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Non-normal type singular integral-differential equations by Riemann-Hilbert approach ☆

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ABSTRACT

This article deals with one class of singular integral-differential equations of non-normal type with convolution and Cauchy principal value integral in class $\{0\}$. By using Fourier transform, this classes of equations are transformed into a Riemann boundary value problem with nodal point, and we prove the existence of solutions and the Noether theory for the equations. For such equations, we propose one method different from classical one, and we obtain the general solutions and the conditions of solvability. All cases about the behaviors of the solution are considered at nodal points. Therefore, the theory of integral equations and the classical Riemann boundary value problems is extended further.

1. Introduction

Musknelishvilli [24] and Litvinchuk [21] studied the singular integral-differential equations in the case of normal type. Bologna [1] considered the solvability of singular integral equations in the class of differentiable function. Wöjcik, etc. [27] extended the results of [1,21,24] to the class of continuous function and the class of discontinuous coefficients. Later on, De-Bonis, etc. [3] discussed singular integral equations of convolution type with Cauchy kernel and constant coefficients. Recently, Li and Ren [10–12,15,20] dealed with some classes of singular integral equations of convolution type with singular kernel and obtained the solvability and the explicit solutions.

For operators containing both Cauchy principal value integral and convolution, the conditions of their Noethericity were discussed in [2,4] in more general cases.

The main aim of this paper is to extend the theory to the following singular integral-differential equations with convolution kernels of the form:

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$$
\begin{aligned}
&\sum_{j=0}^{n}\{a_j\omega^{(j)}(t)+b_j(\pi i)^{-1}\int_{-\infty}^{+\infty}\frac{\omega^{(j)}(s)}{s-t}ds+(2\pi)^{-\frac{1}{2}}\int_{0}^{+\infty}k_{j,1}(t-s)\omega^{(j)}(s)ds \\
&\qquad +(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{0}k_{j,2}(t-s)\omega^{(j)}(s)ds\}=g(t),\quad -\infty<t<+\infty
\end{aligned}
\tag{1.1}
$$

appear frequently in the practical problems, such as fluid dynamics, shell theory, underwater acoustics, theory of elasticity, and quantum mechanics [2,4,15]. In Eq. (1.1), $a_j,\ b_j\ (j=0,1,\ldots,n)$ are constants and b_j not all equal to zero simultaneously. The known functions $k_{j,1}(t),\ k_{j,2}(t), g(t)\in\{0\}$ (for notation $\{0\}$, see the following section 2), and an unknown function $\omega(t)$ as well as its derivatives $\omega^{(j)}(t)(j=1,2,\ldots,n)$ are required to be in $\{0\}$ too. For applications, the problem to find its solutions is very important.

Compared with the classical methods of solution for Eq. (1.1), we give new methods of solution. Via applying Fourier analysis theory, Eq. (1.1) is transformed into boundary value problems of analytic functions with discontinuous coefficients. By means of the classical Riemann boundary value problems, and of the principle of analytic continuation, we prove the existence of the solution and then give the explicit solution under certain conditions, in the case of non-normal type. At the end of the paper, by using the theory of integral equation and linear algebra, we study some properties of the solution at nodes. Thus, the classical theory of integral equations are enriched and generalized. Meanwhile, we also provide the methods of solution for other singular integral-differential equations with convolution, such as Wiener-Hopf equation and dual equations.

2. Definitions and lemmas

In this section we present some definitions and lemmas. It is necessary for us to introduce certain new classes of functions in advance and to point out some of their properties.

Definition 2.1. The Fourier transform $\mathbb{F}$ of $f(t)$ is denoted by

$$
\mathbb{F}[f](x):=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}f(t)e^{ixt}dt \tag{2.1}
$$

and the inverse transform $\mathbb{F}^{-1}$ of $F(x)$ is defined by

$$
\mathbb{F}^{-1}[F](t):=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}F(x)e^{-ixt}dx. \tag{2.2}
$$

Generally, we denote $F(x)=\mathbb{F}[f](x), f(t)=\mathbb{F}^{-1}[F](t)$, respectively.

The Fourier transforms used in this paper understood to be performed in $L^2(\mathbb{R})$ and the functions involved certainly belong to this space.

