



# Four-body central configurations with one pair of opposite sides parallel



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## ABSTRACT

We study four-body central configurations with one pair of opposite sides parallel. We use a novel constraint to write the central configuration equations in this special case, using distances as variables. We prove that, for a given ordering of the mutual distances, a trapezoidal central configuration must have a certain partial ordering of the masses. We also show that if opposite masses of a four-body trapezoidal central configuration are equal, then the configuration has a line of symmetry and it must be a kite. In contrast to the general four-body case, we show that if the two adjacent masses bounding the shortest side are equal, then the configuration must be an isosceles trapezoid, and the remaining two masses must also be equal.

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## 1. Introduction

Let  $P_1, P_2, P_3$ , and  $P_4$  be four points in  $\mathbb{R}^3$  with position vectors  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ , and  $\mathbf{q}_4$ , respectively. Let  $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ , be the distance between the point  $P_i$  and  $P_j$ , and let  $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) \in \mathbb{R}^{12}$ . The center of mass of the system is  $\mathbf{q}_{CM} = \frac{1}{M} \sum_{i=1}^n m_i \mathbf{q}_i$ , where  $M = m_1 + \dots + m_n$  is the total mass. The Newtonian 4-body problem concerns the motion of 4 particles with masses  $m_i \in \mathbb{R}^+$  and positions  $\mathbf{q}_i \in \mathbb{R}^3$ , where  $i = 1, \dots, 4$ . The motion is governed by Newton's law of motion

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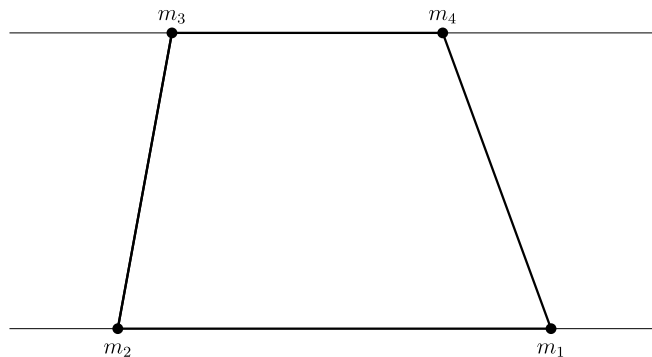


Fig. 1. An example of a trapezoidal central configuration.

$$m_i \ddot{\mathbf{q}}_i = \sum_{j \neq i} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} = \frac{\partial U}{\partial \mathbf{q}_i}, \quad 1 \leq i \leq 4 \quad (1)$$

where  $U(\mathbf{q})$  is the Newtonian potential

$$U(\mathbf{q}) = \sum_{i < j} \frac{m_i m_j}{r_{ij}}, \quad 1 \leq i \leq 4. \quad (2)$$

A *central configuration* (c.c.) of the four-body problem is a configuration  $\mathbf{q} \in \mathbb{R}^{12}$  which satisfies the algebraic equations

$$\lambda m_i (\mathbf{q}_i - \mathbf{q}_{CM}) = \sum_{j \neq i} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}, \quad 1 \leq i \leq n. \quad (3)$$

If we let  $I(\mathbf{q})$  denote the moment of inertia, that is,

$$I(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^n m_i \|\mathbf{q}_i - \mathbf{q}_{CM}\|^2 = \frac{1}{2M} \sum_{1 \leq i < j \leq n} m_i m_j r_{ij}^2,$$

we can write equations (3) as

$$\nabla U(\mathbf{q}) = \lambda \nabla I(\mathbf{q}). \quad (4)$$

Viewing  $\lambda$  as a Lagrange multiplier, a central configuration is simply a critical point of  $U$  subject to the constraint  $I$  equals a constant.

A central configuration is *planar* if the four points  $P_1, P_2, P_3$ , and  $P_4$  lie on the same plane. Equations (3), and (4) also describe planar central configurations provided  $\mathbf{q}_i \in \mathbb{R}^2$  for  $i = 1, \dots, 4$ . We say that a planar configuration is *degenerate* if two or more points coincide, or if more than two points lie on the same line. Non-degenerate planar configurations can be classified as either *concave* or *convex*. A concave configuration has one point which is located strictly inside the convex hull of the other three, whereas a convex configuration does not have a point contained in the convex hull of the other three points. Any convex configuration determines a convex quadrilateral (for a precise definition of quadrilateral see for example [5]). In a planar convex configuration we say that the points are *ordered sequentially* if they are numbered consecutively while traversing the boundary of the corresponding convex quadrilateral. In this paper we are interested in studying *trapezoidal central configurations*, that is, those c.c.'s for which two of the opposite sides are parallel (see Fig. 1). Non-degenerate trapezoidal central configurations are necessarily convex.

The four body problem has a long and distinguished history. In 1900 Dziobek derived equations for central configurations of four bodies with distances as variables [14]. In 1932 McMillan and Bartky used similar equations to obtain many important new results [19]. In 1996 Albouy [1,2] gave a complete classifications of the four-body c.c.'s with equal masses. More recently, in 2006 Hampton and Moeckel [16] proved the finiteness of the number of c.c.'s. Other recent results of note, concerning four-body c.c.'s in the case some of the masses are equal, were obtained by the present author and Perez-Chavela, [20], Albouy, Fu and Sun [4], and Fernandes, Llibre and Mello [15]. Further results for the four-body were recently attained by Cors and Roberts [9], Corbera, Cors and Roberts [8], Deng, Li and Zhang [11,12] and Xie [24], just to mention a few. Particularly important for this paper is the work of Cors and Roberts [9] which provided inspiration for the approach we take here. Additionally, certain bifurcations in the four-body problem, and several planarity conditions and their applications to four-body c.c.'s were obtained by the present author in [21] and [22], respectively.

