



Numerical ranges of weighted composition operators

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ABSTRACT

In this paper, first we consider the numerical range of $C_{\psi,\varphi}$, when $\varphi(z) = rz$ with $|r| \leq 1$. Then we study the numerical range of $C_{\psi,\varphi}$, where φ is an elliptic automorphism. Next, the exact value of the norm of some weighted composition operators are obtained in order to investigate their numerical radii. Finally, we compute the numerical ranges of all hermitian weighted composition operators.

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1. Introduction

Let T be a bounded linear operator on a separable complex Hilbert space H . We write $r(T)$, $r_e(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_e(T)$ for the spectral radius, the essential spectral radius, the spectrum, the point spectrum and the essential spectrum of T , respectively.

Let \mathbb{D} denote the open unit disk in the complex plane, and the Hardy space H^2 consisting of the functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ holomorphic in \mathbb{D} such that $\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$ with $\hat{f}(n)$ denoting the n -th Taylor coefficient of f . The reproducing kernel at w in H^2 is given by $K_w(z) = 1/(1 - \bar{w}z)$. Let k_w denote the normalized reproducing kernel given by $k_w = K_w/\|K_w\|$, where $\|K_w\| = (1 - |w|^2)^{-1/2}$. We recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on \mathbb{D} , with supremum norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. Let φ be an analytic self-map of \mathbb{D} , then the equation $C_\varphi(f) = f \circ \varphi$ defines a composition operator C_φ with inducing map φ . For an analytic function ψ on \mathbb{D} and an analytic self-map φ of \mathbb{D} , the weighted composition operator $C_{\psi,\varphi} : H^2 \rightarrow H^2$ is given by $C_{\psi,\varphi}h = \psi \cdot (h \circ \varphi)$.

For $g \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_g is the operator on H^2 given by $T_g(f) = P(gf)$ for f in H^2 , where P is the orthogonal projection of L^2 onto H^2 .

It is well-known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi = \lambda\alpha_p$, where $|\lambda| = 1$, $|p| < 1$ and $\alpha_p(z) = \frac{p-z}{1-\bar{p}z}$.

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Let $\varphi(z) = (az + b)/(cz + d)$ be a linear-fractional self-map of \mathbb{D} , where $ad - bc \neq 0$. Then $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ maps \mathbb{D} into itself, $g(z) = (-\bar{b}z + \bar{d})^{-1}$ and $h(z) = cz + d$ are in H^∞ . Cowen in [9, Theorem 2] showed that $C_\varphi^* = T_g C_\sigma T_h^*$. The maps σ, g and h are called the *Cowen auxiliary functions*. We know that a is a fixed point of φ if and only if $\frac{1}{\bar{a}}$ is a fixed point of σ . From now on, unless otherwise stated, we assume that σ, h and g are given as above.

We say that φ has a finite angular derivative at a point $\zeta \in \partial\mathbb{D}$ if there is a point $w \in \partial\mathbb{D}$ such that the difference quotient $(\varphi(z) - w)/(z - \zeta)$ has a finite limit, as z tends nontangentially to ζ . This limit is denoted $\varphi'(\zeta)$. For any self-map φ of \mathbb{D} and each positive integer n , we write $\varphi_1 := \varphi$ and $\varphi_{n+1} := \varphi \circ \varphi_n$. We refer to φ_n as the n -th iterate of φ . We denote by φ_0 the identity function on \mathbb{D} . When φ is any analytic self-map of \mathbb{D} , we call $\zeta \in \overline{\mathbb{D}}$ a fixed point of φ if $\lim_{r \rightarrow 1} \varphi(r\zeta) = \zeta$. It is well-known that for an analytic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$, if φ is neither the identity map nor an elliptic automorphism of \mathbb{D} , then there is a point w of $\overline{\mathbb{D}}$ so that the sequence of iterates of φ converges uniformly to w on compact subsets of \mathbb{D} . Moreover, w is the unique fixed point of φ in $\overline{\mathbb{D}}$ for which $|\varphi'(w)| \leq 1$. We say that the unique fixed point w is the Denjoy-Wolff point of φ . Furthermore, if $w \in \mathbb{D}$ is the Denjoy-Wolff point of φ , then $|\varphi'(w)| < 1$.

If $A \subseteq \mathbb{C}$, the convex hull of A , denoted by $\text{Hull}(A)$. If T is a bounded linear operator on a Hilbert space H , the numerical range of T is the set $W(T) = \{\langle Tf, f \rangle : \|f\| = 1\}$. The set $W(T)$ is convex. Its closure contains $\sigma(T)$ and $\sigma_p(T) \subseteq W(T)$. The numerical radius $w(T)$ is given by $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. It is known that $w(T) \leq \|T\| \leq 2w(T)$. There are some interesting papers where the numerical range of composition operators on H^2 was investigated (see [4], [5] and [19]). Moreover, Gunatillake et al. wrote a nice paper about the numerical range of weighted composition operators on H^2 ([15]).

In Section 2, we work on the numerical range of $C_{\psi, \varphi}$, when φ is δ_r -conformal (it will be defined in the first paragraph of Section 2). In [15, Theorem 5.10] the authors claimed that if φ is δ_r -conformal with $-1 \leq r < 0$ and has a fixed point in \mathbb{D} , then 0 is an interior point of $W(C_{\psi, \varphi})$, but we provide a counterexample to [15, Theorem 5.10(1)] in which φ is δ_r -conformal with $-1 \leq r < 0$ and ψ is a fixed function closely related to φ as given in Example 2.1. In Theorem 2.2, we rewrite [15, Theorem 5.10] in the correct form and we clarify the result from [15, Theorem 5.10(1)] by giving a more precise description of the position of 0 in the numerical range of $C_{\psi, \varphi}$. Also we point out that the result from [15, Lemma 3.16] that holds for rotations can be extended to dilations with basically the same proof. Then, in Proposition 2.4, we show that $W(C_{\psi, w\varphi})$ is a disk centered at 0, for $w \in \partial\mathbb{D}$, w not a root of unity, which is an improvement of [15, Theorem 3.17]. Next for a weighted composition operator $C_{\psi, \varphi}$ that φ is an elliptic automorphism with rotation parameter w (it will be defined in the paragraph before the statement of Theorem 2.5), we find a function $\tilde{\psi}$ that $C_{\psi, \varphi}$ and $C_{\tilde{\psi}, w\varphi}$ are unitarily equivalent and in Theorem 2.5, we see that instead of finding $W(C_{\psi, \varphi})$ we should investigate the numerical ranges of $C_{\tilde{\psi}, w\varphi}$.

