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The Brezis–Nirenberg problem for the fractional Laplacian with mixed Dirichlet–Neumann boundary conditions

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ABSTRACT

In this work we study the existence of solutions to the critical Brezis–Nirenberg problem when one deals with the spectral fractional Laplace operator and mixed Dirichlet–Neumann boundary conditions, i.e.,

$$\begin{cases} (-\Delta)^s u = \lambda u + u^{2_s^*-1}, & u > 0 \quad \text{in} \quad \Omega, \\ \qquad u = 0 & \text{on} \quad \Sigma_{\mathcal{D}}, \\ \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Sigma_{\mathcal{N}}, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $\frac{1}{2} < s < 1$, 2_s^* is the critical fractional Sobolev exponent, $0 \leq \lambda \in \mathbb{R}$, ν is the outwards normal to $\partial\Omega$, $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega$, $\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset$, and $\Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma$ is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$.

1. Introduction

For the last decades Dirichlet and Neumann boundary problems associated with elliptic equations as

$$-\Delta u = f(x, u) \tag{1.1}$$

have been widely investigated with different nonlinearities $f(x, u)$. In contrast, mixed Dirichlet–Neumann boundary problems have been much less investigated. Nevertheless, some important results dealing with mixed Dirichlet–Neumann boundary problems associated with (1.1) have been proved over the years. See [1,2,14,15,17–19,23,26,30] among others.

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Problems associated with (1.1), substituting the operator by the fractional Laplacian, have been extensively investigated in the last years, with Dirichlet or Neumann boundary conditions (cf., e.g., [5,6,9,13,11,20,21,27,29,31], among others), but these fractional elliptic problems, once again, have not been so much investigated with mixed Dirichlet–Neumann boundary data, cf. [7,16,21]. Indeed, up to our knowledge, there are no references for mixed Dirichlet–Neumann boundary problems involving the spectral fractional Laplacian operator, which is the one we deal with here. Precisely, we study the Brezis–Nirenberg problem, cf. [10], with the spectral fractional Laplacian operator associated with mixed Dirichlet–Neumann boundary data. A turning point in the history of elliptic boundary problems associated with (1.1) was the seminal paper by Brezis and Nirenberg [10], where the critical power problem for the classical Laplacian with a lower-order perturbation term and a Dirichlet boundary condition was studied. For the pure critical problem it is well known that there is no positive solution when the domain is star-shaped due to a Pohozaev identity, cf. [28]. Nevertheless, Brezis and Nirenberg proved, among other results, that there exists a positive solution when the perturbation is linear, analyzing more carefully the case when the domain is a ball. Since then, there have arisen more than one thousand papers citing [10]. In the fractional setting, Brezis–Nirenberg problems have been also widely investigated. For brevity we just cite some related works dealing only with the fractional Laplacian, cf., e.g., [5,31] for the spectral fractional Laplacian defined in (2.1), and [27,29] for the fractional Laplacian defined by a singular integral in (2.8); both with Dirichlet boundary condition. As we said above, there are no references dealing with problems involving the spectral fractional Laplacian and mixed Dirichlet–Neumann boundary conditions. As a consequence, the main goal of this manuscript is twofold: one is to address for the very first time problems involving spectral fractional Laplacian together with mixed Dirichlet–Neumann boundary conditions, and second to prove existence of a positive solution for the Brezis–Nirenberg problem in this fractional setting with mixed boundary conditions.

The precise problem we study in this work is the following,

$$\begin{cases} (-\Delta)^s u = \lambda u + u^{2_s^*-1} & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega = \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}}, \end{cases} \tag{P_λ}$$

where $\frac{1}{2} < s < 1$, Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 2s$, and mixed Dirichlet–Neumann boundary conditions of the form

$$B(u) = u\chi_{\Sigma_{\mathcal{D}}} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_{\mathcal{N}}}, \tag{1.2}$$

where ν is the outwards normal to $\partial\Omega$, χ_A stands for the characteristic function of a set A, $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_{\mathcal{D}}$ is a closed submanifold of $\partial\Omega$, $\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}}) = \alpha > 0$ where $\mathcal{H}_{N-1}$ is the $(N-1)$-dimensional Hausdorff measure, $\Sigma_{\mathcal{D}} \cap \Sigma_{\mathcal{N}} = \emptyset$, $\Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}} = \partial\Omega$ and $\Sigma_{\mathcal{D}} \cap \overline{\Sigma}_{\mathcal{N}} = \Gamma$ is a smooth $(N-2)$-dimensional submanifold.

For the Dirichlet case ($\mathcal{H}_{N-1}(\Sigma_{\mathcal{N}}) = 0$) it can be seen ([9]) that using a generalized Pohozaev identity, problem (P_λ) has no solution for $\lambda = 0$ and Ω a star-shaped domain. As we will see, in the mixed boundary data case the situation is different.

The classical Pohozaev's identity was extended to the mixed Dirichlet–Neumann boundary data case, involving the classical Laplace operator by Lions–Pacella–Tricarico [25]. Following that ideas, we extend that result to our mixed fractional setting. Precisely, as in [2,14], we will show that taking the mixed Dirichlet–Neumann boundary conditions, in an appropriate way, problem (P_λ) has a solution when $\lambda = 0$, in contrast to the Dirichlet case. Thus, we can include the value $\lambda = 0$ in the existence results. The main result proved in this paper is the following.

Theorem 1.1. *Assume that $\frac{1}{2} < s < 1$ and $N \geq 4s$. Let $\lambda_{1,s}$ be the first eigenvalue of the fractional operator $(-\Delta)^s$ with mixed Dirichlet–Neumann boundary conditions* (1.2)*. Then problem* (P_λ)

(1) *has no solution for* $\lambda \geq \lambda_{1,s}$,
(2) *has at least one solution for* $0 < \lambda < \lambda_{1,s}$,
(3) *has at least one solution for* $\lambda = 0$ *and* $\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}})$ *small enough.*

Note that the range $\frac{1}{2} < s < 1$ is natural for mixed boundary problems in our fractional setting, see Remark 2.2.

Organization of the paper. This manuscript has four more sections. In Section 2 we establish the appropriate functional setting for the study of problem (P_λ), including the definition of an auxiliary problem introduced by Caffarelli and Silvestre, [13], that will help us to overcome some difficulties that appear when we deal with the fractional operator. Following the ideas of [23] and [2], we introduce two constants $\widetilde{S}(\Sigma_{\mathcal{N}})$ and $\widetilde{S}(\Sigma_{\mathcal{D}})$ respectively, that play a similar role to that of the Sobolev constant in the celebrated paper of Brezis and Nirenberg, [10]. In Section 3 we study some useful properties of that constants. Section 4 is devoted to prove Theorem 1.1 and it is divided into two subsections. In Subsection 4.1 we prove the statements (1)–(2) in Theorem 1.1. In Subsection 4.2, we use the constant $\widetilde{S}(\Sigma_{\mathcal{D}})$ to study the existence of solution to problem (P_λ) when we move the boundary conditions in an appropriate way to be specified. These results allow us to prove statement (3) in Theorem 1.1. Finally, in the last section we prove a non-existence result by means of a Pohozaev-type identity.

2. Functional setting and definitions

The definition of the fractional powers of the positive Laplace operator $(-\Delta)$, in a bounded domain Ω with homogeneous mixed Dirichlet–Neumann boundary data, is carried out via the spectral decomposition using the powers of the eigenvalues of $(-\Delta)$ with the same boundary condition. Let (φ_i, λ_i) be the eigenfunctions (normalized with respect to the $L^2(\Omega)$-norm) and eigenvalues of $(-\Delta)$ with homogeneous mixed Dirichlet–Neumann boundary data, then (φ_i, λ_i^s) are the eigenfunctions and eigenvalues of $(-\Delta)^s$ with the same boundary conditions. Thus the fractional operator $(-\Delta)^s$ is well defined in the space of functions that vanish on $\Sigma_{\mathcal{D}}$,

$$H^s_{\Sigma_{\mathcal{D}}}(\Omega) = \left\{ u = \sum_{j\geq 1} a_j \varphi_j \in L^2(\Omega) : \ \|u\|^2_{H_{\Sigma^s_{\mathcal{D}}}(\Omega)} = \sum_{j\geq 1} a_j^2 \lambda_j^s < \infty \right\}.$$

As a direct consequence of the previous definition we get

$$(-\Delta)^s u = \sum_{j\geq 1} a_j \lambda_j^s \varphi_j, \tag{2.1}$$

as well as

$$\|u\|_{H_{\Sigma^s_{\mathcal{D}}}(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}. \tag{2.2}$$

This definition of the fractional powers of the Laplace operator allows us to integrate by parts in the appropriate spaces. A natural definition of energy solution to problem (P_λ) is the following.

Definition 2.1. We say that $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ is a solution of (P_λ) if

$$\int_\Omega (-\Delta)^{s/2} u (-\Delta)^{s/2} \psi dx = \int_\Omega \left(\lambda u + u^{2_s^*-1}\right) \psi dx, \quad \text{for all } \psi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{2.3}$$

The right-hand side of (2.3) is well defined because of the embedding $H^s_{\Sigma_{\mathcal{D}}}(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$ while $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ so $\lambda u + u^{2^*_s-1} \in L^{\frac{2N}{N+2s}} \hookrightarrow \left(H^s_{\Sigma_{\mathcal{D}}}(\Omega)\right)'$. The energy functional associated with problem (P_λ) is

$$I(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{s/2}u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{N-2s}{2N}\int_\Omega u^{\frac{2N}{N-2s}} dx. \tag{2.4}$$

This functional is well defined in $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and critical points of I, defined by (2.4), correspond to solutions of (P_λ).

Remark 2.2. As it was proved in [8], for the range $0 < s \le \frac{1}{2}$, $H^s_0(\Omega) = H^s(\Omega)$, and for $\frac{1}{2} < s < 1$, $H^s_0(\Omega) \subsetneq H^s(\Omega)$. As a consequence, $H^s_{\Sigma_{\mathcal{D}}}(\Omega) = H^s(\Omega)$ for $0 < s \le \frac{1}{2}$. This is the reason why we work here with the fraction $\frac{1}{2} < s < 1$, in which $H^s_{\Sigma_{\mathcal{D}}}(\Omega) \subsetneq H^s(\Omega)$.

