. The proof for $k > 2$ is treated similarly. For $M_1 \sim M_2$, the result follows easily from the Strichartz estimate,

$$\|[e^{it\Lambda^k} P_{M_1} u_0][e^{it\Lambda^k} P_{M_2} v_0]\|_{L^2_t L^2_x} \leq \|e^{it\Lambda^k} P_{M_1} u_0\|_{L^4_t L^4_x} \|e^{it\Lambda^k} P_{M_2} v_0\|_{L^4_t L^4_x} \lesssim \|u_0\|_{L^2_x} \|v_0\|_{L^2_x}.$$

Note that $(4, 4)$ is an admissible pair. Let us consider the case $M_1 \ll M_2$. By duality, it suffices to prove

$$\begin{aligned} &\Big| \iint_{\mathbb{R}^k \times \mathbb{R}^k} G(-|\xi|^k - |\eta|^k, \xi + \eta) \widehat{P_{M_1} u}_0(\xi) \widehat{P_{M_2} v}_0(\eta) d\xi d\eta \Big| \\ &\lesssim (M_1/M_2)^{(k-1)/2} \|G\|_{L^2_\tau L^2_\xi} \|\hat{u}_0\|_{L^2_\xi} \|\hat{v}_0\|_{L^2_\xi}. \end{aligned} \tag{2.6}$$

By renaming the components, we can assume that $|\xi| \sim |\xi_1| \sim M_1$ and $|\eta| \sim |\eta_1| \sim M_2$, where $\xi = (\xi_1, \underline{\xi})$, $\eta = (\eta_1, \underline{\eta})$ with $\underline{\xi}, \underline{\eta} \in \mathbb{R}^{k-1}$. We make a change of variables $\tau = -|\xi|^k - |\eta|^k$, $\vartheta = \xi + \eta$ and $d\tau d\vartheta = J d\xi_1 d\eta$. An easy computation shows that $J = |k(|\eta|^{k-2}\eta_1 - |\xi|^{k-2}\xi_1)| \sim |\eta|^{k-1} \sim M_2^{k-1}$. The Cauchy–Schwarz inequality with the fact that $|\underline{\xi}| \lesssim M$ then yields

$$\begin{aligned} \text{LHS(2.6)} &= \Big| \iiint_{\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{k-1}} G(\tau, \vartheta) \widehat{P_{M_1} u}_0(\xi) \widehat{P_{M_2} v}_0(\eta) J^{-1} d\tau d\vartheta d\underline{\xi} \Big| \\ &\leq \|G\|_{L^2_\tau L^2_\xi} \int_{\mathbb{R}^{k-1}} \Big(\iint_{\mathbb{R} \times \mathbb{R}^k} |\widehat{P_{M_1} u}_0(\xi)|^2 |\widehat{P_{M_2} v}_0(\eta)|^2 J^{-2} d\tau d\vartheta \Big)^{1/2} d\underline{\xi} \\ &\leq \|G\|_{L^2_\tau L^2_\xi} M_1^{(k-1)/2} \Big(\iiint_{\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{k-1}} |\widehat{P_{M_1} u}_0(\xi)|^2 |\widehat{P_{M_2} v}_0(\eta)|^2 J^{-2} d\tau d\vartheta d\underline{\xi} \Big)^{1/2} \\ &\leq \|G\|_{L^2_\tau L^2_\xi} M_1^{(k-1)/2} \Big(\iiint_{\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{k-1}} |\widehat{P_{M_1} u}_0(\xi)|^2 |\widehat{P_{M_2} v}_0(\eta)|^2 J^{-1} d\xi d\eta \Big)^{1/2} \\ &\lesssim \|G\|_{L^2_\tau L^2_\xi} (M_1/M_2)^{(k-1)/2} \|\widehat{P_{M_1} u}_0\|_{L^2_\xi} \|\widehat{P_{M_2} v}_0\|_{L^2_\xi}. \end{aligned}$$

This proves (2.6), and the proof is complete. □

The following result is another application of Lemma 2.2 and Proposition 2.5.

Corollary 2.6. *Let $k \geq 2$, $k \in \mathbb{Z}$ and $u_1, u_2 \in X^{0,1/2+}$ be supported on spatial frequencies $|\xi| \sim M_1, M_2$ respectively. Then for $M_1 \leq M_2$,*

$$\|u_1 u_2\|_{L^2_t L^2_x} \lesssim (M_1/M_2)^{(k-1)/2} \|u_1\|_{X^{0,1/2+}} \|u_2\|_{X^{0,1/2+}}. \tag{2.7}$$

A similar estimate holds for $\overline{u}_1 u_2$ or $u_1 \overline{u}_2$.

2.3. I-operator

For $0 < \gamma < k/2$ and $N \gg 1$, we define the Fourier multiplier I_N by

$$\widehat{I_N f}(\xi) := m_N(\xi) \hat{f}(\xi), \tag{2.8}$$

where m is a smooth, radially symmetric, non-increasing function such that

$$m_N(\xi) := \begin{cases} 1 & \text{if } |\xi| \leq N, \\ (N^{-1}|\xi|)^{\gamma-2} & \text{if } |\xi| \geq 2N. \end{cases} \tag{2.9}$$

For simplicity, we shall drop the N from the notation and write I and m instead of I_N and m_N. The operator I is the identity on low frequencies $|\xi| \leq N$ and behaves like a fractional integral operator of order $k/2 - \gamma$ on high frequencies $|\xi| \geq 2N$. We recall some basic properties of the I-operator in the following lemma.

Lemma 2.7. *Let $q \in (1, \infty)$ and $\gamma \in (0, k/2)$. Then*

$$\|If\|_{L^q_x} \lesssim \|f\|_{L^q_x}, \tag{2.10}$$

$$\|f\|_{H^\gamma_x} \lesssim \|If\|_{H^{k/2}_x} \lesssim N^{k/2-\gamma} \|f\|_{H^\gamma_x}. \tag{2.11}$$

Proof. The estimate (2.10) follows from the fact that m satisfies the Hörmander multiplier condition. For (2.11), we proceed as follows.

$$\begin{aligned} \|f\|^2_{H^\gamma_x} &\lesssim \int\limits_{|\xi|\leq N} \langle\xi\rangle^{2\gamma} |\widehat{If}(\xi)|^2 d\xi + \int\limits_{|\xi|\geq 2N} \langle\xi\rangle^{2\gamma} (N^{-1}|\xi|)^{2(k/2-\gamma)} |\widehat{If}(\xi)|^2 d\xi \\ &\lesssim \int\limits_{|\xi|\leq N} \langle\xi\rangle^{k} |\widehat{If}(\xi)|^2 d\xi + \int\limits_{|\xi|\geq 2N} \langle\xi\rangle^{k} |\widehat{If}(\xi)|^2 d\xi \lesssim \|If\|^2_{H^{k/2}_x}. \end{aligned}$$

