J. Math. Anal. Appl. 471 (2019) 423–447

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stability and moment boundedness of an age-structured model with randomly-varying immigration or harvesting

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A R T I C L E I N F O

Article history:
Received 22 March 2018
Available online 30 October 2018
Submitted by P. Yao

Keywords:
Stochastic age-structured model
Stochastic stability
Moment boundedness
Randomly-varying immigration or harvesting
Characteristic root

A B S T R A C T

In this paper we study the stability and moment boundedness of the solutions to the linear age-structured model with randomly-varying immigration or harvesting. For this model, we study stability and the asymptotic behavior of the first moment and the results are identical to that of the corresponding deterministic age-structured model and the linear age-structured model with the white noise. However, the stability and boundedness of the second moment are complicated and depend on the randomly-varying immigration or harvesting. For the linear age-structured model with randomly-varying immigration or harvesting, we directly prove that the second moment $M(t,a)$ is bounded for $0 \leq t < a$. When $t > a \geq 0$, using the Laplace transform in the framework of Itô–Doob integral, we give the explicit expressions of the second moments $M(t,0)$ and $M(t,a)$ and then establish the sufficient conditions for the second moments to be bounded and unbounded, respectively, through the supreme of the real parts of all characteristic roots. We also study the asymptotic behaviors of the second moments $M(t,0)$ and $M(t,a)$ and give the sufficient condition for stochastically ultimately boundedness of the model.

1. Introduction

The age-structured population model was first proposed by Sharpe and Lotka [20] in 1911 and it has received a great deal of attention all the time. In 1926, introducing the age variable into the fertility and mortality rates, McKendrick [15] obtained the following age-structured population model

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a), & t \geq 0,\ a \geq 0, \\ u(t,0) = \int_0^{+\infty} \beta(a)u(t,a)da, & t \geq 0, \\ u(0,a) = u_0(a), & a \geq 0, \end{cases} \tag{1.1}$$

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[1] Supported by the National Natural Science Foundation of China (No. 11501158) and Anhui Provincial Natural Science Foundation (No. 1608085QA13).

https://doi.org/10.1016/j.jmaa.2018.10.084

where $u(t, a)$ denotes the population at time t in the age-interval $(a, a + da)$; $\mu(a) > 0$ is the death-rate at age a per unit population of age a; $\beta(a) > 0$ is the average number of newborn per unit time produced by an individual of age a.

In 1974, Gurtin and MacCamy [8] established the first nonlinear continuous model and their model had been extensively studied both in specific situations and in possible generalizations ([2,5,7,11,22]).

As well known, stochastic perturbations influence population dynamics. Two common methods [1] to introduce the effect of environmental variability into a population model are: to assume that the parameters in the model satisfy mean-reverting stochastic processes or to assume that the parameters are linear functions of Gaussian white noise processes.

In 2001, Chowdhury and Allen [6] derived a continuous-time age-structured model perturbed by the white noise. In 2004, for the general nonlinear age-structured model with the white noise, Zhang et al. [23] studied the existence and uniqueness of the strong solution and obtained some criteria for the exponential stability. We ([25,26]) also considered the stability and moment boundedness for age-structured model perturbed by the white noise. The results of stochastic dynamics on the stochastic age-structured models driven by the white noises can also be referred to [4,18,19,24].

Allen [1] studied the two methods of introducing the effect of environmental variability into a mathematical model analytically and computationally and showed that mean-reverting processes are more conceptually and biologically realistic than the white noise process in modeling biological systems. The classic mean-reverting processes are the Ornstein–Uhlenbeck processes. Thus the stochastic processes driven by the Ornstein–Uhlenbeck processes (i.e., colored noises) are attracting much attention ([10,13,14]).

Many authors incorporated the harvesting ([9,16,17]) or immigration ([21]) of the population into linear age-structured population model (1.1) to describe the dynamics of an age-structured population with the extraneous gain or loss term $S(t, a)$. The mathematical model for this special phenomenon has the form

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a) + S(t,a), & t \geq 0,\ a \geq 0, \\ u(t,0) = \int_0^{+\infty} \beta(a)u(t,a)da, & t \geq 0, \\ u(0,a) = u_0(a), & a \geq 0. \end{cases} \tag{1.2}$$

In [9,16,21], this term $S(t, a) = E(t)u(t, a)$ is deterministic and dependent of population size $u(t, a)$, but it is not assumed that $S(t, a)$ depends on $u(t, a)$ in [17].

In model (1.2), if the term $S(t, a)$ is assumed that $S(t, a) = \sigma\eta(t)$, where σ is a constant and $\eta(t)$ is the colored noise (usually with zero mean and positive variance, see Section 2) modeled by the Ornstein–Uhlenbeck process [3], then the loss or gain term $S(t, a)$ is independent of age a and population size $u(t, a)$. And $S(t, a) = \sigma\eta(t)$ represents an age-independent randomly varying extraneous loss/gain term, for example, randomly varying immigration/emigration or harvesting/propagation of the population. Thus the special phenomenon models a very specific type of biological processes with a gain/loss term that is asymptotically normal with zero mean and positive variance.

Motivated by the above, in this paper, we are going to study the stochastic stability and moment boundedness of the solutions for the linear age-structured model with a randomly-varying gain/loss term

$$\begin{cases} \dfrac{\partial u(t,a)}{dt} + \dfrac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a) + \sigma\eta(t), & t \geq 0,\ a \geq 0, \\ u(t,0) = \displaystyle\int_0^{+\infty} \beta(a)u(t,a)da, & t \geq 0, \\ u(0,a) = u_0(a), & a \geq 0. \end{cases}$$

In the real age-structured model, the maximum age of the population is a positive real number $a_{max} < +\infty$, hence $u(t,0) = \int_0^{a_{max}} \beta(a)u(t,a)da$. But in this paper, for the convenience of study and calculation, we still assume $a_{max} = +\infty$ (as in [9,16,17,21]) (the detailed reasons see Remark 2.2).

Rest of the paper is organized as follows. In Section 2, we study stabilities and the asymptotic behaviors of the first moments $\mathbb{E}u(t,0)$ and $\mathbb{E}u(t,a)$ $(t > a)$ for the linear age-structured model with a randomly-varying gain/loss term. The stabilities of the first moments are identical to that of the corresponding deterministic age-structured model and the linear age-structured model with the additive white noise. However, the stability and boundedness of the second moment are complicated and depend on the randomly-varying gain/loss term. In Section 3, for the linear age-structured model with the randomly-varying gain/loss term, when $0 \le t < a$, we can directly prove that the second moment $M(t,a)$ is bounded; when $t > a \ge 0$, using the Laplace transform in the sense of Itô–Doob integral [12], we first give the explicit expressions of the second moments $M(t,0)$ and $M(t,a)$, and then establish the sufficient conditions for the boundedness and unboundedness of the second moments $M(t,0)$ and $M(t,a)$, respectively, through the supreme of the real parts of all characteristic roots. We also study the asymptotic behaviors of the second moments $M(t,0)$ and $M(t,a)$ and show that linear age-structured model with the randomly-varying immigration or harvesting is stochastically ultimately bounded when the supreme of the real parts of all characteristic roots is negative in Section 3.

2. The first moment stability of the linear age-structured model with the randomly-varying immigration or harvesting

In this paper, we consider that the linear age-structured model with the randomly-varying immigration or harvesting (i.e., perturbed by the additive colored noise)

$$\begin{cases} \dfrac{\partial u(t,a)}{dt} + \dfrac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a) + \sigma\eta(t), & t \ge 0,\ a \ge 0, \\ u(t,0) = \displaystyle\int_0^{+\infty} \beta(a)u(t,a)da, & t \ge 0, \\ u(0,a) = u_0(a), & a \ge 0, \end{cases} \tag{2.1}$$

where $\mu(a) = \mu$ $(0 < \mu < 1)$, $\beta(a) = e^{-\beta a}$ $(\beta > 0)$, $u_0 \in L^1([0,+\infty),[0,+\infty))$. Here $\eta(t)$ is the colored noise modeled by the Ornstein–Uhlenbeck process, which satisfies the Langevin equation

$$\frac{d\eta(t)}{dt} = -\gamma\eta(t) + \rho\xi(t) \tag{2.2}$$

where $\gamma > 0$, $\rho > 0$ are constants and $\xi(t)$ is a scalar white noise process with $\mathbb{E}(\xi(t)) = 0$ and $\mathbb{E}(\xi(t)\xi(s)) = \delta(t-s)$.

The corresponding stochastic differential equation of Eq. (2.2) is

$$d\eta(t) = -\gamma\eta(t)dt + \rho dW(t), \quad t > 0, \quad \eta(0) = \eta_0, \tag{2.3}$$

where $W(t)$ is a one dimensional Wiener process, η_0 is a random variable independent of Wiener process $W(t)$ and $\mathbb{E}(\eta_0) = 0$, $\mathbb{E}(\eta_0^2) > 0$. Eq. (2.3) has a unique solution

$$\eta(t) = \eta_0 e^{-\gamma t} + \rho \int_0^t e^{-\gamma(t-s)} dW(s). \tag{2.4}$$

Thus the Ornstein–Uhlenbeck process $\eta(t)$ is stationary, $\mathbb{E}(\eta(t)) = \mathbb{E}(\eta_0) = 0$ and its correlation function

$$\mathbb{E}(\eta(t)\eta(s)) = \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma}\right) e^{-\gamma(t+s)} + \frac{\rho^2}{2\gamma} e^{-\gamma|t-s|} \tag{2.5}$$

is exponentially decreasing.

From Theorem 2.1.1 in [2] and the characteristics, model (2.1) has a unique solution satisfying

$$u(t,a) = \begin{cases} u_0(a-t)e^{-\mu t} + \sigma \int\limits_0^t e^{-\mu(t-s)}\eta(s)ds, & 0 \le t \le a, \\ u(t-a,0)e^{-\mu a} + \sigma \int\limits_{t-a}^t e^{-\mu(t-s)}\eta(s)ds, & 0 \le a < t, \end{cases} \tag{2.6}$$

hence

$$\mathbb{E}(u(t,a)) = \begin{cases} u_0(a-t)e^{-\mu t}, & 0 \le t \le a, \\ \mathbb{E}(u(t-a,0))e^{-\mu a}, & 0 \le a < t. \end{cases} \tag{2.7}$$

From (2.7), we find that $\mathbb{E}u(t,a)$ depends on $\mathbb{E}u(t,0)$, thus we first compute $\mathbb{E}u(t,0)$.