In order to overcome some difficulties that appear along several proofs in the paper we use the ideas of Caffarelli and Silvestre, [13], together with those of [9] to give an equivalent definition of the operator $(-\Delta)^s$ defined in a bounded domain by means of an auxiliary problem. Associated with the domain Ω, we consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. We denote with (x,y) points that belong to $\mathcal{C}_\Omega$ and with $\partial_L\Omega = \partial\Omega \times (0,\infty)$ the lateral boundary of the extension cylinder. Given a function $u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega)$, we define its s-extension $w = E_s[u]$ to the cylinder $\mathcal{C}_\Omega$ as the solution of the problem

$$\begin{cases} -div(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ B^*(w) = 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ w(x,0) = u(x) & \text{in } \Omega \times \{y=0\}, \end{cases} \tag{2.5}$$

where

$$B^*(w) = w\chi_{\Sigma^*_{\mathcal{D}}} + \frac{\partial w}{\partial \nu}\chi_{\Sigma^*_{\mathcal{N}}},$$

with $\Sigma^*_{\mathcal{D}} = \Sigma_{\mathcal{D}} \times (0,\infty)$ and $\Sigma^*_{\mathcal{N}} = \Sigma_{\mathcal{N}} \times (0,\infty)$. The extension function belongs to the space

$$\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega) = \overline{\mathcal{C}^\infty_0\left((\Omega \cup \Sigma_{\mathcal{N}}) \times [0,\infty)\right)}^{\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}},$$

equipped with the norm,

$$\|z\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} = \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla z(x,y)|^2 dxdy. \tag{2.6}$$

With that constant κ_s, whose value can be consulted in [9], the extension operator between $H^s_{\Sigma_{\mathcal{D}}}(\Omega)$ and $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ is an isometry, i.e.,

$$\|E_s[\varphi]\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} = \|\varphi\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}, \text{ for all } \varphi \in H^s_{\Sigma_{\mathcal{D}}}(\Omega). \tag{2.7}$$

The key point of the extension function is that it is related to the fractional Laplacian of the original function through the formula

$$\frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y} = (-\Delta)^s u(x).$$

In the case $\Omega = \mathbb{R}^N$ this formulation provides explicit expressions for both the fractional Laplacian and the s-extension in terms of the Riesz and the Poisson kernels, respectively. Namely,

$$w(x,y) = P_y^s * u(x) = c_{N,s} y^{2s} \int\limits_{\mathbb{R}^N} \frac{u(z)}{(|x-z|^2+y^2)^{\frac{N+2s}{2}}} dz$$
$$(-\Delta)^s u(x) = d_{N,s} P.V. \int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}. \tag{2.8}$$

We refer to [9] in order to look up the values of the constants κ_s, $c_{N,s}$ and $d_{N,s}$ as well as the existent relation between them, namely, $2s\kappa_s c_{N,s} = d_{N,s}$. By the arguments above, we can reformulate our problem (P_λ) in terms of the extension problem as follows

$$\begin{cases} -div(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ B^*(w) = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \frac{\partial w}{\partial \nu^s} = \lambda w + w^{2_s^*-1} & \text{in } \Omega \times \{y=0\}. \end{cases} \tag{P_λ^*}$$

An energy solution of this problem is a function $w \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)$ such that

$$\kappa_s \int\limits_{\mathcal{C}_\Omega} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle dxdy = \int\limits_{\Omega} \left(\lambda w(x,0) + w^{2_s^*-1}(x,0) \right) \varphi(x,0) dx,$$

for all $\varphi \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)$. Given $w \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)$ a solution to problem (P_λ^*) the function $u(x) = Tr[w](x) = w(x,0)$ belongs to the space $H^s_{\Sigma_\mathcal{D}}(\Omega)$ and it is an energy solution to problem (P_λ) and vice versa, if $u \in H^s_{\Sigma_\mathcal{D}}(\Omega)$ is a solution to (P_λ) then $w = E_s[u] \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)$ is a solution to (P_λ^*) and, as a consequence, both formulations are equivalent. Finally, the energy functional associated with problem (P_λ^*) is the following,

$$J(w) = \frac{\kappa_s}{2} \int\limits_{\mathcal{C}_\Omega} y^{1-2s} |\nabla w|^2 dxdy - \frac{\lambda}{2} \int\limits_{\Omega} w^2 dx - \frac{N-2s}{2N} \int\limits_{\Omega} w^{2_s^*} dx. \tag{2.9}$$

Plainly, critical points of J in $\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)$ correspond to critical points of I in $H^s_{\Sigma_\mathcal{D}}(\Omega)$. Moreover, minima of J also correspond to minima of I. The proof of this fact is similar to the one of the Dirichlet case, see [5].

Also, in the Dirichlet case, there is a trace inequality [9, Theorem 4.4], i.e.,

$$\int\limits_{\mathcal{C}_\Omega} y^{1-2s} |\nabla z(x,y)|^2 dxdy \geq C \left(\int\limits_{\Omega} |z(x,0)|^r dx \right)^{\frac{2}{r}}, \tag{2.10}$$

for $1 \leq r \leq \frac{2N}{N-2s}$, $N > 2s, z \in \mathcal{X}_0^s(\mathcal{C}_\Omega)$, that turns out to be very useful and by the previous comments this inequality is equivalent to the fractional Sobolev inequality,

$$\int\limits_{\Omega} |(-\Delta)^{s/2} v|^2 dx \geq C \left(\int\limits_{\Omega} |v|^r dx \right)^{\frac{2}{r}}, \tag{2.11}$$

for $1 \leq r \leq \frac{2N}{N-2s}$, $N > 2s, v \in H_0^s(\Omega)$.

Remark 2.3. When $r = 2_s^*$ the best constant in (2.10) will be denoted by $S(s,N)$. This constant is explicit and independent of the domain Ω, and its exact value is given by the following expression,

$$S(s,N) = \frac{2\pi^s \Gamma(1-s) \Gamma(\frac{N+2s}{2}) (\Gamma(\frac{N}{2}))^{\frac{2s}{N}}}{\Gamma(s) \Gamma(\frac{N-2s}{2}) (\Gamma(N))^s}.$$

Since it is not achieved in any bounded domain (see Remarks 2.10-(1)) we have that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla z(x,y)|^2 dxdy \geq S(s,N)\left(\int_{\mathbb{R}^N} |z(x,0)|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{N}}, \; z\in \mathcal{X}^s(\mathbb{R}^{N+1}_+),$$

where $\mathcal{X}^s(\mathbb{R}^{N+1}_+)=\overline{\mathcal{C}^\infty(\mathbb{R}^N\times[0,\infty))}^{\|\cdot\|_{\mathcal{X}^s(\mathbb{R}^{N+1}_+)}}$, with $\|\cdot\|_{\mathcal{X}^s(\mathbb{R}^{N+1}_+)}$ defined as (2.6) replacing $\mathcal{C}_\Omega$ by $\mathbb{R}^{N+1}_+$.

Indeed, in the whole space case the latter inequality is achieved when $z=E_s[u]$ and

$$u(x)=u_\varepsilon(x)=\frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2s}{2}}},$$

with arbitrary $\varepsilon>0$, cf., [9]. Finally, the best constant in (2.11) with $\Omega=\mathbb{R}^N$ is given by $\kappa_s S(s,N)$.

In the mixed boundary data case the situation is quite similar thanks to the fact that we are considering a Dirichlet condition on $\Sigma_{\mathcal{D}}$ with $0<\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}})<\mathcal{H}_{N-1}(\partial\Omega)$, hence there exists a positive constant C such that

$$0<C:=\inf_{\substack{u\in H^s_{\Sigma_{\mathcal{D}}}(\Omega)\\ u\not\equiv 0}}\frac{\|u\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|_{L^{2^*_s}(\Omega)}},$$

so in terms of the extension function,

$$\left(\int_\Omega \varphi^{\frac{2N}{N-2s}}(x,0)dx\right)^{\frac{N-2s}{2N}} \leq C\|\varphi(\cdot,0)\|_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}=C\|E_s[\varphi(\cdot,0)]\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}. \tag{2.12}$$

As we will see below this constant C plays an important role in the proof of Theorem 1.1. With this Sobolev-type inequality in hands we can prove the following result.

Lemma 2.4. *Assume that $\varphi\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$, then there exists a constant $C>0$ such that,*

$$\left(\int_\Omega \varphi^{\frac{2N}{N-2s}}(x,0)dx\right)^{1-\frac{2s}{N}} \leq C\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla\varphi|^2 dxdy. \tag{2.13}$$

Proof. Thanks to (2.12) in order to prove (2.13) it only remains to show the inequality $\|E_s[\varphi(\cdot,0)]\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}\leq \|\varphi\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}$. This inequality is satisfied since, arguing as in [9],

$$\begin{aligned}
\|\varphi\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} &= \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla\varphi|^2 dxdy\\
&= \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla\big(E_s[\varphi(\cdot,0)]+\varphi-E_s[\varphi(\cdot,0)]\big)|^2 dxdy\\
&= \|E_s[\varphi(\cdot,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + \|\varphi - E(\varphi(\cdot,0))\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}\\
&\quad + 2\int_{\mathcal{C}_\Omega} y^{1-2s}\langle\nabla E_s[\varphi(\cdot,0)],\nabla(\varphi-E_s[\varphi(\cdot,0)])\rangle dxdy
\end{aligned}$$

$$
\begin{aligned}
&= \|E_s[\varphi(\cdot,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + \|\varphi - E_s[\varphi(\cdot,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} \\
&\quad + 2\int_\Omega (-\Delta)^s(\varphi(\cdot,0))(\varphi(x,0) - \varphi(x,0))dx \\
&= \|E_s[\varphi(\cdot,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + \|\varphi - E_s[\varphi(\cdot,0)]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)},
\end{aligned}
$$

which concludes the proof. □

Consider now the following quotient

$$Q_\lambda(w) = \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda\|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}},$$

where $w = E_s[u]$, and take

$$S_\lambda(\Omega) = \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)\\ w\not\equiv 0}} \big\{Q_\lambda(w)\big\}. \tag{2.14}$$

If the constant $S_\lambda(\Omega)$ is achieved then problem (P^*_λ) will have at least one solution, and thus problem (P_λ) has also at least one solution, as we will see in the proof of Theorem 1.1. To study the behavior of $Q_\lambda(\cdot)$ we introduce the constants $\widetilde{S}(\Sigma_{\mathcal{N}})$ and $\widetilde{S}(\Sigma_{\mathcal{D}})$ which are inspired in the works [23] and [2] respectively.