Definition 2.2. If $f(x)$ fulfills the following two conditions

(1) $f(x)\in H([-N,N])$ for any $N>0$, where H is the class of Hölder continuous functions;

(2) for any $x_1, x_2 \in \mathbb{R}\backslash[-N, N]$, there exists $B \in \mathbb{R}^+$ such that

$$|f(x_1) - f(x_2)| \leq B|\frac{1}{x_1} - \frac{1}{x_2}|^{\mu}, \quad \mu \in (0,1), \tag{2.3}$$

then we call $f(x) \in \widehat{H}^{\mu}(\mathbb{R})$, or, $f(x) \in \widehat{H}^{\mu}$. For simplification, we also write $f(x) \in \widehat{H}$.

The concepts of classes $\{0\}$ and $\{\{0\}\}$ are introduced as follows.

Definition 2.3. Assume that $F(x)$ satisfies the following conditions:

(1) $F(x) \in \widehat{H}$; (2) $F(x) \in L^2(\mathbb{R})$,

we say that $F(x)$ belongs to class $\{\{0\}\}$, where $L^2(\mathbb{R})$ denotes the space of Lebesgue integrable functions on $\mathbb{R}$ with the standard norm

$$||F||_2 = (\int_{-\infty}^{+\infty} | F(x) |^2 \, dx)^{\frac{1}{2}}.$$

Obviously, $\{\{0\}\} = L^2(\mathbb{R}) \cap \widehat{H} \subset L^2(\mathbb{R}) \cap H$.

Definition 2.4. If $F(x) \in \{\{0\}\}$, we say that $f(t) = \mathbb{F}^{-1}[F](t)$ belongs to class $\{0\}$.

Definition 2.5. The Cauchy principal value integral operator T is introduced as follows

$$(Tf)(t) = \text{P.V.}(\pi i)^{-1} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t} d\tau. \tag{2.4}$$

Definition 2.6. For two functions $f(t)$ and $g(t)$, their convolution is given by formula

$$(f * g)(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} f(t-s)g(s)ds. \tag{2.5}$$

Then it is well known that

$$\mathbb{F}(f * g) = \mathbb{F}f \cdot \mathbb{F}g = FG, \tag{2.6}$$

where $F(x) = \mathbb{F}[f](x), G(x) = \mathbb{F}[g](x)$.

Denote

$$f_{\pm}(t) = \frac{1}{2}(\operatorname{sgn} t \pm 1)f(t), \tag{2.7}$$

obviously, $f(t) = f_+(t) - f_-(t)$.

Lemma 2.1. *Assume that $f^{(j)}(t) \in \{0\}$ $(j = 1, 2, \ldots, n)$. Then for any $j \in \{1, 2, \ldots, n\}$, we have*

$$\mathbb{F}[f_{\pm}^{(j)}](x) = (-ix)^j F^{\pm}(x) - (2\pi)^{-\frac{1}{2}} \sum_{m=0}^{j-1} (-ix)^m f^{(j-m-1)}(0), \tag{2.8}$$

where $F^{\pm}(x) = \mathbb{F}[f_{\pm}](x)$.

Proof. Note that $F^+(\infty) = F^-(\infty) = 0$. By induction on j and integration by parts, the conclusion can be proved.

Lemma 2.2. *Let* $f^{(j)}(t) \in \{0\} (j = 0, 1, \dots, n)$, *then*

$$\mathbb{F}[f^{(j)}](x) = (-ix)^j F(x). \tag{2.9}$$

Proof. By (2.7) and (2.8), we have

$$\mathbb{F}[f^{(j)}](x) = \mathbb{F}[f_+^{(j)}](x) - \mathbb{F}[f_-^{(j)}](x) = (-ix)^j F^+(x) - (-ix)^j F^-(x) = (-ix)^j F(x).$$

Lemma 2.3. *(See [27].) If* $f(t) \in \{0\}$, $\mathbb{F}f(0) = 0$, *then*

$$\mathbb{F}T[f](x) = -sgnx F(x), \tag{2.10}$$

where $F(x) = \mathbb{F}[f](x)$.

Lemma 2.4. *Let* $f^{(j)}(t) \in \{0\} (j = 0, 1 \dots n)$, *then we have*

$$\mathbb{F}[Tf^{(j)}](x) = -(-ix)^j sgnx F(x). \tag{2.11}$$

Proof. By (2.9) and (2.10), we can obtain the conclusion.

Lemma 2.5. *If* $f(t) \in \{0\}$, $\mathbb{F}f(0) = 0$, *then* $T[f](t) \in \{0\}$.

Proof. From $f(t) \in \{0\}$, we have $F(x) \in \{\{0\}\}$. It follows from $F(\infty) = F(0) = 0$ that $F(x)\text{sgn}x \in \{\{0\}\}$. By (2.10), we have $\mathbb{F}Tf \in \{\{0\}\}$, therefore, $Tf \in \{0\}$.