In Section 3, we determine the norms of some weighted composition operators $C_{\psi, \varphi}$ (see Theorem 3.1) and we use Theorem 3.1 to prove that their numerical radii are equal to their norms, when φ has a fixed point on $\partial\mathbb{D}$ (see Theorem 3.2). Finally, we break hermitian weighted composition operators into three cases and we determine their numerical ranges. In Theorem 3.5, we prove that the numerical range of a hermitian isometric weighted composition operator is a closed line segment with endpoints $\frac{-c}{\sqrt{1-|a_0|^2}}$ and $\frac{c}{\sqrt{1-|a_0|^2}}$. Theorem 3.6 computes the numerical range of a compact hermitian weighted composition operator $C_{\psi, \varphi}$ and states that its numerical range depends on the signs of $\psi(w)$ and $\varphi'(w)$, where w is the Denjoy-Wolff point of φ . Moreover, Theorem 3.8 states that the numerical range of a hermitian weighted composition operator $C_{\psi, \varphi}$ for φ a parabolic non-automorphism is an open line segment with endpoints 0 and $\frac{\psi(0)}{1-|\varphi(0)|}$.

2. φ is δ_r -conformal

For $r \in \overline{\mathbb{D}}$, suppose that δ_r is the dilation which is defined by $\delta_r(z) = rz$. We say a map φ is δ_r -conformal if $\varphi = \alpha^{-1} \circ \delta_r \circ \alpha$, where α is an automorphism of \mathbb{D} (in the case that $r \in \mathbb{D}$, Bourdon et al. in [5] called φ

conformal dilation). We know that for some $p \in \mathbb{D}$ and $w \in \partial\mathbb{D}$, $\alpha = \rho_w \circ \alpha_p$, where ρ_w is the rotation which is defined by $\rho_w(z) = wz$. Then it is not hard to see that $\varphi = \alpha_p \circ \rho_{\bar{w}} \circ \delta_r \circ \rho_w \circ \alpha_p = \alpha_p \circ \delta_r \circ \alpha_p$. It is easy to see that $\varphi'(p) = r$. Note that δ_r is a δ_r -dilation because $\delta_r = \alpha_0 \circ \delta_r \circ \alpha_0$. We call φ a positive conformal dilation when $0 < r < 1$. Bourdon et al. in [5, Theorem 4.4] showed that if φ is neither the identity map nor a positive conformal dilation and has a nonzero fixed point in \mathbb{D} , then 0 is an interior point of $W(C_\varphi)$. After that Gunatillake et al. in [15, Theorem 5.10] proved the similar result for weighted composition operators by the methods outlined in [5, Theorem 4.4], but in Theorem 2.2, we rewrite this result with a new idea and also we show that [15, Theorem 5.10] is not correct for φ and ψ in the third part of Theorem 2.2. We must add that [15, Theorem 5.10] was not used in other parts of that paper; apart from this error, as matter of fact nothing can decline the value of this excellent paper. Moreover, if φ is the identity map and $\psi \in H^\infty$, then by [17, Corollary 2], $W(C_{\psi,\varphi}) = \text{Hull}(\psi(\mathbb{D}))$, but in Theorem 2.2 we do not assume that φ is the identity map. Before stating Theorem 2.2, we prefer to give a counterexample for [15, Theorem 5.10].

Example 2.1. Let $\varphi(z) = \frac{1}{2} + \frac{-z/4}{1-z/2}$ and $\psi(z) = \frac{1}{1-z/2}$. By [8, Theorem 2.1], $C_{\psi,\varphi}$ is a hermitian weighted composition operator i.e., $C_{\psi,\varphi}^* = C_{\psi,\varphi}$. It is not hard to see that $\frac{3-\sqrt{5}}{2}$ is the Denjoy-Wolff point of φ . Moreover, $\varphi'(z) = \frac{-1}{(2-z)^2}$. Then $\varphi'(\frac{3-\sqrt{5}}{2}) < 0$. It follows that φ is not a positive conformal dilation. We have $W(C_{\psi,\varphi})$ is a subset of real numbers because $C_{\psi,\varphi}$ is hermitian (see [16, Theorem 1.2-2, p. 7]). It shows that 0 is not an interior point of $W(C_{\psi,\varphi})$, but Gunatillake et al. stated in the first part of [15, Theorem 5.10] that 0 is an interior point of $W(C_{\psi,\varphi})$.

Theorem 2.2. Suppose that $C_{\psi,\varphi}$ is bounded and $\varphi(p) = p$, when $p \in \mathbb{D}$. Assume that $\tilde{\psi} = c \frac{1-\bar{p}z}{1-\bar{p}rz} \circ \alpha_p$, where c is constant and $r \in \overline{\mathbb{D}}$. Then the following statements hold.

- (1) If φ is not δ_r -conformal with $-1 \leq r \leq 1$, then 0 is an interior point of $W(C_{\psi,\varphi})$.
- (2) If $\psi \neq \tilde{\psi}$ and φ is δ_r -conformal with $-1 \leq r \leq 0$, then 0 is an interior point of $W(C_{\psi,\varphi})$.
- (3) If $\psi = \tilde{\psi}$ and φ is δ_r -conformal with $-1 \leq r \leq 0$, then $W(C_{\psi,\varphi})$ is a closed line segment with endpoints cr and c .
- (4) If $\psi = \tilde{\psi}$ and φ is δ_r -conformal with $0 < r < 1$, then $W(C_{\psi,\varphi})$ is a half-open line segment with endpoints 0 and c that $c \in W(C_{\psi,\varphi})$.