Definition 2.5. For $x_0 \in \Sigma_{\mathcal{N}}$ we define the function

$$
\begin{aligned}
\Theta_\lambda \colon \Sigma_{\mathcal{N}} &\to \mathbb{R} \\
x_0 &\mapsto \Theta_\lambda(x_0),
\end{aligned}
$$

by

$$\Theta_\lambda(x_0) = \lim_{\rho\to 0} S_\lambda\big(\Omega_\rho(x_0)\big),$$

where $\Omega_\rho(x_0) = \Omega\cap B_\rho(x_0)$ and the respective infimum in $S_\lambda\big(\Omega_\rho(x_0)\big)$ is taken over the set of functions that vanish on $\Sigma^\rho_{\mathcal{D}} = \partial\Omega_\rho(x_0)\cap\Omega$.

We define the Sobolev constant relative to the Neumann boundary part as

$$\widetilde{S}(\Sigma_{\mathcal{N}}) = \inf_{x_0\in\Sigma_{\mathcal{N}}} \Theta_\lambda(x_0).$$

This constant plays a major role in the existence issues of problem (P_λ), similar of that of the Sobolev constant in the classical Brezis–Nirenberg problem. The next three theorems, which are going to be proved in Section 4, will be useful in the proof of the main result, Theorem 1.1.

Theorem 2.6. *If $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{N}})$ then the infimum (2.14) is achieved.*

As we will see below, the constant $\widetilde{S}(\Sigma_{\mathcal{N}})$ depends only on the regularity of the Neumann boundary part, but it is independent of the Dirichlet boundary part $\Sigma_{\mathcal{D}}$. Since the properties of a Dirichlet problem are quite different from those of a Neumann problem, one would expect that this fact is reflected when we move our boundary conditions, specifically when $\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}}) = \alpha \to 0$, see Lemma 4.7 below. To do so we define the following constant.

Definition 2.7. The Sobolev constant relative to the Dirichlet boundary part is defined by

$$\widetilde{S}(\Sigma_{\mathcal{D}}) = \inf_{\substack{u \in H^s_{\Sigma_{\mathcal{D}}}(\Omega) \\ u \not\equiv 0}} \frac{\|u\|^2_{H^s_{\Sigma_{\mathcal{D}}}(\Omega)}}{\|u\|^2_{L^{2^*_s}(\Omega)}}.$$

Remark 2.8. As it is noted in the proof of Lemma 2.4, the extension function minimizes the $\|\cdot\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}$ norm along all the functions with the same trace on $\{y = 0\}$, thus we can reformulate the definition of $\widetilde{S}(\Sigma_{\mathcal{D}})$ as follows,

$$\widetilde{S}(\Sigma_{\mathcal{D}}) = \inf_{\substack{w \in \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega) \\ w \not\equiv 0}} \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)}}.$$

Arguing in a similar way as in [2, Theorem 2.2] we can prove the following theorem.

Theorem 2.9. *If $\widetilde{S}(\Sigma_{\mathcal{D}}) < 2^{\frac{-2s}{N}}\kappa_s S(s,N)$ then $\widetilde{S}(\Sigma_{\mathcal{D}})$ is attained.*

Remarks 2.10.

(1) This result makes the difference between the Dirichlet boundary condition case and the mixed Dirichlet–Neumann boundary condition case. Note that, by taking $\lambda = 0$ in (P_λ), we have the critical power problem which, in the Dirichlet case, has no positive solution under some geometrical assumptions on Ω, for example, under star-shapeness assumptions on the domain Ω, see [9,28], or under some assumptions on the topology of the domain Ω, see [4], where a non-existence result for domains Ω with trivial topology is established.

(2) In the mixed case, the corresponding Sobolev constant $\widetilde{S}(\Sigma_{\mathcal{D}})$ can be achieved thanks to Theorem 2.9. As we will see, the hypotheses of Theorem 2.9 can be fulfilled by moving the size of the Dirichlet boundary part.

The next result is analogous to that of Theorem 2.6 for the constant relative to the Dirichlet part.

Theorem 2.11. *If $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ then $S_\lambda(\Omega)$ is attained.*

3. Properties of the constants $\widetilde{S}(\Sigma_{\mathcal{N}})$ and $\widetilde{S}(\Sigma_{\mathcal{D}})$

Proposition 3.1. *The constant $\widetilde{S}(\Sigma_{\mathcal{N}})$ does not depend on λ, moreover, if $\Sigma_{\mathcal{N}}$ is a regular $(N-1)$-dimensional submanifold of $\partial\Omega$, then $\widetilde{S}(\Sigma_{\mathcal{N}}) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$.*

We split the proof into several Lemmas.

Lemma 3.2. *The constant $\widetilde{S}(\Sigma_{\mathcal{N}})$ does not depend on λ.*

Proof. Note that by the very definition of $\widetilde{S}(\Sigma_{\mathcal{N}})$ it is enough to prove that $\Theta_\lambda(x_0)$ does not depend on λ, that is $\Theta_\lambda(x_0) = \Theta(x_0) = \lim_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big)$. Since $\lambda \geq 0$, then it is immediate that $\Theta_\lambda(x_0) \leq \lim_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big)$. On the other hand, using Hölder's inequality and the trace inequality (2.13) jointly, we get

$$\|\varphi\|^2_{L^2(\Omega_\rho)} \leq |\Omega_\rho(x_0)|^{\frac{2s}{N}}\|\varphi\|^2_{L^{2^*_s}(\Omega_\rho(x_0))} \leq C|\Omega_\rho(x_0)|^{\frac{2s}{N}}\|E_s[\varphi]\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{\Omega_\rho(x_0)})},$$

thus

$$\Theta_\lambda(x_0) \geq \lim_{\rho\to 0}\left(1-\lambda C|\Omega_\rho(x_0)|^{\frac{2s}{N}}\right) S_0\big(\Omega_\rho(x_0)\big).$$

And the result follows. □

Bearing in mind Lemma 3.2, to prove the last assertion of Proposition 3.1, we need to estimate $S_0\big(\Omega_\rho(x_0)\big) = \inf\big\{Q_0(w) : w \in \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{\Omega_\rho(x_0)})\big\}$. To do so, we use the family of extremal functions of the Sobolev inequality,

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2s}{2}}},$$

and its s-extension, $w_\varepsilon(x) = E_s[u_\varepsilon]$, times a cut-off function as a test function. Note that both functions u_ε and the Poisson kernel (2.8) are self-similar functions, $u_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}}u_1(x)$, and $P^s_y(x) = \frac{1}{y^N}P^s_1\left(\frac{x}{y}\right)$ so the extension family $w_\varepsilon = E_s[u_\varepsilon]$ satisfies

$$w_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right). \tag{3.1}$$

Consider a smooth non-increasing cut-off function $\phi_0(t) \in \mathcal{C}^\infty(\mathbb{R}_+)$, satisfying $\phi_0(t) = 1$ for $0 \leq t \leq \frac{1}{2}$ and $\phi_0(t) = 0$ for $t \geq 1$, and $|\phi_0'(t)| \leq C$ for any $t \geq 0$. Assume, without loss of generality, that $0 \in \Omega$, and define, for some $\rho > 0$ small enough such that $B^+_\rho \subseteq \mathcal{C}_\Omega$, the function $\phi_\rho(x, y) = \phi_0(\frac{r_{xy}}{\rho})$ with $r_{xy} = |(x, y)| = (|x|^2 + y^2)^{\frac{1}{2}}$.

Lemma 3.3. *The family $\{\phi_\rho w_\varepsilon\}$ and its trace on $\{y = 0\}$, $\{\phi_\rho u_\varepsilon\}$, satisfy*

$$\|\phi_\rho w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)} + O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right), \tag{3.2}$$

and

$$\int\limits_\Omega |\phi_\rho u_\varepsilon|^{2^*_s} dx = \|u_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} + O\left(\left(\frac{\varepsilon}{\rho}\right)^{N}\right). \tag{3.3}$$

The proof of this Lemma is similar to the proof of [5, Lemma 3.8] for the Dirichlet boundary conditions. Note that in the Dirichlet case it is not necessary to control the role of the radius of the cut-off function, on the contrary, in the mixed case, by the very definition of the constant $\widetilde{S}(\Sigma_\mathcal{N})$, a careful analysis of the role of that radius is needed. Now we estate the following result proved in [5, Lemma 3.7] that will be useful in the proof of Lemma 3.3.

