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# Free boundary problem of a reaction–diffusion equation with nonlinear convection term

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ABSTRACT

In this paper, we consider the free boundary problem of a reaction diffusion equation with nonlinear convection term in one dimensional space. Our study contains three parts: in the first part we establish the existence and uniqueness of global solution, in the second part we obtain the spreading–vanishing dichotomy, and in third part, we obtain some estimations of the asymptotic speed of free boundaries when spreading happens.



## 1. Introduction

Nonlinear diffusion problems with free boundary conditions are generally used to describe the expansion and propagation of biological species or chemical substances, and the free boundary is used to represent the frontier of this expansion. It has been a central issue in ecological research to explore the law of population expansion of new species or invasive species in new environment. A large amount of empirical evidence shows that many invasive species, which survived in new environment, will expand at a fixed rate after a very short initial stage. A classic example is the rules discovered by Skellam [19] in 1951, which describes the spreading of muskrat in Europe in early twentieth century.

One of the most successful mathematical descriptions of the propagation of species is based on the theory of traveling wave solutions. In 1937, Fisher [11] made use of the equation

$$u_t - du_{xx} = au - bu^2, \quad t > 0, \ x \in \mathbb{R}^1, \tag{1.1}$$

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to study the transmission pattern of advantageous genes, where the function $u$ represents the population density of species that carries advantageous genes. He proved that equation (1.1) admits a solution of the form $u(t,x) = \omega(x - ct)$, if $c \geq 2\sqrt{ad} := c^*$ and $\omega$ satisfies

$$c\omega' + d\omega'' + a\omega - b\omega^2 = 0, \quad \omega(-\infty) = 0, \quad \omega(+\infty) = a/b.$$

He also showed that there exists no such solutions if $c < 2\sqrt{ad}$. Fisher claimed that $c^*$ is the spreading speed for the advantageous genes in his research. The same results were proved by Kolmogorov et al. for a more general class of equations whose nonlinearity is now called Fisher-KPP type. In 1975, Aronson and Weinberger [1] established a more general theory based on the traveling wave solutions. They proved that for any $\varepsilon > 0$, the solution of (1.1) satisfies:

$$\lim_{t\to\infty, |x|\leq(c^*-\varepsilon t)} u(t,x) = a/b, \qquad \lim_{t\to\infty, |x|\geq(c^*+\varepsilon t)} u(t,x) = 0.$$

This means that if an observer travels in the direction of propagation at a speed $c$ which is smaller than $c^*$, then it will find that the population is close to $\frac{a}{b}$, and if his speed is bigger than $c^*$, it would observe that the population is nearly 0. The mathematical results have been extended to higher dimensions in [2] by Aronson and Weinberger.

The mathematical model mentioned above have obtained lots of good results, but the reaction–diffusion equation, which were used in these models to describe the expansion behavior, will force infinite propagation speed. Namely, for any initial population distribution which is nonnegative and is positive somewhere, for $t > 0$ the population density at any location is bigger than 0, which does not conform with the reality. To overcome the difficulty, Du and Lin [7] first began to try to introduce free boundary conditions to study the expansion behavior of biological populations, and they considered the following equations:

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \leq x \leq h_0, \end{cases} \tag{1.2}$$

where $h(t)$ stands for the free boundary, the function $u$ represents population density. The free boundary $h(t)$ satisfies equation $h'(t) = -\mu u_x(t,h(t))$, which is a special case of the well-known Stefan condition and the deduction can refer to [31]. If the right side is free boundary, the left side is fixed boundary condition, then the equation (1.2) describes the expansion behavior of new species or invasive species populations in one dimensional environment, they got the following results:

(1) Equation (1.2) has a unique global solution;
(2) Spreading–vanishing dichotomy and its criterion: if $t \to \infty$, then either $h(t) \to \infty$ and $u(t,x) \to \frac{a}{b}$, which we called spreading, or $h(t) \to h_\infty \leq \frac{\pi}{2}\sqrt{d/a}$ and $u(t,x) \to 0$, which we called vanishing. Furthermore, there exists a positive constant $\mu^*$ that describes the ability of expansion, if $\mu > \mu^*$, then vanishing happens; if $0 \leq \mu \leq \mu^*$, then spreading happens, and $\mu^*$ depends on $u_0$ and $h_0$.
(3) Some basic estimates for the asymptotic spreading speed of two fronts when spreading happens, which means that there exists a positive constant $k_0$ that only depends on $\mu$, and

$$\lim_{t\to\infty} \frac{h(t)}{t} = k_0, \qquad \lim_{t\to\infty} -\frac{g(t)}{t} = k_0.$$

After this pioneering work, a more general reaction diffusion problem with free boundary conditions was considered by Du and Lou [9]:

$$\begin{cases} u_t = u_{xx} + f(u), & t > 0, g(t) < x < h(t), \\ u(t, g(t)) = 0, \ u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ -g(0) = h(0) = h_0, \ u(0, x) = u_0(x), & -h_0 \le x \le h_0. \end{cases} \tag{1.3}$$

They studied the problem with general nonlinearity $f(u)$, including general monostable, bistable and combustion types of $f(u)$. They revealed the relation but different sharp transition natures between vanishing and spreading for these three types of nonlinearities. For instance, if $f(u)$ is bistable or combustion types, then under some initial values, the long-time behavior of solution will have transition state besides spreading and vanishing, which means when $t \to \infty$, then $(g_\infty, h_\infty) = \mathbb{R}$ and $u(t, x)$ converges to some function or some constant locally uniformly. For bistable and combustion nonlinearities, they obtained spreading–vanishing–transition trichotomy and the corresponding criterion, and when spreading happens, for all three kinds of nonlinearities, they estimated the asymptotic spreading speed. Later, Du et al. [5] obtain sharp estimates for the asymptotic spreading speed of two fronts of problem (1.3). Some related works for the single equation can refer to [4,18,21,23,35]. Besides the study of single equation in one dimensional space, lots of free boundary problems in biological mathematics has been studied at the same time, such as the competition model [10, 16,27,28,30], prey–predator model [20,22,26,31] in one dimensional space, single reaction–diffusion equation in high space dimensions [6], competition model [8,33] and prey–predator model [32] in high dimensional space for radially symmetric case.

The expansion behavior of biological populations in the environment is not just caused by simple individual random move. For example, the migration of animals are affected by the spatial distribution of food and water, the movement of bacteria or viruses are affected by the migration of the host, so all these expansion behaviors will have a certain direction. In mathematical models, the convection term is generally used to describe this kind of directional movement. In 2014, Gu et al. [13,14] considered the free boundary problem of convection reaction diffusion equations with linear convection term in one dimensional space:

$$\begin{cases} u_t - u_{xx} + \beta u_x = u(1-u), & t > 0, \ g(t) < x < h(t), \\ u(t, g(t)) = 0, \ u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ -g(0) = h(0) = h_0, \ u(0, x) = u_0(x), & -h_0 \le x \le h_0. \end{cases} \tag{1.4}$$

They showed spreading–vanishing dichotomy and its criterion which depends on initial value, and gave an estimation of the asymptotic spreading speed when the convection coefficient $\beta$ is small. Later, Kaneko and Matsuzawa [17] made a further promotion to this problem. They replaced $u(1-u)$ by general nonlinearity $f(u)$ in (1.4) and established sharp estimation of asymptotic spreading speed for monostable, bistable and combustion types of $f(u)$.

In 2015, Gu et al. [15] studied the long-time behavior of the solution of problem (1.4) under the conditions that reaction term is Fisher-KPP type and convection coefficient $\beta \in (0, +\infty)$. Let $c^*$ represent the asymptotic spreading speed of the solution of problem (1.3) when nonlinearity $f(u)$ is Fisher-KPP type, they proved that there exists a constant $\beta^*$ which depends on $c^*$, if $\beta \ge \beta^*$, then only vanishing happens, if $\beta \in (c^*, \beta^*)$, then the solution will have some new behaviors such as virtual spreading and virtual vanishing. Their works also gave out the sufficient conditions for spreading or vanishing. In the same year, Wang and Zhao [34] studied the convection reaction diffusion equations with linear convection term and mixed free boundary conditions in one dimensional space:

$$\begin{cases} u_t - u_{xx} + \beta u_x = f(u), & t > 0,\ 0 < x < h(t), \\ B[u](t,0) = 0,\ u(t,h(t)) = 0, & t \geq 0, \\ h'(t) = -\mu u_x(t,h(t)), & t \geq 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \leq x \leq h_0. \end{cases} \tag{1.5}$$

The left front is represented by $B[u] = au - bu_x$, where $a, b \geq 0$ and $a + b > 0$, and the nonlinearity $f(u)$ is Fisher-KPP type. In their paper, for the first time, the situation that the convection term coefficient $\beta$ is negative was considered. They revealed that if $\beta \leq c^*$ then only vanishing happens, and they obtained spreading–vanishing dichotomy and its criterion when $|\beta| < c^*$.

In the study mentioned above, the convection term is linear and only dependent of the gradient of the population density. However, in some cases the convection is also affected by population density, which leads to the nonlinear convection. For example, since the total amount of food is limited in a local environment, for a population with the greater density, the more individuals inclined to leave the place to go to a more affluent place for food. Generally, in one-dimensional space the convection reaction diffusion equation with nonlinear term can be written as the following general form:

$$u_t = au_{xx} + (b(u))_x + c(u).$$

In this formulation, $a$ represents the diffusion coefficient, $b(u)$ is a nonlinear convective flux function, $b'(u)$ can be viewed as nonlinear velocity, $c(u)$ denotes reaction term.

In this paper, we mainly study the free boundary problem of a reaction–diffusion equation with nonlinear convection term in one dimensional space:

$$\begin{cases} u_t - du_{xx} + uu_x = u(1-u), & t > 0,\ g(t) < x < h(t), \\ u(t,g(t)) = 0, \quad g' = -\mu u_x(t,g(t)), & t > 0, \\ u(t,h(t)) = 0, \quad h' = -\mu u_x(t,h(t)), & t > 0, \\ -g(0) = h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 \leq x \leq h_0. \end{cases} \tag{1.6}$$

The local existence and uniqueness of the solution are proved by using the theory of parabolic equation and the principle of contraction mapping, then the global existence and uniqueness are obtained. By using the zero number argument, we prove the spreading–vanishing dichotomy and give its criterion by the method of eigenvalue problem and constructing the upper and lower solutions. Finally, we use the semi-waves to estimate the asymptotic spreading speeds of two fronts when spreading happens, and compare these speeds with the asymptotic spreading speeds of fronts of the free boundary problem without convection term.

## 2. Global existence and uniqueness of solutions

In this section, we establish the existence and uniqueness of global solution of (1.6).