Let  $\mathbf{r} = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \in (\mathbb{R}^+)^6$  be a vector of mutual distances. The conditions for which such vector determines a realizable configuration of four bodies in Euclidean space can be expressed by the Cayley–Menger criterion, that we state below. The Cayley–Menger determinant of four points  $P_1, \dots, P_4$  is

$$H(\mathbf{r}) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

A configuration is *geometrically realizable* if and only if the Cayley–Menger determinant of each subconfiguration of two or more points is  $\geq 0$  when the number of points is even, and  $\leq 0$  when it is odd. See, for instance, the book of Blumenthal [7] or Theorem 9.7.3.4 and Exercise 9.14.23 in [6]. Note that an equivalent characterization can be given in terms of Borchardt's quadratic form, see [3,23]. We denote by  $\mathcal{G}$  the set of geometrically realizable configurations. In the remainder of this paper we assume that  $\mathbf{r} \in \mathcal{G}$ .

In the four-body problem the mutual distances are not independent so that describing planar four-body central configurations requires an additional constraint. Following Dziobek [14] it is customary to use the following planarity condition,

**Planarity Condition 1.**  $P_1, P_2, P_3, P_4 \in \mathbb{R}^3$  are coplanar if and only if the Cayley–Menger determinant determined by these four points is 0, that is,  $H(\mathbf{r}) = 0$ .

In this paper we use a different constraint that not only gives planarity of the configuration, but also restricts the configuration to be trapezoidal. This planarity conditions complements the list given in [22]. Our approach parallels the treatment of the co-circular four body problem given by Cors and Roberts in [9]. The new constraint is introduced in Section 2. In Section 3 we study the relationship between the Cayley–Menger constraint and the constraint used in the paper, and show that the gradients of these restrictions are collinear at trapezoidal configurations. We then use this fact to derive the equations for the trapezoidal central configurations. In Section 4 we prove that, for a given ordering of the mutual distances, a trapezoidal central configuration must have a certain partial ordering of the masses. This result is by necessity weaker than the analogous result for co-circular configurations where one obtains a total ordering (see [9]). We also prove that if opposite masses of a four-body trapezoidal central configuration are equal, then the configuration has a line of symmetry and is a kite. This is a special case of the well known result of Albouy, Fu and Sun [4]. A similar result also holds in the case the two adjacent masses bounding the shortest side are equal. In this case the configuration is an isosceles trapezoid, and the remaining two masses

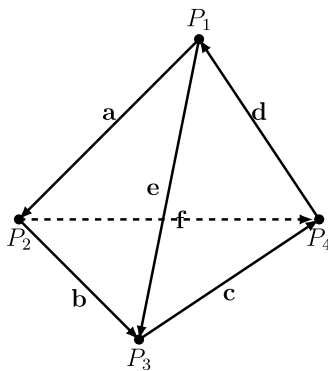


Fig. 2. The points  $P_1, P_2, P_3$ , and  $P_4$  form a tetrahedron in  $\mathbb{R}^3$ .

must also be equal. Finally, we show that, in contrast to the co-circular case, when the two adjacent masses bounding the longest side are equal there are asymmetric solutions.

## 2. Another planarity conditions

Let  $P_1, P_2, P_3$ , and  $P_4$  be four points in  $\mathbb{R}^3$  and let  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ , and  $\mathbf{q}_4$  be their position vectors. In this section we introduce a planarity condition that also constrains the configuration to have one pair of opposite sides parallel. Let

$$\begin{aligned}\mathbf{a} &= \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{b} = \mathbf{q}_3 - \mathbf{q}_2, \quad \mathbf{c} = \mathbf{q}_4 - \mathbf{q}_3, \\ \mathbf{d} &= \mathbf{q}_1 - \mathbf{q}_4, \quad \mathbf{e} = \mathbf{q}_3 - \mathbf{q}_1, \quad \mathbf{f} = \mathbf{q}_4 - \mathbf{q}_2,\end{aligned}$$

then it follows that  $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$ ,  $\mathbf{f} = \mathbf{b} + \mathbf{c}$ , and  $\mathbf{e} = \mathbf{a} + \mathbf{b}$ , see Fig. 2. For convenience, we will also use  $a, b, c, d, e, f$  to denote the mutual distances:

$$a = r_{12}, \quad b = r_{23}, \quad c = r_{34}, \quad d = r_{14}, \quad e = r_{13}, \quad f = r_{24}.$$

In the following lemma we introduce the quantity  $\Delta$  that will be shown to be of great significance for this work.