Lemma 3.4. *[5, Lemma 3.7] The family $w_\varepsilon = w_{\varepsilon,s} = E_s[u_\varepsilon]$ satisfies*

$$|\nabla w_{1,s}(x, y)| \leq C w_{1,s-\frac{1}{2}}(x, y),\ \frac{1}{2} < s < 1,\ (x, y) \in \mathbb{R}^{N+1}_+. \tag{3.4}$$

Proof of Lemma 3.3. We start with the proof of (3.3),

$$\begin{aligned}\int\limits_\Omega |\phi_\rho u_\varepsilon|^{2^*_s} dx = \int\limits_{\mathbb{R}^N} |\phi_\rho u_\varepsilon|^{2^*_s} dx &\geq \int\limits_{|x|<\frac{\rho}{2}} |u_\varepsilon|^{2^*_s} dx \\ &= \|u_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} - \int\limits_{|x|>\frac{\rho}{2}} |u_\varepsilon|^{2^*_s} dx.\end{aligned}$$

Observe that

$$\int\limits_{|x|>\frac{\rho}{2}} |u_\varepsilon|^{2_s^*}dx = \varepsilon^{-N} \int\limits_{|x|>\frac{\rho}{2}} \frac{1}{\left(1+\left(\frac{|x|}{\varepsilon}\right)^2\right)^N}dx$$

$$= \varepsilon^{-N}\int\limits_{\frac{\rho}{2}}^{\infty} \frac{t^{N-1}}{\left(1+\left(\frac{t}{\varepsilon}\right)^2\right)^N}dt = \int\limits_{\frac{\rho}{2\varepsilon}}^{\infty} \frac{s^{N-1}}{(1+s^2)^N}ds$$

$$\le \int\limits_{\frac{\rho}{2\varepsilon}}^{\infty} s^{-N-1}ds = \left(\frac{\varepsilon}{\rho}\right)^N,$$

so we get

$$\int\limits_{\Omega} |\phi_\rho u_\varepsilon|^{2_s^*}dx \ge \|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} + O\left(\left(\frac{\varepsilon}{\rho}\right)^N\right).$$

We continue with the proof of (3.2). The product $\phi_\rho w_\varepsilon$ satisfies

$$\|\phi_\rho w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} \le \|w_\varepsilon\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + \kappa_s \int\limits_{\mathcal{C}_\Omega} y^{1-2s}|w_\varepsilon \nabla\phi_\rho|^2 dxdy + 2\kappa_s \int\limits_{\mathcal{C}_\Omega} y^{1-2s}\langle w_\varepsilon\nabla\phi_\rho, \phi_\rho\nabla w_\varepsilon\rangle dxdy. \tag{3.5}$$

The first term of the right-hand side in (3.5) can be estimated as follows,

$$\int\limits_{\mathcal{C}_\Omega} y^{1-2s}|w_\varepsilon\nabla\phi_\rho|^2dxdy \le \frac{C}{\rho^2}\int\limits_{\{\frac{\rho}{2}\le r_{xy}\le\rho\}} y^{1-2s}w_\varepsilon^2 dxdy$$

$$\le \frac{C}{\rho^2}\varepsilon^{N-2s}\int\limits_{\{\frac{\rho}{2}\le r_{xy}\le\rho\}} y^{1-2s}r_{xy}^{-2(N-2s)}dxdy$$

$$\le \frac{C}{\rho^2}\varepsilon^{N-2s}\int\limits_{\frac{\rho}{2}}^{\rho} s^{1+2s-N}ds$$

$$= O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right),$$

since $0 \le u_\varepsilon(x) \le \varepsilon^{\frac{N-2s}{2}}|x|^{-(N-2s)}$ and the extension of the function $K(x) = |x|^{-(N-2s)}$ is $\tilde{K}(x,y) = (|x|^2+y^2)^{-\frac{N-2s}{2}} = r_{xy}^{-(N-2s)}$.

We end with the estimate of the second term of the right-hand side in (3.5). Applying Cauchy–Schwarz inequality and using (3.1) we get,

$$
\begin{aligned}
&\int\limits_{\mathcal{C}_\Omega} y^{1-2s}\langle w_\varepsilon \nabla \phi_\rho, \phi_\rho \nabla w_\varepsilon\rangle dxdy \\
&\leq \frac{C}{\rho} \int\limits_{\{\frac{\rho}{2}\leq r_{xy}\leq \rho\}} y^{1-2s}|w_\varepsilon(x,y)||\nabla w_\varepsilon(x,y)|dxdy \\
&\leq \frac{C}{\rho}\varepsilon^{-(N-2s)-1} \int\limits_{\{\frac{\rho}{2}\leq r_{xy}\leq \rho\}} y^{1-2s}\left|w_1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|\left|\nabla w_1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|dxdy \\
&= \frac{C}{\rho}\varepsilon \int\limits_{\{\frac{\rho}{2\varepsilon}\leq r_{xy}\leq \frac{\rho}{\varepsilon}\}} y^{1-2s}|w_1(x,y)||\nabla w_1(x,y)|dxdy.
\end{aligned}
\tag{3.6}
$$

Note that for $(x,y) \in \{\frac{\rho}{2\varepsilon} \leq r_{xy} \leq \frac{\rho}{\varepsilon}\}$ we have

$$
\begin{aligned}
w_1(x,y) &= \int\limits_{|z|<\frac{\rho}{4\varepsilon}} P_y^s(x-z)u_1(z)dz + \int\limits_{|z|>\frac{\rho}{4\varepsilon}} P_y^s(x-z)u_1(z)dz \\
&\leq C\left(\frac{\varepsilon}{\rho}\right)^{N+2s} y^{2s} \int\limits_{|z|<\frac{\rho}{4\varepsilon}} u_1(z)dz + C\left(\frac{\varepsilon}{\rho}\right)^{N-2s} \int\limits_{|z|>\frac{\rho}{4\varepsilon}} P_y^s(x-z)dz \\
&\leq C\left(\frac{\varepsilon}{\rho}\right)^{N+2s} y^{2s} \int\limits_{|z|<\frac{\rho}{4\varepsilon}} \frac{1}{|z|^{N-2s}}dz + C\left(\frac{\varepsilon}{\rho}\right)^{N-2s} \int\limits_{\mathbb{R}^N} P_y^s(x-z)dz \\
&\leq C\left(\frac{\varepsilon}{\rho}\right)^{N} y^{2s} + C\left(\frac{\varepsilon}{\rho}\right)^{N-2s} \leq C\left(\frac{\varepsilon}{\rho}\right)^{N-2s}.
\end{aligned}
\tag{3.7}
$$

Using (3.7), (3.6) and (3.4), we get

$$
\begin{aligned}
&\int\limits_{\mathcal{C}_\Omega} y^{1-2s}\langle w_\varepsilon \nabla \phi_\rho, \phi_\rho \nabla w_\varepsilon\rangle dxdy \\
&\leq \frac{C}{\rho}\varepsilon \int\limits_{\{\frac{\rho}{2\varepsilon}\leq r_{xy}\leq \frac{\rho}{\varepsilon}\}} y^{1-2s}\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\left(\frac{\varepsilon}{\rho}\right)^{N-2(s-1/2)} dxdy \\
&\leq c\left(\frac{\varepsilon}{\rho}\right)^{2(1+N-2s)} \int\limits_{\{\frac{\rho}{2\varepsilon}\leq r_{xy}\leq \frac{\rho}{\varepsilon}\}} y^{1-2s}dxdy = O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right).
\end{aligned}
$$

And the proof is complete. □

Lemma 3.5. *Suppose that $\Sigma_{\mathcal{N}}$ is a regular submanifold of $\partial\Omega$, then given $x_0 \in \Sigma_{\mathcal{N}}$ it is satisfied that $\Theta_\lambda(x_0) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$.*

Proof. From Lemma 3.2 we know that $\Theta_\lambda(x_0) = \Theta(x_0) = \lim_{\rho\to 0} S_0\big(\Omega_\rho(x_0)\big)$, also since $\Sigma_{\mathcal{N}}$ is a regular submanifold of $\partial\Omega$, given $x_0 \in \Sigma_{\mathcal{N}}$ we have that,

$$
\lim_{\rho\to 0} \frac{|B_\rho(x_0)\cap\Omega|}{|B_\rho(x_0)|} = \frac{1}{2}. \tag{3.8}
$$

On the other hand, since w_ε is a minimizer of $S(s,N)$, we have

$$S(s,N)=\frac{\int\limits_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_\varepsilon|^2dxdy}{\|u_\varepsilon\|^2_{L^{2^*_s}(\mathbb{R}^N)}}.$$

We take now a cut-off function centered at $x_0\in\Sigma_{\mathcal{N}}$, namely, we take $\psi_\rho(x,y)=\phi_0(\frac{\overline{r}_{xy}}{\rho})$ with $\overline{r}_{xy}=|(x-x_0,y)|=(|x-x_0|^2+y^2)^{\frac{1}{2}}$. Note that $\psi_\rho u_\varepsilon\equiv 0$ on $\partial\Omega_\rho\cap\Omega$. Thanks to (3.2) and (3.3) we can choose $\varepsilon=\rho^\alpha$ with $\alpha>1$ such that

$$\|\phi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)}=\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)}+O\left(\rho^{(\alpha-1)(N-2s)}\right), \tag{3.9}$$

and

$$\|\phi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega)}=\|u_\rho\|^2_{L^{2^*_s}(\mathbb{R}^N)}+O\left(\rho^{(\alpha-1)N}\right), \tag{3.10}$$

where ϕ_ρ is the same cut-off function of Lemma 3.3. Using (3.8)–(3.10), we have that

$$\begin{aligned}\Theta(x_0)&=\lim_{\rho\to 0}S_0\big(\Omega_\rho(x_0)\big)\le\lim_{\rho\to 0}\frac{\|\psi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{\Omega_\rho(x_0)})}}{\|\psi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega_\rho(x_0))}}=\lim_{\rho\to 0}\frac{\|\psi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)}}{\|\psi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega)}}\\&=\lim_{\rho\to 0}\frac{\frac{1}{2}\|\phi_\rho w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)}}{\frac{1}{2^{\frac{2}{2^*_s}}}\|\phi_\rho u_\rho\|^2_{L^{2^*_s}(\Omega)}}=2^{\frac{-2s}{N}}\lim_{\rho\to 0}\frac{\kappa_sS(s,N)+O(\rho^{(\alpha-1)(N-2s)})}{1+O(\rho^{(\alpha-1)N})}\\&=2^{\frac{-2s}{N}}\kappa_sS(s,N).\end{aligned}$$

Finally, we focus on the proof of inequality $\Theta(x_0)\ge 2^{\frac{-2s}{N}}\kappa_sS(s,N)$. To this end we assert the following.

Claim: For $x_0\in\Sigma_{\mathcal{N}}$ we have

$$\Theta_\lambda(x_0)=\Theta(x_0)=\lim_{\rho\to 0}S_0\big(\Omega_\rho(x_0)\big)\ge S_0(B_1^+), \tag{3.11}$$

where B_1^+ is the half ball of radius 1 centered at x_0 with the Neumann boundary part on the flat part of B_1^+ and the Dirichlet boundary part on the closure of the remaining boundary.