This gives the first estimate in (2.11). Similarly,

$$\begin{aligned} \|If\|^2_{H^{k/2}_x} &\lesssim \int\limits_{|\xi|\leq N} \langle\xi\rangle^{k} |\hat{f}(\xi)|^2 d\xi + \int\limits_{|\xi|\geq 2N} \langle\xi\rangle^{k} (N^{-1}|\xi|)^{2(\gamma-k/2)} |\hat{f}(\xi)|^2 d\xi \\ &\lesssim \int\limits_{|\xi|\leq N} \langle\xi\rangle^{2(k/2-\gamma)} \langle\xi\rangle^{2\gamma} |\hat{f}(\xi)|^2 d\xi + \int\limits_{|\xi|\geq 2N} N^{2(k/2-\gamma)} \langle\xi\rangle^{2\gamma} |\hat{f}(\xi)|^2 d\xi \\ &\lesssim N^{2(k/2-\gamma)} \Big(\int\limits_{|\xi|\leq N} \langle\xi\rangle^{2\gamma} |\hat{f}(\xi)|^2 d\xi + \int\limits_{|\xi|\geq 2N} \langle\xi\rangle^{2\gamma} |\hat{f}(\xi)|^2 d\xi \Big) \lesssim N^{2(k/2-\gamma)} \|f\|^2_{H^\gamma_x}. \end{aligned}$$

The proof is complete. □

3. Almost conservation law

As mentioned in the introduction, the equation (NLS$_k$) is locally well-posed in H^γ for any $\gamma > 0$. Moreover, the time of existence depends only on the H^γ_x-norm of the initial data. Thus, the global well-posedness will follows from a global $L^\infty_t H^\gamma_x$ bound of the solution by the usual iterative argument. For H^γ solution with $\gamma \geq k/2$, one can obtain easily the $L^\infty_t H^\gamma_x$ bound of solution using the persistence of regularity and the conserved quantities of mass and energy. But it is not the case for H^γ solution with $\gamma < k/2$ since the energy is no longer conserved. However, it follows from (2.11) that the H^γ_x-norm of the solution u can be controlled by the $H^{k/2}_x$-norm of Iu. It leads to consider the following modified energy functional

$$E(Iu(t)) := \frac{1}{2}\|Iu(t)\|^2_{\dot{H}^{k/2}_x} + \frac{1}{4}\|Iu(t)\|^4_{L^4_x}. \tag{3.12}$$

Since Iu is not a solution to (NLS$_k$), we can expect an energy increment. We have the following "almost conservation law".

Proposition 3.1. *Let $k \geq 3$, $k \in \mathbb{Z}$. Given $k/2 > \gamma > \gamma(k) := \frac{k(4k-1)}{14k-3}$, $N \gg 1$, and initial data $u_0 \in C^\infty(\mathbb{R}^k)$ with $E(Iu_0) \leq 1$, then there exists a $\delta = \delta(\|u_0\|_{L^2_x}) > 0$ so that the solution $u \in C([0,\delta], H^\gamma(\mathbb{R}^k))$ of (NLS$_k$) satisfies*

$$E(Iu(t)) = E(Iu_0) + O(N^{-\gamma_0(k)+}), \tag{3.13}$$

where $\gamma_0(k) := \frac{k(6k-1)}{8k-2}$ for all $t \in [0,\delta]$.

Remark 3.2. This proposition tells us that the modified energy $E(Iu(t))$ decays with respect to the parameter N. We will see in Section 4 that if we can replace the increment $N^{-\gamma_0(k)+}$ in the right hand side of (3.13) with $N^{-\gamma_1(k)+}$ for some $\gamma_1(k) > \gamma_0(k)$, then the global existence can be improved for all $\gamma > \frac{k^2}{2(k+\gamma_1(k))}$. In particular, if $\gamma_1(k) = \infty$, then $E(Iu(t))$ is conserved, and the global well-posedness holds for all $\gamma > 0$.

In order to prove Proposition 3.1, we recall the following interpolation result (see [14, Lemma 12.1]). Let η be a smooth, radial, decreasing function which equals 1 for $|\xi| \leq 1$ and equals $|\xi|^{-1}$ for $|\xi| \geq 2$. For $N \geq 1$ and $\alpha \in \mathbb{R}$, we define the spatial Fourier multiplier J^α_N by

$$\widehat{J^\alpha_N f}(\xi) := (\eta(N^{-1}\xi))^\alpha \hat{f}(\xi). \tag{3.14}$$

The operator J^α_N is a smoothing operator of order α, and it is the identity on the low frequencies $|\xi| \leq N$.

Lemma 3.3 *(Interpolation [14]). Let $\alpha_0 > 0$ and $n \geq 1$. Suppose that $Z, X_1, \dots, X_n$ are translation invariant Banach spaces and T is a translation invariant n-linear operator such that*

$$\|J^\alpha_1 T(u_1, \dots, u_n)\|_Z \lesssim \prod_{i=1}^n \|J^\alpha_1 u_i\|_{X_i},$$

for all $u_1, \dots, u_n$ and all $0 \leq \alpha \leq \alpha_0$. Then one has

$$\|J^\alpha_N T(u_1, \dots, u_n)\|_Z \lesssim \prod_{i=1}^n \|J^\alpha_N u_i\|_{X_i},$$

for all $u_1, \dots, u_n$, all $0 \leq \alpha \leq \alpha_0$, and $N \geq 1$, with the implicit constant independent of N.

Using this interpolation lemma, we are able to prove the following modified version of the usual local well-posed result.

Proposition 3.4. *Let*[1] $\gamma \in (\gamma(k), k/2)$ *and* $u_0 \in H^\gamma(\mathbb{R}^k)$ *be such that* $E(Iu_0) \leq 1$. *Then there is a constant* $\delta = \delta(\|u_0\|_{L^2_x})$ *so that the solution* u *to* (NLS$_k$) *satisfies*

$$\|Iu\|_{X_\delta^{k/2,1/2+}} \lesssim 1. \tag{3.15}$$

Here $X_\delta^{\gamma,b}$ *is the space of restrictions of elements of* $X^{\gamma,b}$ *endowed with the norm*

$$\|u\|_{X_\delta^{\gamma,b}} := \inf\{\|w\|_{X^{\gamma,b}} \mid w_{|[0,\delta]\times\mathbb{R}^k} = u\}. \tag{3.16}$$

Proof. We recall the following estimates involving the $X^{\gamma,b}$ spaces which are proved in the Appendix A. Let $\gamma \in \mathbb{R}$ and $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(t) = 1$ for $t \in [-1, 1]$. One has

$$\|\psi(t)e^{it\Lambda^k}u_0\|_{X^{\gamma,b}} \lesssim \|u_0\|_{H^\gamma_x}, \tag{3.17}$$

$$\Big\|\psi_\delta(t)\int_0^t e^{i(t-s)\Lambda^k}F(s)ds\Big\|_{X^{\gamma,b}} \lesssim \delta^{1-b-b'}\|F\|_{X^{\gamma,-b'}}, \tag{3.18}$$

where $\psi_\delta(t) := \psi(\delta^{-1}t)$ provided $0 < \delta \leq 1$ and

$$0 < b' < 1/2 < b, \quad b + b' < 1. \tag{3.19}$$

Note that the implicit constants are independent of δ. This implies for $0 < \delta \leq 1$ and b, b' as in (3.19) that

$$\|e^{it\Lambda^k}u_0\|_{X_\delta^{\gamma,b}} \lesssim \|u_0\|_{H^\gamma_x}, \tag{3.20}$$

$$\Big\|\int_0^t e^{i(t-s)\Lambda^k}F(s)ds\Big\|_{X_\delta^{\gamma,b}} \lesssim \delta^{1-b-b'}\|F\|_{X_\delta^{\gamma,-b'}}. \tag{3.21}$$