### 2.1. Local existence and uniqueness of the solution

The initial value $u_0$ belongs to $\mathfrak{X}(h_0)$, where

$$\mathfrak{X}(h_0) := \{\phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0, \phi'(-h_0) > 0, \\ \phi'(h_0) < 0, \phi(x) > 0, x \in (-h_0, h_0)\},$$

and $h_0$ is a constant greater than 0.

**Theorem 2.1.** *For any given $u_0$ belongs to $\mathfrak{X}(h_0)$ and any $\alpha \in (0,1)$, there exists a $T > 0$ such that problem (1.6) has a unique solution*

$$(u, h, g) \in C^{\frac{1+\alpha}{2}, 1+\alpha}(D_T) \times C^{1+\frac{\alpha}{2}}([0,T]) \times C^{1+\frac{\alpha}{2}}([0,T]);$$

*moreover,*

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}(D_T)} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C,$$

*where $D_T = \{(t,x) \in \mathbb{R}^2 : \ x \in [g(t),\ h(t)], t \in [0,T]\}$, $C$ and $T$ only depends on $h_0$, $\alpha$ and $\|u_0\|_{C^2[-h_0,h_0]}$.*

**Proof.** The idea of this proof comes [24,25] and we only give the outline. Set

$$y = \frac{2x - g(t) - h(t)}{h(t) - g(t)}, \quad w(t,y) = u\big(t, \frac{1}{2}[(h(t) - g(t))y + h(t) + g(t)]\big).$$

Then (1.6), in $[0,T]$, is equivalent to

$$\begin{cases} w_t - d\rho^2(t)w_{yy} - \zeta(t,y)w_y + \rho(t)ww_y = w(1-w), & 0 < t \le T, \quad |y| < 1, \\ w(t, \pm 1) = 0, & 0 < t \le T, \\ w(0,y) = u_0(h_0 y) := w_0(y), & |y| \le 1, \end{cases} \tag{2.1}$$

$$\begin{cases} g'(t) = -\mu\rho(t)w_y(t,-1), \quad h'(t) = -\mu\rho(t)w_y(t,1), & 0 < t \le T, \\ g(0) = -h_0, \quad h(0) = h_0, \end{cases} \tag{2.2}$$

where

$$\rho(t) = \frac{2}{h(t) - g(t)}, \quad \zeta(t,y) = \frac{h'(t) + g'(t)}{h(t) - g(t)} + \frac{h'(t) - g'(t)}{h(t) - g(t)} y.$$

Denote $g^* = -\mu u_0'(-h_0)$, $h^* = -\mu u_0'(h_0)$. Then $g^* \le 0$ and $h^* \ge 0$. For $0 < T \le \frac{h_0}{2(4+|g^*|+h^*)} := T_1$, we denote $\Delta_T = [0,T] \times [-1,1]$, and

$$\begin{aligned} G_T &= \{g \in C^1([0,T]) : \ g(0) = -h_0,\ g'(0) = g^*,\ \|g' - g^*\|_{C([0,T])} \le 1\}, \\ H_T &= \{h \in C^1([0,T]) : \ h(0) = h_0,\ h'(0) = h^*,\ \|h' - h^*\|_{C([0,T])} \le 1\}. \end{aligned}$$

Clearly, $\mathcal{D}_T = G_T \times H_T$ is a closed convex set of $[C^1([0,T])]^2$. Due to the choice of $T$, we see that for any given $(g,h) \in \mathcal{D}_T$, we can extend $(g,h)$ to new functions, denoted by themselves, such that $(g,h) \in \mathcal{D}^*_{T_1} := G^*_{T_1} \times H^*_{T_1}$, where

$$\begin{aligned} G^*_{T_1} &= \{g \in C^1([0,T_1]) : \ g(0) = -h_0,\ g'(0) = g^*,\ \|g' - g^*\|_{C([0,T_1])} \le 2\}, \\ H^*_{T_1} &= \{h \in C^1([0,T_1]) : \ h(0) = h_0,\ h'(0) = h^*,\ \|h' - h^*\|_{C([0,T_1])} \le 2\}. \end{aligned}$$

Therefore, when $(g,h) \in G_T \times H_T$ we have $(g,h) \in G^*_{T_1} \times H^*_{T_1}$ and

$$|g(t) + h_0| + |h(t) - h_0| \le T_1(\|g'\|_{C([0,T_1])} + \|h'\|_{C([0,T_1])}) \le h_0/2,$$

which implies

$$h_0 \le h(t) - g(t) \le 3h_0, \quad \forall\, t \in [0,T_1].$$

For the given $(g, h) \in \mathcal{D}_T$, we first extend it to $\mathcal{D}^*_{T_1}$, then functions $\rho(t)$ and $\zeta(t, y)$ are known. By the upper and lower solutions method we can show that (2.1) with $T = T_1$ has a unique solution $w = w(t, x; g, h)$ and $w$ is positive by the maximum principle. Take $p > 3$. The $L^p$ theory yields that $w \in W^{1,2}_p(\Delta_{T_1})$ and $\|w\|_{W^{1,2}_p(\Delta_{T_1})} \le C$, where $C$ depends only on $\Lambda := \{h_0, h^*, g^*, \mu, \|u_0\|_{W^2_p(-h_0,h_0)}\}$. The embedding theorem asserts $w \in C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_{T_1})$ and $\|w\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_{T_1})} \le C(T_1^{-1}, \Lambda)$. Therefore,

$$w_y(t, \pm 1) \in C^{\frac{\alpha}{2}}([0, T_1]), \quad \|w_y(t, \pm 1)\|_{C^{\frac{\alpha}{2}}([0,T_1])} \le C(T_1^{-1}, \Lambda).$$

Obviously, when $0 < T \le T_1$, we have

$$\|w_y(t, \pm 1)\|_{C^{\frac{\alpha}{2}}([0,T])} \le C(T_1^{-1}, \Lambda). \tag{2.3}$$

Moreover, the Hopf boundary lemma gives $w_y(t, 1) < 0$ and $w_y(t, -1) > 0$. For such a known function $w$, the problem (2.2) has a unique solution, denoted by $(\hat{g}, \hat{h})$. Obviously

$$\hat{g}, \hat{h} \in C^{1+\frac{\alpha}{2}}([0, T]), \quad \hat{g}'(t) < 0, \quad \hat{h}'(t) > 0.$$

Define $F(g, h) = (\hat{g}, \hat{h})$. Then

$$F : \mathcal{D}_T \to C^{1+\frac{\alpha}{2}}([0, T]).$$

Using (2.3) and (2.2) we have $\|\hat{g}, \hat{h}\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C(T_1^{-1}, \Lambda)$. Thus, we have

$$\|\hat{g}' - g^*\|_{C([0,T])} \le \|\hat{g}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le C(T_1^{-1}, \Lambda) T^{\frac{\alpha}{2}},$$

$$\|\hat{h}' - h^*\|_{C([0,T])} \le \|\hat{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le C(T_1^{-1}, \Lambda) T^{\frac{\alpha}{2}}.$$

If $0 < T \ll 1$, then $(\hat{g}, \hat{h}) \in \mathcal{D}_T$. That is, $F : \mathcal{D}_T \to \mathcal{D}_T$ provided $0 < T \ll 1$.

Next we prove that if $T > 0$ is sufficiently small, then $F$ is a contraction mapping on $D$. For $(g_i, h_i) \in \mathcal{D}_T$ $(i = 1, 2)$, we set $w_i = w(t, x; g_i, h_i)$ and $(\hat{g}_i, \hat{h}_i) = F(g_i, h_i)$. Then it follows from the discussion above that

$$\|w_i\|_{W^{1,2}_p(\Delta_T)} + \|\hat{g}_i, \hat{h}_i\|_{C^{\frac{\alpha}{2}}([0,T])} \le C, \quad i = 1, 2. \tag{2.4}$$

Set

$$w = w_1 - w_2, \quad g = g_1 - g_2, \quad h = h_1 - h_2, \quad \hat{g} = \hat{g}_1 - \hat{g}_2, \quad \hat{h} = \hat{h}_1 - \hat{h}_2$$

and

$$\rho_i(t) = \frac{2}{h_i(t) - g_i(t)}, \quad \zeta_i(t, y) = \frac{h_i'(t) + g_i'(t)}{h_i(t) - g_i(t)} + \frac{h_i'(t) - g_i'(t)}{h_i(t) - g_i(t)} y.$$

Then we have

$$\begin{cases} w_t - d\rho_1^2 w_{yy} + (\rho_1 w_2 - \zeta_1) w_y + (\rho_1 w_{1y} + w_1 + w_2 - 1) w & \\ \quad = \left(\rho_1^2 - \rho_2^2\right) w_{2yy} + (\zeta_1 - \zeta_2) w_{2y} - (\rho_1 - \rho_2) w_2 w_{2y}, & 0 < t \le T, \quad |y| < 1, \\ w(t, \pm 1) = 0, & t \ge 0, \\ w(0, y) = 0, & |y| \le 1, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \hat{g}'(t) = -\mu\rho_1 w_y(t,-1) - \mu(\rho_1-\rho_2)w_{2y}(t,-1), & t \ge 0, \\ \hat{h}'(t) = -\mu\rho_1 w_y(t,1) - \mu(\rho_1-\rho_2)w_{2y}(t,1), & t \ge 0, \\ \hat{g}(0) = \hat{h}(0) = 0. \end{cases} \tag{2.6}$$

Notice (2.4), applying the $L^p$ theory to (2.5) we have $\|w\|_{W_p^{2,1}(\Delta_T)} \le C\|g,\, h\|_{C^1([0,T])}$, where $C$ depends only on $\Lambda$. By similar arguments in the proof of [24, Theorem 1.1], we have

$$[w]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)},\quad [w_y]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)} \le C_1\|w\|_{W_p^{2,1}(\Delta_T)}$$

for some positive constant $C_1$ independent of $T^{-1}$, here $[\cdot]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)}$ is Höder semi-norm. Thus,

$$[w_y]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)} \le CC_1\|g,\, h\|_{C^1([0,T])}.$$

It the follows from (2.6) that

$$[g',\, h']_{C^{\frac{\alpha}{2}}([0,T])} \le \mu\big([\rho_1 w_y(t,\pm1)]_{C^{\frac{\alpha}{2}}([0,T])} + [(\rho_1-\rho_2)w_{2y}(\cdot,\pm1)]_{C^{\frac{\alpha}{2}}([0,T])}\big) \le C_2\|g,\, h\|_{C^1([0,T])}.$$

Thus

$$\|\hat{g},\, \hat{h}\|_{C^1([0,T])} \le C_2 T^{\alpha/2}\|\hat{g},\, \hat{h}\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C_2 T^{\alpha/2}\|g,\, h\|_{C^1([0,T])}.$$

This shows that, when $0 < T \ll 1$, then $F$ is a contraction map on $\mathcal{D}_T$. Hence, the contraction mapping theorem implies that $F$ has a unique fixed point $(g,h)$ in $\mathcal{D}_T$. Thus the Problem (1.6) has a unique local classical solution $(u,g,h)$. $\square$

### *2.2. Global existence and uniqueness of the solution*

To show the solution which obtained above in Theorem 2.1 can be extended to all $t > 0$, we need the following estimate.