To prove the claim, we can argue in a similar way as in [23]. If (3.11) is not true, there exists $\epsilon>0,\ r_0>0$, such that for $0<\rho<r_0$ there exists a function $w_\rho\in\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{\Omega_\rho})$ with $u_\rho=Tr[w_\rho]$ such that

$$\frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{\Omega_\rho})}}{\|u_\rho\|^2_{L^{2^*_s}(\Omega_\rho)}}<S_0(B_1^+)-\epsilon. \tag{3.12}$$

Since x_0 is a regular point, there exists a diffeomorfism T_ρ between $\Omega_\rho(x_0)$ and $B_\rho^+(x_0)$ such that for ρ sufficiently small, $T_\rho(\Sigma^\rho_\mathcal{D})=\partial B_\rho^+\cap\partial B(x_0,\rho)$ and T_ρ transforms $\partial\Omega_\rho\cap\Sigma_\mathcal{N}$ into the flat part of ∂B_ρ^+. Then the function $v_\rho=T_\rho(w_\rho)$ belongs to $\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{B_\rho^+})$ and

$$\frac{\|v_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{B_\rho^+})}}{\|v_\rho(x,0)\|^2_{L^{2^*_s}(B_\rho^+)}}\le C_\rho\frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_{\Omega_\rho})}}{\|u_\rho\|^2_{L^{2^*_s}(\Omega_\rho)}},$$

where C_ρ depends on the diffeomorfism T_ρ and, by the definition of regular point, it can be chosen in such a way that $C_\rho\to 1$ as $\rho\to 0$. Then, for ρ small enough, by (3.12) we have

$$\inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B^+_\rho})\\ w\not\equiv 0}} \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B^+_\rho})}}{\|u_\rho\|^2_{L^{2^*_s}(B^+_\rho)}} < S_0(B^+_1),$$

which is a contradiction because, due to the invariance under scaling, we have

$$\inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B^+_\rho})\\ w\not\equiv 0}} \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B^+_\rho})}}{\|u_\rho\|^2_{L^{2^*_s}(B^+_\rho)}} = \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B^+_1})\\ w\not\equiv 0}} \frac{\|w_\rho\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_{B^+_1})}}{\|u_\rho\|^2_{L^{2^*_s}(B^+_1)}} = S_0(B^+_1).$$

Finally, by (3.2)–(3.3) in Lemma 3.3 it follows that $S_0(B^+_1) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$ and hence $\Theta(x_0) \geq 2^{\frac{-2s}{N}}\kappa_s S(s,N)$. □

Proof of Proposition 3.1. As a consequence of the previous Lemmata we get that if $\Sigma_{\mathcal{N}}$ is a regular submanifold of $\partial\Omega$ then $\widetilde{S}(\Sigma_{\mathcal{N}}) = 2^{\frac{-2s}{N}}\kappa_s S(s,N)$. □

We now turn our attention to the Sobolev constant relative to the Dirichlet part of the boundary $\widetilde{S}(\Sigma_{\mathcal{D}})$. We give an estimate for $\widetilde{S}(\Sigma_{\mathcal{D}})$ similar to that of $\widetilde{S}(\Sigma_{\mathcal{N}})$ in Proposition 3.1.

Proposition 3.6. $\widetilde{S}(\Sigma_{\mathcal{D}}) \leq 2^{\frac{-2s}{N}}\kappa_s S(s,N)$.

Proof. To obtain this estimate we use the extremal functions of the Sobolev inequality and proceed in a similar way as in Proposition 3.1. The lower bound in Proposition 3.1 is due to the fact that the infimum $\widetilde{S}(\Sigma_{\mathcal{N}})$ is taken in the set $\Omega_\rho(x_0)$, on the contrary, for the constant $\widetilde{S}(\Sigma_{\mathcal{D}})$, we do not have such a lower bound by the very definition of $\widetilde{S}(\Sigma_{\mathcal{D}})$. □

4. Proof of main results

4.1. Proof of Theorem 1.1.(1)–(2)

In this subsection we carry out the proof of Theorems 2.6, 2.9 and 2.11 which will be useful in the proof of Theorem 1.1.(1)–(2).

We begin with the upper bound of the parameter λ, i.e., statement (1) in Theorem 1.1.

Lemma 4.1. *Problem (P_λ) has no solution for $\lambda \geq \lambda_{1,s}$, with $\lambda_{1,s}$ the first eigenvalue of $(-\Delta)^s$ with mixed boundary condition.*

Proof. Assume that u is solution to (P_λ) and let φ_1 be a positive first eigenfunction of $(-\Delta)^s$. Taking φ_1 as a test function for (P_λ) we obtain

$$\lambda_{1,s}\int_\Omega u\varphi_1 dx = \int_\Omega (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi_1 dx = \int_\Omega \left(\lambda u + u^{2^*_s-1}\right)\varphi_1 dx > \lambda\int_\Omega u\varphi_1 dx.$$

Therefore, $\lambda < \lambda_{1,s}$. □

Proposition 4.2. *Assume that $0 < \lambda < \lambda_{1,s}$. Then $S_\lambda(\Omega) < 2^{\frac{-2s}{N}}\kappa_s S(s,N) = \widetilde{S}(\Sigma_{\mathcal{N}})$.*

Proof. We recall the following asymptotic identities given in [5, Lemma 3.8],

$$\|\phi_r u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if } N > 4s,\\ C\varepsilon^{2s}\log(1/\varepsilon) + O(\varepsilon^{2s}) & \text{if } N = 4s, \end{cases} \tag{4.1}$$

for some constant $C > 0$, ε small enough and ϕ_r a cut-off function similar to the one in Lemma 3.3. Proceeding in a similar way as in Proposition 3.1, we take a cut-off function centered at a point $x_0 \in \overline{\Sigma}_{\mathcal{N}}$, then using (3.2)–(3.3) and (4.1) jointly, we have the following:

- If $N > 4s$,

$$\begin{aligned} Q_\lambda(\phi_r w_\varepsilon) &\leq 2^{\frac{-2s}{N}} \frac{\kappa_s S(s,N) - \lambda C \varepsilon^{2s} \|u_\varepsilon\|^{-2}_{L^{2^*_s}(\Omega)} + O(\varepsilon^{N-2s})}{1 + O(\varepsilon^N)} \\ &\leq 2^{\frac{-2s}{N}} \kappa_s S(s,N) - \lambda C \varepsilon^{2s} \|u_\varepsilon\|^{-2}_{L^{2^*_s}(\Omega)} + O(\varepsilon^{N-2s}) \\ &< 2^{\frac{-2s}{N}} \kappa_s S(s,N). \end{aligned}$$

- If $N = 4s$ a similar procedure proves that for ε small enough,

$$Q_\lambda(\phi_r w_\varepsilon) \leq 2^{\frac{-2s}{N}} \kappa_s S(s,N) - \lambda C \varepsilon^{2s} \log(1/\varepsilon) \|u_\varepsilon\|^{-2}_{L^{2^*_s}(\Omega)} + O(\varepsilon^{2s}) < 2^{\frac{-2s}{N}} \kappa_s S(s,N). \qquad \square$$

Now we enunciate a concentration–compactness result adapted to our fractional setting with mixed boundary conditions. The proof is a minor variation of that of the concentration–compactness result in [5, Theorem 5.1], which is an adaptation to the fractional setting with Dirichlet boundary conditions of the classical concentration–compactness technique of P.L. Lions, [24]. For the mixed boundary data case involving the classical Laplace operator and Caffarelli–Kohn–Nirenberg weights, [12], a concentration–compactness theorem was proved in [2]. First, we recall the concept of a tight sequence.

Definition 4.3. We say that a sequence $\{y^{1-2s}|\nabla w_n|^2\}_{n\in\mathbb{N}} \subset L^1(\mathcal{C}_\Omega)$ is tight if for any $\eta > 0$ there exists $\rho > 0$ such that

$$\int\limits_{\{y>\rho\}} \int\limits_{\Omega} y^{1-2s} |\nabla w_n|^2 dxdy \leq \eta, \ \forall n \in \mathbb{N}. \tag{4.2}$$

Theorem 4.4 *(Concentration–Compactness). Let $\{w_n\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ be a weakly convergent sequence to w in $\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ such that $\{y^{1-2s}|\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight. Let us denote $u_n = Tr[w_n]$, $u = Tr[w]$ and let μ, ν be two nonnegative measures such that*

$$y^{1-2s}|\nabla w_n|^2 \to \mu, \ \textit{and} \ |u_n|^{2^*_s} \to \nu, \tag{4.3}$$

in the sense of measures. Then, there exist an at most countable set I and points $\{x_i\}_{i\in I} \subset \overline{\Omega}$ such that

(1) $\nu = |u|^{2^*_s} + \sum\limits_{i\in I} \nu_i \delta_{x_i}, \ \nu_i > 0,$

(2) $\mu = y^{1-2s}|\nabla w|^2 + \sum\limits_{i\in I} \mu_i \delta_{x_i}, \ \mu_i > 0,$

(3) $\mu_i \geq \widetilde{S}(\Sigma_{\mathcal{D}}) \nu_i^{\frac{2}{2^*_s}}.$

Using Theorem 4.4 we prove the next result that is analogous to [25, Theorem 2.2].

Theorem 4.5. *Let w_m be a minimizing sequence of $S_\lambda(\Omega)$. Then either w_m is relatively compact or the weak limit, $w \equiv 0$. Even more, in the latter case there exist a subsequence w_m and a point $x_0 \in \overline{\Sigma}_{\mathcal{N}}$ such that*

$$y^{1-2s}|\nabla w_m|^2 \to S_\lambda(\Omega)\delta_{x_0}, \ \textit{and} \ |u_m|^{2^*_s} \to \delta_{x_0}, \tag{4.4}$$

with $u_m = Tr[w_m]$.

Proof. Since $0 \le \lambda < \lambda_{1,s}$ it follows that $0 < S_\lambda(\Omega) \le \widetilde{S}(\Sigma_{\mathcal{D}})$. We distinguish two cases, depending upon if $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ or $S_\lambda(\Omega) = \widetilde{S}(\Sigma_{\mathcal{D}})$:

(1) $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$. In this case we can argue in a similar way as in [5, Prop. 4.2] which in turn is based on the technique of Brezis–Nirenberg.

Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ be a minimizing sequence of $S_\lambda(\Omega)$, and suppose without loss of generality that $w_m \ge 0$ and $\|w_m(\cdot,0)\|_{L^{2^*_s}(\Omega)} = 1$. Clearly, this implies that

$$\|w_m\|_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} \le M, \tag{4.5}$$

then, there exists a subsequence (denoted also by $\{w_m\}$) verifying,

$$\begin{aligned}
&w_m \rightharpoonup w \text{ weakly in } \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega),\\
&w_m(\cdot,0) \to w(\cdot,0) \text{ strongly in } L^q(\Omega),\ 1 \le q < 2^*_s,\\
&w_m(\cdot,0) \to w(\cdot,0) \text{ a.e. in } \Omega.
\end{aligned}$$

Using the weak convergence we get

$$\begin{aligned}
\|w_m\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} &= \|w_m - w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}\\
&\quad + \|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + 2\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\langle \nabla w, \nabla w_m - \nabla w\rangle dxdy\\
&= \|w_m - w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + \|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + o(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
Q_\lambda(w_m) &= \|w_m\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda \|w_m(\cdot,0)\|^2_{L^2(\Omega)}\\
&= \|w_m - w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} + \|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda \|w_m(\cdot,0)\|^2_{L^2(\Omega)} + o(1)\\
&\ge \widetilde{S}(\Sigma_{\mathcal{D}})\|w_m(\cdot,0) - w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda(\Omega)\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + o(1).
\end{aligned}$$

Thus, because of the normalization $\|w_m(\cdot,0)\|_{L^{2^*_s}} = 1$, it follows

$$Q_\lambda(w_m) \ge (\widetilde{S}(\Sigma_{\mathcal{D}}) - S_\lambda(\Omega))\|w_m(\cdot,0) - w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda(\Omega) + o(1).$$

Since $\{w_m\}$ is a minimizing sequence of $S_\lambda(\Omega)$, we obtain

$$o(1) + S_\lambda(\Omega) \ge (\widetilde{S}(\Sigma_{\mathcal{D}}) - S_\lambda(\Omega))\|w_m(\cdot,0) - w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda(\Omega) + o(1).$$

Finally, using that $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ it follows

$$w_m(\cdot,0) \to w(\cdot,0) \text{ in } L^{2^*_s}(\Omega).$$

By a standard lower semi-continuity argument, w is a minimizer for $Q_\lambda(\cdot)$, so we get that the sequence is relatively compact.

(2) $S_\lambda(\Omega) = \widetilde{S}(\Sigma_{\mathcal{D}})$. Let us consider $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ as in the previous case. Thus $\{w_m\}$ is also a minimizing sequence for $\widetilde{S}(\Sigma_{\mathcal{D}})$ and we proceed in a similar way as in [25, Theorem 2.2]. Using Theorem 4.4, we get that either $\{w_m\}$ is relatively compact or the weak limit $w \equiv 0$.

In the first case, $w \not\equiv 0$, by Theorem 4.4 we have

$$\widetilde{S}(\Sigma_{\mathcal{D}}) = \int\limits_{\mathcal{C}_\Omega} d\mu \geq \int\limits_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy + \widetilde{S}(\Sigma_{\mathcal{D}}) \sum_{i\in I} \nu_i^{\frac{2}{2_s^*}},$$

as well as

$$1 = \int\limits_{\Omega} d\nu = \int\limits_{\Omega} |u|^{2_s^*} dx + \sum_{i\in I} \nu_i.$$

By the two expressions above, we get that

$$\begin{aligned}\left(1 - \sum_{i\in I} \nu_i\right)^{\frac{2}{2_s^*}} &= \left(\int\limits_{\Omega} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \\ &\leq \frac{1}{\widetilde{S}(\Sigma_{\mathcal{D}})} \int\limits_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy \qquad (4.6) \\ &\leq \frac{1}{\widetilde{S}(\Sigma_{\mathcal{D}})} \left(\widetilde{S}(\Sigma_{\mathcal{D}}) - \widetilde{S}(\Sigma_{\mathcal{D}}) \sum_{i\in I} \nu_i^{\frac{2}{2_s^*}}\right) \\ &= 1 - \sum_{i\in I} \nu_i^{\frac{2}{2_s^*}},\end{aligned}$$

hence, $\nu_i \leq 1$ $\forall i \in I$. And therefore, by (4.6) the only possibility is $\nu_i = 0$ for all $i \in I$. This leads to

$$\int\limits_{\Omega} |u_m|^{2_s^*} dx \to \int\limits_{\Omega} |u|^{2_s^*} dx,$$

from which we deduce that u_m (and thus $w_m = E_s[u_m]$) is relatively compact.

Now we consider the case $w \equiv 0$ (and thus $u \equiv 0$). In this case by Theorem 4.4 and (4.6) we get

$$\sum_{i\in I} \nu_i = 1, \text{ and } \sum_{i\in I} \nu_i^{\frac{2}{2_s^*}} \leq 1,$$

then we infer that I must be a singleton, i.e.,

$$\nu = \delta_{x_0} \qquad \text{and} \qquad \mu = \widetilde{S}(\Sigma_{\mathcal{D}})\delta_{x_0} = S_\lambda(\Omega)\delta_{x_0},$$

with $x_0 \in \overline{\Omega}$.

To show that $x_0 \in \overline{\Sigma}_{\mathcal{N}}$ we argue by contradiction. If $x_0 \in \Omega \cup \Sigma_{\mathcal{D}}$, we set $\overline{\phi}_r(x, y)$ as a cut-off function centered at $x_0 \in \Omega$, and define the sequence

$$w_{m,r} = w_m \overline{\phi}_r(x, y)$$

and the traces sequence $\{u_{m,r}\} = \{Tr[w_{m,r}]\}$. Then for all $r > 0$

$$\lim_{m\to\infty} \frac{\int\limits_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w_{m,r}|^2 dxdy}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} = \widetilde{S}(\Sigma_{\mathcal{D}}). \qquad (4.7)$$

Note that for r sufficiently small, the sequence $\{w_{m,r}\}$ belongs to $\mathcal{X}_0^s(\mathcal{C}_\Omega)$, then for any $m \in \mathbb{N}$, by Proposition 3.6,

$$\begin{aligned}\lim_{r\to 0}\frac{\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w_{m,r}|^2dxdy}{\|u_{m,r}\|^2_{L^{2_s^*}(\Omega)}} &\geq \inf_{\substack{v\in\mathcal{X}_0^s(\mathcal{C}_\Omega)\\ v\not\equiv 0}}\frac{\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla v|^2dxdy}{\|v(x,0)\|^2_{L^{2_s^*}(\Omega)}}\\ &= \kappa_s S(s,N)\\ &> 2^{\frac{-2s}{N}}\kappa_s S(s,N)\\ &\geq \widetilde{S}(\Sigma_{\mathcal{D}}),\end{aligned}$$

and we reach a contradiction with (4.7). Therefore, $x_0 \in \partial\Omega$. If $x_0 \in \mathring{\Sigma}_{\mathcal{D}}$ arguing as before we reach the same contradiction. As a consequence, $x_0 \in \overline{\Sigma}_{\mathcal{N}}$.

It only remains to prove the tightness condition (4.2) for the minimizing sequence $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$, i.e., there is no evanescence. Since $\{w_m\}$ is a minimizing sequence of $S_\lambda(\Omega)$ then $\{w_m\}$ or a multiple will converge to a critical point of the functional (2.9). Let $\{\tilde{w}_m\}$ be such a sequence, then

$$J(\tilde{w}_m) \to c, \text{ and } J'(\tilde{w}_m) \to 0. \tag{4.8}$$

We proceed now as in [5, Lemma 3.6] which is based on ideas contained in [3]. By contradiction, suppose that there exists $\eta_0 > 0$, and $m_0 \in \mathbb{N}$ such that for any $\rho > 0$ one has, up to a subsequence,

$$\int\limits_{\{y>\rho\}}\int\limits_{\Omega} y^{1-2s}|\nabla\tilde{w}_m|^2dxdy > \eta_0, \ \forall m \geq m_0. \tag{4.9}$$

Fix $\varepsilon > 0$ (to be determined) and let $r > 0$ be such that

$$\int\limits_{\{y>r\}}\int\limits_{\Omega} y^{1-2s}|\nabla\tilde{w}|^2dxdy < \varepsilon.$$

Let $j = \left[\frac{M}{\kappa_s\varepsilon}\right]$ be the integer part with M the constant in (4.5) and $I_k = \{y \in \mathbb{R}^+ : r+k \leq y \leq r+k+1\}$, $k = 0, 1, \ldots, j$. Then

$$\sum_{k=0}^{j}\int\limits_{I_k}\int\limits_{\Omega} y^{1-2s}|\nabla\tilde{w}_m|^2dxdy \leq \int\limits_{\mathcal{C}_\Omega} y^{1-2s}|\nabla\tilde{w}_m|^2dxdy \leq \frac{M}{\kappa_s} < \varepsilon(j+1).$$

Then, there exists $k_0 \in \{0, \ldots, j\}$ such that, up to a subsequence,

$$\int\limits_{I_{k_0}}\int\limits_{\Omega} y^{1-2s}|\nabla\tilde{w}_m|^2dxdy \leq \varepsilon, \ \forall m \geq m_0. \tag{4.10}$$

We set now a regular cut-off function

$$\chi(y) = \begin{cases} 0 & \text{if } y \leq r+k_0,\\ 1 & \text{if } y > r+k_0+1,\end{cases}$$

and we define $v_m(x,y) = \chi(y)\tilde{w}_m(x,y)$. Then, since $v_m(x,0) = 0$, it follows that

$$\begin{aligned}|\langle J'(\tilde{w}_m) - J'(v_m), v_m\rangle| &= \kappa_s \int\limits_{\mathcal{C}_\Omega} y^{1-2s}\langle \nabla(\tilde{w}_m - v_m), \nabla v_m\rangle dxdy \\ &= \kappa_s \int\limits_{I_{k_0}} \int\limits_{\Omega} y^{1-2s}\langle \nabla(\tilde{w}_m - v_m), \nabla v_m\rangle dxdy.\end{aligned}$$