By the Duhamel principle, we have

$$\|Iu\|_{X_\delta^{k/2,b}} = \Big\|e^{it\Lambda^k}Iu_0 + \int_0^t e^{it\Lambda^k}I(|u|^2u)(s)ds\Big\|_{X_\delta^{k/2,b}} \lesssim \|Iu_0\|_{H^{k/2}_x} + \delta^{1-b-b'}\|I(|u|^2u)\|_{X_\delta^{k/2,-b'}}.$$

By the definition of restriction norm (3.16),

$$\|Iu\|_{X_\delta^{k/2,b}} \lesssim \|Iu_0\|_{H^{k/2}_x} + \delta^{1-b-b'}\|I(|w|^2w)\|_{X^{k/2,-b'}},$$

where w agrees with u on $[0,\delta]\times\mathbb{R}^k$ and

$$\|Iu\|_{X_\delta^{k/2,b}} \sim \|Iw\|_{X^{k/2,b}}.$$

Let us assume for the moment that

$$\|I(|w|^2w)\|_{X^{k/2,-b'}} \lesssim \|Iw\|^3_{X^{k/2,b}}. \tag{3.22}$$

[1] See Theorem 1.1 for the definition of $\gamma(k)$.

This implies that

$$\|Iu\|_{X_\delta^{k/2,b}} \lesssim \|Iu_0\|_{H_x^{k/2}} + \delta^{1-b-b'}\|Iu\|^3_{X_\delta^{k/2,b}}.$$

Note that

$$\|Iu_0\|_{H_x^{k/2}} \sim \|Iu_0\|_{\dot{H}_x^{k/2}} + \|Iu_0\|_{L_x^2} \leq 1 + \|u_0\|_{L_x^2}.$$

As $\|Iu\|_{X_\delta^{k/2,b}}$ is continuous in the δ variable, the bootstrap argument (see e.g. [21, Section 1.3]) yields

$$\|Iu\|_{X_\delta^{k/2,b}} \lesssim 1.$$

This proves (3.15). It remains to show (3.22). We will take the advantage of interpolation Lemma 3.3. Note that the I-operator defined in (2.8) is equal to J_N^α defined in (3.14) with $\alpha = k/2-\gamma$. Thus, by Lemma 3.3, (3.22) is proved once there is $\alpha_0 > 0$ so that

$$\|J_1^\alpha(|w|^2w)\|_{X^{k/2,-b'}} \lesssim \|J_1^\alpha w\|^3_{X^{k/2,b}},$$

for all $0 \leq \alpha \leq \alpha_0$. Splitting w to low and high frequency parts $|\xi| \lesssim 1$ and $|\xi| \gg 1$ respectively and using definition of J_1^α, it suffices to show

$$\||w|^2w\|_{X^{\gamma,-b'}} \lesssim \|w\|^3_{X^{\gamma,b}}, \tag{3.23}$$

for all $\gamma \in [\gamma(k), k/2]$. By duality, a Leibniz rule, (3.23) follows from

$$\Big|\iint\limits_{\mathbb{R}\times\mathbb{R}^k} (\langle\Lambda\rangle^\gamma w_1)\overline{w}_2 w_3 w_4 dtdx\Big| \lesssim \|w_1\|_{X^{\gamma,b}}\|w_2\|_{X^{\gamma,b}}\|w_3\|_{X^{\gamma,b}}\|w_4\|_{X^{0,b'}}. \tag{3.24}$$

Note that the last term should be precise as $\|w_4\|_{X^{0,b'}_{\tau=-|\xi|^k}}$ but it does not effect our estimate. Using Hölder's inequality, we can bound the left hand side of (3.24) as

$$\text{LHS(3.24)} \leq \|\langle\Lambda\rangle^\gamma w_1\|_{L_t^4L_x^4}\|w_2\|_{L_t^4L_x^4}\|w_3\|_{L_t^6L_x^6}\|w_4\|_{L_t^3L_x^3}.$$

Since $(4,4)$ is an admissible pair, Corollary 2.4 gives

$$\|\langle\Lambda\rangle^\gamma w_1\|_{L_t^4L_x^4} \lesssim \|w_1\|_{X^{\gamma,b}}, \quad \|w_2\|_{L_t^4L_x^4} \lesssim \|w_2\|_{X^{0,b}} \leq \|w_2\|_{X^{\gamma,b}}.$$

Similarly, Sobolev embedding and Corollary 2.4 yield

$$\|w_3\|_{L_t^6L_x^6} \lesssim \|\langle\Lambda\rangle^{k/6} w_3\|_{L_t^6L_x^3} \lesssim \|w_3\|_{X^{k/6},b} \leq \|w_3\|_{X^{\gamma,b}}.$$

The last estimate comes from the fact that $\gamma > \gamma(k) > k/6$. Finally, we interpolate between $\|w_4\|_{L_t^2L_x^2} = \|w_4\|_{X^{0,0}}$ and $\|w_4\|_{L_t^4L_x^4} \lesssim \|w_4\|_{X^{0,1/2+}}$ to get

$$\|w_4\|_{L_t^3L_x^3} \lesssim \|w_4\|_{X^{0,b'}}.$$

Combing these estimates, we have (3.24). The proof of Proposition 3.4 is now complete. □

We are now able to prove the almost conservation law.