Proof. Let $\varphi(p) = p$. Suppose that $\psi_p := K_p/\|K_p\|$. By [3, Theorem 6], C_{ψ_p,α_p} is unitary, that is, $C_{\psi_p,\alpha_p}^* C_{\psi_p,\alpha_p} = C_{\psi_p,\alpha_p} C_{\psi_p,\alpha_p}^* = I$. Let σ , g and h be the Cowen auxiliary functions for α_p . Since $T_h^* T_{\psi_p}^* = (T_{\psi_p} h)^* = 1/\|K_p\|$, $\sigma = \alpha_p$ and $C_{\alpha_p}^* = T_g C_{\alpha_p} T_h^*$, we have

$$C_{\psi_p,\alpha_p}^* C_{\psi,\varphi} C_{\psi_p,\alpha_p} = C_{\alpha_p}^* T_{\psi_p}^* T_{\psi} C_{\varphi} T_{\psi_p} C_{\alpha_p} = \frac{1}{\|K_p\|} T_{g \cdot \psi \circ \alpha_p \cdot \psi_p \circ \varphi \circ \alpha_p} C_{\alpha_p \circ \varphi \circ \alpha_p}.$$

Since unitary equivalence does not change the numerical range, $W(C_{\psi,\varphi}) = W(C_{q,\alpha_p \circ \varphi \circ \alpha_p})$, where $q = \frac{1}{\|K_p\|} g \cdot \psi \circ \alpha_p \cdot \psi_p \circ \varphi \circ \alpha_p$. It is easy to see that $\alpha_p \circ \varphi \circ \alpha_p$ fixes zero. After some computation, we can see that if φ is δ_r -conformal with $-1 \leq r \leq 1$, then $\frac{1}{\|K_p\|} g \cdot \psi \circ \alpha_p \cdot \psi_p \circ \varphi \circ \alpha_p = \frac{1-\bar{p}rz}{1-\bar{p}z} \cdot \psi \circ \alpha_p$. Hence in this case, $W(C_{\psi,\varphi}) = W(C_{q,rz})$, where $q = \frac{1-\bar{p}rz}{1-\bar{p}z} \cdot \psi \circ \alpha_p$.

- (1) Since $W(C_{\psi,\varphi}) = W(C_{q,\alpha_p \circ \varphi \circ \alpha_p})$, the result follows from [15, Lemma 5.5].
- (2) Note that $\psi = \tilde{\psi}$ if and only if $q \equiv c$. The conclusion follows from [15, Lemma 5.6] and the fact that $W(C_{\psi,\varphi}) = W(C_{q,\alpha_p \circ \varphi \circ \alpha_p})$.
- (3) If $\psi = \tilde{\psi}$, then $q \equiv c$. We can see that $W(C_{\psi,\varphi}) = cW(C_{rz})$. The result follows immediately from [19, Proposition 2.1] and [19, Proposition 2.2] (see also [5, p. 417]).
- (4) By the argument stated in part (3) and [19, Proposition 2.2], it is clear. \square

In the second part of [15, Theorem 5.10], the numerical range of $C_{\psi,\varphi}$ was found, when φ is constant. We must state that Theorem 2.2 also works for constant functions φ . Note that if $\varphi \equiv p$, then φ is δ_r -conformal with $r = 0$ and $\varphi = \alpha_p \circ \delta_0 \circ \alpha_p$. If $r = 0$, then $\hat{\psi} = \mu K_p$, where μ is constant and $\hat{\psi}$ was defined in the statement of Theorem 2.2. Therefore, by the third part of Theorem 2.2, if $\varphi \equiv p$ and $\psi \equiv \mu K_p$, then $0 \in W(C_{\psi,\varphi})$, but 0 is not an interior point of $W(C_{\psi,\varphi})$. Moreover, if $\varphi \equiv p$ and $\psi \neq \mu K_p$, then by the second part of Theorem 2.2, 0 is an interior point of $W(C_{\psi,\varphi})$. Furthermore, by the proof of Theorem 2.2, we saw that instead of finding the numerical range of weighted composition operator with δ_r -conformal composition map, we should investigate the numerical range of weighted composition operator $C_{\psi,\varphi}$ with $\varphi(z) = rz$ for $r \in \overline{\mathbb{D}}$. Gunatillake et al. in the third section of [15] studied $W(C_{\psi,\varphi})$, where φ is a rotation. In the following remark we give information about sets contained in $W(C_{\psi,\varphi})$, when φ is a dilation. Since for any constant α , $W(C_{\alpha\psi,\varphi}) = \alpha W(C_{\psi,\varphi})$, in the next remark, we assume that $\psi(0) = 1$.

Remark 2.3. Let $\psi(z) = 1 + \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots$ and $\varphi(z) = rz$ for $r \in \mathbb{D}$. Then by the similar proof which was seen in [15, Lemma 3.16], for a nonnegative integer n and a positive integer m , $W(C_{\psi,\varphi})$ contains the ellipse with foci r^n and r^{n+m} , and major axis $\sqrt{|r^n - r^{n+m}|^2 + |r^n \hat{\psi}_m|^2}$ and minor axis $|r^n \hat{\psi}_m|$. Moreover, for $0 < r < 1$, if there are a nonnegative integer n and a positive integer m that

$$\frac{r^n + r^{n+m}}{2} < \frac{\sqrt{|r^n - r^{n+m}|^2 + |r^n \hat{\psi}_m|^2}}{2}, \quad (1)$$

then 0 is an interior point of $W(C_{\psi,\varphi})$. After some calculations, we can see that Equation (1) is equivalent to $r^m < (\frac{|\hat{\psi}_m|}{2})^2$. Hence we see that for $0 < r < 1$ if there is a positive integer m such that $r^m < (\frac{|\hat{\psi}_m|}{2})^2$, then 0 is an interior point of $W(C_{\psi,\varphi})$.

In [15, Theorem 3.17], the disks centered at 0 were found which contained in $W(C_{\psi,\varphi})$ for $\varphi(z) = wz$ and w not a root of unity. In the following proposition, we show that in this case $W(C_{\psi,\varphi})$ is a disk centered at 0.

Proposition 2.4. Let $\psi \in H^\infty$. Suppose that $|w| = 1$ and w is not a root of unity. Then $W(C_{\psi,wz})$ is a disk centered at 0.

Proof. Assume that w is not a root of unity. For an arbitrary function $f \in H^2$ with $\|f\| = 1$, we see

$$\langle C_{\psi,wz} T_z f, T_z f \rangle = \langle T_\psi T_{wz} C_{wz} f, T_z f \rangle = \langle T_z^* T_{wz} T_\psi C_{wz} f, f \rangle = w \langle C_{\psi,wz} f, f \rangle.$$

We know that T_z is an isometry. Hence $wW(C_{\psi,wz}) \subseteq W(C_{\psi,wz})$. Since the set $\{w^n : n = 0, 1, \dots\}$ is dense in the unit circle, $\overline{W(C_{\psi,wz})}$ is a closed disk centered at 0. The result follows from the convexity of $W(C_{\psi,wz})$. \square

Suppose that φ is an elliptic automorphism with the fixed point $p \in \mathbb{D}$. We know that φ must have the form $\varphi = \alpha_p \circ \delta_w \circ \alpha_p$, where $|w| = 1$, $\varphi'(p) = w$ and δ_w is the rotation. We call w the rotation parameter of φ . In [4, Theorem 4.1] Bourdon et al. showed that for an elliptic automorphism φ with a rotation parameter w which is not a root of unity, $\overline{W(C_\varphi)}$ is a disk centered at the origin. In the following theorem we establish that this result holds for weighted composition operator. Furthermore, Gunatillake et al. in the third section of [15] computed $W(C_{\psi,\varphi})$ with rotational composition maps; Theorem 2.5 shows that those interesting results which were obtained in the third section of [15] will be useful in order to investigate $W(C_{\psi,\varphi})$, when φ is an elliptic automorphism.