Since $u(t,0) = \int\limits_0^{+\infty} \beta(a)u(t,a)da$, from (2.6), we have

$$\begin{aligned} u(t,0) &= \int\limits_0^t \beta(a)u(t,a)da + \int\limits_t^{+\infty} \beta(a)u(t,a)da \\ &= \int\limits_0^t \beta(a)u(t-a,0)e^{-\mu a}da + \sigma \int\limits_0^t \beta(a) \int\limits_{t-a}^t e^{-\mu(t-s)}\eta(s)dsda \\ &\quad + e^{-\mu t} \int\limits_t^{+\infty} \beta(a)u_0(a-t)da + \sigma \int\limits_t^{+\infty} \beta(a)da \int\limits_0^t e^{-\mu(t-s)}\eta(s)ds. \end{aligned} \tag{2.8}$$

Define

$$\begin{aligned} J(t) = \sigma \int\limits_0^t \beta(a) \int\limits_{t-a}^t e^{-\mu(t-s)}\eta(s)dsda + e^{-\mu t} \int\limits_t^{+\infty} \beta(a)u_0(a-t)da \\ + \sigma \int\limits_t^{+\infty} \beta(a)da \int\limits_0^t e^{-\mu(t-s)}\eta(s)ds. \end{aligned}$$

Then

$$u(t,0) = \int\limits_0^t e^{-(\mu+\beta)(t-s)}u(s,0)ds + J(t), \quad t \ge 0.$$

Thus we get

$$\mathbb{E}u(t,0) = \int_0^t e^{-(\mu+\beta)(t-s)}\mathbb{E}u(s,0)ds + \mathbb{E}\big(J(t)\big), \tag{2.9}$$

where

$$\mathbb{E}\big(J(t)\big) = e^{-\mu t}\int_t^{+\infty} \beta(a)u_0(a-t)da = e^{-(\mu+\beta)t}\int_0^{+\infty} e^{-\beta a}u_0(a)da \geq 0$$

and $\lim\limits_{t\to+\infty} \mathbb{E}\big(J(t)\big) = 0$.

Taking the Laplace transform on the both sides of (2.9), we have

$$\begin{aligned}\mathcal{L}\left(\mathbb{E}u(t,0)\right)(\lambda) &= \int_0^{+\infty} e^{-\lambda t}\left(\int_0^t e^{-(\mu+\beta)(t-s)}\mathbb{E}u(s,0)ds\right)dt + \int_0^{+\infty} e^{-\lambda t}\mathbb{E}\big(J(t)\big)dt \\ &= \mathcal{L}\left(\mathbb{E}u(t,0)\right)(\lambda)\mathcal{L}\left(e^{-(\mu+\beta)t}\right)(\lambda) + \mathcal{L}\left(\mathbb{E}\big(J(t)\big)\right)(\lambda),\end{aligned}$$

that is,

$$\mathcal{L}\left(\mathbb{E}u(t,0)\right)(\lambda) = \frac{\mathcal{L}(\mathbb{E}J)(\lambda)}{1-\mathcal{L}(e^{-(\mu+\beta)t})(\lambda)} = \frac{\int_0^{+\infty} e^{-\beta a}u_0(a)da}{\lambda+\mu+\beta-1}.$$

The above equation implies that the characteristic equation for $\mathbb{E}u(t,0)$ is

$$\mathcal{L}(e^{-(\mu+\beta)t})(\lambda) = 1, \tag{2.10}$$

which is the same as the characteristic equations (2.5) and (3.6) in [26] for $u(t,0)$ in the deterministic and stochastic age-structured model when $\mu(a) = \mu$, $\beta(a) = e^{-\beta a}$ and $A = T = +\infty$.

Let

$$\alpha_1 = \sup\left\{\mathrm{Re}(\lambda) : \mathcal{L}(e^{-(\mu+\beta)t})(\lambda) = 1, \lambda \in \mathbb{C}\right\}. \tag{2.11}$$

Then $\alpha_1 = 1-\mu-\beta$ (it is identical to formula (3.7) in [26]) and

$$\mathcal{L}\left(\mathbb{E}u(t,0)\right)(\lambda) = \frac{1}{\lambda-\alpha_1}\int_0^{+\infty} e^{-\beta a}u_0(a)da.$$

By the inverse Laplace transform, we have

$$\mathbb{E}u(t,0) = e^{\alpha_1 t}\int_0^{+\infty} e^{-\beta a}u_0(a)da, \quad t \geq 0. \tag{2.12}$$

Thus we know that when $\alpha_1 < 0$, the first moment $\mathbb{E}u(t,0)$ of the boundary $u(t,0)$ satisfies $\lim\limits_{t\to+\infty} \mathbb{E}u(t,0) = 0$; when $\alpha_1 = 0$, the first moment $\mathbb{E}u(t,0)$ is a positive constant; when $\alpha_1 > 0$, $\lim\limits_{t\to+\infty} \mathbb{E}u(t,0) = +\infty$.

From (2.7) and (2.12), we obtain

$$\mathbb{E}(u(t,a)) = \begin{cases} u_0(a-t)e^{-\mu t}, & 0 \le t \le a, \\ e^{-\mu a} e^{\alpha_1 t} \int_0^{+\infty} e^{-\beta a} u_0(a) da, & 0 \le a < t. \end{cases} \tag{2.13}$$

Hence from (2.13), when $0 \le t \le a$, the first moment $\mathbb{E}u(t,a)$ of the solution $u(t,a)$ of model (2.1) is bounded. For any fixed $a \ge 0$ and $t > a \ge 0$, we have the following results of the asymptotic behavior for $\mathbb{E}u(t,a)$.

Theorem 2.1. *For any fixed $a \ge 0$, when $t > a \ge 0$, the first moment $\mathbb{E}u(t,a)$ of $u(t,a)$ satisfies*

(1) *when $\alpha_1 < 0$, $\lim_{t\to+\infty} \mathbb{E}u(t,a) = 0$;*
(2) *when $\alpha_1 > 0$, $\lim_{t\to+\infty} \mathbb{E}u(t,a) = +\infty$;*
(3) *when $\alpha_1 = 0$, $\mathbb{E}u(t,a) = e^{-\mu a} \int_0^{+\infty} e^{-\beta a} u_0(a) da$ is bounded.*

Thus the asymptotic behaviors of $\mathbb{E}u(t,0)$ and $\mathbb{E}u(t,a)$ are separately identical with the age-structured model perturbed by the white noise (see Lemma 3.3 and Theorem 3.4 in [26]). Therefore for the asymptotic behaviors of $u(t,0)$ and $u(t,a)$ in the sense of the first moments, the colored noise perturbation and the white noise perturbation have the same effect on the linear age-structured model.

When $\sigma = 0$ in model (2.1), it is a deterministic equation and its stationary solution $u(t;a) \equiv 0$ of Eq. (2.1) is locally asymptotically stable if $\alpha_1 = 1 - \mu - \beta < 0$ (see Theorem 2.4 in [26]). The result (1) of Theorem 2.1 indicates that for any fixed $a \ge 0$, when $\alpha_1 < 0$, the population $u(t,a)$ is extinct in mean when $t \to +\infty$. Thus for the deterministic age-structured model, the dynamics in the sense of mean have not been changed drastically by the randomly-varying gain/loss term (i.e., the additive colored noise).

In the end of this section, we explain the reasons that we assume the ideal maximum age $a_{max} = +\infty$.

Remark 2.2. In the real age-structured model, the maximum age of the population is a positive real number $a_{max} < +\infty$, hence $u(t,0) = \int_0^{a_{max}} \beta(a)u(t,a)da$. Then from (2.6), when $0 \le t \le a_{max}$,

$$\begin{aligned} u(t,0) = &\int_0^t e^{-(\mu+\beta)a} u(t-a,0)da + \sigma \int_0^t e^{-\beta a} \int_{t-a}^t e^{-\mu(t-s)} \eta(s) ds da \\ &+ e^{-\mu t} \int_t^{a_{max}} e^{-\beta a} u_0(a-t) da + \sigma \int_t^{a_{max}} e^{-\beta a} da \int_0^t e^{-\mu(t-s)} \eta(s) ds; \end{aligned} \tag{2.14}$$

when $t > a_{max}$,

$$\begin{aligned} u(t,0) &= \int_0^{a_{max}} e^{-(\mu+\beta)a} u(t-a,0)da + \sigma \int_0^{a_{max}} e^{-\beta a} \int_{t-a}^t e^{-\mu(t-s)} \eta(s) ds da \\ &= \int_{t-a_{max}}^t e^{-(\mu+\beta)(t-a)} u(a,0)da + \sigma \int_0^{a_{max}} e^{-\beta a} \int_{t-a}^t e^{-\mu(t-s)} \eta(s) ds da \\ &= \int_0^t e^{-(\mu+\beta)(t-a)} u(a,0)da - e^{-(\mu+\beta)a_{max}} \int_0^{t-a_{max}} e^{-(\mu+\beta)(t-a_{max}-a)} u(a,0)da \end{aligned}$$

$$+\sigma \int_0^{a_{max}} e^{-\beta a} \int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)dsda. \tag{2.15}$$

In (2.14), the maximum of t is a_{max}, we can not study the asymptotic behaviors of $u(t,0)$ and $u(t,a)$ when $t \to +\infty$. The first two terms of (2.15) are convolutions, thus we can apply the same method (Laplace transform) to study $\mathbb{E}(u(t,0))$ as in the case of (2.8).

Therefore for the convenience of study and calculation, we assume $a_{max} = +\infty$ (as in [9,16,17,21]) in this paper.

3. Boundedness of the second moment for the linear age-structured model with the randomly-varying immigration or harvesting

In this section, we will investigate the boundedness and the asymptotic behavior of the second moment for the linear age-structured model with a randomly-varying gain/loss term (2.1).

From (2.6) and (2.7), we have

$$u(t,a) - \mathbb{E}u(t,a) = \begin{cases} \sigma \int_0^t e^{-\mu(t-s)}\eta(s)ds, & 0 \le t \le a, \\ [u(t-a,0) - \mathbb{E}u(t-a,0)]\, e^{-\mu a} & \\ \qquad +\sigma \int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds, & 0 \le a < t. \end{cases} \tag{3.1}$$

When $0 \le t \le a$, the second moment of the solution $u(t,a)$ of model (2.1) is

$$M(t,a) \triangleq \mathbb{E}(u(t,a) - \mathbb{E}u(t,a))^2 = \sigma^2 \mathbb{E}\left(\int_0^t e^{-\mu(t-s)}\eta(s)ds\right)^2. \tag{3.2}$$

When $t > a \ge 0$, the second moment of the solution $u(t,a)$ of model (2.1) is

$$\begin{aligned} M(t,a) &= \mathbb{E}(u(t,a) - \mathbb{E}u(t,a))^2 \\ &= e^{-2\mu a} M(t-a,0) + \sigma^2 \mathbb{E}\left(\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right)^2 \\ &\quad +2\sigma e^{-\mu a}\mathbb{E}\left(\big[u(t-a,0) - \mathbb{E}u(t-a,0)\big] \int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right), \end{aligned} \tag{3.3}$$

where $M(t,0) \triangleq \mathbb{E}\left[u(t,0) - \mathbb{E}u(t,0)\right]^2$.