**Lemma 1.** Let  $\Delta = \frac{1}{2} \|\mathbf{a} \times \mathbf{c}\|$ , then, with the above definitions, the following equation holds

$$4\Delta^2 = a^2 c^2 - \frac{1}{4}(b^2 + d^2 - e^2 - f^2)^2.$$

**Proof.** Clearly,

$$4\Delta^2 = (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{c} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})^2 = a^2 c^2 - (\mathbf{a} \cdot \mathbf{c})^2.$$

But

$$\begin{aligned}2(\mathbf{a} \cdot \mathbf{c}) &= 2 \cdot (\mathbf{f} + \mathbf{d}) \cdot (\mathbf{d} + \mathbf{e}) = 2\mathbf{d} \cdot (\mathbf{f} + \mathbf{d}) + 2\mathbf{e} \cdot (\mathbf{f} + \mathbf{d}) \\ &= 2\mathbf{d} \cdot (\mathbf{b} - \mathbf{e}) + 2\mathbf{e} \cdot (\mathbf{f} + \mathbf{d}) = 2\mathbf{b} \cdot \mathbf{d} - 2\mathbf{e} \cdot \mathbf{f} \\ &= (\mathbf{d} + \mathbf{b}) \cdot (\mathbf{d} + \mathbf{b}) - \mathbf{d} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{b} - (\mathbf{e} - \mathbf{f}) \cdot (\mathbf{e} - \mathbf{f}) + \mathbf{e} \cdot \mathbf{e} + \mathbf{f} \cdot \mathbf{f} \\ &= e^2 + f^2 - d^2 - b^2 + (\mathbf{d} + \mathbf{b}) \cdot (\mathbf{d} + \mathbf{b}) - (\mathbf{b} + 2\mathbf{c} + \mathbf{d})^2\end{aligned}$$

$$\begin{aligned}
&= e^2 + f^2 - d^2 - b^2 - \mathbf{c} \cdot (\mathbf{b} + \mathbf{d}) - 4\mathbf{c} \cdot \mathbf{c} \\
&= e^2 + f^2 - d^2 - b^2 - 4c^2 + 4\mathbf{c} \cdot (\mathbf{a} + \mathbf{c}) \\
&= e^2 + f^2 - d^2 - b^2 + 4\mathbf{a} \cdot \mathbf{c}.
\end{aligned}$$

It follows that

$$2(\mathbf{a} \cdot \mathbf{c}) = d^2 + b^2 - e^2 - f^2.$$

Hence,

$$4\Delta^2 = a^2c^2 - \frac{1}{4}(e^2 + f^2 - b^2 - d^2)^2. \quad \square$$

In the case of a planar configuration  $\Delta$  can be interpreted as the absolute value of the difference of the areas of the triangles whose bases are the sides  $\mathbf{b}$  and  $\mathbf{d}$  of a convex quadrilateral, and whose vertices coincide with the intersection of the diagonals (see [17] page 208). Note that  $\Delta$  can also be viewed as the area of a crossed quadrilateral (see [10]).

There are two ways to obtain a planarity condition from this. One is to impose that  $\Delta$  is equal to the absolute value of the difference of the areas  $A_3$  and  $A_4$  (or the absolute value of the difference between  $A_1$  and  $A_2$ ). Here  $A_i$  is the area of the triangle whose vertices contain all bodies except for the  $i$ th body. The second approach, which is the one we take here, is to impose that  $\Delta = 0$ .

**Planarity Condition 2.** Suppose  $\Delta = \frac{1}{2}\|\mathbf{a} \times \mathbf{c}\|$ . Then,  $\Delta = 0$  if and only if  $\mathbf{a}$  and  $\mathbf{c}$  are parallel and the configuration is planar.

**Proof.** Clearly, if  $\|\mathbf{a} \times \mathbf{c}\| = 0$  the vectors  $\mathbf{a}$  and  $\mathbf{c}$  are parallel, in which case the configuration is planar because the four points lie on two parallel lines. Conversely, if the configuration is planar with  $\mathbf{a}$  and  $\mathbf{c}$  parallel, then  $\|\mathbf{a} \times \mathbf{c}\| = 0$ .  $\square$

Note that the above condition can be written explicitly in terms of mutual distances as  $a^2c^2 = \frac{1}{4}(e^2 + f^2 - b^2 - d^2)^2$ , or

$$(2ac + e^2 + f^2 - b^2 - d^2)(2ac - e^2 - f^2 + b^2 + d^2) = 0. \quad (5)$$

For the remainder of this paper we will assume that any trapezoidal configuration satisfying the planarity condition above is ordered sequentially so that  $r_{12}, r_{34}$  are the lengths of the bases of the trapezoid,  $r_{23}$  and  $r_{14}$  are the lengths of legs, and  $r_{13}$  and  $r_{24}$  are the lengths of the diagonals. In this case one has

$$(2ac - e^2 - f^2 + b^2 + d^2) = 0, \quad (6)$$

which is known as a necessary and sufficient condition for a convex quadrilateral with consecutive sides  $a, b, c, d$  and diagonals  $e, f$  to be a trapezoid with parallel sides  $a$  and  $c$ . See for example [18].

To double check that for realizable configurations equation (6) implies planarity, we proceed as follows. Substituting  $e^2 = 2ac - f^2 + b^2 + d^2$  into the Cayley–Menger determinant yields

$$-2(r_{12}^2r_{34} + r_{12}r_{23}^2 - r_{12}r_{24}^2 - r_{12}r_{34}^2 - r_{14}^2r_{34} + r_{24}^2r_{34})^2 \leq 0.$$

By the Cayley–Menger criterion, this implies that for the mutual distances vector  $\mathbf{r}$  to correspond to a realizable configuration one must have

$$r_{12}^2 r_{34} + r_{12} r_{23}^2 - r_{12} r_{24}^2 - r_{12} r_{34}^2 - r_{14}^2 r_{34} + r_{24}^2 r_{34} = 0, \quad (7)$$

which in turn implies that  $H(\mathbf{r}) = 0$ , and leads to the formulas (19) and (20) for the diagonals of a trapezoid.

We remark that if one imposes Ptolemy's condition to study co-circular configurations, as done in [9], it is possible to see that any realizable configuration satisfying Ptolemy's must be planar as a consequence of the Cayley–Menger criterion.