Theorem 2.5. Suppose that φ is an elliptic automorphism with the fixed point $p \in \mathbb{D}$ and the rotation parameter w . Assume that $\psi \in H^\infty$. Then $W(C_{\psi,\varphi}) = W(T_{\psi \circ \alpha_p, \frac{1-\overline{p}wz}{1-\overline{p}z}} C_{wz})$. Moreover, if w is not a root of unity, then $W(C_{\psi,\varphi})$ is a disk centered at 0.

Proof. By the proof of Theorem 2.2, $W(C_{\psi,\varphi}) = W(T_{\psi \circ \alpha_p, \frac{1-\overline{p}wz}{1-\overline{p}z}} C_{wz})$. Furthermore if w is not a root of unity, then the result follows from the preceding proposition. \square

Bourdon et al. in [5] showed that for any analytic self-map φ which is not the identity, 0 lies in the closure of $W(C_\varphi)$. After that Gunatillake et al. in [15] proved that the similar result holds for weighted composition operators. Moreover, Gunatillake et al. in [15] stated that for a weighted composition operator $C_{\psi,\varphi}$, $\psi(p)$ belongs to $W(C_{\psi,\varphi})$, when $p \in \mathbb{D}$ is the fixed point of φ . In Proposition 2.6, we show that $2\psi(\zeta)/(1+\varphi'(\zeta))$ belongs to the closure of $W(C_{\psi,\varphi})$, when $\zeta \in \partial\mathbb{D}$ is the fixed point of φ .

It is well-known and easy to prove that $C_{\psi,\varphi}^*(K_w) = \overline{\psi(w)}K_{\varphi(w)}$ for any $w \in \mathbb{D}$; in the proof of Proposition 2.6, we use this fact.

Proposition 2.6. Suppose that φ is an analytic self-map of \mathbb{D} with the fixed point ζ in $\partial\mathbb{D}$ and $\psi \in H^\infty$. If φ has a finite angular derivative at ζ , then $2\psi(\zeta)/(2|\varphi'(\zeta)| + 1 - \overline{\varphi'(\zeta)})$ belongs to the closure of $W(C_{\psi,\varphi})$. In particular, if $\varphi'(\zeta) = 1$, then $\psi(\zeta)$ belongs to the closure of $W(C_{\psi,\varphi})$.

Proof. For $0 < r < 1$, we can see that

$$\langle C_{\psi,\varphi}^* k_{r\zeta}, k_{r\zeta} \rangle = (1 - |r\zeta|^2) \overline{\psi(r\zeta)} \langle K_{\varphi(r\zeta)}, K_{r\zeta} \rangle = \overline{\psi(r\zeta)} \frac{1 - |r\zeta|^2}{1 - \overline{\varphi(r\zeta)}r\zeta}.$$

By the Julia-Carathéodory Theorem, we have

$$\begin{aligned} \lim_{r \rightarrow 1} \frac{1 - \overline{\varphi(r\zeta)}r\zeta}{1 - |r\zeta|^2} &= \lim_{r \rightarrow 1} \frac{1 - |\varphi(r\zeta)|^2}{1 - |r\zeta|^2} + \lim_{r \rightarrow 1} \frac{|\varphi(r\zeta)|^2 - \overline{\varphi(r\zeta)}r\zeta}{1 - |r\zeta|^2} \\ &= |\varphi'(\zeta)| + \lim_{r \rightarrow 1} \overline{\varphi(r\zeta)} \frac{\varphi(r\zeta) - r\zeta}{1 - |r\zeta|^2} \\ &= |\varphi'(\zeta)| + \overline{\zeta} \lim_{r \rightarrow 1} \left(\frac{\varphi(r\zeta) - \zeta}{1 - r^2} + \frac{\zeta - r\zeta}{1 - r^2} \right) \\ &= |\varphi'(\zeta)| + \frac{\overline{\zeta}}{2} \left(\zeta \lim_{r \rightarrow 1} \frac{\varphi(r\zeta) - \zeta}{\zeta - r\zeta} + \zeta \right) \\ &= |\varphi'(\zeta)| + \frac{\overline{\zeta}}{2} (-\varphi'(\zeta)\zeta + \zeta) \\ &= |\varphi'(\zeta)| + \frac{1 - \varphi'(\zeta)}{2} \\ &= \frac{2|\varphi'(\zeta)| + 1 - \varphi'(\zeta)}{2}. \end{aligned}$$

We know that $W(C_{\psi,\varphi}^*) = \{\overline{z} : z \in W(C_{\psi,\varphi})\}$ and this yields the desired result. \square

The function g is called an inner function if it is a bounded analytic function on the unit disk such that $\lim_{r \rightarrow 1} |g(re^{i\theta})| = 1$ almost everywhere.

Bourdon et al. in [5, p. 439] showed that if φ has a Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$ with $\varphi'(\zeta) < 1$, then 0 is an interior point of $W(C_\varphi)$. In Remark 2.7, we add some conditions on φ and ψ and we investigate when 0 lies in the interior of $W(C_{\psi,\varphi})$.

Remark 2.7. Suppose that φ is analytic in a neighborhood of the closed unit disk, maps the unit disk to itself, and has Denjoy-Wolff point ζ on the unit circle with $\varphi'(\zeta) < 1$. Assume that φ is not inner. Suppose that ζ is the only fixed point of φ in the closed unit disk. Assume that $\psi \in H^\infty$ is continuous at ζ and $\psi(\zeta) \neq 0$. By [11, Exercise 7.5.2], [10, Theorem 4] and [10, Corollary 16], $\sigma_p(C_{\psi,\varphi}) = \sigma_p(\psi(\zeta)C_\varphi)$. Invoking [11, Lemma 7.24],

$$\{\lambda : |\psi(\zeta)|\varphi'(\zeta)^{1/2} < |\lambda| < |\psi(\zeta)|\varphi'(\zeta)^{-1/2}\} \subseteq \sigma_p(C_{\psi,\varphi}).$$

Since $\sigma_p(C_{\psi,\varphi}) \subseteq W(C_{\psi,\varphi})$, we can see that $W(C_{\psi,\varphi})$ contains the open disk with radius $|\psi(\zeta)|\varphi'(\zeta)^{-1/2}$ centered at 0. Proposition 2.6 implies that $2\psi(\zeta)/(1 + \varphi'(\zeta))$ belongs to the closure of $W(C_{\psi,\varphi})$ (note that $\varphi'(\zeta) > 0$). It is easy to see that $2\varphi'(\zeta)^{1/2} < 1 + \varphi'(\zeta)$ and so $\frac{2|\psi(\zeta)|}{1+\varphi'(\zeta)} < |\psi(\zeta)|\varphi'(\zeta)^{-1/2}$. Since $W(C_{\psi,\varphi})$ contains the open disk with radius $|\psi(\zeta)|\varphi'(\zeta)^{-1/2}$ centered at 0, $2\psi(\zeta)/(1 + \varphi'(\zeta)) \in W(C_{\psi,\varphi})$.