3.1. The second moment $M(t,a)$ of the solution $u(t,a)$ when $t \le a$

When $0 \le t \le a$, from (2.5), applying Itô integral, some calculation yields

$$0 \le M(t,a) = \mathbb{E}\left(\sigma \int_0^t e^{-\mu(t-s)}\eta(s)ds\right)^2 = \sigma^2 \int_0^t e^{-\mu(t-s)} \int_0^t e^{-\mu(t-l)}\mathbb{E}\left(\eta(s)\eta(l)\right) dlds$$

$$
\begin{aligned}
&= \sigma^2 \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma} \right) \left(\int_0^t e^{-\mu(t-s)} e^{-\gamma s} ds \right)^2 \\
&\quad + \frac{\sigma^2 \rho^2}{2\gamma} \int_0^t e^{-\mu(t-s)} \int_0^t e^{-\mu(t-l)} e^{-\gamma|s-l|} dl ds.
\end{aligned} \tag{3.4}
$$

Thus when $t \leq a$, we have the following results on the boundedness for the second moment $M(t, a)$ of model (2.1).

Theorem 3.1. *For every fixed $a \geq 0$, when $t \leq a$, the second moment $M(t, a)$ of model (2.1) is bounded and*

(1) *when $\mu \neq \gamma$,*

$$
M(t, a) = \sigma^2 \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma} \right) \frac{\left(e^{-\mu t} - e^{-\gamma t}\right)^2}{(\mu - \gamma)^2} + \frac{\sigma^2 \rho^2}{2\gamma} \left[\frac{1}{\mu(\mu + \gamma)} + \frac{e^{-2\mu t}}{\mu(\mu - \gamma)} - \frac{2e^{-(\mu+\gamma)t}}{\mu^2 - \gamma^2} \right];
$$

(2) *when $\mu = \gamma$,*

$$
M(t, a) = \sigma^2 \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma} \right) t^2 e^{-2\gamma t} + \frac{\sigma^2 \rho^2}{2\gamma} \left[\frac{1}{2\gamma^2} - \left(\frac{t}{\gamma} + \frac{1}{2\gamma^2} \right) e^{-2\gamma t} \right].
$$

Proof. (1) When $\mu \neq \gamma$, we obtain

$$
\left(\int_0^t e^{-\mu(t-s)} e^{-\gamma s} ds \right)^2 = e^{-2\mu t} \left(\int_0^t e^{(\mu-\gamma)s} ds \right)^2 = \frac{\left(e^{-\mu t} - e^{-\gamma t}\right)^2}{(\mu - \gamma)^2} \tag{3.5}
$$

and

$$
\begin{aligned}
\int_0^t e^{-\mu(t-l)} e^{-\gamma|s-l|} dl &= \int_0^s e^{-\mu(t-l)} e^{-\gamma(s-l)} dl + \int_s^t e^{-\mu(t-l)} e^{-\gamma(l-s)} dl \\
&= e^{-\mu t} e^{-\gamma s} \int_0^s e^{(\mu+\gamma)l} dl + e^{-\mu t} e^{\gamma s} \int_s^t e^{(\mu-\gamma)l} dl \\
&= \frac{2\gamma e^{-\mu(t-s)}}{\gamma^2 - \mu^2} - \frac{e^{-\mu t} e^{-\gamma s}}{\mu + \gamma} + \frac{e^{-\gamma(t-s)}}{\mu - \gamma}.
\end{aligned}
$$

Hence when $\mu \neq \gamma$, we get

$$
\begin{aligned}
\int_0^t e^{-\mu(t-s)} \int_0^t e^{-\mu(t-l)} e^{-\gamma|s-l|} dl ds &= \int_0^t e^{-\mu(t-s)} \left(\frac{2\gamma e^{-\mu(t-s)}}{\gamma^2 - \mu^2} - \frac{e^{-\mu t} e^{-\gamma s}}{\mu + \gamma} + \frac{e^{-\gamma(t-s)}}{\mu - \gamma} \right) ds \\
&= \frac{1}{\mu(\mu + \gamma)} + \frac{e^{-2\mu t}}{\mu(\mu - \gamma)} - \frac{2e^{-(\mu+\gamma)t}}{\mu^2 - \gamma^2}.
\end{aligned} \tag{3.6}
$$

Thus from (3.4)–(3.6), when $\mu \neq \gamma$, we obtain

$$M(t,a) = \sigma^2 \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma} \right) \frac{\left(e^{-\mu t} - e^{-\gamma t}\right)^2}{(\mu-\gamma)^2} + \frac{\sigma^2 \rho^2}{2\gamma} \left[\frac{1}{\mu(\mu+\gamma)} + \frac{e^{-2\mu t}}{\mu(\mu-\gamma)} - \frac{2e^{-(\mu+\gamma)t}}{\mu^2-\gamma^2} \right],$$

which implies that the second moment $M(t)$ of model (2.1) is bounded for every fixed $a \geq 0$ and $t \leq a$.

(2) When $\mu = \gamma$, we have

$$\left(\int_0^t e^{-\mu(t-s)} e^{-\gamma s} ds \right)^2 = e^{-2\mu t} \left(\int_0^t e^{(\mu-\gamma)s} ds \right)^2 = t^2 e^{-2\mu t} = t^2 e^{-2\gamma t} \tag{3.7}$$

and

$$\begin{aligned} \int_0^t e^{-\mu(t-l)} e^{-\gamma|s-l|} dl &= e^{-\mu t} e^{-\gamma s} \int_0^s e^{(\mu+\gamma)l} dl + e^{-\mu t} e^{\gamma s} \int_s^t dl \\ &= \frac{e^{-\mu t} e^{\mu s} - e^{-\mu t} e^{-\gamma s}}{\mu+\gamma} + (t-s) e^{-\mu t} e^{\gamma s} \\ &= \frac{e^{-\gamma t} e^{\gamma s} - e^{-\gamma t} e^{-\gamma s}}{2\gamma} + (t-s) e^{-\gamma t} e^{\gamma s}. \end{aligned}$$

Hence when $\mu = \gamma$, we get

$$\begin{aligned} \int_0^t e^{-\mu(t-s)} \int_0^t e^{-\mu(t-l)} e^{-\gamma|s-l|} dl ds &= \int_0^t e^{-\gamma(t-s)} \left(\frac{e^{-\gamma t} e^{\gamma s} - e^{-\gamma t} e^{-\gamma s}}{2\gamma} + (t-s) e^{-\gamma t} e^{\gamma s} \right) ds \\ &= \frac{1}{2\gamma^2} - \left(\frac{t}{\gamma} + \frac{1}{2\gamma^2} \right) e^{-2\gamma t}. \end{aligned} \tag{3.8}$$

Thus from (3.7) and (3.8), when $\mu = \gamma$, we know that

$$M(t,a) = \sigma^2 \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma} \right) t^2 e^{-2\gamma t} + \frac{\sigma^2 \rho^2}{2\gamma} \left[\frac{1}{2\gamma^2} - \left(\frac{t}{\gamma} + \frac{1}{2\gamma^2} \right) e^{-2\gamma t} \right],$$

which indicates that the second moment $M(t)$ of model (2.1) is bounded for every fixed $a \geq 0$ and $t \leq a$. Therefore this theorem is proved. □

3.2. The second moment $M(t,0)$ of the boundary $u(t,0)$

When $t > a$, from (3.3), we know that $M(t,a)$ depends on $M(t-a,0)$, thus we first consider the boundedness of the second moment $M(t,0)$ for the boundary $u(t,0)$, and then study the boundedness and the asymptotic behavior of the second moment $M(t,a)$ for $t > a$.

From (2.8) and (2.9), we obtain

$$\begin{aligned} u(t,0) - \mathbb{E}u(t,0) &= \int_0^t \beta(a) e^{-\mu a} \left[u(t-a,0) - \mathbb{E}u(t-a,0) \right] da \\ &\quad + \sigma \int_t^\infty \beta(a) da \int_0^t e^{-\mu(t-s)} \eta(s) ds + \sigma \int_0^t \beta(a) \int_{t-a}^t e^{-\mu(t-s)} \eta(s) ds da \end{aligned}$$

$$= \int_0^t \beta(a) e^{-\mu a} \left[u(t-a,0) - \mathbb{E}u(t-a,0) \right] da + \sigma F(t), \tag{3.9}$$

where

$$\begin{aligned}
F(t) &= \int_t^\infty \beta(a)da \int_0^t e^{-\mu(t-s)}\eta(s)ds + \int_0^t \beta(a) \int_{t-a}^t e^{-\mu(t-s)}\eta(s)dsda \\
&= \int_t^\infty \beta(a)da \int_0^t e^{-\mu(t-s)}\eta(s)ds \\
&\quad + \int_0^t \beta(a) \left(\int_0^t e^{-\mu(t-s)}\eta(s)ds - \int_0^{t-a} e^{-\mu(t-s)}\eta(s)ds \right) da \\
&= \int_0^\infty \beta(a)da \int_0^t e^{-\mu(t-s)}\eta(s)ds - \int_0^t \beta(a) \int_0^{t-a} e^{-\mu(t-s)}\eta(s)dsda \\
&= \frac{1}{\beta} \int_0^t e^{-\mu(t-s)}\eta(s)ds - \int_0^t e^{-(\mu+\beta)a} \int_0^{t-a} e^{-\mu(t-a-s)}\eta(s)dsda \\
&= \frac{1}{\beta} A(t) - B(t)
\end{aligned} \tag{3.10}$$

with

$$A(t) = \int_0^t e^{-\mu(t-s)}\eta(s)ds, \quad B(t) = \int_0^t e^{-(\mu+\beta)a} A(t-a)da = \int_0^t e^{-(\mu+\beta)(t-a)} A(a)da.$$

In the following, we will investigate the expression of $u(t,0) - \mathbb{E}u(t,0)$ according to (3.9). For this purpose, we first study the properties of $F(t)$ and get the following lemma.