**Lemma 2.2.** *Let $(u,h,g)$ be a solution of (1.6) defined for $t \in (0,T_0)$ for some $T_0 > 0$, then there exist constants $C_1$ and $C_2$ independent of $T_0$ such that*

$$0 < u(t,x) \le C_1,\ 0 < h'(t),\ -g'(t) \le C_2,$$

*for $g(t) < x < h(t),\ t \in (0,T_0)$.*

**Proof.** Firstly, the maximum principle gives

$$u \le \max\{1, \max_{[-h_0,h_0]} u_0\}.$$

Defining

$$\Omega_{M_1} = \{(t,x):\ 0<t<T_0,\ h(t)-M_1^{-1} < x < h(t)\},$$
$$\Omega_{M_2} = \{(t,x):\ 0<t<T_0,\ g(t) < x < g(t)+M_2^{-1}\}.$$

We construct an auxiliary function $\omega_1(t,x) := C_1[2M_1(h(t)-x) - M_1^2(h(t)-x)^2]$ on $\Omega_{M_1}$. Clearly,

$$\omega_{1,t} = 2M_1C_1h'(t)[1 - M_1(h(t) - x)] \geq 0,$$
$$\omega_{1,x} = C_1[-2M_1 + 2M_1^2(h(t) - x)], \quad \omega_{1,xx} = -2M_1^2C_1,$$

then

$$\begin{aligned}\omega_1\omega_{1,x} &= C_1^2[2M_1(h(t) - x) - M_1^2(h(t) - x)^2][-2M_1 + 2M_1^2(h(t) - x)] \\ &= C_1^2M_1^2[-4(h(t) - x) + 6M_1(h(t) - x)^2 - 2M_1^2(h(t) - x)^3].\end{aligned}$$

Let $h(t) - x = h$, then $0 < h < M_1^{-1}$ holds over $\Omega_{M_1}$. By $\omega_1\omega_{1,x} = 0$, we obtain $h = 0$ or $h = \frac{-6M_1 \pm \sqrt{36M_1^2 - 32M_1^2}}{-4M_1^2}$, in other words, $h = \frac{2}{M_1}$ or $h = \frac{1}{M_1}$. Denote $\omega_1\omega_{1,x} = H(h)$, then

$$H'(h) = C_1^2M_1^2(-4 + 12M_1h - 6M_1^2h^2).$$

By solving $H'(h) = 0$, we have $h = \frac{-12M_1 \pm \sqrt{144M_1^2 - 96M_1^2}}{-12M_1^2} = (1 \pm \frac{\sqrt{3}}{3})\frac{1}{M_1}$. Since $H(h) = 0$ when $h = 0, \frac{1}{M_1}, \frac{2}{M_1}$. Using the properties of cubic function, we know that $H(h)$ reaches the minimum at $h = (1 - \frac{\sqrt{3}}{3})$ for $0 < h < M_1^{-1}$, and $H(1 - \frac{\sqrt{3}}{3}) = -\frac{4}{9}\sqrt{3}C_1^2M_1$.

Direct calculations show that $\omega_{1,t} - d\omega_{1,xx} \geq 2dC_1M_1^2$, and $\omega_1(1 - \omega_1) < C_1$. It follows that

$$2dC_1M_1^2 - \frac{4}{9}\sqrt{3}C_1^2M_1 - C_1 \geq 0,$$

if

$$M_1 \geq \frac{\frac{4}{9}\sqrt{3}C_1^2 + \sqrt{\frac{48}{81}C_1^4 + 8dC_1^2}}{4dC_1}.$$

Then, we have

$$\omega_{1,t} - d\omega_{1,xx} + \omega_1\omega_{1,x} \geq \omega_1(1 - \omega_1),$$
$$\omega_1(t, h(t) - M_1^{-1}) = C_1 \geq u(t, h(t) - M_1^{-1}),\ \omega_1(t, h(t)) = 0 = u(t, h(t)).$$

Now if we can find $M_1$, which is independent of $T_0$, such that $\omega_1(0, x) \geq u_0(x)$ for $x \in [h_0 - M_1^{-1}, h_0]$, then we can apply the maximum principle to $\omega_1 - u$.

By the definition of $\omega_1$, we have

$$\omega_1(0, x) = C_1[2M_1(h_0 - x) - M_1^2(h_0 - x)^2],$$

if $x \in [h_0 - M_1^{-1}, h_0 - (2M_1)^{-1}]$, then $\omega_1(0, x) \geq \frac{3}{4}C_1$; if $x \in [h_0 - (2M_1)^{-1}, h_0]$, then

$$\omega_{1,x}(0, x) = C_1[-2M_1 + 2M_1^2(h_0 - x)] \leq -C_1M_1.$$

Setting

$$M_1 = \max\left\{\frac{4}{3}\frac{\|u_0\|_{C^1[-h_0,h_0]}}{C_1}, \quad \frac{\frac{4}{9}\sqrt{3}C_1^2 + \sqrt{\frac{48}{81}C_1^4 + 8dC_1^2}}{4dC_1}\right\},$$

then $\omega_{1,x}(0, x) \leq u_0'(x)$ for $x \in [h_0 - (2M_1)^{-1}, h_0]$. Since $\omega_1(0, h_0) = u_0(h_0) = 0$, the above inequality implies that $\omega_1(0, x) \geq u_0(x)$ for $x \in [h_0 - (2M_1)^{-1}, h_0]$. Moreover, for $x \in [h_0 - M_1^{-1}, h_0 - (2M_1)^{-1}]$, we have

$$\frac{u_0(x) - u_0(h_0)}{\frac{1}{M_1}} \le \|u_0\|_{C^1[-h_0,h_0]},$$

then $u_0(x) \le \frac{3}{4}C_1$.

By the above discussion we know that $u_0(x) \le \omega_1(0,x)$ for $x \in [h_0 - M_1^{-1}, h_0]$, therefore by the maximum principle we obtain

$$u_x(t,h(t)) \ge \omega_{1,x}(t,h(t)),$$

for $(t,x) \in [0,T_0] \times [h(t) - M_1^{-1}, h(t)]$. Since $\omega_{1,x}(t,h(t)) = -2M_1C_1$, so

$$h'(t) = -\mu u_x(t,h(t)) \le 2\mu M_1 C_1.$$

Define an auxiliary function $\omega_2(t,x) := C_1[2M_2(x-g(t)) - M_2^2(x-g(t))^2]$ on $\Omega_{M_2}$, direct calculations show that

$$\omega_{2,t} = 2M_2C_1(-g'(t))[1 - M_2(x-g(t))] \ge 0,$$
$$\omega_{2,x} = C_1[2M_2 - 2M_2^2(x-g(t))], \quad \omega_{2,xx} = -2M_2^2C_1,$$

then

$$\begin{aligned}\omega_2\omega_{2,x} &= C_1^2[2M_2(x-g(t)) - M_2^2(x-g(t))^2][2M_2 - 2M_2^2(x-g(t))] \\ &= C_1^2M_2^2[4(x-g(t)) + 6M_2(x-g(t))^2 + 2M_2^2(x-g(t))^3].\end{aligned}$$

Let $x - g(t) = g$, then $0 < g < M_2^{-1}$ holds over $\Omega_{M_2}$. By $\omega_2\omega_{2,x} = 0$ we obtain $g = 0$ or $g = \frac{1}{M_2}$ or $g = \frac{2}{M_2}$. If $0 < g < \frac{1}{M_2}$, then $\omega_2\omega_{2,x} > 0$, thus

$$\omega_{2,t} - d\omega_{2,xx} + \omega_2\omega_{2,x} \ge 2dC_1M_2^2$$

holds over $\Omega_{M_2}$. Since $\omega_2(1-\omega_2) < C_1$, then

$$\omega_{2,t} - d\omega_{2,xx} + \omega_2\omega_{2,x} \ge \omega_2(1-\omega_2),$$

if $M_2^2 \ge \frac{1}{2d}$. By the definition of $\omega_2$, we have

$$\omega_2(0,x) = C_1[2M_2(x-g_0) - M_2^2(x-g_0)^2],$$

if $x \in [g_0 + (2M_2)^{-1}, g_0 + M_2^{-1}]$, then $\omega_2(0,x) \ge \frac{3}{4}C_1$; and if $x \in [g_0, g_0 + (2M_2)^{-1}]$, then

$$\omega_{2,x}(0,x) = C_1[2M_2 - 2M_2^2(x-g_0)] \ge C_1M_2.$$

Take

$$M_2 = \max\left\{\frac{4}{3}\frac{\|u_0\|_{C^1[-h_0,h_0]}}{C_1}, \quad \sqrt{\frac{1}{2d}}\right\}.$$

By the similarly discussion, we know that $u_0(x) \le \omega_2(0,x)$ for $x \in [g_0, g_0 + M_2^{-1}]$. Using the maximum principle, we have

$$u_x(t,g(t)) \le \omega_{2,x}(t,g(t))$$

for $(t, x) \in [0, T_0] \times [g(t), g(t) + M_2^{-1}]$. Since $\omega_{2,x}(t, g(t)) = 2M_2C_1$, then

$$-g'(t) = \mu u_x(t, g(t)) \leq 2\mu M_2 C_1.$$

Take $C_2 = \max\{\mu M_1 C_1, \mu M_2 C_1\}$, this completes the proof. $\square$

**Theorem 2.3.** *For all $t \in (0, \infty)$, the solution of problem (1.6) exists and is unique.*

**Proof.** Let $[0, T_{\max})$ be the maximum time interval of solution. By Theorem 2.1 we know that $T_{\max} > 0$. Now we show that $T_{\max} = \infty$.