Moreover, by the Cauchy–Schwarz inequality, (4.10) and the compact inclusion of the space $H^1\left(I_{k_0} \times \Omega, y^{1-2s}dxdy\right)$ into $L^2\left(I_{k_0} \times \Omega, y^{1-2s}dxdy\right)$, it follows that

$$\begin{aligned}&|\langle J'(\tilde{w}_m) - J'(v_m), v_m\rangle| \\ &\le \kappa_s \left(\int\limits_{I_{k_0}} \int\limits_{\Omega} y^{1-2s}|\nabla(\tilde{w}_m - v_m)|^2 dxdy\right)^{1/2} \left(\int\limits_{I_{k_0}} \int\limits_{\Omega} y^{1-2s}|\nabla v_m|^2 dxdy\right)^{1/2} \\ &\le C\kappa_s\varepsilon.\end{aligned}$$

Finally, by (4.8),

$$|\langle J'(v_m), v_m\rangle| \le C\kappa_s\varepsilon + o(1),$$

thus, for m big enough

$$\int\limits_{\{y>r+k_0+1\}} \int\limits_{\Omega} y^{1-2s}|\nabla w_m|^2 dxdy \le \int\limits_{\mathcal{C}_\Omega} \int\limits_{\Omega} y^{1-2s}|\nabla v_m|^2 dxdy \le \frac{\langle J'(v_m), v_m\rangle}{\kappa_s} \le C\varepsilon,$$

which contradicts (4.9). Then, the proof of Theorem 4.5 is complete. □

Remark 4.6. Note that the proof of Theorem 2.11 was done in the first part of the proof of Theorem 4.5.

Now we prove Theorems 2.6, 2.9.

Proof of Theorem 2.9. Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_\mathcal{D}}(\mathcal{C}_\Omega)$ be a minimizing sequence of $\widetilde{S}(\Sigma_\mathcal{D})$ and w its weak limit. By Theorem 4.5, $\{w_m\}$ is relatively compact, and consequently the infimum is achieved, or $w \equiv 0$ and

$$y^{1-2s}|\nabla w_n|^2 \to \mu\delta_{x_0}, \text{ and } |u_n|^{2^*_s} \to \nu\delta_{x_0},$$

with $x_0 \in \overline{\Sigma}_\mathcal{N}$. Indeed, we can assume, without loss of generality, that $\mu = \widetilde{S}(\Sigma_\mathcal{D})$ and $\nu = 1$. With the same notation as in the proof of Theorem 4.5, we consider the functions

$$w_{m,r} = w_m\overline{\phi}_r(x, y) \tag{4.11}$$

with $\overline{\phi}_r(x, y)$ a smooth cut-off function centered at $x_0 \in \overline{\Sigma}_\mathcal{N}$. Clearly, (4.11) satisfies (4.7). Since $\Sigma_\mathcal{N}$ is smooth, for r small enough, the sequence $\{u_{m,r}\} \subset H^s_{\Sigma^r_\mathcal{D}}(\Omega_r)$, or equivalently, the sequence $\{w_{m,r}\} \subset \mathcal{X}^s_{\Sigma^r_\mathcal{D}}(\mathcal{C}_{\Omega_r})$ thus, by Proposition 3.6,

$$\lim_{r\to 0} \frac{\int\limits_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w_{m,r}|^2 dxdy}{\|u_{m,r}\|^2_{L^{2^*_s}(\Omega)}} \ge 2^{\frac{-2s}{N}}\kappa_s S(s, N) > \widetilde{S}(\Sigma_\mathcal{D}),$$

which contradicts (4.7). Then the only possibility is that $\{w_m\}$ is relatively compact, which proves the assertion. □

Proof of Theorem 2.6. Let $\{w_m\} \subset \mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)$ be a minimizing sequence for $S_\lambda(\Omega)$ and w its weak limit. Thus, either $\{w_m\}$ is relatively compact and consequently the infimum is achieved or by Theorem 4.5, (4.4) holds up to a subsequence. For that sequence we consider the functions $w_{m,r} = w_m \overline{\phi}_r(x,y)$, with $\overline{\phi}_r(x,y)$ a smooth cut-off function centered at $x_0 \in \overline{\Sigma}_{\mathcal{N}}$ as in (4.11). On the one hand, $\{w_{m,r}\}$ and its trace $\{u_{m,r}\}$ satisfy

$$\frac{\|w_{m,r}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda \|u_{m,r}\|^2_{L^2(\Omega)}}{\|u_{m,r}\|^2_{L^{2^*_s}(\Omega)}} \to S_\lambda(\Omega), \quad \text{as} \quad m \to \infty, \tag{4.12}$$

for any $r > 0$. On the other, by the definition of $\widetilde{S}(\Sigma_{\mathcal{N}})$ we have

$$\lim_{r\to 0} \frac{\|w_{m,r}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda \|u_{m,r}\|^2_{L^2(\Omega)}}{\|u_{m,r}\|^2_{L^{2^*_s}(\Omega)}} \geq \widetilde{S}(\Sigma_{\mathcal{N}}),$$

which contradicts (4.12) since we are supposing $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$. Hence $\{w_m\}$ is relatively compact. □

Proof of Theorem 1.1-(2). By Theorem 2.6, it follows immediately the existence of a solution to problem (P_λ) whenever we have $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{N}})$, which is guaranteed by Proposition 4.2 if $0 < \lambda < \lambda_{1,s}$. Also, there exists a solution when $S_\lambda(\Omega) < \widetilde{S}(\Sigma_{\mathcal{D}})$ by Theorem 2.11.

Specifically, by Theorem 2.6 and Proposition 4.2, if $0 < \lambda < \lambda_{1,s}$ there exists a minimizer function $\tilde{w}$ with $\tilde{u} = Tr[\tilde{w}]$ satisfying

$$\|\tilde{w}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda \|\tilde{u}\|^2_{L^2(\Omega)} = S_\lambda(\Omega) \|\tilde{u}\|^2_{L^{2^*_s}(\Omega)}.$$

Taking $w = \tilde{w}/\|\tilde{u}\|^2_{L^{2^*_s}(\Omega)}$ and its trace $u = \tilde{u}/\|\tilde{u}\|^2_{L^{2^*_s}(\Omega)}$,

$$\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} - \lambda \|u\|^2_{L^2(\Omega)} = S_\lambda(\Omega). \tag{4.13}$$

Thus w is a minimizer of $S_\lambda(\Omega)$ constrained to the sphere $\|u\|_{L^{2^*_s}(\Omega)} = 1$. Without loss of generality we can assume $w \geq 0$, otherwise we take $|w|$ instead. Or equivalently, w is a critical point of the functional Q_λ constrained to $\|u\|^2_{L^{2^*_s}(\Omega)} = 1$, then thanks to (2.2) and (2.7), such a critical point is a non-negative solution to equation

$$(-\Delta)^s u - \lambda u = \tau u^{2^*_s - 1} \ \text{ in } \ \Omega,$$

where $\tau \in \mathbb{R}$ is a Lagrange multiplier. Moreover $\tau = S_\lambda(\Omega) > 0$ since $\lambda < \lambda_{1,s}$. Thus, it follows that defining $v = ku$, it is a non-negative solution to the equation in (P_λ) for $k = (S_\lambda(\Omega))^{\frac{1}{2^*_s - 2}}$. Indeed, since by the maximum principle, see [22, Corollary 2.3.9], we have that $v > 0$ in Ω, then v is a solution to (P_λ). □

To complete the proof of Theorem 1.1 it only remains to prove statement (3) in Theorem 1.1. This will be done in the next subsection.

4.2. Moving the boundary conditions. Proof of Theorem 1.1-(3)

Let us consider the following eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda_{1,s}(\alpha) u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \Sigma_{\mathcal{D}}(\alpha), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_{\mathcal{N}}(\alpha), \end{cases} \tag{EP_α}$$

with the following hypotheses:

B_1: $\Omega \subset \mathbb{R}^N$ is a regular bounded domain.
B_2: $\Sigma_{\mathcal{D}}(\alpha)$ and $\Sigma_{\mathcal{N}}(\alpha)$ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Sigma_{\mathcal{D}}(\alpha) \cup \Sigma_{\mathcal{N}}(\alpha) = \partial\Omega$, $\Sigma_{\mathcal{D}}(\alpha) \cap \Sigma_{\mathcal{N}}(\alpha) = \emptyset$, and the interphase $\Gamma(\alpha) = \Sigma_{\mathcal{D}}(\alpha) \cap \overline{\Sigma}_{\mathcal{N}}(\alpha)$ is a $(N-2)$-dimensional submanifold.
B_3: $\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}}(\alpha)) = \alpha$, $\Sigma_{\mathcal{D}}(\alpha_1) \subseteq \Sigma_{\mathcal{D}}(\alpha_2)$ for any $0 < \alpha_1 \leq \alpha_2 < \mathcal{H}_{N-1}(\partial\Omega)$.

Following [14, Lemma 4.1] we have the next result.

Lemma 4.7. *Let u_α be a positive solution to problem (EP_α) and suppose hypotheses B_1–B_3. Then we obtain,*

$$\lambda_{1,s}(\alpha) \to 0, \ \text{as } \alpha \to 0.$$

Proof. By the definition of the fractional operator $(-\Delta)^s$, we have that the eigenvalue $\lambda_{1,s}(\alpha) = \lambda_{1,1}^s(\alpha)$.
By [14, Lemma 4.1], we have that $\lambda_{1,1}(\alpha) \to 0$ as $\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}}(\alpha)) = \alpha \to 0$. Then the result follows. □

Remark 4.8. We would like to point out Theorem 8 by Denzler; see [18], in which it is proved that

$$\sup_{0<\alpha<|\partial\Omega|} \{\lambda_1(\Sigma_{\mathcal{D}}) : |\Sigma_{\mathcal{D}}| = \alpha\} = \lambda_1(|\partial\Omega|).$$

This means that there exist configurations of the distribution of $\Sigma_{\mathcal{D}}$, and $\Sigma_{\mathcal{N}}$ such that [14, Lemma 4.1] and hence Lemma 4.7 do not apply.

The next proposition is the analogous to [2, Proposition 2.1] for our fractional setting.