Lemma 3.2. *The first moment of $F(t)$ is $\mathbb{E}(F(t)) = 0$ and the second moment $\mathbb{E}(F(t))^2$ is bounded and satisfies*

$$\mathbb{E}(F(t))^2 \le \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma} \right) \left(\frac{1}{\mu^2\beta^2} + \frac{1}{\mu^2(\mu+\beta)^2} \right).$$

Proof. From the expressions of $A(t)$ and $B(t)$ and (3.10), we know that $\mathbb{E}(A(t)) = \mathbb{E}(B(t)) = 0$, which implies that $\mathbb{E}(F(t)) = 0$.

From (3.10), we have

$$\mathbb{E}(F(t))^2 = \frac{1}{\beta^2}\mathbb{E}(A^2(t)) + \mathbb{E}(B^2(t)) - \frac{2}{\beta}\mathbb{E}(A(t)B(t)). \tag{3.11}$$

By (2.5), for any $t, s > 0$, we get

$$\mathbb{E}(\eta(t)\eta(s)) = \mathbb{E}(\eta_0^2)e^{-\gamma(t+s)} + \frac{\rho^2}{2\gamma} \left(e^{-\gamma|t-s|} - e^{-\gamma(t+s)} \right),$$

which indicates that

$$0 \le \mathbb{E}(\eta(t)\eta(s)) \le \mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma} \tag{3.12}$$

since $\gamma > 0$. Thus, from (3.10) and (3.12), for any $t, s > 0$, we obtain

$$\begin{aligned} 0 \le \mathbb{E}(A(t)A(s)) &= \mathbb{E}\left(\int_0^t e^{-\mu(t-l)}\eta(l)dl \int_0^s e^{-\mu(s-p)}\eta(p)dp\right) \\ &= \int_0^t e^{-\mu(t-l)} \int_0^s e^{-\mu(s-p)}\mathbb{E}\left(\eta(l)\eta(p)\right) dldp \\ &\le \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \int_0^t e^{-\mu(t-l)}dl \int_0^s e^{-\mu(s-p)}dp \\ &= \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \frac{(1-e^{-\mu t})(1-e^{-\mu s})}{\mu^2}. \end{aligned} \tag{3.13}$$

Hence

$$\mathbb{E}(A^2(t)) \le \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \frac{(1-e^{-\mu t})^2}{\mu^2}. \tag{3.14}$$

By the expression of $B(t)$ and (3.13), we get

$$\begin{aligned} \mathbb{E}(B^2(t)) &= \mathbb{E}\left(\int_0^t e^{-(\mu+\beta)(t-s)}A(s)ds \int_0^t e^{-(\mu+\beta)(t-a)}A(a)da\right) \\ &= \int_0^t e^{-(\mu+\beta)(t-s)} \int_0^t e^{-(\mu+\beta)(t-a)}\mathbb{E}(A(s)A(a))dads \\ &\le \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \frac{1}{\mu^2} \left(\int_0^t e^{-(\mu+\beta)(t-a)}da\right)^2 \\ &= \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \frac{\left(1-e^{-(\mu+\beta)t}\right)^2}{\mu^2(\mu+\beta)^2} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} 0 \le \mathbb{E}(A(t)B(t)) &= \int_0^t e^{-(\mu+\beta)a}\mathbb{E}\left(A(t)A(t-a)\right) da \\ &\le \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \frac{1}{\mu^2} \int_0^t e^{-(\mu+\beta)a}da \\ &= \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right) \frac{1-e^{-(\mu+\beta)t}}{\mu^2(\mu+\beta)}. \end{aligned} \tag{3.16}$$

Thus, from (3.11) and (3.14)–(3.16), we obtain

$$\begin{aligned}\mathbb{E}(F(t))^2 &\le \frac{1}{\beta^2}\mathbb{E}(A^2(t)) + \mathbb{E}(B^2(t))\\ &\le \frac{1}{\beta^2}\left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right)\frac{(1-e^{-\mu t})^2}{\mu^2} + \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right)\frac{\left(1-e^{-(\mu+\beta)t}\right)^2}{\mu^2(\mu+\beta)^2}\\ &\le \left(\mathbb{E}(\eta_0^2) + \frac{\rho^2}{2\gamma}\right)\left(\frac{1}{\mu^2\beta^2} + \frac{1}{\mu^2(\mu+\beta)^2}\right),\end{aligned}$$

which implies $\mathbb{E}(F(t))^2$ is bounded. Hence this lemma is proved. $\square$

Now we want to use the Laplace transform in the framework of Itô–Doob integral [12] to study $u(t,0) - \mathbb{E}u(t,0)$. From Lemma 3.2, we can take the Laplace transform in the sense of Itô–Doob integral on $F(t)$. Here we first give the properties of the Laplace transform $\mathcal{L}(F)$ of $F(t)$ for preparation.

Lemma 3.3. *When* $\mathrm{Re}(\lambda) > 0$*, the Laplace transform* $\mathcal{L}(F)$ *of* $F(t)$ *satisfies*

$$\mathcal{L}(F(t))(\lambda) = \frac{\eta_0 + \rho\lambda\mathcal{L}(W(t))(\lambda)}{\beta(\lambda+\mu+\beta)(\lambda+\gamma)}. \tag{3.17}$$

Then $\mathbb{E}\left(\mathcal{L}(F(t))(\lambda)\right) = 0$ *and*

$$\lim_{\substack{|\lambda|\to+\infty\\ \mathrm{Re}(\lambda)>0}} \mathcal{L}(F(t))(\lambda) = 0 \quad \textit{in probability.}$$

Proof. Taking the Laplace transform in the framework of Itô–Doob integral on both sides of (3.10), when $\mathrm{Re}(\lambda) > 0$, we obtain

$$\mathcal{L}(F(t)) = \frac{1}{\beta}\mathcal{L}(A(t))(\lambda) - \mathcal{L}(B(t))(\lambda) \tag{3.18}$$

and

$$\begin{aligned}\mathcal{L}(A(t))(\lambda) &= \mathcal{L}(e^{-\mu t})(\lambda)\cdot\mathcal{L}(\eta(t))(\lambda) = \frac{\mathcal{L}(\eta(t))(\lambda)}{\lambda+\mu},\\ \mathcal{L}(B(t))(\lambda) &= \mathcal{L}(e^{-(\mu+\beta)t})(\lambda)\cdot\mathcal{L}(A(t))(\lambda) = \frac{\mathcal{L}(A(t))(\lambda)}{\lambda+\mu+\beta}.\end{aligned} \tag{3.19}$$

By Itô's formula, we get

$$d(e^{\gamma s}W(s)) = \gamma e^{\gamma s}W(s)ds + e^{\gamma s}dW(s),$$

which implies that

$$e^{\gamma t}W(t) - 0 = \gamma\int_0^t e^{\gamma s}W(s)ds + \int_0^t e^{\gamma s}dW(s),$$

hence

$$\int_0^t e^{-\gamma(t-s)}dW(s) = W(t) - \gamma \int_0^t e^{-\gamma(t-s)}W(s)ds. \tag{3.20}$$

From (2.4), we have

$$\eta(t) = \eta_0 e^{-\gamma t} + \rho\left(W(t) - \gamma \int_0^t e^{-\gamma(t-s)}W(s)ds\right).$$

Thus when $\mathrm{Re}(\lambda) > 0$,

$$\begin{aligned}\mathcal{L}(\eta(t)) &= \eta_0\mathcal{L}(e^{-\gamma t}) + \rho\left[\mathcal{L}(W(t)) - \gamma\mathcal{L}(e^{-\gamma t})\mathcal{L}(W(t))\right] \\ &= \frac{\eta_0}{\lambda+\gamma} + \rho\left[\mathcal{L}(W(t)) - \frac{\gamma}{\lambda+\gamma}\mathcal{L}(W(t))\right] = \frac{\eta_0 + \rho\lambda\mathcal{L}(W(t))}{\lambda+\gamma}.\end{aligned}$$

Therefore when $\mathrm{Re}(\lambda) > 0$,

$$\mathcal{L}(A(t))(\lambda) = \frac{1}{\lambda+\mu}\cdot\mathcal{L}(\eta(t))(\lambda) = \frac{\eta_0 + \rho\lambda\mathcal{L}(W(t))}{(\lambda+\mu)(\lambda+\gamma)}. \tag{3.21}$$

From (3.18), (3.19) and (3.21), for $\mathrm{Re}(\lambda) > 0$, we obtain

$$\mathcal{L}(F(t))(\lambda) = \frac{\lambda+\mu}{\beta(\lambda+\mu+\beta)}\mathcal{L}(A(t)) = \frac{\eta_0 + \rho\lambda\mathcal{L}(W(t))}{\beta(\lambda+\mu+\beta)(\lambda+\gamma)},$$

thus (3.17) holds. Hence $\mathbb{E}\left(\mathcal{L}(F(t))(\lambda)\right) = 0$ since $\mathbb{E}(\eta_0) = \mathbb{E}\left(\mathcal{L}(W(t))(\lambda)\right) = 0$.

Let $\bar{\lambda}$ be the complex conjugate of λ. Then when $\mathrm{Re}(\lambda) > 0$, we have

$$\begin{aligned}\mathbb{E}\left|\mathcal{L}(W(t))(\lambda)\right|^2 &= \mathbb{E}\left(\int_0^{+\infty} e^{-\lambda t}W(t)dt \int_0^{+\infty} e^{-\bar{\lambda}s}W(s)ds\right) \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda t}e^{-\bar{\lambda}s}\mathbb{E}(W(t)W(s))dsdt \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda t}e^{-\bar{\lambda}s}(t\wedge s)dsdt \\ &= \int_0^{+\infty} e^{-\lambda t}\left(\int_0^t se^{-\bar{\lambda}s}ds + \int_t^{+\infty} te^{-\bar{\lambda}s}ds\right)dt \\ &= \frac{1}{\lambda\bar{\lambda}(\lambda+\bar{\lambda})} = \frac{1}{2\mathrm{Re}(\lambda)|\lambda|^2},\end{aligned}$$

hence when $\mathrm{Re}(\lambda) > 0$,

$$\lim_{|\lambda|\to+\infty} \mathbb{E}\left|\mathcal{L}(W(t))(\lambda)\right|^2 = 0. \tag{3.22}$$