Proof of Proposition 3.1. By the assumption $E(Iu_0) \leq 1$, Proposition 3.4 shows that there exists $\delta = \delta(\|u_0\|_{L^2_x})$ such that the solution u to (NLS$_k$) satisfies (3.15). We firstly note that the usual energy satisfies

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \mathrm{Re} \int_{\mathbb{R}^k} \overline{\partial_t u(t,x)}(|u(t,x)|^2 u(t,x) + \Lambda^k u(t,x))dx \\ &= \mathrm{Re} \int_{\mathbb{R}^k} \overline{\partial_t u(t,x)}(|u(t,x)|^2 u(t,x) + \Lambda^k u(t,x) + i\partial_t u(t,x))dx = 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}E(Iu(t)) &= \mathrm{Re} \int_{\mathbb{R}^k} \overline{I\partial_t u(t,x)}(|Iu(t,x)|^2 Iu(t,x) + \Lambda^k Iu(t,x) + i\partial_t Iu(t,x))dx \\ &= \mathrm{Re} \int_{\mathbb{R}^k} \overline{I\partial_t u(t,x)}(|Iu(t,x)|^2 Iu(t,x) - I(|u(t,x)|^2 u(t,x)))dx.\end{aligned}$$

Here the second line follows by applying I to both sides of (NLS$_k$). Integrating in time and applying the Parseval formula, we obtain

$$E(Iu(t)) - E(Iu_0) = \mathrm{Re} \int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \Big(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\Big)\widehat{\overline{I\partial_t u}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)dt.$$

Here $\int_{\sum_{j=1}^4 \xi_j=0}$ denotes the integration with respect to the hyperplane's measure $\delta_0(\xi_1+\ldots+\xi_4)d\xi_1\ldots d\xi_4$. Using that $iI\partial_t u = -\Lambda^k Iu - I(|u|^2 u)$, we have

$$|E(Iu(t)) - E(Iu_0)| \leq \mathrm{Term}_1 + \mathrm{Term}_2,$$

where

$$\mathrm{Term}_1 = \Big|\int_0^\delta \int_{\sum_{j=1}^4 \xi_j=0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\overline{\Lambda^k Iu}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)dt\Big|,$$

and

$$\mathrm{Term}_2 = \Big|\int_0^\delta \int_{\sum_{j=1}^4 \xi_j=0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\overline{I(|u|^2u)}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)dt\Big|,$$

with

$$\mu(\xi_2,\xi_3,\xi_4) := 1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}.$$

Our purpose is to prove

$$\mathrm{Term}_1 + \mathrm{Term}_2 \lesssim N^{-\gamma_0(k)+}.$$

Let us consider the first term (Term$_1$). To do so, we decompose $u = \sum_{M \geq 1} P_M u =: \sum_{M \geq 1} u_M$ with the convention $P_1 u := P_{\leq 1} u$ and write Term$_1$ as a sum over all dyadic pieces. By the symmetry of μ in ξ_2, ξ_3, ξ_4 and the fact that the bilinear estimate (2.7) allows complex conjugations on either factors, we may assume that $M_2 \geq M_3 \geq M_4$. Thus,

$$\mathrm{Term}_1 \lesssim \sum_{\substack{M_1, M_2, M_3, M_4 \geq 1 \\ M_2 \geq M_3 \geq M_4}} A(M_1, M_2, M_3, M_4),$$

where

$$A(M_1, M_2, M_3, M_4) := \Big| \int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \mu(\xi_2, \xi_3, \xi_4) \widehat{\overline{\Lambda^k I u_{M_1}}}(\xi_1) \widehat{I u_{M_2}}(\xi_2) \widehat{\overline{I u_{M_3}}}(\xi_3) \widehat{I u_{M_4}}(\xi_4) dt \Big|.$$

For simplifying the notation, we will drop the dependence of M_1, M_2, M_3, M_4 and write A instead of $A(M_1, M_2, M_3, M_4)$. In order to have Term$_1 \lesssim N^{-\gamma_0(k)+}$, it suffices to prove

$$A \lesssim N^{-\gamma_0(k)+} M_2^{0-}. \tag{3.25}$$

To show (3.25), we will break the frequency interactions into three cases due to the comparison of N with M_j. It is worth to notice that $M_1 \lesssim M_2$ due to the fact that $\sum_{j=1}^4 \xi_j = 0$.

Case 1. $N \gg M_2$. In this case, we have $|\xi_2|, |\xi_3|, |\xi_4| \ll N$ and $|\xi_2 + \xi_3 + \xi_4| \leq N$, hence

$$m(\xi_2 + \xi_3 + \xi_4) = m(\xi_2) = m(\xi_3) = m(\xi_4) = 1 \text{ and } \mu(\xi_2, \xi_3, \xi_4) = 0.$$

Thus (3.25) holds trivially.

Case 2. $M_2 \gtrsim N \gg M_3 \geq M_4$. Since $\sum_{j=1}^4 \xi_j = 0$, we get $M_1 \sim M_2$. We also have from the mean value theorem that

$$|\mu(\xi_2, \xi_3, \xi_4)| = \Big| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)} \Big| \lesssim \frac{|\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)|}{m(\xi_2)} \lesssim \frac{M_3}{M_2}.$$

The pointwise bound, Hölder's inequality, Plancherel theorem and bilinear estimate (2.7) yield

$$\begin{aligned}
A &\lesssim \frac{M_3}{M_2} \|\Lambda^k I u_{M_1} I u_{M_3}\|_{L^2_t L^2_x} \|I u_{M_2} I u_{M_4}\|_{L^2_t L^2_x} \\
&\lesssim \frac{M_3}{M_2} \Big(\frac{M_3}{M_1}\Big)^{(k-1)/2} \Big(\frac{M_4}{M_2}\Big)^{(k-1)/2} M_1^k \prod_{j=1}^4 \|I u_{M_j}\|_{X^{0,1/2+}} \\
&\lesssim \frac{M_3}{M_2} \Big(\frac{M_3}{M_1}\Big)^{(k-1)/2} \Big(\frac{M_4}{M_2}\Big)^{(k-1)/2} \frac{M_1^{k/2}}{M_2^{k/2} \langle M_3 \rangle^{k/2} \langle M_4 \rangle^{k/2}} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{k/2,1/2+}} \\
&= \Big(\frac{M_3}{N}\Big)^{1/2} \Big(\frac{M_1}{M_2}\Big)^{1/2} \Big(\frac{N}{M_2}\Big)^{k-} N^{-(k-1/2)+} M_2^{0-} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{k/2,1/2+}} \\
&\lesssim N^{-(k-1/2)+} M_2^{0-} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{k/2,1/2+}}.
\end{aligned} \tag{3.26}$$

Using (3.15) and the fact that $\gamma_0(k) < k - 1/2$ for $k \geq 3$, $k \in \mathbb{Z}$, we have (3.25).

Case 3. $M_2 \geq M_3 \gtrsim N$. In this case, we simply bound

$$|\mu(\xi_2, \xi_3, \xi_4)| \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}.$$

Here we use that $m(\xi_1) \gtrsim m(\xi_2)$ and $m(\xi_3) \leq m(\xi_4) \leq 1$ due to the fact that $M_1 \lesssim M_2$ and $M_3 \geq M_4$.