Assuming $T_{\max} < \infty$, by Lemma 2.2, there exist constants $C_1$ and $C_2$ independent of $T_{\max}$ such that for any $t \in [0, T_{\max})$ and $x \in [g(t), h(t)]$,

$$\begin{gathered} 0 \leq u(t, x) \leq C_1, \quad h_0 \leq h(t) \leq h_0 + C_2 t, \quad 0 \leq h'(t) \leq C_2, \\ h_0 - C_2 t \leq g(t) \leq h_0, \quad 0 \leq -g'(t) \leq C_2. \end{gathered}$$

Fix $\delta_0 \in (0, T_{\max})$ and $M > T_{\max}$. Using $L_p$ theory, Sobolev imbedding theorem and the Schauder estimates for parabolic equations, we can find a positive constant $C_3$ which depends on $\delta_0$, $M$, $C_1$ and $C_2$ such that $\|u(t, x)\|_{C^2[g(t), h(t)]} \leq C_3$ for $t \in [\delta_0, T_{\max})$. It then follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ which depends on $C_1$, $C_2$ and $C_3$ such that the solution of problem (1.6) with initial moment $T_{\max} - \frac{\tau}{2}$ can be extended uniquely to the moment $T_{\max} - \frac{\tau}{2} + \tau$. But this contradicts the definition of $T_{\max}$. The proof is complete. $\square$

## 3. Some estimates on solutions of (1.6)

**Theorem 3.1.** *Let $(u, g, h)$ be the unique global solution of (1.6). Then there exists a positive constant $C$, depends only on $\Lambda$, $g_\infty$ and $h_\infty$ such that*

$$\|u(t, \cdot)\|_{C^{1+\alpha}([g(t),\, h(t)])} \leq C, \quad \forall\, t \geq 1; \qquad \|g', h'\|_{C^{\alpha/2}([1,\infty))} \leq C. \tag{3.1}$$

**Proof.** The proof is the same as those of [23, Theorem 2.1] and [29, Theorem 2.2]. When $h_\infty - g_\infty < \infty$, along the arguments in the proof of [23, Theorem 2.1] step by step we can get (3.1). When $h_\infty - g_\infty = \infty$, the estimate (3.1) can be obtained by use the same method of [29, Theorem 2.2]. $\square$

**Lemma 3.2** *([29, Lemma 4.1]). Let $d$, $C$, $\mu$ and $m_0$ be positive constants, $w \in W_p^{1,2}((0, T) \times (0, m(t)))$ for some $p > 1$ and any $T > 0$, and $w_x \in C([0, \infty) \times [0, m(t)])$, $m \in C^1([0, \infty))$. If $(w, m)$ satisfies*

$$\begin{cases} w_t - d v_{xx} \geq -Cw, & t > 0, \quad 0 < x < m(t), \\ w \geq 0, & t > 0, \quad x = 0, \\ w = 0, \quad m'(t) \geq -\mu w_x, & t > 0, \quad x = m(t), \\ w(0, x) = w_0(x) \geq, \not\equiv 0, & x \in (0, m_0), \\ m(0) = m_0, \end{cases}$$

*and*

$$\begin{gathered} \lim_{t\to\infty} m(t) = m_\infty < \infty, \quad \lim_{t\to\infty} m'(t) = 0, \\ \|w(t, \cdot)\|_{C^1[0,\, m(t)]} \leq M, \quad \forall\, t > 1 \end{gathered}$$

*for some constant $M > 0$. Then*

$$\lim_{t\to\infty} \max_{0\le x\le m(t)} w(t,x) = 0.$$

**Theorem 3.3.** *Assume that $(u,h,g)$ is the time-global solution of (1.6). If $I_\infty$ is a finite interval, then*

$$\lim_{t\to+\infty} \|u\|_{C^1[g(t),h(t)]} = 0. \tag{3.2}$$

**Proof.** When $I_\infty$ is finite, by Theorem 3.1 and Lemma 3.2 we have $\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0$. So, $\lim_{t\to\infty} \max_{-1\le y\le 1} w(t,y) = 0$, where $w(t,y) = u\big(t, \frac{1}{2}[(h(t)-g(t))y + h(t) + g(t)]\big)$. The estimate (3.1) implies $\|w(t,\cdot)\|_{C^{1+\alpha}([-1,1])} \le C$ for all $t \ge 1$. Using the compact arguments we can get $\lim_{t\to+\infty} \|w(t,\cdot)\|_{C^1[-1,1]} = 0$, which implies (3.2). □

## 4. The spreading–vanishing dichotomy

We first give the sufficient conditions for spreading and vanishing.

**Lemma 4.1.** *Let $(u,h,g)$ be the unique solution of (1.6). If $h_\infty < \infty,\ g_\infty > -\infty$, then $h_\infty - g_\infty \le \pi\sqrt{d}$.*

**Proof.** Assume on the contrary that $h_\infty - g_\infty > \pi\sqrt{d}$. Then there is $\varepsilon > 0$ such that $h_\infty - g_\infty > \pi\sqrt{d/(1-\varepsilon)}$. Noticing (3.2), there exists $T_\varepsilon \gg 1$ such that $h(T_\varepsilon) - g(T_\varepsilon) > \pi\sqrt{d/(1-\varepsilon)}$ and $|u_x| < \varepsilon$ for all $t > T_\varepsilon$ and $g(t) \le x \le h(t)$. Thus $u$ satisfies

$$\begin{cases} u_t - du_{xx} \ge u(1-\varepsilon-u), & t > T_\varepsilon,\ g(t) < x < h(t), \\ u(t,g(t)) = 0, \quad g' = -\mu u_x(t,g(t)), & t \ge T_\varepsilon, \\ u(t,h(t)) = 0, \quad h' = -\mu u_x(t,h(t)), & t \ge T_\varepsilon. \end{cases}$$

As $h(T_\varepsilon) - g(T_\varepsilon) > \pi\sqrt{d/(1-\varepsilon)}$, by the comparison principle and the results of [7] we have $h_\infty - g_\infty = \infty$, which yields a contradiction. This completes the proof. □

By [3, Theorem A.1 and Theorem A.2] and [7, Lemma 3.5 and Lemma 5.7], the following comparison theorem holds:

**Lemma 4.2.** *Let $(u,h,g)$ be the unique solution of (1.6). Suppose that $T \in (0,\infty)$, $\bar{h},\ \bar{g} \in C^1([0,T])$, $\bar{u} \in C(\bar{D}_T) \cap C^{1,2}(D_T)$, where $D_T = \{(t,x) : 0 < t \le T, \bar{g}(t) < x < \bar{h}(t)\}$, and*

$$\begin{cases} \bar{u}_t - d\bar{u}_{xx} + \bar{u}\,\bar{u}_x \ge \bar{u}(1-\bar{u}), & t > 0,\ \bar{g}(t) < x < \bar{h}(t), \\ \bar{u}(t,\bar{g}(t)) = \bar{u}(t,\bar{h}(t)) = 0, & t \ge 0, \\ \bar{g}'(t) \le -\mu\bar{u}_x(t,\bar{g}(t)), & t \ge 0, \\ \bar{h}'(t) \ge -\mu\bar{u}_x(t,\bar{h}(t)), & t \ge 0. \end{cases}$$

*If $[-h_0,h_0] \subseteq [\bar{g}(0),\bar{h}(0)]$ and $u_0(x) \le \bar{u}(0,x)$ in $[-h_0,h_0]$, then*

$$g(t) \ge \bar{g}(t), h(t) \le \bar{h}(t),\ \ \forall\, t \in [0,T]; \quad u(t,x) \le \bar{u}(t,x),\ \ \forall\, t \in [0,T],\ x \in [g(t),h(t)].$$

**Lemma 4.3.** *Assume $h_0 < \frac{\pi}{2}\sqrt{d}$. Then there exists an $0 < \mu_0 \ll 1$, such that $(g_\infty, h_\infty)$ is finite for all $0 < \mu \le \mu_0$.*

**Proof.** In view of $h_0 < \frac{\pi}{2}\sqrt{d}$, there exists sufficient small $\delta > 0$ such that

$$\left(\frac{\pi}{h_0(1+\delta)}\right)^2 \geq \frac{4}{d}\left(1+\delta+\frac{\delta^2 h_0}{\mu}\right).$$

Obviously, there exists $s > 0$ such that

$$s\cos\frac{\pi x}{2h_0(1+\delta/2)} \geq u_0(x), \quad -h_0(1+\delta/2) \leq x \leq h_0(1+\delta/2).$$

Let $k(t) = h_0(1+\delta-\frac{\delta}{2}e^{-\delta t})$ and

$$w(t,x) := se^{-\delta t}\cos\left(\frac{\pi x}{2k(t)}\right), \quad t \geq 0,\ -k(t) \leq x \leq k(t).$$

Then $w(t,\pm k(t)) = 0$, $k(0) = h_0(1+\frac{\delta}{2}) > h_0$, and for $t \geq 0$, we have $h_0(1+\frac{\delta}{2}) \leq k(t) \leq h_0(1+\delta)$, $k'(t) = h_0\frac{\delta^2}{2}e^{-\delta t}$. Direct calculations show that

$$\begin{aligned}
w_t &= -\delta w - se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi x}{-2k^2(t)}\right)k'(t),\\
w_x &= -se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi}{2k(t)}\right),\\
w_{xx} &= -se^{-\delta t}\cos\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi}{2k(t)}\right)^2 = -\left(\frac{\pi}{2k(t)}\right)^2 w.
\end{aligned}$$

For $t \geq 0$, $-k(t) \leq x \leq k(t)$, we have

$$-se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi x}{-2k^2(t)}\right)k'(t) = se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi x}{2k^2(t)}\right)h_0\frac{\delta^2}{2}e^{-\delta t} \geq 0.$$

Thus,

$$\begin{aligned}
&w_t - dw_{xx} + ww_x - w(1-w)\\
&\geq d\left(\frac{\pi}{2k(t)}\right)^2 w - \delta w + w\left[-se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi}{2k(t)}\right)\right] - w(1-w)\\
&= \left[d\left(\frac{\pi}{2k(t)}\right)^2 - \delta - se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi}{2k(t)}\right) + w - 1\right]w.
\end{aligned}$$

By the formulation of $w$, we know that if $t \geq 0$, $-k(t) \leq x \leq k(t)$, then $0 \leq w \leq s$, and

$$se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi}{2k(t)}\right) \geq -s\frac{\pi}{2h_0(1+\frac{\delta}{2})}.$$

Therefore

$$\begin{aligned}
&d\left(\frac{\pi}{2k(t)}\right)^2 - \delta - se^{-\delta t}\sin\left(\frac{\pi x}{2k(t)}\right)\left(\frac{\pi}{2k(t)}\right) + w - 1\\
&\geq d\left(\frac{\pi}{2k(t)}\right)^2 - \delta - s\frac{\pi}{2h_0(1+\frac{\delta}{2})} - 1
\end{aligned}$$

$$\geq d\left(\frac{\pi}{2k(t)}\right)^2 - \delta - \frac{\delta^2 h_0}{2\mu} - 1$$
$$\geq \left(\frac{\pi}{4h_0(1+\delta)}\right)^2 - \delta - \frac{\delta^2 h_0}{2\mu} - 1 \geq 0.$$