From (3.17) and Chebyshev's inequality, for any $\varepsilon > 0$, we obtain

$$\mathbb{P}\left(\left|\mathcal{L}(F(t))(\lambda)\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\mathbb{E}\left|\mathcal{L}(F(t))(\lambda)\right|^2 = \frac{\mathbb{E}\left|\eta_0 + \rho\lambda\mathcal{L}(W(t))\right|^2}{\varepsilon^2\beta^2|(\lambda+\mu+\beta)(\lambda+\gamma)|^2}$$
$$\leq \frac{2\mathbb{E}(\eta_0)^2 + 2\rho^2|\lambda|^2\mathbb{E}\left|\mathcal{L}(W(t))\right|^2}{\varepsilon^2\beta^2|(\lambda+\mu+\beta)(\lambda+\gamma)|^2},$$

thus, by (3.22), we get

$$\lim_{\substack{|\lambda|\to+\infty \\ \mathrm{Re}(\lambda)>0}} \mathcal{L}(F(t))(\lambda) = 0 \quad \text{in probability.}$$

Therefore this lemma is proved. $\square$

Now from (3.9), Lemmas 3.2–3.3 and [12], for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, taking Laplace transform in framework of Itô–Doob integral on the both sides of Eq. (3.9) yields that

$$\mathcal{L}(u(t,0) - \mathbb{E}u(t,0)) = \mathcal{L}(e^{-(\mu+\beta)t}) \cdot \mathcal{L}(u(t,0) - \mathbb{E}u(t,0)) + \sigma\mathcal{L}(F(t)),$$

which implies that when $\mathcal{L}(e^{-(\mu+\beta)t}) \neq 1$,

$$\mathcal{L}(u(t,0) - \mathbb{E}u(t,0)) = \frac{\sigma\mathcal{L}(F(t))}{1 - \mathcal{L}(e^{-(\mu+\beta)t})}. \tag{3.23}$$

Thus $\mathcal{L}(u(t,0) - \mathbb{E}u(t,0))(\lambda)$ only has poles, which are the roots of the characteristic equation

$$\mathcal{L}(e^{-(\mu+\beta)t})(\lambda) = 1 \tag{3.24}$$

which is the same as the characteristic equations (2.10).

Hence, for the linear age-structured model with a randomly-varying gain/loss term (2.1), the characteristic equations for the first and the second moments are identical with that of the deterministic linear age-structured model and the linear age-structured model with the additive white noise (see model (4.2) in [26]).

Let α_2 be the supreme of the real parts of all characteristic roots of Eq. (3.24), thus $\alpha_2 = 1 - \mu - \beta = \alpha_1$ is the unique and simple real solution of Eq. (3.24).

In the following lemma, using the inverse Laplace transform, we will give the expression of $u(t,0) - \mathbb{E}u(t,0)$ depending on $e^{\alpha_2 t}$.

Lemma 3.4. *Let $\alpha_2 = 1 - \mu - \beta$. The expression of $u(t,0) - \mathbb{E}u(t,0)$ is that*

(1) *when $\alpha_2 \neq -\gamma$,*

$$u(t,0) - \mathbb{E}u(t,0) = \frac{\sigma\eta_0}{\beta(\alpha_2+\gamma)}\left(e^{\alpha_2 t} - e^{-\gamma t}\right) + \frac{\sigma\rho}{\beta(\alpha_2+\gamma)}\int_0^t \left(e^{\alpha_2(t-s)} - e^{-\gamma(t-s)}\right) dW(s); \tag{3.25}$$

(2) *when $\alpha_2 = -\gamma$,*

$$u(t,0) - \mathbb{E}u(t,0) = \frac{\sigma\eta_0}{\beta} te^{-\gamma t} + \frac{\sigma\rho}{\beta}\int_0^t (t-s)e^{-\gamma(t-s)} dW(s). \tag{3.26}$$

Proof. From (3.17) and (3.23), when $\mathrm{Re}(\lambda) > 0$, we have

$$\begin{aligned}\mathcal{L}(u(t,0) - \mathbb{E}u(t,0))(\lambda) &= \frac{\sigma(\lambda+\mu+\beta)}{\lambda+\mu+\beta-1}\cdot\frac{\eta_0+\rho\lambda\mathcal{L}(W(t))}{\beta(\lambda+\mu+\beta)(\lambda+\gamma)}\\ &= \frac{\sigma\eta_0+\sigma\rho\lambda\mathcal{L}(W(t))}{\beta(\lambda-\alpha_2)(\lambda+\gamma)}.\end{aligned}\tag{3.27}$$

(1) When $\alpha_2 \neq -\gamma$, it is easy to see that

$$\frac{1}{(\lambda-\alpha_2)(\lambda+\gamma)} = \frac{1}{\alpha_2+\gamma}\left(\frac{1}{\lambda-\alpha_2}-\frac{1}{\lambda+\gamma}\right)$$

and

$$\frac{\lambda}{(\lambda-\alpha_2)(\lambda+\gamma)} = \frac{1}{\alpha_2+\gamma}\left(\frac{\alpha_2}{\lambda-\alpha_2}+\frac{\gamma}{\lambda+\gamma}\right);$$

moreover, when $\mathrm{Re}(\lambda) > \max\{\alpha_2, -\gamma\}$,

$$\mathcal{L}\left(e^{\alpha_2 t}\right)(\lambda) = \frac{1}{\lambda-\alpha_2}, \quad \mathcal{L}\left(e^{-\gamma t}\right)(\lambda) = \frac{1}{\lambda+\gamma}.$$

Thus, when $\mathrm{Re}(\lambda) > \max\{0, \alpha_2, -\gamma\}$, taking the inverse of the Laplace transform on both sides of (3.27), we obtain

$$\begin{aligned}u(t,0) - \mathbb{E}u(t,0) = {}&\frac{\sigma\eta_0}{\beta(\alpha_2+\gamma)}\left(e^{\alpha_2 t}-e^{-\gamma t}\right)\\ &+\frac{\sigma\rho}{\beta(\alpha_2+\gamma)}\left(\alpha_2\int_0^t e^{\alpha_2(t-s)}W(s)ds+\gamma\int_0^t e^{-\gamma(t-s)}W(s)ds\right).\end{aligned}$$

By (3.20), we have

$$\alpha_2\int_0^t e^{\alpha_2(t-s)}W(s)ds = -W(t)+\int_0^t e^{\alpha_2(t-s)}dW(s).$$

Thus

$$\begin{aligned}u(t,0) - \mathbb{E}u(t,0) &= \frac{\sigma\eta_0}{\beta(\alpha_2+\gamma)}\left(e^{\alpha_2 t}-e^{-\gamma t}\right)+\frac{\sigma\rho}{\beta(\alpha_2+\gamma)}\Bigg(-W(t)\\ &\quad+\int_0^t e^{\alpha_2(t-s)}dW(s)+W(t)-\int_0^t e^{-\gamma(t-s)}dW(s)\Bigg)\\ &= \frac{\sigma\eta_0}{\beta(\alpha_2+\gamma)}\left(e^{\alpha_2 t}-e^{-\gamma t}\right)+\frac{\sigma\rho}{\beta(\alpha_2+\gamma)}\int_0^t\left(e^{\alpha_2(t-s)}-e^{-\gamma(t-s)}\right)dW(s).\end{aligned}$$

(2) When $\alpha_2 = -\gamma$, from (3.27), when $\mathrm{Re}(\lambda) > 0$, we have

$$\mathcal{L}(u(t,0) - \mathbb{E}u(t,0))(\lambda) = \frac{\sigma\eta_0}{\beta(\lambda+\gamma)^2}+\frac{\sigma\rho\lambda}{\beta(\lambda+\gamma)^2}\mathcal{L}(W(t)).\tag{3.28}$$

Obviously,

$$\frac{\lambda}{(\lambda+\gamma)^2}=\frac{1}{\lambda+\gamma}-\frac{\gamma}{(\lambda+\gamma)^2},\quad \mathcal{L}\left(te^{-\gamma t}\right)(\lambda)=\frac{1}{(\lambda+\gamma)^2}\quad (\mathrm{Re}(\lambda)>-\gamma).$$

Thus, when $\mathrm{Re}(\lambda)>\max\{0,-\gamma\}=0$, taking the inverse of the Laplace transform on both sides of (3.28), we obtain

$$\begin{aligned}u(t,0)-\mathbb{E}u(t,0)&=\frac{\sigma\eta_0}{\beta}te^{-\gamma t}+\frac{\sigma\rho}{\beta}\left(\int_0^t e^{-\gamma(t-s)}W(s)ds-\gamma\int_0^t(t-s)e^{-\gamma(t-s)}W(s)ds\right)\\&=\frac{\sigma\eta_0}{\beta}te^{-\gamma t}+\frac{\sigma\rho}{\beta}\left((1-\gamma t)\int_0^t e^{-\gamma(t-s)}W(s)ds+\gamma\int_0^t se^{-\gamma(t-s)}W(s)ds\right).\end{aligned}\tag{3.29}$$

By Itô's formula, we get

$$d(se^{\gamma s}W(s))=e^{\gamma s}W(s)ds+\gamma se^{\gamma s}W(s)ds+se^{\gamma s}dW(s),$$

which implies that

$$te^{\gamma t}W(t)-0=\int_0^t e^{\gamma s}W(s)ds+\gamma\int_0^t se^{\gamma s}W(s)ds+\int_0^t se^{\gamma s}dW(s),$$

that is,

$$\gamma\int_0^t se^{-\gamma(t-s)}W(s)ds=tW(t)-\int_0^t e^{-\gamma(t-s)}W(s)ds-\int_0^t se^{-\gamma(t-s)}dW(s).\tag{3.30}$$

By (3.20), we have

$$\int_0^t e^{-\gamma(t-s)}W(s)ds=\frac{1}{\gamma}\left(W(t)-\int_0^t e^{-\gamma(t-s)}dW(s)\right),\tag{3.31}$$

thus from (3.30), we get

$$\begin{aligned}\gamma\int_0^t se^{-\gamma(t-s)}W(s)ds&=tW(t)+\frac{1}{\gamma}\left(-W(t)+\int_0^t e^{-\gamma(t-s)}dW(s)\right)-\int_0^t se^{-\gamma(t-s)}dW(s)\\&=\left(t-\frac{1}{\gamma}\right)W(t)+\frac{1}{\gamma}\int_0^t e^{-\gamma(t-s)}dW(s)-\int_0^t se^{-\gamma(t-s)}dW(s).\end{aligned}\tag{3.32}$$

From (3.29), (3.31) and (3.32), we obtain

$$u(t,0)-\mathbb{E}u(t,0)=\frac{\sigma\eta_0}{\beta}te^{-\gamma t}+\frac{\sigma\rho}{\beta}\left((1-\gamma t)\int_0^t e^{-\gamma(t-s)}W(s)ds+\gamma\int_0^t se^{-\gamma(t-s)}W(s)ds\right)$$

$$= \frac{\sigma\eta_0}{\beta} t e^{-\gamma t} + \frac{\sigma\rho}{\beta}\Bigg[(1-\gamma t)\frac{1}{\gamma}\left(W(t) - \int_0^t e^{-\gamma(t-s)} dW(s)\right)$$

$$+ \left(t - \frac{1}{\gamma}\right) W(t) + \frac{1}{\gamma}\int_0^t e^{-\gamma(t-s)} dW(s) - \int_0^t s e^{-\gamma(t-s)} dW(s)\Bigg]$$

$$= \frac{\sigma\eta_0}{\beta} t e^{-\gamma t} + \frac{\sigma\rho}{\beta}\int_0^t (t-s) e^{-\gamma(t-s)} dW(s).$$

Hence the lemma is complete. □

Thus, from Lemma 3.4, we have the asymptotic behavior of the second moment $M(t,0)$ for the boundary $u(t,0)$ of model (2.1).