By Lemma 2.5, it is trivial to see that T maps $\{0\}$ into $\{0\}$ and $T^2 = I$ (identity).

3. The solvability of equation (1.1)

In this section, we discuss the solutions and the solvable conditions to Eq. (1.1). In order to solve Eq. (1.1), we may write it as

$$\sum_{j=0}^{n} \{a_j \omega^{(j)}(t) + b_j T\omega^{(j)}(t) + k_{j,1} * \omega_+^{(j)}(t) + k_{j,2} * \omega_-^{(j)}(t)\} = g(t), \quad -\infty < t < +\infty. \tag{3.1}$$

By using the Bekya regularization method [8,22], Eq. (3.1) can be directly solved, namely, Eq. (3.1) is transformed to the classical Fredholm integral equations. In this paper, we shall apply the theory of complex analysis, Fourier transform and bilinear transform to solve Eq. (3.1). Here, our methods of solution is novel and effective, and the methods mentioned here may be also used to solve other singular integral-differential equations with convolution.

By (2.8)–(2.10), we take Fourier transforms on both sides of (3.1) and obtain the following Riemann-Hilbert problem (R-HP):

$$\Omega^+(x) = P(x)\Omega^-(x) + Q(x), \quad -\infty < x < +\infty, \tag{3.2}$$

where

$$Q(x) = \frac{G(x) + (2\pi)^{-\frac{1}{2}} \sum_{j=1}^{n} \sum_{m=0}^{j-1} (-ix)^m (K_{j,1}(x) - K_{j,2}(x))}{\sum_{j=0}^{n} (a_j - b_j \text{sgn } x + K_{j,1}(x))(-ix)^j};$$

$$P(x) = \frac{\sum_{j=0}^{n} (a_j - b_j \text{sgn } x + K_{j,2}(x))(-ix)^j}{\sum_{j=0}^{n} (a_j - b_j \text{sgn } x + K_{j,1}(x))(-ix)^j};$$

$$\Omega(x) = \mathbb{F}[\omega](x),\ G(x) = \mathbb{F}[g](x),\ K_{j,p}(x) = \mathbb{F}[k_{j,p}](x), \quad p \in \{1, 2\},\ j \in \{0, 1, 2, \ldots, n\}.$$

Since $k_{j,p}(t), g(t) \in \{0\}$, therefore their Fourier transforms belong to the class $\{\{0\}\}$. It is easy to see that (3.2) is a R-HP on the real axis.

Consider the linear fractional transformation

$$z = -\frac{i\xi}{\xi + i}, \tag{3.3}$$

and (3.3) maps the real axis $\mathbb{R}$ onto a circle $\Gamma : |\xi + \frac{i}{2}| = \frac{1}{2}$, which surrounds an interior region Σ^+ and an exterior region Σ^-, and maps the upper half-plane Z^+ and lower half-plane Z^- onto the conformal Σ^+, Σ^-, respectively.

Denote

$$\Omega(z) = F(\xi), \quad G(z) = C(\xi), \quad K_{j,p}(z) = E_{j,p}(\xi), \quad p \in \{1, 2\}, j \in \{0, 1, 2, \ldots, n\}.$$

Then (3.2) is readily reduced to the following R-HP

$$F^+(\tau) = U(\tau)F^-(\tau) + W(\tau), \qquad \tau \in \Gamma, \tag{3.4}$$

where

$$U(\tau) = P(x), \quad W(\tau) = Q(x);$$

$$U(\tau) = \frac{U_2(\tau)}{U_1(\tau)}, \quad W(\tau) = \frac{1}{U_1(\tau)}[C(\tau) + (2\pi)^{-\frac{1}{2}} \sum_{j=1}^{n} \sum_{m=0}^{j-1} (-\frac{\tau}{\tau + i})^m (E_{j,1}(\tau) - E_{j,2}(\tau))];$$

$$U_p(\tau) = \sum_{j=0}^{n} [a_j - b_j \delta(\tau) + E_{j,p}(\tau)](-\frac{\tau}{\tau + i})^j; \quad p \in \{1, 2\}, j \in \{0, 1, 2, \ldots, n\},$$

$$\delta(\tau) = \begin{cases} 1, & \tau \in \Gamma_1, \\ -1, & \tau \in \Gamma_2. \end{cases}$$

Γ_1, Γ_2 are the left half and the right half circles of Γ, respectively, and $\Gamma = \Gamma_1 \cup \Gamma_2$.