In other words, for $t \geq 0$, $-k(t) \leq x \leq k(t)$,

$$w_t - dw_{xx} + ww_x - w(1-w) \geq 0.$$

Take $\mu_0 = \delta^2 h_0^2/(\pi s)$. Then we have, when $0 < \mu \leq \mu_0$,

$$-\mu w_x(t, k(t)) = \mu s e^{-\delta t}\frac{\pi}{2k(t)} \leq h_0 \frac{\delta^2}{2} e^{-\delta t} = k'(t),$$
$$-\mu w_x(t, -k(t)) = -\mu s e^{-\delta t}\frac{\pi}{2k(t)} \geq -h_0 \frac{\delta^2}{2} e^{-\delta t} = -k'(t).$$

By the comparison principle (Lemma 4.2),

$$-h_0(1+\delta) < -k(t) \leq g(t) < h(t) \leq k(t) < h_0(1+\delta).$$

So $I_\infty$ is a finite interval. □

**Lemma 4.4.** *Let $C > 0$ be a constant. For any given constants $\bar{h}_0, H > 0$, and any function $\bar{u}_0 \in C^2([0, \bar{h}_0])$ satisfying $\bar{u}_0(0) = \bar{u}_0(\bar{h}_0) = 0$ and $\bar{u}_0 > 0$ in $(0, \bar{h}_0)$, there exists $\mu^0 > 0$ so that when $\mu \geq \mu^0$ and $(\bar{u}, \bar{h})$ satisfies*

$$\begin{cases} \bar{u}_t - d\bar{u}_{xx} + \bar{u}\bar{u}_x \geq -C\bar{u}, & t > 0, \quad 0 < x < \bar{h}(t), \\ \bar{u}(t,0) = \bar{u}(t, \bar{h}(t)) = 0, & t > 0, \\ \bar{h}'(t) = -\mu \bar{u}_x(t, \bar{h}(t)), & t > 0, \\ \bar{u}(0,x) = \bar{u}_0(x), \quad \bar{h}(0) = \bar{h}_0, & 0 < x < \bar{h}_0, \end{cases}$$

*we must have $\liminf_{t\to\infty} \bar{h}(t) > H$.*

**Proof.** This proof is the same as that of [26, Lemma 3.2]. For the convenience to readers we shall give the details. By the results of §2 we see that the problem

$$\begin{cases} v_t - dv_{xx} + vv_x = -Cv, & t > 0, \quad 0 < x < r(t), \\ v(t,0) = v(t, r(t)) = 0, & t > 0, \\ r'(t) = -\mu v_x(t, r(t)), & t > 0, \\ v(0,x) = \bar{u}_0(x), \quad r(0) = \bar{h}_0, & 0 < x < \bar{h}_0 \end{cases} \tag{4.1}$$

admits a unique global solution $(v, r)$ and $r'(t) > 0$ for $t > 0$. By Lemma 4.2,

$$\bar{u}(t,x) \geq v(t,x), \quad \bar{h}(t) \geq r(t), \ \forall\, t \geq 0, \ x \in [0, r(t)]. \tag{4.2}$$

In what follows, we are going to prove that for all large $\mu$,

$$r(2) \geq H. \tag{4.3}$$

To the end, we first choose a smooth function $\underline{r}(t)$ with $\underline{r}(0) = \bar{h}_0/2$, $\underline{r}'(t) > 0$ and $\underline{r}(2) = H$. We then discuss the following initial-boundary value problem

$$\begin{cases} \underline{v}_t - d\underline{v}_{xx} + \underline{v}\,\underline{v}_x = -C\underline{v}, & t > 0, \quad 0 < x < \underline{r}(t), \\ \underline{v}(t,0) = \underline{v}(t,\underline{r}(t)) = 0, & t > 0, \\ \underline{v}(0,x) = \underline{v}_0(x), & 0 < x < \bar{h}_0/2. \end{cases} \tag{4.4}$$

Here, for the smooth initial value $\underline{v}_0$, we require

$$\begin{cases} 0 < \underline{v}_0(x) \le \bar{u}_0(x), \quad x \in (0, \bar{h}_0/2), \\ \underline{v}_0(0) = \underline{v}_0(\bar{h}_0/2) = 0, \quad \underline{v}_0'(\bar{h}_0/2) < 0. \end{cases} \tag{4.5}$$

The standard theory for parabolic equations ensures that (4.4) has a unique positive solution $\underline{v}$, and $\underline{v}_x(t, \underline{r}(t)) < 0$ for all $t \in [0,2]$ due to the Hopf boundary lemma. According to our choice of $\underline{r}(t)$ and $\underline{v}_0(x)$, there is a constant $\mu^0 > 0$ such that, for all $\mu \ge \mu^0$,

$$\underline{r}'(t) \le -\mu\underline{v}_x(t, \underline{r}(t)), \quad \forall\, 0 \le t \le 2. \tag{4.6}$$

On the other hand, for the problem (4.1), we can establish the comparison principle analogous with lower solution to Lemma 4.2 by the same argument. Thus, note that $\underline{r}(0) = \bar{h}_0/2 < r(0)$, it follows from (4.1) and (4.4)–(4.6) that

$$v(t,x) \ge \underline{v}(t,x), \quad r(t) \ge \underline{r}(t), \quad \forall\, t \in [0,2], \ x \in [0, \underline{r}(t)].$$

Which particularly implies $r(2) \ge \underline{r}(2) = H$, and so (4.3) holds true. Making use of (4.2) and (4.3) we then obtain

$$\lim_{t\to\infty} \bar{h}(t) \ge \lim_{t\to\infty} r(t) > r(2) \ge H.$$

The proof is complete. □

In view of Lemmas 4.4 and 4.2 we have

**Corollary 4.5.** *Assume $h_0 < \frac{\pi}{2}\sqrt{d}$. Then there exists an $\mu^0 > 0$, such that $(g_\infty, h_\infty) = \mathbb{R}$ for all $\mu \ge \mu^0$.*

**Theorem 4.6.** *Assume $h_0 < \frac{\pi}{2}\sqrt{d}$. Then there exists an $\mu^* > 0$ such that $(g_\infty, h_\infty) = \mathbb{R}$ when $\mu > \mu^*$, and $(g_\infty, h_\infty)$ is finite when $0 < \mu \le \mu^*$.*

**Proof.** Noticing Lemma 4.3 and Corollary 4.5, by use of the continuity method, we can prove this theorem. Please refer to the proof of [7, Theorem 3.9] for details. □

## 5. The estimates for asymptotic spreading speeds

In this section, we estimate the spreading speeds of $h(t)$ and $g(t)$ when spreading happens by using upper and lower solutions constructed by semi-waves.

If $(c, q(z))$ satisfies

$$\begin{cases} q'' - (\frac{c-q}{d})q' + \frac{q(1-q)}{d} = 0, & z \in (0, +\infty), \\ q(0) = 0, \quad q(+\infty) = 1, \quad q(z) > 0, & z \in (0, +\infty), \end{cases} \tag{5.1}$$

we call $q(z)$ a semi-wave with speed $c$. The first equation of (5.1) is equivalent to the following system:

$$\begin{cases} q' = p, \\ p' = \frac{c-q}{d}p - \frac{q(1-q)}{d}. \end{cases} \tag{5.2}$$

A solution $q(z)$ of (5.1) corresponds to a trajectory $(q(z), p(z))$ of (5.2) in the $q - p$ phase plane, such a trajectory starts from the point $(0, \omega)$ and ends at the point $(1, 0)$ when $z \to +\infty$, where $\omega = q'(0) > 0$. For any point satisfies $p \neq 0$, the trajectory has a slope

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{c-q}{d} - \frac{q(1-q)}{dp}. \tag{5.3}$$

For any $c \geq 0$, the critical points of the system (5.2) are $(0, 0)$ and $(1, 0)$. By direct calculations, we know that the linearized system of (5.2) at point $(0, 0)$ is

$$\begin{cases} q' = p, \\ p' = \frac{-q}{d} + \frac{c}{d}p \end{cases}$$

and the corresponding eigenvalues are

$$\lambda_0^{\pm} = \frac{c \pm \sqrt{c^2 - 4d}}{2d}.$$

The linearized system of (5.2) at point $(1, 0)$ is

$$\begin{cases} q' = p, \\ p' = \frac{q}{d} + \frac{(c-1)}{d}p \end{cases}$$

and the corresponding eigenvalues are

$$\lambda_1^{\pm} = \frac{(c-1) \pm \sqrt{(c-1)^2 + 4d}}{2d}.$$

Therefore $(1, 0)$ is a saddle point, and $(0, 0)$ is a center if $c = 0$, or a unstable spiral point if $0 < c < 2\sqrt{d}$, or a unstable nodal point if $c \geq 2\sqrt{d}$. By the theory of ordinary differential equations, there are two trajectories of the system (5.2) that approach $(1, 0)$ from $q < 1$. One of them has slope $\frac{(c-1)-\sqrt{(c-1)^2+4d}}{2d} < 0$ at point $(1, 0)$, will be denoted as $T_r^c$. Some part of $T_r^c$ lies in the set $S := \{(q, p) : 0 \leq q \leq 1,\ p \geq 0\}$ and contains the point $(1, 0)$, this part expressed by a function $p = P_r^c(q)$, $q \in [q_c, 1]$, where $q_c \in [0, 1)$, $P_r^c(q) > 0$ in $(q_c, 1)$, and the point $(q_c, P_r^c(q_c))$ lies on the boundary of the set $S$. The function $p = P_r^c(q)$ satisfies (5.3) and $(P_r^c(1))' = \frac{(c-1)-\sqrt{(c-1)^2+4d}}{2d} < 0$ at the point $(1, 0)$. If $q_c > 0$, then $P_r^c(q_c) = 0$. When $q$ decreases from 1, the function $P_r^c(q)$ stays positive and approaches 0 if $q$ decreases to $q_c$. According to the formulation of (5.3), checking the sign of $(P_r^c(q))'$ we can see that this cannot happen except $q_c \leq 0$, so $T_r^c$ lies in the set $S := \{(q, p) : 0 \leq q \leq 1,\ p \geq 0\}$ and pass through points $(1, 0)$ and $(0, P_r^c(0))$, where $P_r^c(0) \geq 0$.

Consider the dependence of the solution of the following problem on variable $c$:

$$\begin{cases} \frac{\mathrm{d}P_i}{\mathrm{d}q} + \frac{f(q)}{dP_i} = \frac{1}{d}(c_i - q), \qquad q \in (\alpha, \beta), \\ P_i(\beta) = \beta_i, \qquad i = 1, 2, \end{cases} \tag{5.4}$$

where $0 \leq \alpha < \beta \leq 1$.