Theorem 3.5. *Let $u(t,0)$ be the boundary of model (2.1) and $\alpha_2 = 1-\mu-\beta$. Then*

(i) *the second moment $M(t,0)$ of the boundary $u(t,0)$ is bounded if $\alpha_2 < 0$, and it approaches to $\frac{\sigma^2\rho^2}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)}$ exponentially as t tends to $+\infty$;*
(ii) *the second moment $M(t,0)$ of the boundary $u(t,0)$ is unbounded if $\alpha_2 > 0$ and $\alpha_2 \neq \gamma$, and it approaches to $+\infty$ exponentially as t tends to $+\infty$.*

Proof. When $\alpha_2 \neq -\gamma$, by (3.25), we obtain

$$M(t,0) = \mathbb{E}\left[u(t,0) - \mathbb{E}u(t,0)\right]^2$$

$$= \mathbb{E}\left[\frac{\sigma\eta_0}{\beta(\alpha_2+\gamma)}\left(e^{\alpha_2 t} - e^{-\gamma t}\right) + \frac{\sigma\rho}{\beta(\alpha_2+\gamma)}\int_0^t \left(e^{\alpha_2(t-s)} - e^{-\gamma(t-s)}\right) dW(s)\right]^2$$

$$= \frac{\sigma^2\mathbb{E}\left(\eta_0^2\right)}{\beta^2(\alpha_2+\gamma)^2}(e^{\alpha_2 t} - e^{-\gamma t})^2 + \frac{\sigma^2\rho^2}{\beta^2(\alpha_2+\gamma)^2}\int_0^t \left(e^{\alpha_2(t-s)} - e^{-\gamma(t-s)}\right)^2 ds \tag{3.33}$$

since η_0 is a random variable independent of Wiener process $W(t)$. When $\alpha_2 \neq \gamma$, we get

$$\int_0^t \left(e^{\alpha_2(t-s)} - e^{-\gamma(t-s)}\right)^2 ds = \int_0^t e^{2\alpha_2(t-s)} ds + \int_0^t e^{-2\gamma(t-s)} ds - 2\int_0^t e^{\alpha_2(t-s)} e^{-\gamma(t-s)} ds$$

$$= \int_0^t e^{2\alpha_2 s} ds + \int_0^t e^{-2\gamma s} ds - 2e^{(\alpha_2-\gamma)t}\int_0^t e^{-(\alpha_2-\gamma)s} ds$$

$$= \frac{e^{2\alpha_2 t} - 1}{2\alpha_2} + \frac{1 - e^{-2\gamma t}}{2\gamma} + \frac{2\left(1 - e^{(\alpha_2-\gamma)t}\right)}{\alpha_2 - \gamma}. \tag{3.34}$$

Hence from (3.33) and (3.34), when $\alpha_2 \neq \pm\gamma$, we have

$$M(t,0) = \frac{\sigma^2\mathbb{E}\left(\eta_0^2\right)}{\beta^2(\alpha_2+\gamma)^2}(e^{\alpha_2 t} - e^{-\gamma t})^2 + \frac{\sigma^2\rho^2}{\beta^2(\alpha_2+\gamma)^2}\left[\frac{e^{2\alpha_2 t}-1}{2\alpha_2} + \frac{1-e^{-2\gamma t}}{2\gamma} + \frac{2\left(1-e^{(\alpha_2-\gamma)t}\right)}{\alpha_2-\gamma}\right]. \tag{3.35}$$

When $\alpha_2 = -\gamma$, by (3.26), we have

$$\begin{aligned} M(t,0) &= \mathbb{E}\left[u(t,0) - \mathbb{E}u(t,0)\right]^2 = \mathbb{E}\left[\frac{\sigma\eta_0}{\beta}te^{-\gamma t} + \frac{\sigma\rho}{\beta}\int_0^t (t-s)e^{-\gamma(t-s)}dW(s)\right]^2 \\ &= \frac{\sigma^2\mathbb{E}(\eta_0^2)}{\beta}t^2e^{-2\gamma t} + \frac{\sigma^2\rho^2}{\beta^2}\int_0^t s^2e^{-2\gamma s}ds \\ &= \frac{\sigma^2\mathbb{E}(\eta_0^2)}{\beta}t^2e^{-2\gamma t} + \frac{\sigma^2\rho^2}{\beta^2}\left[\frac{1}{4\gamma^3} - \left(\frac{t^2}{2\gamma} + \frac{t}{2\gamma^2} + \frac{1}{4\gamma^3}\right)e^{-2\gamma t}\right]. \end{aligned} \tag{3.36}$$

(i) When $\alpha_2 < 0$ and $\alpha_2 \neq -\gamma$, from (3.35), we know that the second moment $M(t,0)$ is bounded and

$$\begin{aligned} \lim_{t\to+\infty} M(t,0) &= \frac{\sigma^2\rho^2}{\beta^2(\alpha_2+\gamma)^2}\left[\frac{-1}{2\alpha_2} + \frac{1}{2\gamma} + \frac{2}{\alpha_2-\gamma}\right] \\ &= \frac{\sigma^2\rho^2}{\beta^2(\alpha_2+\gamma)^2}\frac{(\alpha_2+\gamma)^2}{2\alpha_2\gamma(\alpha_2-\gamma)} = \frac{\sigma^2\rho^2}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)} \end{aligned} \tag{3.37}$$

exponentially since $\alpha_2 < 0$ and $\gamma > 0$.

When $\alpha_2 = -\gamma$, formula (3.36) implies that the second moment $M(t,0)$ is bounded and

$$\lim_{t\to+\infty} M(t,0) = \frac{\sigma^2\rho^2}{4\beta^2\gamma^3} \tag{3.38}$$

exponentially since $\gamma > 0$.

From (3.37) and (3.38), we have

$$\lim_{\alpha_2\to-\gamma}\frac{\sigma^2\rho^2}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)} = \frac{\sigma^2\rho^2}{4\beta^2\gamma^3},$$

thus if we permit $\alpha_2 = -\gamma$ in (3.37), then the right-hand sides of (3.37) and (3.38) are the same. Therefore when $\alpha_2 < 0$, the second moment $M(t,0)$ is bounded and

$$\lim_{t\to+\infty} M(t,0) = \frac{\sigma^2\rho^2}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)}.$$

(ii) If $\alpha_2 > 0$, then $\alpha_2 \neq -\gamma$. Thus when $0 < \alpha_2 \neq \gamma$, from (3.35), we obtain

$$M(t,0) = \frac{\sigma^2\mathbb{E}\left(\eta_0^2\right)}{\beta^2(\alpha_2+\gamma)^2}(e^{\alpha_2 t} - e^{-\gamma t})^2 + \frac{\sigma^2\rho^2}{\beta^2(\alpha_2+\gamma)^2}\times \left(\frac{e^{\alpha_2 t}\left[(\alpha_2-\gamma)e^{\alpha_2 t} - 4\alpha_2 e^{-\gamma t}\right]}{2\alpha_2(\alpha_2-\gamma)} + \frac{3\alpha_2+\gamma}{2\alpha_2(\alpha_2-\gamma)} + \frac{1-e^{-2\gamma t}}{2\gamma}\right),$$

thus

$$\lim_{t\to+\infty} M(t,0) = +\infty$$

exponentially and the second moment $M(t,0)$ is unbounded. Hence the theorem is completed. □

3.3. The second moment $M(t,a)$ when $t > a \geq 0$

From now on, we will study the boundedness and asymptotic behavior of the second moment $M(t,a)$ of the solution $u(t,a)$ of model (2.1) when $t > a$.

From Theorem 3.1, we know that when $t \leq a$, the second moment $M(t,a)$ of model (2.1) is bounded for the two cases $\mu \neq \gamma$ and $\mu = \gamma$. Here for convenience of the computation, we assume that $\mu \neq \gamma$ in the following. Thus we have the following results on the boundedness and the asymptotic behavior for the second moment $M(t,a)$ when $t > a \geq 0$ and the stochastically ultimate boundedness for the model (2.1).

Theorem 3.6. *Let $u(t,a)$ be the solution of model (2.1) and $\alpha_2 = 1 - \mu - \beta$. Assume that $\mu \neq \gamma$. Then for any fixed $a \geq 0$, when $t > a$,*

(1) $\lim_{a\to 0} M(t,a) = M(t,0)$;
(2) *if $\alpha_2 < 0$, then the second moment $M(t,a)$ is bounded and there exist two nonnegative functions $C_1(a), C_2(a)$ such that*

$$\lim_{t\to+\infty} M(t,a) = \begin{cases} C_1(a), & \alpha_2 \neq -\gamma, \\ C_2(a), & \alpha_2 = -\gamma \end{cases}$$

exponentially and

$$\lim_{a\to 0} C_1(a) = \lim_{a\to 0} C_2(a) = \lim_{t\to+\infty} M(t,0);$$

moreover, the solution $u(t,a)$ of model (2.1) is stochastically ultimate bounded.
(3) *if $\alpha_2 > 0$ and $\alpha_2 \neq \gamma, \mu$, then the second moment $M(t,a)$ is unbounded and it approaches to $+\infty$ exponentially for any fixed $a \geq 0$ as t tends to $+\infty$.*

Proof. Denote that

$$C(t,a) \triangleq \mathbb{E}\left(\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right)^2$$

and

$$D(t,a) \triangleq \mathbb{E}\left(\left[u(t-a,0) - \mathbb{E}u(t-a,0)\right]\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right),$$

where $t > a \geq 0$. Thus from (3.3), when $t > a \geq 0$, we know that the second moment $M(t,a)$ of model (2.1) is

$$M(t,a) = e^{-2\mu a}M(t-a,0) + \sigma^2 C(t,a) + 2\sigma e^{-\mu a}D(t,a),$$

where $M(t,0) \triangleq \mathbb{E}\left[u(t,0) - \mathbb{E}u(t,0)\right]^2$ (see (3.35) and (3.36)).