Proposition 4.9. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Given a family $\{\Sigma_{\mathcal{D}}(\alpha) : 0 < \alpha < \mathcal{H}_{N-1}(\partial\Omega)\}$ satisfying hypotheses B_1-B_3, there exists a positive constant α_0 such that for any $\alpha < \alpha_0$, $\widetilde{S}(\Sigma_{\mathcal{D}}(\alpha))$ is attained.*

Proof. We only have to check that hypotheses of Theorem 2.9 are satisfied. To do so, we use the Hölder inequality together with Lemma 4.7 as follows. By Hölder's inequality,

$$\begin{aligned} \widetilde{S}(\Sigma_{\mathcal{D}}(\alpha)) &= \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}(\alpha)}}(\mathcal{C}_\Omega)\\ w\not\equiv 0}} \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)}} \\ &\leq |\Omega|^{\frac{2s}{N}} \inf_{\substack{w\in\mathcal{X}^s_{\Sigma_{\mathcal{D}(\alpha)}}(\mathcal{C}_\Omega)\\ w\not\equiv 0}} \frac{\|w\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)}}{\|w(\cdot,0)\|^2_{L^2(\Omega)}} \\ &= |\Omega|^{\frac{2s}{N}} \lambda_{1,s}(\alpha). \end{aligned} \tag{4.14}$$

Applying Lemma 4.7 into (4.14), we have that there exists $\alpha_0 > 0$ such that $\widetilde{S}(\Sigma_{\mathcal{D}}(\alpha)) < 2^{\frac{-2s}{N}} \kappa_s S(s,N)$ for any $\alpha < \alpha_0$. Hence, by Theorem 2.9 the result follows. □

We complete now the proof of Theorem 1.1.

Proof of Theorem 1.1-(3). Since $S_\lambda(\Omega) = \widetilde{S}(\Sigma_{\mathcal{D}})$ for $\lambda = 0$, the existence of solution to problem (P_0) is equivalent to the attainability of $\widetilde{S}(\Sigma_{\mathcal{D}})$. Thus, letting α sufficiently small, by Proposition 4.9 there exists a minimizer function $\tilde{w}$ with $\tilde{u} = Tr[\tilde{w}]$ satisfying

$$\|\tilde{w}\|^2_{\mathcal{X}^s_{\Sigma_{\mathcal{D}}}(\mathcal{C}_\Omega)} = \widetilde{S}(\Sigma_{\mathcal{D}}) \|\tilde{u}\|^2_{L^{2^*_s}(\Omega)},$$

and we are done. □

5. A nonexistence result: Pohozaev-type identity

This last part deals with a non-existence result relying on a Pohozaev-type identity. Notice that by Theorem 1.1-(3) we have the existence of solution to the following critical problem,

$$\begin{cases} (-\Delta)^s u = u^{2_s^*-1} & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega = \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}}, \end{cases} \tag{5.1}$$

provided $\alpha = \mathcal{H}_{N-1}(\Sigma_{\mathcal{D}})$ is small enough, in contrast to the non-existence results for the Dirichlet boundary data case and Ω a star-shaped domain, see Pohozaev [28], in the classical setting or [9] for the fractional case under the same geometrical hypotheses. Nevertheless, and in spite of Theorem 1.1-(3), proceeding in a similar way as in [25,23] we are going to show a Pohozaev-type identity for our fractional mixed Dirichlet–Neumann problems that provides us a non-existence result under appropriate assumptions on the geometry of Ω, $\Sigma_{\mathcal{D}}$, $\Sigma_{\mathcal{N}}$.

Let us consider the problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega = \Sigma_{\mathcal{D}} \cup \Sigma_{\mathcal{N}}. \end{cases} \tag{P_f}$$

We have the following result.

Theorem 5.1. *Suppose that u is a solution of problem (P_f), $w = E_s[u]$ and f is a continuous function with primitive F. Then the following Pohozaev-type identity holds,*

$$\begin{aligned} &(N-2s)\int_{\Omega} uf(u)dx - 2N\int_{\Omega} F(u)dx \\ &= \kappa_s \int_{\Sigma_{\mathcal{N}}^*} y^{1-2s}|\nabla w|^2\langle x,\nu\rangle d\sigma(x,y) - \kappa_s \int_{\Sigma_{\mathcal{D}}^*} y^{1-2s}|\nabla w|^2\langle x,\nu\rangle d\sigma(x,y) \\ &\quad -2\int_{\Sigma_N} F(u)\langle x,\nu\rangle d\sigma(x), \end{aligned} \tag{5.2}$$

where ν denotes the outwards normal vector to $\partial\Omega$.

Proof. Since $w = E_s[u]$ is a solution of problem

$$\begin{cases} -div(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ B^*(w) = 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ \frac{\partial w}{\partial \nu^s} = f(u) & \text{in } \Omega, \end{cases} \tag{P_f^*}$$

multiplying the equation of (P_f^*) by $\varphi(x,y)$ and integrating by parts we get

$$\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}\nabla w \nabla\varphi dxdy = \int_{\Omega} \varphi(x,0)f(u)dx + \kappa_s \int_{\Sigma_{\mathcal{D}}^*} \varphi y^{1-2s}\langle\nabla w,\nu^*\rangle d\sigma(x,y). \tag{5.3}$$

With ν^* the outwards normal vector to $\partial_L\mathcal{C}_\Omega$. We take $\varphi(x,y) = \langle (x,y), \nabla w\rangle$ and note that $\langle \nabla w, \nu^*\rangle = |\nabla w|$ on $\Sigma_{\mathcal{D}}^*$, as well that, by construction, the outwards normal vector ν^* to the lateral boundary $\partial_L\mathcal{C}_\Omega$ verifies $\nu^* = (\nu, 0)$ with ν the outwards normal vector to $\partial\Omega$. Then, we find,

$$\frac{2s-N}{2}\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy + \frac{1}{2}\kappa_s \int_{\partial_L\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 \langle x,\nu\rangle d\sigma(x,y) =$$
$$\int_{\Sigma_{\mathcal{N}}} F(u)\langle x,\nu\rangle d\sigma(x) - N\int_\Omega F(u)dx + \kappa_s \int_{\Sigma_{\mathcal{D}}^*} y^{1-2s}|\nabla w|^2\langle x,\nu\rangle d\sigma(x,y),$$

which proves (5.2). $\square$

As a consequence we obtain a non-existence result for problem (P_f).

Corollary 5.2. *Assume the hypotheses of Theorem 5.1 and suppose there exists $x_0 \in \Omega$ such that $\langle x - x_0, \nu\rangle = 0$ on $\Sigma_{\mathcal{N}}$ and $\langle x - x_0, \nu\rangle > 0$ on $\Sigma_{\mathcal{D}}$. If f and F satisfy the inequality $(N-2s)tf(t) - 2NF(t) \geq 0$, then problem (P_f) has no solution.*

This result highlights the difference between a mixed boundary condition problem and a Dirichlet one as well as the relevance of the geometry of Ω and the decomposition of $\partial\Omega$ into $\Sigma_{\mathcal{D}}$ and $\Sigma_{\mathcal{N}}$ in the existence issues.

As an example, let us consider the critical power problem (5.1) with Ω defined as follows. Given A_α a smooth submanifold of the unit sphere $\mathbb{S}^{N-1}$ such that $\mathcal{H}_{N-1}(A_\alpha) = \alpha$, we set $\Omega = \{tx : x \in A_\alpha, 0 < t < R\}$, $\Sigma_{\mathcal{D}} = \{x \in \overline{\Omega} : |x| = R\}$ and $\Sigma_{\mathcal{N}} = \partial\Omega \backslash \Sigma_{\mathcal{D}}$.

We consider a smooth perturbation $\widetilde{\Omega}$ where the vertex $x_0 = \overline{0}$ and the corners of Ω are regularized, such that $|\widetilde{\Omega}\backslash\Omega|$ is small enough. Set $\widetilde{\Sigma}_{\mathcal{D}} = \Sigma_{\mathcal{D}}$ and $\widetilde{\Sigma}_{\mathcal{N}} = \partial\widetilde{\Omega}\backslash\widetilde{\Sigma}_{\mathcal{D}}$. Then, $\langle x,\nu\rangle = 0$ on $\widetilde{\Sigma}_{\mathcal{N}}\backslash T_\rho$ and $\langle x,\nu\rangle \neq 0$ on $\widetilde{\Sigma}_{\mathcal{N},\rho} = \widetilde{\Sigma}_{\mathcal{N}} \cap T_\rho$ with $T_\rho = B_\rho(0) \cup \{x \in \mathbb{R}^N : R - \rho < |x| < R\}$ and some $\rho > 0$ small enough, as well as $\langle x,\nu\rangle > 0$ on $\widetilde{\Sigma}_{\mathcal{D}}$. Since we can approximate the cone Ω arbitrarily by means of $\widetilde{\Omega}$, we can let ρ be sufficiently small in order to obtain a contradiction with the Pohozaev identity, namely

$$\begin{aligned}&\frac{N-2s}{N}\int_{\widetilde{\Sigma}_{\mathcal{N},\rho}} |u|^{2_s^*}\langle x,\nu\rangle d\sigma \\ &= \kappa_s \int_{\widetilde{\Sigma}^*_{\mathcal{N},\rho}} y^{1-2s}|\nabla w|^2\langle x,\nu\rangle d\sigma + R\kappa_s \int_{\widetilde{\Sigma}^*_{\mathcal{D}}} y^{1-2s}|\nabla w|^2 d\sigma.\end{aligned} \tag{5.4}$$

Thus, no solution to the problem (5.1) exists on $\widetilde{\Omega}$.

Remark 5.3. If we move the boundary conditions in the example above, letting $\mathcal{H}_{N-1}(\Sigma_{\mathcal{D}}) \to 0$, by means of Theorem 1.1-(3) we get the existence of solution to problem (5.1) on the perturbed cone $\widetilde{\Omega}$. This is not in contradiction with the previous arguments, because by this procedure, points that belonged to the Dirichlet boundary part for which we had $\langle x,\nu\rangle > 0$, start to contribute to the integral involving the Neumann part of the boundary in (5.4), and hence Theorem 1.1-(3) and Corollary 5.2 agree.

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