### 3. Trapezoidal c.c.'s equations

In this section we give a derivation of the trapezoidal c.c.'s equations that mirrors the approach of Cors and Roberts [9] for the co-circular problem. Let  $\mathcal{F} \subset \mathcal{G}$  denote the set of geometrically realizable  $\mathbf{r}$  satisfying  $F(\mathbf{r}) = 0$ . As a consequence of Planarity Condition 2, it seems natural to use the condition  $F = 4\Delta^2 = 0$  to look for central configurations with opposite sides parallel. It is important to note, however, that imposing this condition one might find solutions that are critical points within the submanifold  $F = 0$ , but not central configurations. To exclude this possibility we need to show that the level sets  $\{F = 0\}$  and  $\{H = 0\}$  meet tangentially for any  $\mathbf{r} \in \mathcal{F}$ . This is done in the next lemma, which shows that  $\nabla F$  and  $\nabla H$  are parallel at any point of  $\mathcal{F}$ :

**Lemma 2.** *For any trapezoidal central configuration, that is, for any  $\mathbf{r} \in \mathcal{F}$*

$$\nabla H(\mathbf{r}) = 8h^2 \nabla F(\mathbf{r}),$$

where  $h$  is the distance between the parallel sides of the trapezoid. In other words, on the set of geometrically realizable vectors for which both  $H$  and  $F$  vanish, the gradients of these two functions are parallel.

**Proof.** Let  $\Delta_i$  be the oriented area of the triangle whose vertices contain all bodies except for the  $i$ th body. For a quadrilateral ordered sequentially, we have  $\Delta_1, \Delta_3 > 0$  and  $\Delta_2, \Delta_4 < 0$ . The derivatives of the Cayley–Menger determinant at planar c.c.'s are given by the following formula due to Dziobek [14]

$$\frac{\partial H}{\partial r_{ij}^2}(\mathbf{r}) = -32\Delta_i\Delta_j.$$

As it was observed by Cors and Roberts [9] the minus sign in the above equation is not included in Dziobek's original paper [14] nor in several later works on four-body central configurations, including one of my own papers [20].

In a trapezoid the areas  $|\Delta_i|$  take the form:

$$|\Delta_1| = |\Delta_2| = \frac{1}{2}r_{34}h, \quad |\Delta_3| = |\Delta_4| = \frac{1}{2}r_{12}h$$

where  $h$  is the height, that is, the distances between the opposite parallel sides. If the parallel sides have different lengths (i.e.,  $r_{12} \neq r_{34}$ ) the height of a trapezoid can be expressed in terms of mutual distances as follows:

$$h = \frac{\sqrt{(a-c+d+e)(-a+c+d+e)(a+c-d+e)(a+c+d-e)}}{2|a-c|}.$$

If  $r_{12} = r_{34}$ , then the trapezoid reduces to a parallelogram, in which case, since the area is  $A = r_{12}h$ , we have

$$h = \frac{r_{12}}{A}$$

and  $A$  is given by Bretschneider's formula for the area of a quadrilateral, that is,

$$A = \frac{1}{2} \sqrt{e^2 f^2 - \frac{1}{4} (b^2 + d^2 - a^2 - c^2)^2}.$$

In any case, since

$$\frac{\partial H}{\partial r_{ij}}(\mathbf{r}) = \frac{\partial H}{\partial r_{ij}^2}(\mathbf{r}) \cdot \frac{d(r_{ij}^2)}{dr_{ij}} = -64r_{ij} \Delta_i \Delta_j$$

we find that at a trapezoidal central configuration

$$\nabla H(\mathbf{r}) = 16r_{12}r_{34}h^2(r_{34}, -r_{13}, r_{14}, r_{23}, -r_{24}, r_{12}),$$

where  $h$  is defined above. On the other hand, the gradient of  $F(\mathbf{r})$  at a trapezoidal configuration is,

$$\nabla F(\mathbf{r}) = 2r_{12}r_{34}(r_{34}, -r_{13}, r_{14}, r_{23}, -r_{24}, r_{12}).$$

Comparing the two gradients above shows that  $\nabla F$  and  $\nabla H$  are parallel, and concludes the proof.  $\square$

This result is not entirely surprising in view of Planarity Condition 2 and the discussion that follows it. This is because  $\{F = 0\}$  and  $\{H = 0\}$  are codimension one level surfaces, while trapezoidal central configurations have codimension 2. We have shown, however, that to describe the trapezoidal configurations it is enough to use the conditions  $F(\mathbf{r}) = 0$  and  $H(\mathbf{r}) \leq 0$ . This type of conditions, generically, describe a codimension 1 object. Since trapezoidal configurations have codimension 2, however, it is reasonable to expect that  $\{F = 0\}$  and  $\{H = 0\}$  meet tangentially at any  $\mathbf{r}$  for which both  $H$  and  $F$  vanish. A similar situation arises in the co-circular problem.

Note that a isosceles trapezoid central configurations is both a trapezoidal and co-circular. Therefore taking the above lemma together with Lemma 2.1 in [9] implies that on the set of geometrically realizable vectors for which  $H$ ,  $F$  and  $P = r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24}$  vanish the gradients of these three functions are parallel. Thus, the codimension one level surfaces defined by the equations  $H = 0$ ,  $F = 0$ , and  $P = 0$  meet tangentially at the isosceles trapezoid configurations.