Hence we first calculate the two formulas $C(t,a)$ and $D(t,a)$. In the following proofs, we always assume that $\mu \neq \gamma$. We need 3 steps to prove the results (1)–(3).

Step 1: computation of $C(t,a)$. By (2.5), we get

$$C(t,a) = \int_{t-a}^{t} e^{-\mu(t-s)} \int_{t-a}^{t} e^{-\mu(t-l)} \mathbb{E}\left(\eta(s)\eta(l)\right) dl ds$$
$$= \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma}\right) \left(\int_{t-a}^{t} e^{-\mu(t-s)} e^{-\gamma s} ds\right)^2$$
$$+ \frac{\rho^2}{2\gamma} \int_{t-a}^{t} e^{-\mu(t-s)} \int_{t-a}^{t} e^{-\mu(t-l)} e^{-\gamma|s-l|} dl ds. \tag{3.39}$$

When $t > a$ and $\mu \neq \gamma$, it is easy to see that

$$\left(\int_{t-a}^{t} e^{-\mu(t-s)} e^{-\gamma s} ds\right)^2 = e^{-2\mu t} \left(\int_{t-a}^{t} e^{(\mu-\gamma)s} ds\right)^2 = \frac{e^{-2\gamma t}\left(1 - e^{(\gamma-\mu)a}\right)^2}{(\mu-\gamma)^2} \tag{3.40}$$

and

$$\int_{t-a}^{t} e^{-\mu(t-l)} e^{-\gamma|s-l|} dl = \int_{t-a}^{s} e^{-\mu(t-l)} e^{-\gamma(s-l)} dl + \int_{s}^{t} e^{-\mu(t-l)} e^{-\gamma(l-s)} dl$$
$$= e^{-\mu t} e^{-\gamma s} \int_{t-a}^{s} e^{(\mu+\gamma)l} dl + e^{-\mu t} e^{\gamma s} \int_{s}^{t} e^{(\mu-\gamma)l} dl$$
$$= \frac{2\gamma e^{-\mu(t-s)}}{\gamma^2 - \mu^2} - \frac{e^{\gamma(t-s)} e^{-(\mu+\gamma)a}}{\mu+\gamma} + \frac{e^{-\gamma(t-s)}}{\mu-\gamma}.$$

Hence when $t > a$, we have

$$\int_{t-a}^{t} e^{-\mu(t-s)} \int_{t-a}^{t} e^{-\mu(t-l)} e^{-\gamma|s-l|} dl ds$$
$$= \int_{t-a}^{t} e^{-\mu(t-s)} \left(\frac{2\gamma e^{-\mu(t-s)}}{\gamma^2 - \mu^2} - \frac{e^{\gamma(t-s)} e^{-(\mu+\gamma)a}}{\mu+\gamma} + \frac{e^{-\gamma(t-s)}}{\mu-\gamma}\right) ds$$
$$= \frac{1}{\mu(\mu+\gamma)} + \frac{e^{-2\mu a}}{\mu(\mu-\gamma)} - \frac{2e^{-(\mu+\gamma)a}}{\mu^2 - \gamma^2}. \tag{3.41}$$

Thus by (3.39)–(3.41), when $t > a$, we obtain

$$C(t,a) = \left(\mathbb{E}(\eta_0^2) - \frac{\rho^2}{2\gamma}\right) \frac{e^{-2\gamma t}\left(1 - e^{(\gamma-\mu)a}\right)^2}{(\mu-\gamma)^2}$$
$$+ \frac{\rho^2}{2\gamma} \left(\frac{1}{\mu(\mu+\gamma)} + \frac{e^{-2\mu a}}{\mu(\mu-\gamma)} - \frac{2e^{-(\mu+\gamma)a}}{\mu^2 - \gamma^2}\right). \tag{3.42}$$

Step 2: calculation of $D(t,a)$. When $\alpha_2 \neq -\gamma$ and $t > a$, from (3.25),

$$D(t,a) = \frac{\sigma\left(e^{\alpha_2 t} - e^{-\gamma t}\right)}{\beta(\alpha_2+\gamma)}\mathbb{E}\left(\eta_0 \int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right) + \frac{\sigma\rho}{\beta(\alpha_2+\gamma)} \times$$
$$\mathbb{E}\Big[\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds \int_0^t \left(e^{\alpha_2(t-s)} - e^{-\gamma(t-s)}\right) dW(s)\Big]. \tag{3.43}$$

By (2.4), we obtain

$$\mathbb{E}\left(\eta_0 \int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right) = \mathbb{E}(\eta_0^2)e^{-\mu t}\int_{t-a}^{t} e^{(\mu-\gamma)s}ds = \frac{\mathbb{E}(\eta_0^2)e^{-\gamma t}(1-e^{-(\mu-\gamma)a})}{\mu-\gamma} \tag{3.44}$$

and when $\alpha_2 \neq \pm\gamma$ and $\alpha_2 \neq \mu$,

$$\begin{aligned}
&\mathbb{E}\left(\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds \int_0^t e^{\alpha_2(t-s)}dW(s)\right)\\
&= \mathbb{E}\left(\int_{t-a}^{t} e^{-\mu(t-s)}\left[\eta_0 e^{-\gamma s} + \rho\int_0^s e^{-\gamma(s-l)}dW(l))dl\right] ds \int_0^t e^{\alpha_2(t-p)}dW(p)\right)\\
&= \rho\int_{t-a}^{t} e^{-\mu(t-s)}\mathbb{E}\left(\int_0^s e^{-\gamma(s-l)}dW(l))dl \int_0^t e^{\alpha_2(t-p)}dW(p)\right)ds\\
&= \rho\int_{t-a}^{t} e^{-\mu(t-s)}\frac{e^{\alpha_2 t}(e^{-\alpha_2 s} - e^{-\gamma s})}{\gamma-\alpha_2}\\
&= \frac{\rho(1-e^{-(\mu-\alpha_2)a})}{(\gamma-\alpha_2)(\mu-\alpha_2)} - \frac{\rho e^{(\alpha_2-\gamma)t}(1-e^{-(\mu-\gamma)a})}{(\gamma-\alpha_2)(\mu-\gamma)}.
\end{aligned} \tag{3.45}$$

Similarly, when $\mu \neq \gamma$, we get

$$\mathbb{E}\left(\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds \int_0^t e^{-\gamma(t-s)}dW(s)\right) = \frac{\rho(1-e^{-(\mu+\gamma)a})}{2\gamma(\mu+\gamma)} - \frac{\rho e^{-2\gamma t}(1-e^{-(\mu-\gamma)a})}{2\gamma(\mu-\gamma)}. \tag{3.46}$$

Hence from (3.43)–(3.46), when $\alpha_2 \neq \pm\gamma$ and $\alpha_2 \neq \mu$, we obtain

$$\begin{aligned}
D(t,a) &= \frac{\sigma\mathbb{E}(\eta_0^2)e^{-\gamma t}(1-e^{-(\mu-\gamma)a})\left(e^{\alpha_2 t} - e^{-\gamma t}\right)}{\beta(\alpha_2+\gamma)(\mu-\gamma)}\\
&\quad+\frac{\sigma\rho^2}{\beta(\alpha_2+\gamma)}\Big[\frac{1-e^{-(\mu-\alpha_2)a}}{(\gamma-\alpha_2)(\mu-\alpha_2)} - \frac{e^{(\alpha_2-\gamma)t}(1-e^{-(\mu-\gamma)a})}{(\gamma-\alpha_2)(\mu-\gamma)}\\
&\quad-\frac{1-e^{-(\mu+\gamma)a}}{2\gamma(\mu+\gamma)} + \frac{e^{-2\gamma t}(1-e^{-(\mu-\gamma)a})}{2\gamma(\mu-\gamma)}\Big].
\end{aligned} \tag{3.47}$$

When $\alpha_2 = -\gamma$ and $\mu \neq \gamma$, from (3.26), we get

$$D(t,a) = \frac{\sigma t e^{-\gamma t}}{\beta}\mathbb{E}\left(\eta_0 \int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds\right) + \frac{\sigma\rho}{\beta}\mathbb{E}\Big[\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds \int_0^t (t-s)e^{-\gamma(t-s)}dW(s)\Big] \tag{3.48}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\int_{t-a}^{t} e^{-\mu(t-s)}\eta(s)ds \int_0^t (t-s)e^{-\gamma(t-s)}dW(s)\right) \\
&= \rho e^{-(\mu+\gamma)t}\int_{t-a}^{t} e^{\mu s}e^{-\gamma s}\mathbb{E}\left(\int_0^s e^{\gamma l}dW(l)\int_0^t (t-p)e^{\gamma p}dW(p)\right)ds \\
&= \rho e^{-(\mu+\gamma)t}\int_{t-a}^{t} e^{(\mu-\gamma)s}\int_0^s (t-l)e^{2\gamma l}dlds \\
&= \rho e^{-(\mu+\gamma)t}\int_{t-a}^{t} e^{(\mu-\gamma)s}\left(\frac{(2\gamma t+1-2\gamma s)e^{2\gamma s}}{4\gamma^2} - \frac{2\gamma t+1}{4\gamma^2}\right)ds \\
&= \frac{\rho(2\gamma t+1)(1-e^{-(\mu+\gamma)a})}{4\gamma^2(\mu+\gamma)} - \frac{\rho[t-(t-a)e^{-(\mu+\gamma)a}]}{2\gamma(\mu+\gamma)} \\
&\quad + \frac{\rho(1-e^{-(\mu+\gamma)a})}{2\gamma(\mu+\gamma)^2} - \frac{\rho(2\gamma t+1)e^{-2\gamma t}(1-e^{-(\mu-\gamma)a})}{4\gamma^2(\mu-\gamma)} \\
&= \frac{\rho[1-(1+2\gamma a)e^{-(\mu+\gamma)a}]}{4\gamma^2(\mu+\gamma)} + \frac{\rho(1-e^{-(\mu+\gamma)a})}{2\gamma(\mu+\gamma)^2} - \frac{\rho(2\gamma t+1)e^{-2\gamma t}(1-e^{-(\mu-\gamma)a})}{4\gamma^2(\mu-\gamma)}.
\end{aligned} \tag{3.49}$$

Thus from (3.44), (3.48) and (3.49) when $\alpha_2 = -\gamma$, we obtain

$$\begin{aligned}
D(t,a) = {} & \frac{\sigma\mathbb{E}(\eta_0^2)te^{-2\gamma t}(1-e^{-(\mu-\gamma)a})}{\beta(\mu-\gamma)} + \frac{\sigma\rho^2}{\beta}\Big[\frac{1-(1+2\gamma a)e^{-(\mu+\gamma)a}}{4\gamma^2(\mu+\gamma)} \\
& + \frac{1-e^{-(\mu+\gamma)a}}{2\gamma(\mu+\gamma)^2} - \frac{(2\gamma t+1)e^{-2\gamma t}(1-e^{-(\mu-\gamma)a})}{4\gamma^2(\mu-\gamma)}\Big].
\end{aligned} \tag{3.50}$$

Step 3: the proofs of the results (1)–(3).