Subcase 3a. $M_2 \gg M_3 \gtrsim N$. We see that $M_1 \sim M_2$ since $\sum_{j=1}^4 \xi_j = 0$. The pointwise bound, Hölder's inequality, Plancherel theorem and bilinear estimate (2.7) again give

$$\begin{aligned} A &\lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \|\overline{\Lambda^k I u_{M_1}} I u_{M_4}\|_{L^2_t L^2_x} \|I u_{M_2} \overline{I u_{M_3}}\|_{L^2_t L^2_x} \\ &\lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \Big(\frac{M_4}{M_1}\Big)^{(k-1)/2} \Big(\frac{M_3}{M_2}\Big)^{(k-1)/2} \frac{M_1^{k/2}}{M_2^{k/2} M_3^{k/2} \langle M_4 \rangle^{k/2}} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{k/2,1/2+}}. \end{aligned}$$

Thanks to (3.15), we only need to show

$$\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \Big(\frac{M_4}{M_1}\Big)^{(k-1)/2} \Big(\frac{M_3}{M_2}\Big)^{(k-1)/2} \frac{M_1^{k/2}}{M_2^{k/2} M_3^{k/2} \langle M_4 \rangle^{k/2}} \lesssim N^{-\gamma_0(k)+} M_2^{0-}. \tag{3.27}$$

Remark that the function $m(\lambda)\lambda^\alpha$ is increasing, and $m(\lambda)\langle\lambda\rangle^\alpha$ is bounded below for any $\alpha + \gamma - k/2 > 0$ due to

$$(m(\lambda)\lambda^\alpha)' = \begin{cases} \alpha\lambda^{\alpha-1} & \text{if } 1 \leq \lambda \leq N, \\ N^{k/2-\gamma}(\alpha+\gamma-k/2)\lambda^{\alpha+\gamma-k/2-1} & \text{if } \lambda \geq 2N. \end{cases}$$

We shall shortly choose an appropriate value of α, says $\alpha(k)$, so that

$$m(M_4)\langle M_4\rangle^{\alpha(k)} \gtrsim 1, \quad m(M_3)M_3^{\alpha(k)} \gtrsim m(N)N^{\alpha(k)} = N^{\alpha(k)}. \tag{3.28}$$

Using that $m(M_1) \sim m(M_2)$, we have

$$\begin{aligned} \text{LHS(3.27)} &\lesssim \frac{M_3^{\alpha(k)-1/2} \langle M_4\rangle^{\alpha(k)-1/2} M_1^{1/2}}{m(M_3)M_3^{\alpha(k)} m(M_4)\langle M_4\rangle^{\alpha(k)} M_2^{k-1/2}} \\ &\lesssim \frac{1}{N^{\alpha(k)} M_2^{k-2\alpha(k)}} \Big(\frac{M_3}{M_2}\Big)^{\alpha(k)-1/2} \Big(\frac{\langle M_4\rangle}{M_2}\Big)^{\alpha(k)-1/2} \Big(\frac{M_1}{M_2}\Big)^{1/2} \\ &\lesssim N^{-(k-\alpha(k))+} M_2^{0-}. \end{aligned}$$

Therefore, if we choose $\alpha(k)$ so that $\gamma_0(k) = k - \alpha(k)$ or $\alpha(k) = k - \gamma_0(k) = \frac{k(2k-1)}{8k-2}$, then we get (3.25). Note that $\alpha(k) + \gamma(k) - k/2 \geq 0$ for $k \geq 3$, $k \in \mathbb{Z}$, hence (3.28) holds.

Subcase 3b. $M_2 \sim M_3 \gtrsim N$. In this case, we see that $M_1 \lesssim M_2$. Arguing as in Subcase 3a, we obtain

$$\begin{aligned} A &\lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \|\overline{\Lambda^k I u_{M_1}} I u_{M_2}\|_{L^2_t L^2_x} \|I u_{M_3} \overline{I u_{M_4}}\|_{L^2_t L^2_x} \\ &\lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \Big(\frac{M_1}{M_2}\Big)^{(k-1)/2} \Big(\frac{M_4}{M_3}\Big)^{(k-1)/2} \frac{\langle M_1\rangle^{k/2}}{M_2^{k/2} M_3^{k/2} \langle M_4 \rangle^{k/2}} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{k/2,1/2+}}. \end{aligned}$$

As in Subcase 3a, our aim is to prove

$$\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)}\Big(\frac{M_1}{M_2}\Big)^{(k-1)/2}\Big(\frac{M_4}{M_3}\Big)^{(k-1)/2}\frac{\langle M_1\rangle^{k/2}}{M_2^{k/2}M_3^{k/2}\langle M_4\rangle^{k/2}} \lesssim N^{-\gamma_0(k)+}M_2^{0-}. \tag{3.29}$$

We use (3.28) to get

$$\begin{aligned}
\text{LHS(3.29)} &\lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)\langle M_4\rangle^{1/2}M_3^{k-1/2}} \\
&\lesssim \frac{m(M_1)M_2^{\alpha(k)}\langle M_4\rangle^{\alpha(k)-1/2}}{m(M_2)M_2^{\alpha(k)}m(M_3)M_3^{\alpha(k)}m(M_4)\langle M_4\rangle^{\alpha(k)}M_3^{k-\alpha(k)-1/2}} \\
&\lesssim \frac{1}{N^{2\alpha(k)}}\Big(\frac{M_2}{M_3}\Big)^{\alpha(k)}\Big(\frac{\langle M_4\rangle}{M_3}\Big)^{\alpha(k)-1/2}\frac{1}{M_3^{k-3\alpha(k)}} \\
&\lesssim N^{-(k-\alpha(k))+}M_2^{0-}.
\end{aligned}$$

Choosing $\alpha(k)$ as in Subcase 3a, we get (3.25).

We now consider the second term (Term_2). We again decompose u in dyadic frequencies, $u=\sum_{M\geq 1}u_M$. By the symmetry, we can assume that $M_2\geq M_3\geq M_4$. We can assume further that $M_2\gtrsim N$ since $\mu(\xi_2,\xi_3,\xi_4)$ vanishes otherwise. Thus,

$$\text{Term}_2 \lesssim \sum_{\substack{M_1,M_2,M_3,M_4\geq 1\\ M_2\geq M_3\geq M_4}} B(M_1,M_2,M_3,M_4),$$

where

$$B(M_1,M_2,M_3,M_4) := \Big|\int_0^\delta \int_{\sum_{j=1}^4 \xi_j=0} \mu(\xi_2,\xi_3,\xi_4)\overline{\widehat{P_{M_1}I(|u|^2u)}}(\xi_1)\widehat{Iu_{M_2}}(\xi_2)\overline{\widehat{Iu_{M_3}}}(\xi_3)\widehat{Iu_{M_4}}(\xi_4)dt\Big|.$$

As for the Term_1, we will use the notation B instead of $B(M_1,M_2,M_3,M_4)$. Using the trivial bound

$$|\mu(\xi_2,\xi_3,\xi_4)| \lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)},$$

Hölder's inequality and Plancherel theorem, we bound

$$B \lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)}\|P_{M_1}I(|u|^2u)\|_{L^2_tL^2_x}\|Iu_{M_2}\|_{L^4_tL^4_x}\|Iu_{M_3}\|_{L^4_tL^4_x}\|Iu_{M_4}\|_{L^\infty_tL^\infty_x}.$$