**Lemma 5.1.** *Assume there exists $\delta > 0$ such that $f(q) \geq 0$ if $\beta - \delta < q < \beta$, $P_i(q)$ satisfies (5.4) when $\alpha < q < \beta$ and $P_i(q)$ is positive, then $P_1(q) > P_2(q)$ when $c_1 < c_2,\ \beta_1 \geq \beta_2$.*

**Proof.** By (5.4) we have

$$\frac{\mathrm{d}(P_1 - P_2)}{\mathrm{d}q} - \frac{f(q)}{dP_1P_2}(P_1 - P_2) = \frac{1}{d}(c_1 - c_2).$$

Multiplying two sides of the last equation by

$$h(q) := \exp\{-\int\limits_{\beta-\frac{\delta}{2}}^{q} \frac{f(t)}{dP_1(t)P_2(t)}\mathrm{d}t\}$$

and $G(q) = (P_1(q) - P_2(q))h(q)$, we obtain

$$\frac{\mathrm{d}G}{\mathrm{d}q} = \frac{1}{d}(c_1 - c_2)h(q), \qquad q \in (\alpha, \beta).$$

Define

$$K = \int\limits_{\beta-\frac{\delta}{2}}^{\beta} \frac{f(t)}{dP_1(t)P_2(t)}\mathrm{d}t.$$

Then if $K$ is divergent, we have $\lim\limits_{q\to\beta^-} G(q) = 0$. If $K$ is convergent, then

$$\lim_{q\to\beta^-} G(q) \geq 0; \quad \frac{\mathrm{d}G}{\mathrm{d}q} < 0, \quad q \in (\alpha, \beta).$$

Thus $G(q) > 0$, in other words, $P_1(q) > P_2(q)$ for $q \in (\alpha, \beta)$. $\square$

It follows from the Application 8.10 of the theory of nonlinear convection reaction diffusion equation [12] that if $d \geq \frac{1}{4}$ and $c \geq 2\sqrt{d}$, or if $0 < d < \frac{1}{4}$ and $c \geq 2d + \frac{1}{2}$, then the equation

$$u_t - du_{xx} + uu_x = u(1 - u),$$

admits a traveling-wave solution $u(t, x) = U(ct - x)$ connecting 0 and 1 with wave speed $c$, which is unique modulo translation.

For any $d > 0$, when $c \geq 2\sqrt{d}$, the equation

$$u_t - du_{xx} - uu_x = u(1 - u),$$

admits a traveling-wave solution from 0 to 1 with wave speed $c$ of the form $u = U(ct - x)$, and the traveling-wave solution is unique modulo translation.

If $d \geq \frac{1}{4}$, then define $c_0^r := 2\sqrt{d}$. If $0 < d < \frac{1}{4}$, then define $c_0^r := 2d + \frac{1}{2}$. We have the following theorem:

**Theorem 5.2.** *The trajectory $p = P_r^{c_0^r}(q),\ q \in (0, 1)$ corresponds to a solution $q_0(z)$ of the following problem with $c = c_0^r$:*

$$\begin{cases} q'' - (\frac{c-q}{d})q' + \frac{q(1-q)}{d} = 0, & z \in \mathbb{R}^1, \\ q(-\infty) = 0, \quad q(+\infty) = 1, \quad q(z) > 0, \ z \in \mathbb{R}^1, \end{cases}$$

*and $q_0(z)$ is unique modulo translation. Moreover, for any $c \geq c_0^r$, the above problem admits a unique solution. However, no such solution exists if $c \in [0, c_0^r)$.*

By Theorem 5.2, we know that for any $c \geq c_0 = c_0^r$, $P_r^c(0) = 0$, and $P_r^c(q) > 0$ in $(0, 1)$. In order to prove the existence of a semi-wave, we need the following lemma.

**Lemma 5.3.** *If $0 \leq c_1 < c_2 < c_0^r$, then $P_r^{c_1} > P_r^{c_2}$ for $q \in [0, 1)$, and for any $\bar{c} \geq 0$, $\lim\limits_{c \to \bar{c}} P_r^c = P_r^{\bar{c}}$ uniformly in $[0, 1]$. Moreover, for any $c \geq c_0 = c_0^r$, $P_r^c(0) = 0$, and $P_r^c(q) > 0$ in $(0, 1)$.*

**Proof.** By $(P_r^c(1))' = \frac{(c-1)-\sqrt{(c-1)^2+4d}}{2d}$, we have $(P_r^{c_1}(1))' < (P_r^{c_2}(1))'$ when $0 \leq c_1 < c_2$. Since $P_r^{c_1}(1) = P_r^{c_2}(1) = 0$, for sufficient small $\delta > 0$, we can see that $P_r^{c_1}(1 - \delta) > P_r^{c_2}(1 - \delta)$, and $P_r^{c_1}(q) > P_r^{c_2}(q)$ in $(1 - \delta, 1)$.

We claim that $q_{c_2} \geq q_{c_1}$. Otherwise, since $q(1 - q) > 0$ for $q \in (0, 1)$, define $\alpha = q_{c_1}, \beta = 1 - \delta$, then it follows from Lemma 5.1 that $P_r^{c_1}(q) > P_r^{c_2}(q)$ in $(q_{c_1}, 1)$. However, by the definition of $q_{c_1}$ and $q_{c_2}$ we can find that $P_r^{c_1}(q_{c_1}) = 0$, this contradicts to $P_r^{c_2}(q_{c_1}) > 0$.

Let $\alpha = q_{c_1}$, $\beta = 1 - \delta$. Using Lemma 5.1, we know that $P_r^{c_1}(q) > P_r^{c_2}(q)$ in $(q_{c_2}, 1)$. Since $q_{c_2} \leq 0$, we have $P_r^{c_1}(q) > P_r^{c_2}(q)$ in $(0, 1)$. It follows from Theorem 5.2 that $q_{c_2} < 0$ for $0 \leq c_1 < c_2 < c_0^r$, therefore, $P_r^{c_1}(q) > P_r^{c_2}(q) > 0$ in $[0, 1)$.

Summarizing the above results, we can see that $P_r^c(q)$ is nonincreasing in variable $c$ for $q \in [0, 1]$, so for any $\bar{c} \geq 0$, if $c$ increases to $\bar{c}$, then $P_r^c(q)$ converges monotonically to some $R(q)$ in $[0, 1]$, where $R(q)$ represents the trajectory of (5.2) with $c = \bar{c}$ that approaches $(1, 0)$ from $q < 1$, and its slope is negative at $(1, 0)$. By the uniqueness of $T_r^{\bar{c}}$, $R(q)$ must coincides with $P_r^{\bar{c}}(q)$. In a similar way, we can show that when $c$ decreases to $\bar{c}$, $P_r^c(q)$ converges monotonically to $P_r^{\bar{c}}(q)$ in $[0, 1]$. □

Now we prove the uniqueness of the semi-wave.

**Theorem 5.4.** *For any $\mu > 0$, there exists a unique $c_r^* = c_{r,\mu}^* \in (0, c_0^r)$ such that $P_r^{c_r^*}(0) = \frac{c_r^*}{\mu}$, and the problem (5.1) admits a unique solution $(c_r^*, q_r^*)$ satisfying $q'(0) = \frac{c_r^*}{\mu}$. Moreover, if $c_{r,\mu}^*$ increases in variable $\mu$, then $\lim\limits_{\mu \to \infty} c_{r,\mu}^* = c_0^r$.*

**Proof.** By the results of Lemma 5.3, for any $c \in [0, c_0^r)$, $P_r^c(0) > 0$. In particular, $P_r'(0)$ is strictly decreasing in $c \in [0, c_0^r]$ and $P_r^{c_0^r}(0) = 0$.

Define a continuous function

$$\xi(c) = \xi_\mu(c) := P_r^c(0) - \frac{c}{\mu}, \quad c \in [0, c_0^r].$$

We know that the function $\xi(c)$ is strictly decreasing in $c \in [0, c_0^r]$, $\xi(0) = P_r^0(0) > 0$, $\xi(c_0^r) = -\frac{c_0^r}{\mu} < 0$. Thus there exists a unique $c_r^* = c_{r,\mu}^* \in (0, c_0^r)$ such that $\xi(c_r^*) = 0$.

We can view $(c_{r,\mu}^*, \frac{c_{r,\mu}^*}{\mu})$ as the unique intersection point of the decreasing curve $y = P_r^c(0)$ and the increasing curve $y = \frac{c}{\mu}$ in $[0, c_0^r]$, it is obviously that $c_{r,\mu}^*$ increases to $c_0^r$ when $\mu$ increases to $\infty$.

For $q \in [0, 1)$, the curve $p = P_r^{c_r^*}(q)$ corresponds to a trajectory of (5.2), denoted by $(q_r^*(z), p_r^*(z))$, $z \in [0, \infty)$, that connects the regular point $(0, P_r^{c_r^*}(0))$ and the equilibrium $(1, 0)$. $(c_r^*, q_r^*)$ solves (5.1) with $c = c_r^*$ and $q'(0) = \frac{c_r^*}{\mu}$. If $(c, q)$ is another solution of (5.1) satisfies $q'(0) = \frac{c}{\mu}$, then it corresponds to a trajectory of (5.2) that connects $(0, \frac{c}{\mu})$ and $(1, 0)$ in set $S$. Since for each $c \geq 0$, there is only one trajectory connecting $(1,0)$ in $S$, it coincides with $p = P_r^c(q)$ for $q \in [0,1)$. Thus we have $P_r^c(0) = \frac{c}{\mu}$, and hence $c = c_r^*$, it follows that $q = q_r^*$. $\square$

To estimate the asymptotic spreading speed of the left free boundary $g(t)$, we need to consider another semi-wave

$$\begin{cases} q'' - (\frac{c+q}{d})q' + \frac{q(1-q)}{d} = 0, & z \in (0,+\infty), \\ q(0) = 0, \quad q(+\infty) = 1, \quad q(z) > 0, & z \in (0,+\infty). \end{cases} \tag{5.5}$$

The first equation of (5.5)) is equivalent to the following system:

$$\begin{cases} q' = p, \\ p' = \frac{c+q}{d}p - \frac{q(1-q)}{d}. \end{cases} \tag{5.6}$$

For any point satisfies $p \neq 0$, the trajectory has a slope

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{c+q}{d} - \frac{q(1-q)}{dp}. \tag{5.7}$$

For any $c \geq 0$, the critical points of (5.6) are $(0,0)$ and $(1,0)$, by direct calculations, the linearized system of (5.6) at $(0,0)$ is

$$\begin{cases} q' = p, \\ p' = \frac{-q}{d} + \frac{c}{d}p \end{cases}$$

and the corresponding eigenvalues are

$$\lambda_0^{\pm} = \frac{c \pm \sqrt{c^2 - 4d}}{2d}.$$

The linearized system of (5.6) at $(1,0)$ is

$$\begin{cases} q' = p, \\ p' = \frac{q}{d} + \frac{(c+1)}{d}p \end{cases}$$

and the corresponding eigenvalues are

$$\lambda_1^{\pm} = \frac{(c+1) \pm \sqrt{(c+1)^2 + 4d}}{2d}.$$

Thus, $(1,0)$ is a saddle point, and $(0,0)$ is a center if $c = 0$, or a unstable spiral point if $0 < c < 2\sqrt{d}$, or a unstable nodal if $c \geq 2\sqrt{d}$. Similar to the discussion of semi-wave (5.1), let $p = P_l^c(q)$ represent the part of trajectory $T_l^c$ that lies in $S := \{(q,p) : 0 \leq q \leq 1,\ p \geq 0\}$ and passes through $(1,0)$ and $(0, P_l^c(0))$, where $P_l^c(0) \geq 0$.