A straightforward consequence of Lemma 2 is that  $\nabla H$  is unnecessary when using the Lagrange multiplier method to locate trapezoidal central configurations. Hence, we can use the condition  $F = 0$  to obtain trapezoidal central configurations as follows:

**Proposition 1.** *Assuming the bodies are sequentially ordered, a trapezoidal central configuration is a critical point of the function*

$$U + \lambda M(I - I_0) + \sigma F, \quad (8)$$

satisfying  $I - I_0 = 0$ ,  $F = 0$  and  $H = 0$ , where  $\lambda$  and  $\sigma$  are Lagrange multipliers.

Taking derivatives with respect to  $r_{ij}^2$ , and absorbing the  $\frac{1}{2}$  multiple into the Lagrange multiplier  $\sigma$ , we find that the condition for a planar extrema is

$$m_i m_j (\lambda - r_{ij}^{-3}) + \sigma \frac{\partial F}{\partial r_{ij}^2} = 0, \quad 1 \leq i < j \leq 4 \quad (9)$$

$$I - I_0 = 0, \quad F = 0.$$

Writing (9) explicitly yields

$$m_1 m_2 (r_{12}^{-3} - \lambda) = \sigma r_{34}^2 \quad m_3 m_4 (r_{34}^{-3} - \lambda) = \sigma r_{12}^2 \quad (10)$$

$$m_1 m_3 (r_{13}^{-3} - \lambda) = -\frac{1}{2} \sigma R \quad m_2 m_4 (r_{24}^{-3} - \lambda) = -\frac{1}{2} \sigma R \quad (11)$$

$$m_1 m_4 (r_{14}^{-3} - \lambda) = \frac{1}{2} \sigma R \quad m_2 m_3 (r_{23}^{-3} - \lambda) = \frac{1}{2} \sigma R, \quad (12)$$

where  $R = (r_{13}^2 + r_{24}^2 - r_{14}^2 - r_{23}^2)$ , together with  $I - I_0 = 0$  and  $F = 0$ . Since  $F = 0$ , and we are assuming the ordering of the bodies described in the previous sections, then equation (6) is verified and hence it follows that  $R = 2r_{12}r_{34}$ . Then, the previous system of equations takes the form

$$m_1 m_2 (r_{12}^{-3} - \lambda) = \sigma r_{34}^2 \quad m_3 m_4 (r_{34}^{-3} - \lambda) = \sigma r_{12}^2 \quad (13)$$

$$m_1 m_3 (r_{13}^{-3} - \lambda) = -\sigma r_{12} r_{34} \quad m_2 m_4 (r_{24}^{-3} - \lambda) = -\sigma r_{12} r_{34} \quad (14)$$

$$m_1 m_4 (r_{14}^{-3} - \lambda) = \sigma r_{12} r_{34} \quad m_2 m_3 (r_{23}^{-3} - \lambda) = \sigma r_{12} r_{34}. \quad (15)$$

The equations have been grouped in pairs so that when they are multiplied together the product of the right-hand sides is  $\sigma^2 r_{34}^2 r_{12}^2$ . Consequently, the right hand sides are identical on the configurations satisfying  $F = 0$ . This yields the well-known relation of Dziobek [14]

$$(r_{12}^{-3} - \lambda)(r_{34}^{-3} - \lambda) = (r_{13}^{-3} - \lambda)(r_{24}^{-3} - \lambda) = (r_{14}^{-3} - \lambda)(r_{23}^{-3} - \lambda), \quad (16)$$

which is required of any planar 4-body central configuration (not only c.c.'s with parallel opposite sides).

Eliminating  $\lambda$  from equation (16) and factoring gives the important relation

$$(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) = (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3). \quad (17)$$

Assuming the six mutual distances determine an actual configuration in the plane, this equation is necessary and sufficient for the existence of a four-body planar central configuration. Further restrictions are needed to ensure that the masses are positive.

Reasoning as in [9] it is possible to show that positivity of the masses implies that each side of the quadrilateral is shorter in length than either diagonal, and that the shortest exterior side must lie opposite the longest. Then, the longest side will be either one of the parallel sides or one of the remaining exterior sides. In the former case suppose  $r_{14}$  is the longest exterior side, then we have that  $r_{23}$  is the shortest, and thus

$$|r_{14} - r_{23}| > |r_{34} - r_{12}|.$$

However, four lengths can constitute the consecutive sides of a non-parallelogram trapezoid, with  $r_{12}$  and  $r_{34}$  the lengths of the parallel sides, only when

$$|r_{14} - r_{23}| < |r_{34} - r_{12}| < r_{14} + r_{23},$$

which contradicts the previous inequality. A similar reasoning shows that  $r_{23}$  cannot be the longest exterior side. Hence, in a trapezoidal central configuration, one of the legs cannot be the longest exterior side.

In the latter case, without any loss of generality, we can label the bodies so that  $r_{12}$  is the longest exterior side-length. Then, positivity of the masses implies that

$$r_{13}, r_{24} > r_{12} \geq r_{14}, r_{23} \geq r_{34}.$$



With an appropriate relabeling it is also possible to assume  $r_{14} \geq r_{23}$  (see [9]). This choice imposes  $r_{13} \geq r_{24}$ , and thus

$$r_{13} \geq r_{24} > r_{12} \geq r_{14} \geq r_{23} \geq r_{34}. \quad (18)$$

To prove the relation between the diagonals, recall that the lengths of the diagonals in a trapezoid are given by (see [18]):

$$r_{13} = \sqrt{r_{12}r_{34} - \frac{r_{34}r_{23}^2 - r_{12}r_{14}^2}{r_{12} - r_{34}}} \quad (19)$$

$$r_{24} = \sqrt{r_{12}r_{34} - \frac{r_{34}r_{14}^2 - r_{12}r_{23}^2}{r_{12} - r_{34}}}. \quad (20)$$

The second of these equations can be obtained by solving (7) for  $r_{24}$ , and the first can also be obtained in a similar manner. Using the equations for the diagonals we obtain

$$r_{13}^2 - r_{24}^2 = \frac{(r_{14}^2 - r_{23}^2)(r_{34} + r_{12})}{r_{12} - r_{34}} \geq 0,$$

since  $r_{12} > r_{34}$ , and  $r_{14} \geq r_{23}$ .