(1) From the expressions of $C(t,a)$ and $D(t,a)$, when $t > a$, we have

$$\lim_{a\to 0} C(t,a) = \lim_{a\to 0} D(t,a) = 0,$$

thus

$$\lim_{a\to 0} M(t,a) = \lim_{a\to 0}\left[e^{-2\mu a}M(t-a,0) + \sigma^2 C(t,a) + 2\sigma e^{-\mu a}D(t,a)\right] = M(t,0).$$

Note that we can also obtain the result $\lim_{a\to 0} M(t,a) = M(t,0)$ from (3.42), (3.47) and (3.50).

(2) If $\alpha_2 < 0$, then $\alpha_2 \neq \mu$, when $\alpha_2 \neq -\gamma$ and $t > a$, by (3.42), (3.47) and Theorem 3.5 (i), we get

$$\begin{aligned}\lim_{t\to+\infty} M(t,a) &= \lim_{t\to+\infty}\left[e^{-2\mu a}M(t-a,0)+\sigma^2 C(t,a)+2\sigma e^{-\mu a}D(t,a)\right]\\ &= \frac{\sigma^2\rho^2 e^{-2\mu a}}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)}+\frac{\sigma^2\rho^2}{2\gamma}\left(\frac{1}{\mu(\mu+\gamma)}+\frac{e^{-2\mu a}}{\mu(\mu-\gamma)}-\frac{2e^{-(\mu+\gamma)a}}{\mu^2-\gamma^2}\right)\\ &\quad+\frac{2\sigma^2\rho^2 e^{-\mu a}}{\beta(\alpha_2+\gamma)}\left[\frac{1-e^{-(\mu-\alpha_2)a}}{(\gamma-\alpha_2)(\mu-\alpha_2)}-\frac{1-e^{-(\mu+\gamma)a}}{2\gamma(\mu+\gamma)}\right]\triangleq C_1(a)\end{aligned}$$

exponentially for any fixed $a \geq 0$ since $\gamma > 0$, which implies that the second moment $M(t,a)$ is bounded and

$$\lim_{a\to0}\lim_{t\to+\infty} M(t,a) = \lim_{a\to0} C_1(a) = \frac{\sigma^2\rho^2}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)} = \lim_{t\to+\infty} M(t,0).$$

When $\alpha_2 = -\gamma$ and $\mu \neq \gamma$, by Theorem 3.5 (i), (3.42) and (3.50), we obtain

$$\begin{aligned}\lim_{t\to+\infty} M(t,a) &= \frac{\sigma^2\rho^2 e^{-2\mu a}}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)}+\frac{\sigma^2\rho^2}{2\gamma}\left(\frac{1}{\mu(\mu+\gamma)}+\frac{e^{-2\mu a}}{\mu(\mu-\gamma)}-\frac{2e^{-(\mu+\gamma)a}}{\mu^2-\gamma^2}\right)\\ &\quad+\frac{2\sigma^2\rho^2 e^{-\mu a}}{\beta}\left[\frac{1-(1+2\gamma a)e^{-(\mu+\gamma)a}}{4\gamma^2(\mu+\gamma)}+\frac{1-e^{-(\mu+\gamma)a}}{2\gamma(\mu+\gamma)^2}\right]\triangleq C_2(a)\end{aligned}$$

exponentially for any fixed $a \geq 0$ since $\gamma > 0$, which indicates that the second moment $M(t,a)$ is bounded and

$$\lim_{a\to0}\lim_{t\to+\infty} M(t,a) = \lim_{a\to0} C_2(a) = \frac{\sigma^2\rho^2}{2\alpha_2\beta^2\gamma(\alpha_2-\gamma)} = \lim_{t\to+\infty} M(t,0).$$

By Chebyshev's inequality, for any $\delta > 0$, we have

$$\mathbb{P}\left(\left|u(t,a)-\mathbb{E}(u(t,a))\right|\geq\delta\right)\leq\frac{Var(u(t,a))}{\delta^2}=\frac{M(t,a))}{\delta^2}.$$

When $\alpha_1 = \alpha_2 < 0$, for any fixed $a \geq 0$, $\lim\limits_{t\to+\infty}\mathbb{E}(u(t,a)) = 0$, and then for any $\varepsilon \in (0,1)$, there is a positive constant $\delta = \sqrt{\frac{\max\{C_1(a),C_2(a)\}}{\varepsilon}}$ such that for any initial $u_0 \in L^1([0,+\infty),[0,+\infty))$, the solution $u(t,a)$ of model (2.1) satisfies

$$\lim_{t\to+\infty}\mathbb{P}\left(\left|u(t,a)\right|\geq\delta\right)=\lim_{t\to+\infty}\mathbb{P}\left(\left|u(t,a)-\mathbb{E}(u(t,a))\right|\geq\delta\right)\leq\lim_{t\to+\infty}\frac{M(t,a)}{\delta^2}\leq\varepsilon,$$

that is

$$\lim_{t\to+\infty}\mathbb{P}\left(\left|u(t,a)\right|\leq\delta\right)\geq 1-\varepsilon,$$

which implies that the solution $u(t,a)$ of model (2.1) is stochastically ultimate bounded.

(3) If $\alpha_2 > 0$ and $\alpha_2 \neq \gamma,\mu$, from Theorem 3.5 (ii), (3.42) and (3.47), we have

$$\begin{aligned}&\lim_{t\to+\infty} M(t,a)\\ &= +\infty+\frac{\sigma^2\rho^2}{2\gamma}\left(\frac{1}{\mu(\mu+\gamma)}+\frac{e^{-2\mu a}}{\mu(\mu-\gamma)}-\frac{2e^{-(\mu+\gamma)a}}{\mu^2-\gamma^2}\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\sigma^2\rho^2 e^{-\mu a}}{\beta(\alpha_2+\gamma)}\left[\frac{1-e^{-(\mu-\alpha_2)a}}{(\gamma-\alpha_2)(\mu-\alpha_2)}-\frac{1-e^{-(\mu+\gamma)a}}{2\gamma(\mu+\gamma)}\right]\\
&+\lim_{t\to+\infty}\frac{2\sigma^2\mathbb{E}(\eta_0^2)e^{-\mu a}e^{(\alpha_2-\gamma)t}(1-e^{-(\mu-\gamma)a})}{\beta(\alpha_2+\gamma)(\mu-\gamma)}-\frac{2\sigma^2\rho^2 e^{-\mu a}}{\beta(\alpha_2+\gamma)}\lim_{t\to+\infty}\frac{e^{(\alpha_2-\gamma)t}(1-e^{-(\mu-\gamma)a})}{(\gamma-\alpha_2)(\mu-\gamma)}\\
=&+\infty+\frac{\sigma^2\rho^2}{2\gamma}\left(\frac{1}{\mu(\mu+\gamma)}+\frac{e^{-2\mu a}}{\mu(\mu-\gamma)}-\frac{2e^{-(\mu+\gamma)a}}{\mu^2-\gamma^2}\right)\\
&+\frac{2\sigma^2\rho^2 e^{-\mu a}}{\beta(\alpha_2+\gamma)}\left[\frac{1-e^{-(\mu-\alpha_2)a}}{(\gamma-\alpha_2)(\mu-\alpha_2)}-\frac{1-e^{-(\mu+\gamma)a}}{2\gamma(\mu+\gamma)}\right]\\
&+\frac{\sigma^2}{\beta(\alpha_2+\gamma)}\lim_{t\to+\infty}e^{(\alpha_2-\gamma)t}\left[\frac{2\mathbb{E}(\eta_0^2)e^{-\mu a}(1-e^{-(\mu-\gamma)a})}{\mu-\gamma}+\frac{2\rho^2e^{-\mu a}(1-e^{-(\mu-\gamma)a})}{(\alpha_2-\gamma)(\mu-\gamma)}\right]\\
=&+\infty
\end{aligned}
$$

exponentially for any fixed $a \geq 0$ since the last limit term is 0 when $0 < \alpha_2 < \gamma$ or $+\infty$ when $\alpha_2 > \gamma$, thus the second moment $M(t, a)$ is unbounded. Therefore the proof is proved. □

Effects of environmental fluctuations are measured in terms of the second moments, which indicates the dynamical behaviors of stochastic model systems along with the fluctuations of population distribution around their mean values. From Theorem 3.6, for any fixed $a \geq 0$, when $t > a$ and $\alpha_2 < 0$, the second moment $M(t, a)$ is bounded, thus we know that the solution $u(t, a)$ of model (2.1) is stochastically ultimate bounded, which is important for stochastic population model. The finite second moment indicates that the environmental fluctuation plays a significant role in the dynamical behavior exhibited by the linear age-structured model with randomly varying immigration/emigration or harvesting/propagation of the population (i.e., the additive colored noise).

For the results in Sections 2 and 3, we know that the dynamics have not been changed drastically by the randomly varying immigration/emigration or harvesting/propagation of the population when the deterministic age-structured model is asymptotically stable. Specially, from Section 2, we know that when $\alpha_1 = \alpha_2 = 1 - \mu - \beta < 0$, the first moment $\mathbb{E}u(t, a)$ approaches to 0 as t tends to $+\infty$ for any $a \geq 0$. By Theorem 3.6, when $t > a \geq 0$ and $\alpha_2 = 1 - \mu - \beta < 0$, the second moment $M(t, a)$ is bounded. However, $M(t, a)$ is non-zero as t tends to $+\infty$ for any $a \geq 0$, which indicates that it is the limitation of the additive colored noise in a population model.

When the extraneous gain or loss term $S(t, a) = [\sigma_0 + \sigma_2\eta(t)]u(t, a)$ in model (1.2) (i.e., the age-dependent population model is perturbed by the state-dependent colored noise), where $\eta(t)$ is the Ornstein–Uhlenbeck process, the sufficient conditions for the boundedness of the first and second moments and the dependence of the moments on the state-dependent colored noise are not clear now. This question will be considered in the future.

Acknowledgment

We would like to thank the referee for his/her valuable comments and suggestions that greatly improve the presentation of this work.

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