For any $d > 0$, define $c_0^l := 2\sqrt{d}$. Similarly as (5.1), we have the following propositions.

**Proposition 5.5.** *If $0 \leq c_1 < c_2 < c_0^l$, then $P_l^{c_1} > P_l^{c_2}$ for $q \in [0,1)$, and for any $\bar{c} \geq 0$, $\lim_{c \to \bar{c}} P_l^c = P_l^{\bar{c}}$ uniformly in $[0,1]$. Moreover, for any $c \geq c_0 = c_0^l$, $P_l^c(0) = 0$, and $P_l^c(q) > 0$ in $(0,1)$.*

**Proposition 5.6.** *For any $\mu > 0$, there exists a unique $c_l^* = c_{l,\mu}^* \in (0, c_0^l)$ such that $P_l^{c^*}(0) = \frac{c_l^*}{\mu}$, and (5.5) admits a unique solution $(c_l^*, q_l^*)$ satisfying $q'(0) = \frac{c_l^*}{\mu}$. Moreover, if $c_{l,\mu}^*$ increases in variable $\mu$, then $\lim_{\mu\to\infty} c_{l,\mu}^* = c_0^l$.*

Next, we shall estimate the spreading speeds of the free boundaries $h(t)$ and $g(t)$.

We make some suitable perturbations of $f(u) = u(1-u)$, and the corresponding semi-waves can be used to construct upper and lower solutions of (1.6). For any small $\varepsilon > 0$, let

$$\underline{f}_\varepsilon(u) := f(u) - \frac{\varepsilon}{1-\varepsilon}u^2 = u(1 - \frac{1}{1-\varepsilon}u),$$
$$\bar{f}_\varepsilon(u) := f(u) + \frac{\varepsilon}{1+\varepsilon}u^2 = u(1 - \frac{1}{1+\varepsilon}u).$$

We can see that $\underline{f}_\varepsilon(u)$ is strictly decreasing in $\varepsilon$, $\bar{f}_\varepsilon(u)$ is strictly increasing in $\varepsilon$, $\underline{f}_\varepsilon(u)$ has exactly two zeros 0 and $1-\varepsilon$, and $\bar{f}_\varepsilon(u)$ has two zeros 0 and $1+\varepsilon$.

When $f(q) = q(1-q)$ replaced by $\underline{f}_\varepsilon(q)$, the problem (5.1) admits a unique solution $(\underline{c}_r^*, \underline{q}_r^*)$ satisfying $\mu(\underline{q}_r^*)'(0) = \underline{c}_r^*$, where $\underline{c}_r^* \in (0, c_0^r)$. When $f(q) = q(1-q)$ replaced by $\bar{f}_\varepsilon(q)$, the problem (5.1) admits a unique solution $(\bar{c}_r^*, \bar{q}_r^*)$ satisfying $\mu(\bar{q}_r^*)'(0) = \bar{c}_r^*$, where $\bar{c}_r^* \in (0, c_0^r)$. $\underline{c}_r^*$ and $\bar{c}_r^*$ depend on $\varepsilon$.

Similarly, when $f(q) = q(1-q)$ replaced by $\underline{f}_\varepsilon(q)$, the problem (5.5) has a unique solution $(\underline{c}_l^*, \underline{q}_l^*)$ such that $\mu(\underline{q}_l^*)'(0) = \underline{c}_l^*$, where $\underline{c}_l^* \in (0, c_0^l)$. When $f(q) = q(1-q)$ replaced by $\bar{f}_\varepsilon(q)$, the problem (5.5) admits a unique solution $(\bar{c}_l^*, \bar{q}_l^*)$ such that $\mu(\bar{q}_l^*)'(0) = \bar{c}_l^*$, where $\bar{c}_l^* \in (0, c_0^l)$.

**Proposition 5.7.** *The following conclusions hold.*

$$\underline{c}_r^* < c_r^* < \bar{c}_r^*, \quad \lim_{\varepsilon\to 0} \underline{c}_r^* = \lim_{\varepsilon\to 0} \bar{c}_r^* = c_r^*.$$
$$\underline{c}_l^* < c_l^* < \bar{c}_l^*, \quad \lim_{\varepsilon\to 0} \underline{c}_l^* = \lim_{\varepsilon\to 0} \bar{c}_l^* = c_l^*.$$

**Proof.** We first prove $\underline{c}_r^* < c_r^*$. For any $c \in [0, c_0^r)$, $p = \underline{P}_{r,\varepsilon}^c$ is a solution of (5.3) with $f(q) = q(1-q)$ replaced by $\underline{f}_\varepsilon(q)$, it corresponds to the trajectory passing through $(1-\varepsilon, 0)$, and $p > 0$ in $(0, 1-\varepsilon)$. Similarly to Lemma 5.3, we have $\underline{P}_{r,\varepsilon}^c(0) > 0$ for $c \in [0, c_0^r)$. Moreover, because $\underline{f}_\varepsilon(q) < f(q)$ for $q \in (0, 1-\varepsilon]$, $\underline{P}_{r,\varepsilon}^c(1-\varepsilon) = 0$ and $P_r^c(1-\varepsilon) > 0$, we claim that $\underline{P}_{r,\varepsilon}^c < P_r^c$ for $q \in (0, 1-\varepsilon]$. Otherwise, there exists a $q_1 \in (0, 1-\varepsilon)$ such that $\underline{P}_{r,\varepsilon}^c(q_1) = P_r^c(q_1) = Q_1$ and

$$\frac{\mathrm{d}\underline{P}_{r,\varepsilon}^c(q_1)}{\mathrm{d}q} < \frac{\mathrm{d}P_r^c(q_1)}{\mathrm{d}q}.$$

On the other hand, by

$$\frac{\mathrm{d}\underline{P}_{r,\varepsilon}^c(q_1)}{\mathrm{d}q} = \frac{c-q_1}{d} - \frac{\underline{f}_\varepsilon(q_1)}{dQ_1}, \qquad \frac{\mathrm{d}P_r^c(q_1)}{\mathrm{d}q} = \frac{c-q_1}{d} - \frac{f(q_1)}{dQ_1},$$

we have

$$\underline{f}_\varepsilon(q_1) = dQ_1\left[\frac{c-q_1}{d} - \frac{\mathrm{d}\underline{P}_{r,\varepsilon}^c(q_1)}{\mathrm{d}q}\right], \qquad f(q_1) = dQ_1\left[\frac{c-q_1}{d} - \frac{\mathrm{d}P_r^c(q_1)}{\mathrm{d}q}\right],$$

it follows that $f(q_1) < \underline{f}_\varepsilon(q_1)$, however, this contradicts to the fact that $\underline{f}_\varepsilon(q) < f(q)$ for any $q \in (0, 1-\varepsilon]$. Thus, $\underline{P}_{r,\varepsilon}^c < P_r^c$ for $q \in (0, 1-\varepsilon]$.

Next, we prove that for any $c \in [0, c_0^r)$, the following conclusion holds:

$$0 < \underline{P}_{r,\varepsilon}^c(0) < P_r^c(0). \tag{5.8}$$

Otherwise $\underline{P}_{r,\varepsilon}^c(0) = P_r^c(0)$. Define $\eta(q) = P_r^c(q) - \underline{P}_{r,\varepsilon}^c(q)$, we can see that $\eta$ satisfies

$$\begin{aligned}\eta'(q) &= -\frac{f(q)}{P_r^c(q)} + \frac{\underline{f}_\varepsilon(q)}{\underline{P}_{r,\varepsilon}^c(q)} = \frac{1}{P_r^c \underline{P}_{r,\varepsilon}^c}(\underline{f}_\varepsilon P_r^c - f \underline{P}_{r,\varepsilon}^c) \\ &< \frac{\underline{f}_\varepsilon}{P_r^c \underline{P}_{r,\varepsilon}^c}(P_r^c - \underline{P}_{r,\varepsilon}^c) := a(q)\eta(q).\end{aligned}$$

Since $a(q) > 0$ in $(0, 1-\varepsilon)$ and $\eta(0) = 0$, we have $\eta(q) < 0$ for $q \in (0, 1-\varepsilon)$. This contradicts to $\underline{P}_{r,\varepsilon}^c < P_r^c$ for $q \in (0, 1-\varepsilon]$!

Define

$$\underline{\zeta}(c) := \underline{P}_{r,\varepsilon}^c(0) - \frac{c}{\mu}.$$

By (5.8) we have $\underline{\zeta}(c) < \zeta(c)$ for $c \in [0, c_0^r)$ and $\underline{P}_{r,\varepsilon}^{c_0^r}(0) = 0$. Since $\underline{\zeta}(c)$ and $\zeta(c)$ are decreasing functions for $c \in [0, c_0^r)$, and

$$\underline{\zeta}(c_0^r) = \zeta(c_0^r) = -\frac{c_0^r}{\mu}.$$

By the definitions of $\underline{c}_r^*$ and $c_r^*$, we have $\underline{\zeta}(\underline{c}_r^*) = \zeta(c_r^*) = 0$, and hence, $\underline{c}_r^* < c_r^*$. In a similar way, we can prove $c_r^* < \bar{c}_r^*$.