Lemma 3.5. *We have*

$$\|P_{M_1}I(|u|^2u)\|_{L^2_tL^2_x} \lesssim \frac{1}{\langle M_1\rangle^{k/2}}\|Iu\|^3_{X^{k/2,1/2+}}, \tag{3.30}$$

$$\|Iu_{M_j}\|_{L^4_tL^4_x} \lesssim \frac{1}{\langle M_j\rangle^{k/2}}\|Iu_{M_j}\|_{X^{k/2,1/2+}}, \quad j=2,3, \tag{3.31}$$

$$\|Iu_{M_4}\|_{L^\infty_tL^\infty_x} \lesssim \|Iu_{M_4}\|_{X^{k/2,1/2+}}. \tag{3.32}$$

Proof. The estimate (3.30) is in turn equivalent to

$$\|\langle\Lambda\rangle^{k/2}P_{M_1}I(|u|^2u)\|_{L^2_tL^2_x} \lesssim \|Iu\|^3_{X^{k/2,1/2+}}.$$

Since $\langle\Lambda\rangle^{k/2} I$ obeys a Leibniz rule, it suffices to prove

$$\|P_{M_1}((\langle\Lambda\rangle^{k/2} Iu_1)u_2u_3)\|_{L_t^2L_x^2} \lesssim \prod_{j=1}^{3} \|Iu_j\|_{X^{k/2,1/2+}}. \tag{3.33}$$

The Littlewood–Paley theorem and Hölder's inequality imply

$$\text{LHS(3.33)} \lesssim \| \langle\Lambda\rangle^{k/2} Iu_1\|_{L_t^4L_x^4}\|u_2\|_{L_t^8L_x^8}\|u_3\|_{L_t^8L_x^8}.$$

We have from Strichartz estimate (2.5) that

$$\| \langle\Lambda\rangle^{k/2} Iu_1\|_{L_t^4L_x^4} \lesssim \| \langle\Lambda\rangle^{k/2} Iu_1\|_{X^{0,1/2+}} = \|Iu_1\|_{X^{k/2,1/2+}}.$$

Combining Sobolev embedding and Strichartz estimate (2.5) yield

$$\|u_2\|_{L_t^8L_x^8} \lesssim \| \langle\Lambda\rangle^{k/4} u_2\|_{L_t^8L_x^{8/3}} \lesssim \| \langle\Lambda\rangle^{k/4} u_2\|_{X^{0,1/2+}} \lesssim \|Iu_2\|_{X^{k/2,1/2+}},$$

where the last estimate follows from (2.11). Similarly for $\|u_3\|_{L_t^8L_x^8}$. This shows (3.33). The estimate (3.31) follows easily from Strichartz estimate. For (3.32), we use Sobolev embedding and Strichartz estimate to get

$$\|Iu_{M_4}\|_{L_t^\infty L_x^\infty} \lesssim \| \langle\Lambda\rangle^{k/2} Iu_{M_4}\|_{L_t^\infty L_x^2} \lesssim \| \langle\Lambda\rangle^{k/2} Iu_{M_4}\|_{X^{0,1/2+}} = \|Iu_{M_4}\|_{X^{k/2,1/2+}}.$$

The proof is complete. $\square$

We use Lemma 3.5 to bound

$$B \lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \frac{1}{\langle M_1\rangle^{k/2}\langle M_2\rangle^{k/2}\langle M_3\rangle^{k/2}} \|Iu\|_{X^{k/2,1/2+}} \prod_{j=2}^{4} \|Iu_{M_j}\|_{X^{k/2,1/2+}},$$

with $M_2 \geq M_3 \geq M_4$ and $M_2 \gtrsim N$. Using (3.15), the estimate (3.25) follows once we have

$$\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \frac{1}{\langle M_1\rangle^{k/2}\langle M_2\rangle^{k/2}\langle M_3\rangle^{k/2}} \lesssim N^{-\gamma_0(k)+}M_2^{0-}. \tag{3.34}$$

We now break the frequency interactions into two cases: $M_2 \sim M_3$ and $M_2 \sim M_1$ since $\sum_{j=1}^4 \xi_j = 0$.

Case 1. $M_2 \sim M_3$, $M_2 \geq M_3 \geq M_4$ and $M_2 \gtrsim N$. We see that

$$\begin{aligned}\text{LHS(3.34)} &\sim \frac{m(M_1)}{(m(M_2))^2 m(M_4)} \frac{1}{\langle M_1\rangle^{k/2}\langle M_2\rangle^{k}} \lesssim \frac{m(M_1)}{N^{2\alpha(k)} m(M_4) \langle M_1\rangle^{k/2}\langle M_2\rangle^{k-2\alpha(k)}} \\ &\lesssim \frac{1}{N^{2\alpha(k)}} \frac{1}{m(M_4)\langle M_2\rangle^{k-2\alpha(k)}} \lesssim \frac{1}{N^{2\alpha(k)}} \frac{1}{M_2^{k-3\alpha(k)}} \lesssim N^{-(k-\alpha(k))+}M_2^{0-}.\end{aligned}$$

Here we use that $m(M_2)\langle M_2\rangle^{\alpha(k)} \geq m(N)N^{\alpha(k)} = N^{\alpha(k)}$, $m(M_1) \lesssim \langle M_1\rangle^{k/2}$ and that $m(y)\langle x\rangle^{\alpha(k)} \gtrsim 1$ for all $1 \leq y \leq x$.

Case 2. $M_2 \sim M_1$, $M_2 \geq M_3 \geq M_4$ and $M_2 \gtrsim N$. We have

$$\begin{aligned}\text{LHS(3.34)} &\lesssim \frac{1}{m(M_3)m(M_4)} \frac{1}{\langle M_2\rangle^k \langle M_3\rangle^{k/2}} \\ &\lesssim \frac{1}{m(M_3)\langle M_3\rangle^{\alpha(k)}} \frac{1}{m(M_4)\langle M_2\rangle^{\alpha(k)}} \frac{1}{\langle M_2\rangle^{k-\alpha(k)}\langle M_3\rangle^{k/2-\alpha(k)}} \\ &\lesssim N^{-(k-\alpha(k))+} M_2^{0-}.\end{aligned}$$

Here we use again $m(M_3)\langle M_3\rangle^{\alpha(k)}, m(M_4)\langle M_2\rangle^{\alpha(k)} \gtrsim 1$. By choosing $\alpha(k)$ as in Subcase 3a, we prove (3.34). The proof of Proposition 3.1 is now complete. □

Remark 3.6. Let us now comment on the choices of $\alpha(k)$ and $\gamma_0(k)$. As mentioned in Remark 3.2, if the increment of the modified energy is $N^{-\gamma_0(k)}$, then we can show (see Section 4, after (4.39)) that the global well-posedness holds for data in $H^\gamma(\mathbb{R}^k)$ with $\gamma > \frac{k^2}{2(k+\gamma_0(k))} =: \gamma(k)$. We learn from (3.26) that $\gamma_0(k) \leq k - 1/2$, hence $\gamma(k) \geq \frac{k^2}{4k-1}$. On the other hand, in Subcase 3a, we need $\alpha(k) + \gamma - k/2 > 0$ and $\alpha(k) = k - \gamma_0(k)$. Since $\gamma > \gamma(k)$, we have $\alpha(k) + \gamma - k/2 > \alpha(k) + \gamma(k) - k/2 \geq \alpha(k) + \frac{k^2}{4k-1} - \frac{k}{2}$. We thus choose $\alpha(k) := \frac{k}{2} - \frac{k^2}{4k-1} = \frac{k(2k-1)}{8k-2}$, hence $\gamma_0(k) = k - \alpha(k) = \frac{k(6k-1)}{8k-2}$.