3. The functions φ and ψ have the same denominator

Note that if φ is a linear-fractional non-automorphism self-map of \mathbb{D} and $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then by [18, Corollary 2.2], $C_{\psi,\varphi} - \psi(\zeta)C_\varphi$ is compact. Thus, $\sigma_e(C_{\psi,\varphi}) = \sigma_e(\psi(\zeta)C_\varphi)$ and we use this fact frequently in this section.

Cowen in [9] provided an exact value of the norm of composition operators generated by self-maps of the disk of the form $\varphi(z) = az + b$ for $|a| + |b| \leq 1$. In the next theorem, we extend [9, Theorem 3] and give another proof. It is interesting that some important weighted composition operators on H^2 which have been studied recently satisfy the hypotheses of Theorem 3.1 (see [3, Theorem 10], [3, Theorem 6], [8, Theorem 2.1], [13, Proposition 2.9] and [12, Theorem 2.5]). We use Theorem 3.1 in the proof of Theorem 3.2 in order to find some weighed composition operators $C_{\psi,\varphi}$ such that $\|C_{\psi,\varphi}\| = w(C_{\psi,\varphi})$.

Theorem 3.1. Suppose that $\varphi(z) = \frac{az+b}{cz+d}$ is a linear-fractional non-automorphism self-map of \mathbb{D} . Assume that $\psi(z) = \frac{\mu}{cz+d}$, where μ is constant. Then the following are true.

- (a) If $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then $\|C_{\psi,\varphi}\| = |\mu|/(c\zeta + d)(-\bar{b}\eta + \bar{d})|^{1/2}$.
- (b) If $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, then $\|C_{\psi,\varphi}\| = |\mu|/|-\bar{b}ap - |b|^2 + \bar{d}cp + |d|^2|^{1/2}$, where $p \in \mathbb{D}$ is the fixed point of $\sigma \circ \varphi$, σ one of the Cowen auxiliary functions.

Proof. Since $\|C_{\psi,\varphi}\|^2 = \|C_{\psi,\varphi}C_{\psi,\varphi}^*\| = r(C_{\psi,\varphi}C_{\psi,\varphi}^*)$, by Cowen's adjoint formula, we have

$$\|C_{\psi,\varphi}\|^2 = \|T_\psi C_\varphi T_g C_\sigma T_h^* T_\psi^*\| = |\mu| \|T_{\psi \cdot g \circ \varphi} C_{\sigma \circ \varphi}\| = |\mu| r(T_{\psi \cdot g \circ \varphi} C_{\sigma \circ \varphi}).$$

(a) Assume that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. By [18, Proposition 3.5] and [18, Corollary 2.2], $r_e(T_{\psi \cdot g \circ \varphi} C_{\sigma \circ \varphi}) = |\psi(\zeta)g(\eta)|$ (note that by [18, Corollary 2.2], $T_{\psi \cdot g \circ \varphi} C_{\sigma \circ \varphi} - \psi(\zeta)g(\eta)C_{\sigma \circ \varphi}$ is compact). Moreover, [1, Lemma 4.1], [7, Corollary 6.2] and [18, Proposition 3.5] imply that for each $\lambda \in \sigma_p(T_{\psi \cdot g \circ \varphi} C_{\sigma \circ \varphi})$, $|\lambda| \leq |\psi(\zeta)g(\eta)|$. From [6, Proposition 6.7, p. 210] and [6, Proposition 4.4, p. 359], we infer that $\|C_{\psi,\varphi}\|^2 = |\mu\psi(\zeta)g(\eta)|$.

(b) Suppose that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Since $C_{\psi,\varphi}C_{\psi,\varphi}^*$ is compact ([11, p. 129]), by [6, Lemma 5.9, p. 48], $\|C_{\psi,\varphi}C_{\psi,\varphi}^*\| = \sup\{|\lambda| : \lambda \in \sigma_p(\overline{\mu}T_{\psi \cdot g \circ \varphi} C_{\sigma \circ \varphi})\}$. Since $\sigma \circ \varphi$ must have Denjoy-Wolff point $p \in \mathbb{D}$ and $|(\sigma \circ \varphi)'(p)| < 1$, by [14, Theorem 1], $\|C_{\psi,\varphi}\|^2 = |\mu\psi(p)g(p)|$. After some calculation, the result follows. \square

A linear-fractional self-map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ which has a unique fixed point $\zeta \in \partial\mathbb{D}$ is parabolic. The linear-fractional transformation $\tau(z) := (1 + \bar{\zeta}z)/(1 - \bar{\zeta}z)$ takes the unit disk onto the right half-plane Π and takes ζ to ∞ . Set $\phi := \tau \circ \varphi \circ \tau^{-1}$. Therefore, ϕ is a linear-fractional self-map of Π which fixes only

∞ , so it must have the form $\phi(z) = z + t$ for some complex number t , where $\operatorname{Re} t \geq 0$. Let us call t the translation number of φ . It is not hard to see that

$$\varphi(z) = \frac{(2-t)z + t\zeta}{2+t-t\zeta z}. \quad (2)$$

When $\operatorname{Re} t = 0$ we have a parabolic automorphism; otherwise the map is not an automorphism. Also in [20, p. 3] Shapiro showed that among the linear-fractional transformations fixing $\zeta \in \partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$.