Hence, without loss of generality we can restrict our analysis to the set

$$\Omega = \{\mathbf{r} \in (\mathbb{R}^+)^6 : r_{13} \geq r_{24} > r_{12} \geq r_{14} \geq r_{23} \geq r_{34}\}.$$

From the different ratios of two masses that can be derived from equations (13)–(15), we obtain the following set of equations:

$$\frac{m_1}{m_2} = -\frac{r_{23}^{-3} - r_{24}^{-3}}{r_{13}^{-3} - r_{14}^{-3}} \quad \frac{m_1}{m_3} = \frac{r_{34}(r_{23}^{-3} - r_{34}^{-3})}{r_{12}(r_{12}^{-3} - r_{14}^{-3})} \quad (21)$$

$$\frac{m_1}{m_4} = -\frac{r_{34}(r_{24}^{-3} - r_{34}^{-3})}{r_{12}(r_{12}^{-3} - r_{13}^{-3})} \quad \frac{m_2}{m_3} = -\frac{r_{34}(r_{13}^{-3} - r_{34}^{-3})}{r_{12}(r_{12}^{-3} - r_{24}^{-3})} \quad (22)$$

$$\frac{m_2}{m_4} = \frac{r_{34}(r_{14}^{-3} - r_{34}^{-3})}{r_{12}(r_{12}^{-3} - r_{23}^{-3})} \quad \frac{m_3}{m_4} = -\frac{r_{14}^{-3} - r_{24}^{-3}}{r_{13}^{-3} - r_{23}^{-3}}. \quad (23)$$

#### 4. Some applications of the trapezoidal c.c. equations

In this section we apply equations (13)–(15), (17) and (21)–(23) to obtain a few interesting results. The next two propositions, which have very simple proofs, study the case where two pairs of masses are equal.

**Proposition 2.** *If  $m_1 = m_2$  and  $m_3 = m_4$ , then the corresponding trapezoidal central configuration must be an isosceles trapezoid.*

**Proof.** By using the two equations (14) it follows that the diagonals are equal, that is,  $r_{13} = r_{24}$ . To conclude the proof it is enough to observe that a trapezoid with diagonals of equal length is an isosceles trapezoid.  $\square$

**Proposition 3.** *If  $m_1 = m_3$  and  $m_2 = m_4$ , then the corresponding trapezoidal central configuration must be a rhombus.*

**Proof.** By using the two equations (15) it follows that  $r_{14} = r_{23}$ . It follows that the quadrilateral is either a isosceles trapezoid or a rhombus.

If it is an isosceles trapezoid the diagonals have equal length, and from equation (14) it follows that the masses are all equal. From the first of equation (22) it follows that the bases are equal, that is  $r_{12} = r_{34}$ . Moreover, from equations (13) and (14) we have that all the exterior sides are equal. The quadrilateral is then a square. In either case the quadrilateral is a rhombus.  $\square$

Cors and Roberts [9] proved that, in any co-circular configuration with a given ordering of the mutual distances, the masses must be ordered in a precise fashion, that is, the set of masses  $\{m_1, m_2, m_3, m_4\}$  is totally ordered. We now want to obtain an similar result in the case of trapezoidal central configurations. Because of the different geometry, however, it turns out that the set of masses is not totally ordered. In fact, in this case, it is only possible to obtain the following weaker result

**Theorem 1.** *Any trapezoid central configuration in  $\Omega$  satisfies*

$$m_3 \leq m_4 \leq m_2 \quad \text{and} \quad m_3 \leq m_1.$$

Before proving this we present two important inequalities needed in the proof of the theorem.

**Lemma 3.** *Let  $\phi : I \rightarrow \mathbb{R}$  be a decreasing differentiable function on an interval  $I \subset \mathbb{R}$ . Suppose that  $x_1 \leq x_2 \leq x_3 \leq x_4$ , then*

$$\frac{\phi(x_2) - \phi(x_3)}{\phi(x_1) - \phi(x_4)} \leq 1.$$

**Proof.** The proof is straightforward. Since  $x_1 \leq x_2 \leq x_3 \leq x_4$  and  $\phi$  is decreasing, then  $\phi(x_1) \geq \phi(x_2) \geq \phi(x_3) \geq \phi(x_4)$ . It follows that  $\phi(x_1) - \phi(x_4) \geq \phi(x_2) - \phi(x_3)$ . Since both sides of the inequality are positive we obtain

$$\frac{\phi(x_2) - \phi(x_3)}{\phi(x_1) - \phi(x_4)} \leq 1,$$

which is our claim.  $\square$

**Lemma 4.** *Any trapezoidal central configuration in  $\Omega$  satisfies*

$$\frac{r_{23}^2}{r_{14}^2} \geq \frac{r_{34}}{r_{12}},$$

where the equality sign holds if, and only if, the trapezoid is a parallelogram with  $r_{12} = r_{34}$  and  $r_{23} = r_{14}$ .