4. The proof of Theorem 1.1

We now are able to show the global existence given in Theorem 1.1. We only consider positive time, the negative one is treated similarly. The conservation of mass and Lemma 2.7 give

$$\|u(t)\|^2_{H^\gamma_x} \lesssim \|Iu(t)\|^2_{H^{k/2}_x} \sim \|Iu(t)\|^2_{\dot{H}^{k/2}_x} + \|Iu(t)\|^2_{L^2_x} \lesssim E(Iu(t)) + \|u_0\|^2_{L^2_x}. \tag{4.35}$$

By density argument, we may assume that $u_0 \in C^\infty_0(\mathbb{R}^k)$. Let u be a global solution to (NLS$_k$) with initial data u_0. As $E(Iu_0)$ is not necessarily small, we will use the scaling (1.1) to make the energy of rescaled initial data small in order to apply the almost conservation law given in Proposition 3.1. Let $\lambda > 0$ and u_λ be as in (1.1). We have

$$E(Iu_\lambda(0)) = \frac{1}{2}\|Iu_\lambda(0)\|^2_{\dot{H}^{k/2}_x} + \frac{1}{4}\|Iu_\lambda(0)\|^4_{L^4_x}. \tag{4.36}$$

We then estimate

$$\|Iu_\lambda(0)\|^2_{\dot{H}^{k/2}_x} \lesssim N^{2(k/2-\gamma)}\|u_\lambda(0)\|^2_{\dot{H}^\gamma_x} = N^{2(k/2-\gamma)}\lambda^{-2\gamma}\|u_0\|^2_{\dot{H}^\gamma_x},$$

and

$$\|Iu_\lambda(0)\|^4_{L^4_x} \lesssim \|u_\lambda(0)\|^4_{L^4_x} = \lambda^{-k}\|u_0\|^4_{L^4_x} \lesssim \lambda^{-k}\|u_0\|^4_{H^\gamma_x}.$$

Note that $\gamma > \gamma(k) \geq k/4$ allows us to use Sobolev embedding in the last inequality. Thus, (4.36) gives for $\lambda \gg 1$,

$$E(Iu_\lambda(0)) \lesssim (N^{2(k/2-\gamma)}\lambda^{-2\gamma} + \lambda^{-k})(1 + \|u_0\|_{H^\gamma_x})^4 \leq C_0 N^{2(k/2-\gamma)}\lambda^{-2\gamma}(1 + \|u_0\|_{H^\gamma_x})^4.$$

We now choose

$$\lambda := N^{\frac{k/2-\gamma}{\gamma}}\left(\frac{1}{2C_0}\right)^{-\frac{1}{2\gamma}}(1 + \|u_0\|_{H^\gamma_x})^{\frac{2}{\gamma}} \tag{4.37}$$

so that $E(Iu_\lambda(0)) \leq 1/2$. We then apply Proposition 3.1 for $u_\lambda(0)$. Note that we may reapply this proposition until $E(Iu_\lambda(t))$ reaches 1, that is at least $C_1 N^{\gamma_0(k)-}$ times. Therefore,

$$E(Iu_\lambda(C_1 N^{\gamma_0(k)-}\delta)) \sim 1. \tag{4.38}$$

Now given any $T \gg 1$, we choose $N \gg 1$ so that

$$T \sim \frac{N^{\gamma_0(k)-}}{\lambda^k} C_1 \delta.$$

Using (4.37), we see that

$$T \sim N^{\frac{2(\gamma_0(k)+k)\gamma - k^2}{2\gamma}-}. \tag{4.39}$$

Here $\gamma > \gamma(k) = \frac{k^2}{2(\gamma_0(k)+k)}$, hence the power of N is positive and the choice of N makes sense for arbitrary $T \gg 1$. Next, using (1.1), a direct computation shows

$$E(Iu(t)) = \lambda^k E(Iu_\lambda(\lambda^k t)).$$

Thus, we have from (4.37), (4.38) and (4.39) that

$$\begin{aligned} E(Iu(T)) &= \lambda^k E(Iu_\lambda(\lambda^k T)) = \lambda^k E(Iu_\lambda(C_1 N^{\gamma_0(k)-}\delta)) \\ &\sim \lambda^k \leq N^{\frac{k(k/2-\gamma)}{\gamma}} \sim T^{\frac{k(k-2\gamma)}{2(\gamma_0(k)+k)\gamma-k^2}+}. \end{aligned}$$

This shows that there exists $C_2 = C_2(\|u_0\|_{H^\gamma_x}, \delta)$ such that

$$E(Iu(T)) \leq C_2 T^{\frac{k(k-2\gamma)}{2(\gamma_0(k)+k)\gamma-k^2}+},$$

for $T \gg 1$. This together with (4.35) show that

$$\|u(T)\|_{H^\gamma_x} \lesssim C_3 T^{\frac{k(k-2\gamma)}{2(2(\gamma_0(k)+k)\gamma-k^2)}+} + C_4,$$

where C_3, C_4 depend only on $\|u_0\|_{H^\gamma_x}$. The proof of Theorem 1.1 is complete.

Acknowledgments

The author would like to express his deep thanks to his wife – Uyen Cong for her encouragement and support. He also would like to thank his supervisor Prof. Jean-Marc BOUCLET for the kind guidance and constant encouragement. He also would like to thank the reviewers for their helpful comments and suggestions, which helped improve the manuscript.

Appendix A. Linear estimate in $X^{\gamma,b}$ spaces

In this section, we will give the proof of linear estimates (3.17) and (3.18) which is essentially given in [17]. The estimate (3.17) follows from the fact that

$$\|u\|_{X^{\gamma,b}} = \|e^{-it\Lambda^k} u\|_{H^b_t H^\gamma_x}. \tag{A.1}$$

Indeed, we have

$$\|\psi(t)e^{it\Lambda^k}u_0\|_{X^{\gamma,b}} = \|e^{-it\Lambda^k}\psi(t)e^{it\Lambda^k}u_0\|_{H^b_t H^\gamma_x} = \|\psi\|_{H^b_t}\|u_0\|_{H^\gamma_x} \lesssim \|u_0\|_{H^\gamma_x}.$$

For (3.18), we firstly remark that it is a consequence of the following estimate

$$\Big\|\psi_\delta(t)\int_0^t g(s)ds\Big\|_{H^b_t} \lesssim \delta^{1-b-b'}\|g\|_{H_t^{-b'}}. \tag{A.2}$$

In fact, using (A.1), it suffices to prove

$$\Big\|\psi_\delta(t)\int_0^t G(s)ds\Big\|_{H^b_t H^\gamma_x} \lesssim \|G\|_{H_t^{-b'}H^\gamma_x}. \tag{A.3}$$