A linear-fractional self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is hyperbolic if it has a fixed point $\zeta \in \overline{\mathbb{D}}$ and the other fixed point outside \mathbb{D} . The map φ is an automorphism if and only if two fixed points lie on $\partial\mathbb{D}$. Assume that $\zeta \in \partial\mathbb{D}$ is the Denjoy-Wolff point of φ . Then $\varphi'(\zeta) < 1$. It is easy to see that $\phi(w) = (\tau \circ \varphi \circ \tau^{-1})(w) = rw + t$, where $r = 1/\varphi'(\zeta)$ and $\operatorname{Re} t \geq 0$. Hence $\varphi(z) = \frac{(1+r-t)z + \zeta(r+t-1)}{(r-t-1)\zeta z + 1+r+t}$.

Theorem 3.2. Assume that $\varphi(z) = \frac{az+b}{cz+d}$ is a linear-fractional non-automorphism self-map of \mathbb{D} with the fixed point $\zeta \in \partial\mathbb{D}$. Suppose that $\psi(z) = \mu/(cz+d)$, where μ is constant. Then $w(C_{\psi,\varphi}) = \|C_{\psi,\varphi}\|$.

Proof. The proof is broken into three cases.

(a) Suppose that φ is parabolic with translation number t . Then $\varphi(z) = \frac{(2-t)z+t\zeta}{(2+t)-t\zeta z}$, $g(z) = \frac{1}{-t\zeta z+2+t}$ and $\psi(z) = \frac{\mu}{2+t-t\zeta z}$, where g is one of the Cowen auxiliary functions for φ . Theorem 3.1 shows that $\|C_{\psi,\varphi}\| = |\mu|/2$. Applying [18, Corollary 2.2] and [2, Theorem 2.8 (iv)], we have $\sigma_e(C_{\psi,\varphi}) = \{\psi(\zeta)e^{-\beta t} : \beta \geq 0\} \cup \{0\}$. Then $r(C_{\psi,\varphi}) \geq |\psi(\zeta)| = |\mu|/2$. We see that $w(C_{\psi,\varphi}) = \|C_{\psi,\varphi}\|$ because $r(C_{\psi,\varphi}) \leq w(C_{\psi,\varphi}) \leq \|C_{\psi,\varphi}\|$.

(b) Suppose that φ is hyperbolic with Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$. Then $\varphi(z) = \frac{(1+r-t)z + \zeta(r+t-1)}{(r-t-1)\zeta z + 1+r+t}$, $\psi(z) = \frac{\mu}{(r-t-1)\zeta z + 1+r+t}$ and the Cowen auxiliary function $g(z) = \frac{1}{-\zeta(r+t-1)z + 1+r+t}$. Theorem 3.1 shows that $\|C_{\psi,\varphi}\| = |\mu|/(2\sqrt{r})$. We have $\psi(\zeta) = \mu/(2r)$ and as we saw in the paragraph before Theorem 3.2, $r = 1/\varphi'(\zeta)$, so we may apply [18, Corollary 2.2] and [2, Theorem 2.8 (iii)] to conclude that $r_e(C_{\psi,\varphi}) = \frac{|\mu|}{2\sqrt{r}}$ and so $\|C_{\psi,\varphi}\| = w(C_{\psi,\varphi})$.

(c) Let φ be hyperbolic with Denjoy-Wolff point $w \in \mathbb{D}$ and another fixed point $\zeta \in \partial\mathbb{D}$. We know that $\|C_{\psi,\varphi}\| = \|C_{\psi,\varphi}^*\| = \|T_g C_\sigma T_{h\psi}^*\| = |\mu| \|T_g C_\sigma\|$ because $h\psi \equiv \mu$. One can easily see that the Cowen auxiliary function σ is hyperbolic with Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$ and σ has the form which was described before. Then $\sigma(z) = \frac{(1+r-t)z + \zeta(r+t-1)}{(r-t-1)\zeta z + 1+r+t}$, where $r = \sigma'(\zeta)^{-1}$. By the Cowen adjoint formula, we obtain that $\varphi(z) = \frac{(1+r-t)z - (r-t-1)\zeta}{-\zeta(r+t-1)z + 1+r+t}$ and so $\psi(z) = \frac{\mu}{-\zeta(r+t-1)z + 1+r+t}$ and $\psi(\zeta) = \mu/2$. By [18, Proposition 3.6], $r = \varphi'(\zeta)$. It is easy to see that $C_{g,\sigma}$ satisfies the hypotheses of Theorem 3.1, so $\|C_{\psi,\varphi}\| = |\mu|/(2\sqrt{r})$. We can see that [18, Corollary 2.2] and [2, Theorem 2.8 (ii)] show that $r_e(C_{\psi,\varphi}) = |\mu|/(2\sqrt{r})$. Then $w(C_{\psi,\varphi}) = |\mu|/(2\sqrt{r})$. \square

We know that if P and R are two operators, then $W(P \oplus R) = \operatorname{Hull}(W(P) \cup W(R))$ (see [15, p. 460]). We use this fact in the following corollary.

Corollary 3.3. Assume that $\varphi(z) = \frac{az}{cz+d}$ is a self-map of \mathbb{D} , where a, c, d are constant, $c \neq 0$ and $d \neq 0$. Then the half-open line segment $[0, 1)$ is in the interior of $W(C_\varphi)$ and $1 \in W(C_\varphi)$.

Proof. We know that zH^2 is a reducing subspace for C_φ . Suppose that $\psi(z) = \frac{a}{cz+d}$. We claim that $C_{\psi,\varphi}$ is unitarily equivalent to $C_\varphi|_{zH^2}$. It may be seen as follows. Let zf be an arbitrary element in zH^2 , we have

$$T_z C_{\psi,\varphi} T_z^*(zf) = T_z C_{\psi,\varphi}(f) = \frac{az}{cz+d} \cdot f \circ \varphi = \varphi \cdot f \circ \varphi = (z \cdot f) \circ \varphi = C_\varphi|_{zH^2}(zf).$$

We see that $C_\varphi = I \oplus C_\varphi|_{zH^2}$. Then $W(C_\varphi) = \operatorname{Hull}(W(I) \cup W(C_\varphi|_{zH^2})) = \operatorname{Hull}(\{1\} \cup W(C_{\psi,\varphi}))$. The point 0 is the fixed point of φ . We claim that φ is not δ_r -conformal. We may see it as follows. Assume that

φ is δ_r -conformal. Then $\varphi = \alpha_0 \circ rz \circ \alpha_0$ for some $r \in \overline{\mathbb{D}}$ and so $\varphi(z) = rz$ which is a contradiction. The first part of Theorem 2.2 shows that 0 is an interior point of $W(C_\varphi)$. For each $0 \leq x < 1$, x is an interior point of $W(C_\varphi)$ because $1 \in W(C_\varphi)$, $W(C_\varphi)$ is convex and 0 is an interior point of $W(C_\varphi)$. \square

Example 3.4. (a) Suppose that $\varphi(z) = \frac{(1-i)z+1+i}{3+i-(1+i)z}$. By Equation (2), it is not hard to see that φ is a parabolic non-automorphism with the fixed point 1 and the translation number $1+i$. Let $\psi(z) = \frac{1}{3+i-(1+i)z}$. Invoking Theorems 3.1 and 3.2, we have $\|C_{\psi,\varphi}\| = w(C_{\psi,\varphi}) = 1/2$.