**Proof.** If  $r_{12} \neq r_{34}$ , then we can use equation (20). Since  $r_{24} > r_{12}$ , we have that

$$r_{12}r_{34}(r_{12} - r_{34}) - r_{34}r_{14}^2 + r_{12}r_{23}^2 > r_{12}^2(r_{12} - r_{34}),$$

or  $-r_{34}r_{14}^2 > r_{12}(r_{34}^2 + r_{12}^2 - r_{23}^2 - 2r_{12}r_{34})$ . Using condition (6), that is,  $2r_{12}r_{34} = r_{13}^2 + r_{24}^2 - r_{23}^2 - r_{14}^2$  we obtain

$$r_{34}r_{14}^2 < r_{12}(r_{13}^2 + r_{24}^2 - r_{34}^2 - r_{12}^2 - r_{14}^2) \leq r_{12}r_{23}^2,$$

where we used Euler's quadrilateral inequality  $r_{13}^2 + r_{24}^2 \leq r_{12}^2 + r_{14}^2 + r_{23}^2 + r_{34}^2$  (see [13], or any good book on Euclidean geometry for a proof). It follows that  $\frac{r_{23}^2}{r_{14}^2} > \frac{r_{34}^2}{r_{12}^2}$ .

If  $r_{12} = r_{34}$  then the trapezoid degenerates to a parallelogram and thus  $r_{23} = r_{14}$ . It follows that  $1 = \frac{r_{23}^2}{r_{14}^2} \geq \frac{r_{34}^2}{r_{12}^2} = 1$ , which completes the proof.  $\square$

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** We first prove that  $m_3 \leq m_4$ . Using the second of equations (23) we have

$$\frac{m_3}{m_4} = \frac{r_{14}^{-3} - r_{24}^{-3}}{r_{23}^{-3} - r_{13}^{-3}}.$$

Let  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$  be defined by  $\phi(x) = x^{-3}$ , then  $\phi$  is a decreasing function. An application of Lemma 3 with  $x_1 = r_{23}$ ,  $x_2 = r_{14}$ ,  $x_3 = r_{24}$ , and  $x_4 = r_{13}$ , yields that

$$\frac{\phi(x_2) - \phi(x_3)}{\phi(x_1) - \phi(x_4)} = \frac{r_{14}^{-3} - r_{24}^{-3}}{r_{23}^{-3} - r_{13}^{-3}} \leq 1.$$

It follows that,

$$m_3 \leq m_4. \quad (24)$$

Next, we verify that  $m_2 \geq m_4$ . Multiplying the second equation in (22) and the second equation in (23), we find

$$\begin{aligned} \frac{m_2}{m_4} &= \frac{r_{12}^2 r_{23}^3}{r_{34}^2 r_{14}^3} \cdot \frac{(r_{13}^3 - r_{34}^3)(r_{14}^3 - r_{24}^3)}{(r_{12}^3 - r_{24}^3)(r_{13}^3 - r_{23}^3)} \\ &= \left( \frac{r_{12} r_{23}^2}{r_{34} r_{14}^2} \right)^2 \cdot \frac{r_{14}}{r_{23}} \cdot \frac{(r_{13}^3 - r_{34}^3)}{(r_{13}^3 - r_{23}^3)} \cdot \frac{(r_{24}^3 - r_{14}^3)}{(r_{24}^3 - r_{12}^3)}. \end{aligned}$$

All the fractions in the equation above are greater than or equal to one. In fact, the first fraction is greater than or equal to one by Lemma 4, the second fraction because  $r_{14} \geq r_{23}$ , and the last two fractions because  $r_{23} \geq r_{34}$  and  $r_{12} \geq r_{14}$ , respectively. This shows that  $m_2 \geq m_4$ .

Finally, using equations (21) and (17) we find that

$$\frac{m_1}{m_3} = \frac{r_{14}^3}{r_{23}^3} \cdot \frac{r_{12}^2}{r_{34}^2} \cdot \frac{r_{23}^3 - r_{34}^3}{r_{12}^3 - r_{14}^3} = \frac{r_{14}^3}{r_{23}^3} \cdot \frac{r_{12}^2}{r_{34}^2} \cdot \frac{(r_{24}^3 - r_{34}^3)}{(r_{24}^3 - r_{14}^3)} \cdot \frac{(r_{13}^3 - r_{23}^3)}{(r_{13}^3 - r_{12}^3)}.$$

All the fractions in the equation above are greater than or equal to one. To me more precise, the first two fractions are greater than or equal to one because  $r_{14} \geq r_{23}$  and  $r_{12} \geq r_{34}$ , respectively. The last two fractions because  $r_{14} \geq r_{34}$  and  $r_{12} \geq r_{23}$ , respectively. Hence,  $m_1 \geq m_3$ , and the proof is complete.  $\square$

**Remark.** From numerical experiments it appears that  $m_1$  can be larger or smaller than either  $m_2$  and  $m_4$ . For example choosing

$$\begin{aligned} r_{13} &= 9.7414781617108145730, & r_{24} &= 8.7500000000000000, \\ r_{12} &= 8 & r_{14} &= 7.52080447824566090, \\ r_{23} &= 7.1064329749865061893, & r_{34} &= 4.0246879466945716437, \end{aligned}$$

we have  $m_1/m_2 = 1.0194571510769873907$  and  $m_1/m_4 = 7.9942119368105807422$ , also these distances satisfy condition (17).