Next we prove $\lim_{\varepsilon \to 0} \underline{c}_r^* = c_r^*$. Since $\underline{f}_\varepsilon(q)$ is monotonically decreasing in $\varepsilon$, by Lemma 5.1 we know that for any $\delta \in (0,1)$, $\underline{P}_{r,\varepsilon}^c(q)$ is monotonically decreasing in $\varepsilon$ for $q \in [0, 1-\delta]$, and $\underline{P}_{r,\varepsilon}^c < P_r^c$. Therefore, for any $\delta \in (0,1)$, $\underline{P}_{r,\varepsilon}^c(q)$ converges to some function $R(q)$ in $[0, 1-\delta]$ as $\varepsilon \to 0$. Since $p = R(q)$ corresponds to a trajectory of (5.2) that lies in $S$ and approaches $(1,0)$, we have $R(q) \equiv P_r^c$. Consequently, if $\varepsilon \to 0$, then $\underline{P}_{r,\varepsilon}^c(0) \to P_r^c(0)$, which means $\lim_{\varepsilon \to 0} \underline{c}_r^* = c_r^*$. In a similar way as above, we can prove $\lim_{\varepsilon \to 0} \bar{c}_r^* = c_r^*$, $\underline{c}_l^* < c_l^* < \bar{c}_l^*$, and $\lim_{\varepsilon \to 0} \underline{c}_l^* = \lim_{\varepsilon \to 0} \bar{c}_l^* = c_l^*$. $\square$

**Theorem 5.8.** *Let $(u, h, g)$ be a solution of (1.6) for which spreading happens, then the asymptotic spreading speed of the left free boundary $h(t)$ is $c_r^*$:*

$$\lim_{t \to \infty} \frac{h(t)}{t} = c_r^*.$$

*The asymptotic spreading speed of the right free boundary $g(t)$ is $c_l^*$:*

$$\lim_{t \to \infty} -\frac{g(t)}{t} = c_l^*.$$

**Proof.** For any small $\varepsilon > 0$, define

$$\underline{\omega}(t, x) = \underline{q}_r^*(\underline{c}_r^* t - x), \quad x \in [0, \underline{c}_r^* t].$$

Since $(\underline{c}_r^*, \underline{q}_r^*)$ satisfies

$$\begin{cases} q'' - (\frac{c-q}{d})q' + \frac{\underline{f}_\varepsilon(q)}{d} = 0, & z \in (0, +\infty), \\ q(0) = 0, \quad q(+\infty) = 1 - \varepsilon, \quad q(z) > 0, \quad q'(0) = \frac{c}{\mu}, & z \in (0, +\infty), \end{cases}$$

then for $t > 0$ and $x \in [0, \underline{c}_r^* t]$, we have

$$\underline{\omega}(t, x) \leq 1 - \varepsilon, \quad \underline{c}_r^* = -\mu \underline{\omega}_x(t, \underline{c}_r^* t).$$

Since spreading happens, we have $\lim_{t\to\infty} u(t, x) = 1$ locally uniformly in $\mathbb{R}$. Then there exists a $T > 0$ such that

$$u(t, 0) > 1 - \varepsilon \text{ for } t > T.$$

By comparison principle, $(\underline{\omega}(t, x), \underline{c}_r^* t)$ is a lower solution of (1.6) on $\{(t, x) : x \in [0, \underline{c}_r^* t], t > T\}$, and for $\{(t, x) : x \in [0, \underline{c}_r^* t], t > T\}$, we have:

$$\underline{c}_r^* t \leq h(t + T), \quad \underline{\omega}(t, x) \leq u(t + T, x).$$

This implies that

$$\liminf_{t\to\infty} \frac{h(t)}{t} \geq \underline{c}_r^*. \tag{5.9}$$

Solve the problem

$$\begin{cases} \eta'(t) = f(\eta), \quad t > 0, \\ \eta(0) = \|u_0\|_\infty + 1. \end{cases}$$

We can get

$$\eta(t) = \left(1 - \frac{\|u_0\|_\infty}{\|u_0\|_\infty + 1} e^{-t}\right)^{-1}.$$

By comparison principle we can see that $u(t, x) \leq \eta(t)$ in $[g(t), h(t)]$ for $t > 0$. Therefore, for any small $\varepsilon > 0$, there exists a $\bar{T} > 0$ such that

$$u(t, x) \leq 1 + \frac{\varepsilon}{2}, \quad x \in [0, h(t)], \quad t \geq \bar{T}.$$

Recall that $(\bar{c}_r^*, \bar{q}_r^*)$ is the unique solution of (5.1) with $f(u) = u(1 - u)$ replaced by $\bar{f}_\varepsilon(u)$, which satisfies $\mu(\bar{q}_r^*)'(0) = \bar{c}_r^*$ and $\bar{q}_r^*(\infty) = 1 + \varepsilon$. Therefore, there exists $\bar{x} > h(\bar{T})$ such that

$$u(\bar{T}, x) \leq 1 + \frac{\varepsilon}{2} < \bar{q}_r^*(\bar{x} - x), \quad x \in [0, h(\bar{T})].$$

Let

$$w(t, x) = \bar{q}_r^*(\bar{c}_r^* t + \bar{x} - x), \quad x \in [0, \bar{c}_r^* t + \bar{x}], \quad t > 0.$$

We can see that $(w, \bar{c}_r^* t + \bar{x})$ is a upper solution of (1.6) over $\{(t, x) : x \in [0, h(t + \bar{T})], \ t > 0\}$. Thus for $\{(t, x) : x \in [0, h(t + \bar{T})], \ t > 0\}$, we have

$$h(t + \bar{T}) \leq \bar{c}_r^* t + \bar{x}, \quad u(t + \bar{T}, x) \leq w(t, x).$$

This implies

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le \bar{c}_r^*. \tag{5.10}$$

Because (5.9) and (5.10) hold for any small $\varepsilon > 0$, we get

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_r^*.$$

In a similar way as above, we can prove

$$\lim_{t\to\infty} -\frac{g(t)}{t} = c_l^*.$$

This completes the proof. □

Now we try to use phase plane analysis to compare $c_l^*$ and $c_r^*$.

Let $(P)$ represent the problem (1.6) without convection term $uu_x$. The semi-wave corresponding to problem $(P)$ is

$$\begin{cases} q'' - (\frac{c}{d})q' + \frac{q(1-q)}{d} = 0, & z \in (0, +\infty), \\ q(0) = 0, \quad q(+\infty) = 1, \quad q(z) > 0, & z \in (0, +\infty). \end{cases}$$

Consider the problem

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{c}{d} - \frac{q(1-q)}{dp}, \quad p(1) = 0. \tag{5.11}$$

According to the results of [9], we know that for any $c \in [0, 2\sqrt{d})$, (5.11) admits a solution $(c, P_c(q))$, and $P_c(0)$ decreases to $P_{c_0}(0) = 0$ as $c$ increases to $c_0 = 2\sqrt{d}$. Moreover, (5.11) has a unique solution $(c^*, P_{c^*}(q))$ satisfying $P_{c^*}(0) = \frac{c^*}{\mu}$, where $c^*$ is the leftward and rightward spreading speed of problem $(P)$ for which spreading happens.

**Theorem 5.9.** *For any $\mu > 0$, we have $c_l^* < c^* < c_r^*$. Particularly, if $d \ge \frac{1}{4}$, then $c_l^* = c^* = c_r^*$ as $\mu \to \infty$.*

**Proof.** Since $P_r^c(0)$, $P_l^c(0)$ and $P_c(0)$ all depend on variable $c$, for $c \in [0, 2\sqrt{d})$ we define

$$\gamma_r(c) := P_r^c(0), \quad \gamma_l(c) := P_l^c(0), \quad \gamma(c) := P_c(0).$$

If $d \ge \frac{1}{4}$ and $c = 2\sqrt{d}$, then $P_r^c(0) = P_l^c(0) = P_c(0) = 0$. If $0 < d < \frac{1}{4}$, since $2d + \frac{1}{2} > 2\sqrt{d}$, then for $c = 2\sqrt{d}$ we have $P_l^c(0) = P_c(0) = 0$ and $P_r^c(0) > 0$.

Denote $f(q) = q(1-q)$, for any $c \in [0, 2\sqrt{d})$, similar to the proof of Lemma 5.1, we define

$$P_c(q) - P_l^c(q) - \frac{f(q)}{dP_c(q)P_l^c(q)}(P_c(q) - P_l^c(q)) = -\frac{q}{d}.$$

Multiplying two sides of the last equation by

$$h(q) := \exp\left\{-\int_{1-\delta}^{q} \frac{f(t)}{dP_c(t)P_l^c(t)}\mathrm{d}t\right\}$$

and defining $G_l(q) = (P_c(q) - P_l^c(q))h(q)$, we have

$$\frac{\mathrm{d}G}{\mathrm{d}q} = -\frac{q}{d}h(q), \qquad q \in [0,1).$$

Fix small $\delta > 0$, let

$$K = \int_{1-\delta}^{1-\frac{\delta}{2}} \frac{f(t)}{dP_c(t)P_l^c(t)}\mathrm{d}t.$$

According the formulation of $(P_c(1))'$ and $(P_l^c(1))'$ we know that $P_l^c(q) < P_c(q)$ for $q \in [1-\frac{\delta}{2}, 1)$. If $K$ is divergent, then

$$\lim_{q \to (1-\frac{\delta}{2})^-} G_l(q) = 0.$$

If $K$ is convergent, then since $-\frac{q}{d} < 0$ for $q \in (0,1)$, we have

$$\lim_{q \to (1-\frac{\delta}{2})^-} G_l(q) \geq 0; \quad \frac{\mathrm{d}G}{\mathrm{d}q} < 0, \quad q \in (0, 1-\frac{\delta}{2}).$$

Thus $G(q) > 0$ for $q \in [0,1)$, which further implies that

$$P_c(q) > P_l^c(q), \quad q \in [0,1).$$

In particular, we have

$$P_c(0) > P_l^c(0). \tag{5.12}$$

Similarly, consider

$$P_r^c(q) - P_c(q) = -\frac{f(q)}{dP_c(q)P_r^c(q)}(P_r^c(q) - P_c(q)) = -\frac{q}{d}.$$

By similar discussion as above, we have

$$P_r^c(0) > P_c(0). \tag{5.13}$$

Since (5.12) and (5.13) hold for any $c \in [0, 2\sqrt{d})$, we can see that if $c \in [0, 2\sqrt{d})$, then

$$\gamma_l(c) < \gamma(c) < \gamma_r(c).$$

For any fixed $\mu > 0$, these three curves intersect the straight line $\gamma = \frac{c}{\mu}$ at points $(c_l^*, \frac{c_l^*}{\mu})$, $(c^*, \frac{c^*}{\mu})$ and $(c_r^*, \frac{c_r^*}{\mu})$. It is clearly that for any $\mu > 0$,

$$c_l^* < c^* < c_r^*.$$

Moreover, since $\gamma_l\left(2\sqrt{d}\right) = \gamma\left(2\sqrt{d}\right) = \gamma_r\left(2\sqrt{d}\right) = 0$ if $d \geq \frac{1}{4}$, we have $c_l^* = c^* = c_r^*$ as $\mu \to \infty$ for $d \geq \frac{1}{4}$. $\square$

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