(b) Let $\varphi(z) = \frac{(1+i)z-(i-1)}{-(3-i)z+(5-i)}$. It is easy to see that the fixed points of φ are 1 and $\frac{1-i}{3-i}$. Then φ is a hyperbolic non-automorphism. Let $\psi(z) = \frac{1}{-(3-i)z+(5-i)}$. We infer from Theorems 3.1 and 3.2, $\|C_{\psi,\varphi}\| = w(C_{\psi,\varphi}) = 1/\sqrt{8}$.

Cowen et al. in [8] characterized all hermitian weighted composition operators on H^2 . They showed that $C_{\psi,\varphi}$ is hermitian if and only if $\psi(z) = \frac{c}{1-\overline{a_0}z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a_0}z}$, where $a_0 = \varphi(0)$, $a_1 = \varphi'(0)$, $c = \psi(0)$ and a_1 and c are real. Moreover, [8, Corollary 2.3] states that φ maps \mathbb{D} into itself if and only if $|a_0| < 1$ and $-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2$. In order to find the numerical ranges of hermitian weighted composition operators on H^2 , we break the problem into three cases $a_1 = -1 + |a_0|^2$ (φ is an automorphism and $C_{\psi,\varphi}$ is a multiple of a hermitian isometric weighted composition operator), $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$ ($C_{\psi,\varphi}$ is a compact hermitian weighted composition operator) and $a_1 = (1 - |a_0|)^2$ (φ is a parabolic non-automorphism). In the rest of this section, we assume that a_0 , a_1 and c are as above. Note that if $C_{\psi,\varphi}$ is hermitian and $a_1 \neq |a_0|^2 - 1$, then φ and ψ satisfy the hypotheses of Theorem 3.1 and $\|C_{\psi,\varphi}\|$ can be computed by Theorem 3.1. Moreover, in the case that φ is a parabolic non-automorphism, Theorem 3.2 shows that $\|C_{\psi,\varphi}\| = w(C_{\psi,\varphi}) = \frac{|c|}{1-|a_0|}$ and also $W(C_{\psi,\varphi})$ will be found in Theorem 3.8. Furthermore, in the case that $C_{\psi,\varphi}$ is a compact hermitian weighted composition operator, Theorem 3.2 cannot compute $w(C_{\psi,\varphi})$, because by [8, p. 5778], φ maps the closed unit disk into the open unit disk. Hence we state Theorem 3.6 in order to find the numerical range of $C_{\psi,\varphi}$, when $C_{\psi,\varphi}$ is a compact hermitian weighted composition operator.

Theorem 3.5. *If $C_{\psi,\varphi}$ is a multiple of a hermitian isometric weighted composition operator, then $W(C_{\psi,\varphi})$ is a closed line segment with endpoints $\frac{-c}{\sqrt{1-|a_0|^2}}$ and $\frac{c}{\sqrt{1-|a_0|^2}}$.*

Proof. From [8, Theorem 3.1], $\frac{\sqrt{1-|a_0|^2}}{c}C_{\psi,\varphi}$ is a hermitian isometric weighted composition operator with spectrum $\{-1, 1\}$. It is easy to see that the closure of $W(\frac{\sqrt{1-|a_0|^2}}{c}C_{\psi,\varphi})$ is equal to $[-1, 1]$ (see [16, Theorem 1.4-4, p. 16]). By the fact that $\sigma_p(\frac{\sqrt{1-|a_0|^2}}{c}C_{\psi,\varphi}) \subseteq W(\frac{\sqrt{1-|a_0|^2}}{c}C_{\psi,\varphi})$ and -1 and 1 are the eigenvalues of $\frac{\sqrt{1-|a_0|^2}}{c}C_{\psi,\varphi}$ (see [8, Theorem 3.1]), we have $W(\frac{\sqrt{1-|a_0|^2}}{c}C_{\psi,\varphi}) = [-1, 1]$. It shows that $W(C_{\psi,\varphi})$ is the closed line segments with endpoints $\frac{c}{\sqrt{1-|a_0|^2}}$ and $\frac{-c}{\sqrt{1-|a_0|^2}}$. \square

Theorem 3.6. *Let $C_{\psi,\varphi}$ be a compact hermitian weighted composition operator. Suppose that φ has a Denjoy-Wolff point $w \in \mathbb{D}$. Then one of the following holds.*

- (1) *If $\psi(w) > 0$ and $\varphi'(w) > 0$, then $W(C_{\psi,\varphi}) = (0, \psi(w)]$.*
- (2) *If $\psi(w) > 0$ and $\varphi'(w) < 0$, then $W(C_{\psi,\varphi}) = [\psi(w)\varphi'(w), \psi(w)]$.*
- (3) *If $\psi(w) < 0$ and $\varphi'(w) < 0$, then $W(C_{\psi,\varphi}) = [\psi(w), \psi(w)\varphi'(w)]$.*
- (4) *If $\psi(w) < 0$ and $\varphi'(w) > 0$, then $W(C_{\psi,\varphi}) = [\psi(w), 0)$.*

Proof. By the proof of [8, Theorem 4.1],

$$\sigma_p(C_{\psi,\varphi}) = \{\psi(w), \psi(w)\varphi'(w), \dots, \psi(w)(\varphi'(w))^j, \dots\}$$

and $\sigma(C_{\psi,\varphi}) = \sigma_p(C_{\psi,\varphi}) \cup \{0\}$. We infer from [16, Theorem 1.2-2, p. 7] and [16, Theorem 1.4-4, p. 16] that $\text{Hull}(\sigma(C_{\psi,\varphi})) = \overline{W(C_{\psi,\varphi})} \subseteq (-\infty, +\infty)$. The result follows easily from the fact that $\sigma_p(C_{\psi,\varphi}) \subseteq W(C_{\psi,\varphi})$ and [16, Theorem 1.5-5, p. 20]. \square

Lemma 3.7. *Suppose that φ is a parabolic non-automorphism and $\psi \in H^\infty$ is nonzero. If $C_{\psi,\varphi}$ is hermitian, then $C_{\psi,\varphi}$ has no eigenvalues.*