On the other hand if

$$\begin{aligned} r_{13} &= 12.129061710615553753, & r_{24} &= 9.5117033174926140565, \\ r_{12} &= 8 & r_{14} &= 7.8020830551846857406, \\ r_{23} &= 7.6549229903601603027, & r_{34} &= 7.3822682494734852600, \end{aligned}$$

we have  $m_1/m_2 = 0.69074480337446980353$  and  $m_1/m_4 = 0.87696321790891338292$ , also these distances satisfy condition (17).

We now show that, as a consequence of Theorem 1, if two of the masses are equal, there are strong restrictions on the shape of the allowed central configuration.

**Proposition 4.** *For any trapezoidal c.c. in  $\Omega$ , if either  $m_2 = m_4$  or  $m_1 = m_3$  the configuration is a rhombus and the remaining two masses are necessarily equal. If  $m_3 = m_4$ , then the configuration is an isosceles trapezoid and the other two masses are necessarily equal.*

**Proof.** First we consider the case  $m_1 = m_3$ . By the proof of Theorem 1 we have that  $m_1 \geq m_3$  and the expression

$$\frac{m_1}{m_3} = \frac{r_{14}^3}{r_{23}^3} \cdot \frac{r_{12}^2}{r_{34}^2} \cdot \frac{(r_{24}^3 - r_{34}^3)}{(r_{24}^3 - r_{14}^3)} \cdot \frac{(r_{13}^3 - r_{23}^3)}{(r_{13}^3 - r_{12}^3)}$$

is the product of three numbers greater than or equal to one. If  $m_3 = m_1$ , it follows that each of the fractions in the equation above is equal to one. From the first two fractions we obtain  $r_{14} = r_{23}$  and  $r_{12} = r_{34}$ , respectively. From the last two fractions we obtain  $r_{14} = r_{34}$  and  $r_{12} = r_{23}$ , respectively. This yields a rhombus. Moreover, comparing the two equation (15) we find that  $m_2 = m_4$ .

Second we consider  $m_2 = m_4$ . From the proof of Theorem 1, we have that  $m_2 \geq m_4$  and

$$\frac{m_2}{m_4} = \left( \frac{r_{12}r_{23}^2}{r_{34}r_{14}^2} \right)^2 \cdot \frac{r_{14}}{r_{23}} \cdot \frac{(r_{13}^3 - r_{34}^3)}{(r_{13}^3 - r_{23}^3)} \cdot \frac{(r_{24}^3 - r_{14}^3)}{(r_{24}^3 - r_{12}^3)}.$$

This expression is the product of four numbers greater than or equal to one. If we require  $m_2 = m_4$ , then from the second fraction we obtain  $r_{14} = r_{23}$ . Moreover, the first fraction now reduces to  $(r_{12}/r_{34})^2$ , and thus we must have  $r_{12} = r_{34}$ . The last two fractions give  $r_{23} = r_{34}$  and  $r_{14} = r_{12}$ , yielding again a rhombus.

Third, suppose  $m_3 = m_4$ . From equation (23), we have that

$$\frac{m_3}{m_4} = \frac{r_{24}^{-3} - r_{14}^{-3}}{r_{13}^{-3} - r_{23}^{-3}}.$$

If  $m_3 = m_4$  this implies that  $r_{24}^{-3} - r_{14}^{-3} = r_{13}^{-3} - r_{23}^{-3}$ . Consequently,

$$r_{24}^{-3} - r_{13}^{-3} = r_{14}^{-3} - r_{23}^{-3}.$$

If  $r_{23} < r_{14}$ , then  $r_{23}^{-3} > r_{14}^{-3}$  and the right-hand side of equation above is negative, contradicting the fact that  $r_{13} \geq r_{24}$ . Hence,  $r_{23} = r_{14}$  and  $r_{13} = r_{24}$ . Since a trapezoid with equal diagonals is an isosceles trapezoid, the configuration is an isosceles trapezoid. The equality of the masses  $m_1 = m_2$  then follows from the first of equations (21). This completes the proof.  $\square$

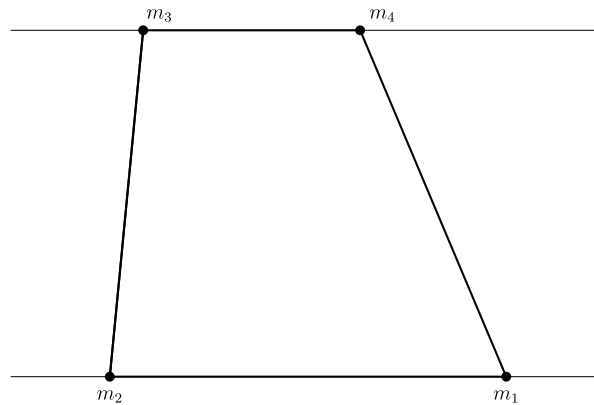


Fig. 3. An example of a non-symmetric trapezoidal central configuration with  $m_1 = m_2$ .

**Remark.** Note that  $m_1 = m_2$  does not imply that the configuration is an isosceles trapezoid. For instance, a non-symmetric central configuration with  $m_1 = m_2$  can be found numerically to be

$$\begin{aligned} r_{13} &= 10.13318587483539368, & r_{24} &= 8.63262460668978253, \\ r_{12} &= 8, & r_{14} &= 7.59545875301365884, \\ r_{23} &= 7.03230033956929474, & r_{34} &= 4.37871386495945262, \end{aligned}$$

see Fig. 3.

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