We now apply (A.2) for $g(s) = \widehat{G}(s,\xi)$ with ξ fixed to have

$$\Big\|\psi_\delta(t)\int_0^t \widehat{G}(s,\xi)ds\Big\|_{H^b_t} \lesssim \delta^{1-b-b'}\|\widehat{G}(t,\xi)\|_{H_t^{-b'}}, \tag{A.4}$$

where $\widehat{\cdot}$ is the spatial Fourier transform. If we denote

$$H(t,x) := \psi_\delta(t)\int_0^t G(s,x)ds,$$

then (A.4) becomes

$$\|\widehat{H}(t,\xi)\|_{H^b_t} \lesssim \delta^{1-b-b'}\|\widehat{G}(t,\xi)\|_{H_t^{-b'}}.$$

Squaring the above estimate, multiplying both sides with $\langle\xi\rangle^{2\gamma}$ and integrating over $\mathbb{R}^k$, we obtain (A.3). It remains to prove (A.2). To do so, we write

$$\begin{aligned}\psi_\delta(t)\int_0^t g(s)ds &= \psi_\delta(t)\int_{\mathbb{R}}\Big(\int_0^t e^{i\tau s}ds\Big)\hat{g}(\tau)d\tau = \psi_\delta(t)\int_{\mathbb{R}}\frac{e^{it\tau}-1}{i\tau}\hat{g}(\tau)d\tau \\ &= \psi_\delta(t)\sum_{k\geq 1}\frac{t^k}{k!}\int_{|\delta\tau|\leq 1}(i\tau)^{k-1}\hat{g}(\tau)d\tau - \psi_\delta(t)\int_{|\delta\tau|\geq 1}(i\tau)^{-1}\hat{g}(\tau)d\tau \\ &+ \psi_\delta(t)\int_{|\delta\tau|\geq 1}(i\tau)^{-1}e^{it\tau}\hat{g}(\tau)d\tau =: I + II + III.\end{aligned}$$

Let us consider the first term. The Cauchy–Schwarz inequality gives

$$\|I\|_{H^b_t} \leq \sum_{k\geq 1}\frac{1}{k!}\|t^k\psi_\delta\|_{H^b_t}\delta^{1-k}\|g\|_{H_t^{-b'}}\Big(\int_{|\delta\tau|\leq 1}\langle\tau\rangle^{2b'}d\tau\Big)^{1/2}.$$

Using that $t^k\psi_\delta(t) = \delta^k\varphi_k(\delta^{-1}t)$ where $\varphi_k(t) = t^k\psi(t)$, we have

$$\|t^k\psi_\delta\|_{H^b_t} = \delta^k\|\varphi_k(\delta^{-1}t)\|_{H^b_t} = \delta^k\Big(\int_{\mathbb{R}}\langle\tau\rangle^{2b}\delta^2|\hat{\varphi}_k(\delta\tau)|^2d\tau\Big)^{1/2} \lesssim \delta^k\delta^{1/2-b}\|\varphi_k\|_{H^b_t}.$$

We also have

$$\int\limits_{|\delta\tau|\leq 1} \langle\tau\rangle^{2b'} d\tau = \int\limits_{|\tau|\leq 1} \left\langle\delta^{-1}\tau\right\rangle^{2b'} \delta^{-1} d\tau \lesssim \delta^{-1-2b'},$$

since $b' < 1/2$. This implies

$$\|I\|_{H^b_t} \lesssim \sum_{k\geq 1} \frac{1}{k!} \delta^k \delta^{1/2-b} \delta^{1-k} \|g\|_{H^{-b'}_t} \delta^{-1/2-b'} \lesssim \delta^{1-b-b'} \|g\|_{H^{-b'}_t}.$$

Similarly, we have

$$\|II\|_{H^b_t} \lesssim \|\psi_\delta\|_{H^b_t} \|g\|_{H^{-b'}_t} \Big(\int\limits_{|\delta\tau|\geq 1} |\tau|^{-2} \langle\tau\rangle^{2b'} d\tau \Big)^{1/2} \lesssim \delta^{1-b-b'} \|g\|_{H^{-b'}_t},$$

by using that $\|\psi_\delta\|_{H^b_t} \lesssim \delta^{1/2-b} \|\psi\|_{H^b_t} \lesssim \delta^{1/2-b}$ and

$$\int\limits_{|\delta\tau|\geq 1} |\tau|^{-2} \langle\tau\rangle^{2b'} d\tau = \int\limits_{|\tau|\geq 1} |\delta^{-1}\tau|^{-2} \left\langle\delta^{-1}\tau\right\rangle^{2b'} \delta^{-1} d\tau \leq \delta^{1-2b'} \int\limits_{|\tau|\geq 1} |\tau|^{-2} \langle\tau\rangle^{2b'} d\tau \lesssim \delta^{1-2b'}.$$

Here $b' < 1/2$ hence $2(1-b') > 1$ implies the last integral is convergent. We finally treat the third term as follows. Set

$$J(t) := \int\limits_{|\delta\tau|\geq 1} (i\tau)^{-1} \hat{g}(\tau) e^{it\tau} d\tau.$$

We see that

$$\hat{J}(\zeta) = \int\limits_{|\delta\tau|\geq 1} (i\tau)^{-1} \hat{g}(\tau) \delta_0(\zeta-\tau) d\tau,$$

where δ_0 is the Dirac delta function. This yields that

$$\begin{aligned} \|J\|_{H^b_t} = \Big(\int \langle\zeta\rangle^{2b} |\hat{J}(\zeta)|^2 d\zeta \Big)^{1/2} &= \Big(\int\limits_{|\delta\tau|\geq 1} \langle\tau\rangle^{2b} |\tau|^{-2} |\hat{g}(\tau)|^2 d\tau \Big)^{1/2} \\ &\leq \|g\|_{H^{-b'}_t} \sup_{|\delta\tau|\geq 1} |\tau|^{-1} \langle\tau\rangle^{b+b'} \lesssim \delta^{1-b-b'} \|g\|_{H^{-b'}_t}. \end{aligned}$$

Similarly,

$$\|J\|_{L^2_t} \lesssim \delta^{1-b'} \|g\|_{H^{-b'}_t}.$$

Thus, the Young's inequality gives

$$\|III\|_{H^b_t} = \| \langle\tau\rangle^b (\hat{\psi}_\delta \star \hat{J}) \|_{L^2_\tau} \lesssim \| |\tau|^b \hat{\psi}_\delta \|_{L^1_\tau} \|\hat{J}\|_{L^2_\tau} + \|\hat{\psi}_\delta\|_{L^1_\tau} \| \langle\tau\rangle^b \hat{J} \|_{L^2_\tau} \lesssim \delta^{1-b-b'} \|g\|_{H^{-b'}_t}.$$

Here we use the fact that $\langle\tau\rangle^b \lesssim |\tau-\zeta|^b + \langle\zeta\rangle^b$ to have the first estimate. This completes the proof.

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