Proof. Suppose that $C_{\psi,\varphi}$ is hermitian and has an eigenvalue α with eigenvector $q \in H^2$ for α . Since ψ is not the zero function, $\{a \in \mathbb{D} : \psi(a) = 0\}$ has no limit point in \mathbb{D} . If $\alpha = 0$, then $\psi \cdot q \circ \varphi$ is the zero function. It shows that $q \circ \varphi$ must be the zero function. Since φ is nonconstant, by the Open Mapping Theorem, q vanishes on a nonempty open subset of \mathbb{D} . Hence q must be the zero function which is a contradiction. Assume that t is the translation number of φ . By [7, Corollary 6.2], for each $\beta \geq 0$, $e^{-\beta t}$ is an eigenvalue of C_φ with eigenvector $f_\beta(z) = e^{-\beta(\frac{1+z}{1-z})}$. Moreover, Cowen in [7, Corollary 6.2] showed that $f_\beta \in H^\infty$. Now we claim that for each $\beta \geq 0$, $\alpha e^{-\beta t}$ is an eigenvalue of $C_{\psi,\varphi}$ with eigenvector $q f_\beta$. We have $C_{\psi,\varphi} q = \alpha q$ and $C_\varphi f_\beta = e^{-\beta t} f_\beta$. Since $f_\beta \in H^\infty$, we have $q f_\beta \in H^2$. We see

$$T_\psi C_\varphi (q f_\beta) = \psi \cdot (q \circ \varphi) \cdot (f_\beta \circ \varphi) = (\psi \cdot q \circ \varphi) \cdot (e^{-\beta t} f_\beta) = (\alpha q) \cdot (e^{-\beta t} f_\beta) = \alpha e^{-\beta t} (q f_\beta),$$

so $q f_\beta$ is an eigenvector for $C_{\psi,\varphi}$ with eigenvalue $\alpha e^{-\beta t}$. Since $C_{\psi,\varphi}$ is hermitian, [6, Proposition 5.7, p. 47] shows that if $\beta_1 \neq \beta_2$, then $q f_{\beta_1}$ and $q f_{\beta_2}$ are orthogonal which is a contradiction because H^2 is separable. \square

Assume that $C_{\psi,\varphi}$ is a hermitian weighted composition operator and φ is a parabolic non-automorphism. Then by [8], $\varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0} z}$, $\psi(z) = \frac{c}{1 - \overline{a_0} z}$ and $a_1 = (1 - |a_0|)^2$, where $0 < |a_0| < 1$. It is easy to see that $a_0/|a_0|$ is the fixed point of φ . By Equation (2), $\varphi(0) = \frac{ta_0}{(2+t)|a_0|}$, where t is the translation number of φ . Hence $a_0 = \frac{ta_0}{(2+t)|a_0|}$. It shows that t is a positive number $\frac{2|a_0|}{1-|a_0|}$. In the following theorem, we study $W(C_{\psi,\varphi})$, where $C_{\psi,\varphi}$ is a hermitian weighted composition operator and φ is a parabolic non-automorphism.

Theorem 3.8. *Suppose that $C_{\psi,\varphi}$ is a hermitian weighted composition operator and φ is a parabolic non-automorphism. Then $W(C_{\psi,\varphi})$ is an open line segment with endpoints 0 and $\frac{c}{1-|a_0|}$.*

Proof. First assume that $c > 0$. As we saw in the paragraph before the statement of Theorem 3.8, $\varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0} z}$, $\psi(z) = \frac{c}{1 - \overline{a_0} z}$ and $a_1 = (1 - |a_0|)^2$ and the translation number of φ is positive. Since $a_0 + \frac{(1-|a_0|)^2 z}{1 - \overline{a_0} z}$ is parabolic with the fixed point $a_0/|a_0|$ and the positive translation number t , by Lemma 3.7, [6, Proposition 4.6, p. 359], [2, Theorem 2.8 (iv)] (or [7, Corollary 6.2]) and [18, Corollary 2.2], $\sigma(C_{\psi,\varphi}) = \{\psi(\frac{a_0}{|a_0|})z : z \in \sigma_e(C_\varphi)\} = [0, \frac{c}{1-|a_0|}]$. It follows from [16, Theorem 1.4-4, p. 16] that $\overline{W(C_{\psi,\varphi})} = [0, \frac{c}{1-|a_0|}]$. Lemma 3.7 and [16, Theorem 1.5-5, p. 20] show that 0 and $\frac{c}{1-|a_0|}$ are not in $W(C_{\psi,\varphi})$. Hence $W(C_{\psi,\varphi}) = (0, \frac{c}{1-|a_0|})$.

In the case that $c < 0$, the idea of the proof is the same. \square

The hermitian weighted composition operators fall into three categories described in [8] which was stated in the paragraph before Theorem 3.5. In the following example, we give examples for these three categories.

Example 3.9. Suppose that $a_0 = 1/2$.

(a) If $\varphi(z) = \frac{1}{2} + \frac{\frac{-3}{4}z}{1 - \frac{1}{2}z}$ and $\psi(z) = \frac{1}{1 - \frac{1}{2}z}$, then $C_{\psi,\varphi}$ is a multiple of a hermitian isometric weighted composition operator. Then by Theorem 3.5, $W(C_{\psi,\varphi}) = \left[\frac{-2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right]$.

(b) Let $\varphi(z) = \frac{1}{2} + \frac{-\frac{1}{2}z}{1-\frac{1}{2}z}$ and $\psi(z) = \frac{1}{1-\frac{1}{2}z}$. Then $C_{\psi,\varphi}$ is a compact hermitian weighted composition operator. It is not hard to see that $\frac{7-\sqrt{33}}{4}$ is the Denjoy-Wolff point of φ and $\varphi'(\frac{7-\sqrt{33}}{4}) = \frac{-32}{(1+\sqrt{33})^2} < 0$ and $\psi(\frac{7-\sqrt{33}}{4}) = \frac{8}{1+\sqrt{33}} > 0$. Then Theorem 3.6 shows that $W(C_{\psi,\varphi}) = \left[\frac{-256}{(1+\sqrt{33})^3}, \frac{8}{1+\sqrt{33}} \right]$.

(c) Let $\varphi(z) = \frac{1}{2} + \frac{\frac{1}{4}z}{1-\frac{1}{2}z}$ and $\psi(z) = \frac{1}{1-\frac{1}{2}z}$. The map φ is a parabolic non-automorphism. Theorem 3.8 implies that $W(C_{\psi,\varphi}) = (0, 2)$.

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