Note that Eqs. (3.1), (3.2), and (3.4) are equivalent to each other. We now solve (3.4) in the case of non-normal type, that is, $U(\tau)$ has some zero-points and pole-points on Γ. In fact, this case includes that in Refs. [21,24] as a special case, thus the results in this paper generalize ones in Refs. [1,3,21,24,27].

Let $U_1(\tau)$ have some zero points u_j $(j \in \{1, 2 \ldots, s\})$ with the orders α_j $(j \in \{1, 2 \ldots, s\})$ respectively. $U_2(\tau)$ has some zero points v_j $(j \in \{1, 2 \ldots, l\})$ with the orders β_j $(j \in \{1, 2 \ldots, l\})$ respectively. $U_1(\tau)$ and $U_2(\tau)$ have common and the same order zero points ϱ_j $(j \in \{1, 2 \ldots, q\})$ with the orders r_j $(j \in \{1, 2 \ldots, q\})$ respectively.

Again let

$$\Pi_1(\tau) = \prod_{j=1}^{s} (\tau - u_j)^{\alpha_j}, \quad \Pi_2(\tau) = \prod_{j=1}^{l} (\tau - v_j)^{\beta_j},$$

$$\sum_{j=1}^{s}\alpha_j = N_1, \quad \sum_{j=1}^{l}\beta_j = N_2, \quad \sum_{j=1}^{q}r_j = N_3.$$

Then (3.4) can be written as

$$F^+(\tau) = \frac{\Pi_2(\tau)}{\Pi_1(\tau)}U_0(\tau)F^-(\tau) + W(\tau), \quad \tau \in \Gamma, \tag{3.5}$$

where $U(\tau) = \frac{\Pi_2(\tau)}{\Pi_1(\tau)}U_0(\tau)$, and $U_0(\tau) \neq 0$ on $\Gamma_1 \cup \Gamma_2$. Since $\omega(t) \in \{0\}$, we have $\Omega(x) \in \{\{0\}\}$ and hence $F(\tau) \in H$ on Γ. From our previous discussions, we know that $F(\tau)$ is continuous and bounded near ϱ_j $(j \in \{1, 2 \dots, q\})$, thus the following Noether conditions of solvability must be satisfied

$$\{C(\tau) + (2\pi)^{-\frac{1}{2}}\sum_{j=1}^{n}[\sum_{m=0}^{j-1}(-\frac{\tau}{\tau+i})^m](E_{j,1}(\tau) - E_{j,2}(\tau))\}^{(k)}|_{\tau=\varrho_j} = 0 \tag{3.6}$$

for any $k \in \{0, 1, \dots, r_j - 1\}$, $j \in \{1, 2, \dots, q\}$.

Therefore, it follows from our previous discussions that the derivatives $C(\tau), E_{j,p}(\tau)$ $(p = 1, 2)$ must exist until order $r_j - 1$ $(j \in \{1, 2, \dots, q\})$ in the neighborhood of ϱ_j, and all order derivatives satisfy Hölder conditions.

In view of the values of $a_j \pm b_j$ $(j \in \{0, 1, 2, \dots, n\})$, without loss of generality, we only discuss the following two cases.

Case (1):

$$a_j - b_j \neq 0, \; a_j + b_j \neq 0, \quad j = 0, 1, 2, \dots, n; \tag{3.7}$$

and case (2):

$$a_j - b_j = 0, \; a_j + b_j \neq 0, \quad j = 0, 1, 2, \dots, n. \tag{3.8}$$

Other cases can be transformed to the cases (1) and (2), or similar to the discussion in [11,12,20].

3.1. *Solution of R-HP (3.5) under conditions (3.7)*

Under conditions (3.7), we know that (3.5) is a R-HP with discontinuous coefficients and nodal point $\tau = 0$.

Denote

$$\gamma_0 = \alpha_0 + i\beta_0 = \frac{1}{2\pi i}\{\log U_0(+0) - \log U_0(-0)\}, \tag{3.9}$$

in which we have taken the definite branch of $\log U_0(t)$.

Then we choose an integer μ, the index of (3.5), such that $0 \leq \alpha = \alpha_0 - \mu < 1$, and denote

$$\lambda = \gamma_0 - \mu = \alpha + i\beta_0. \tag{3.10}$$

Let

$$\Upsilon(\xi) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\log U(\tau)}{\tau - \xi}d\tau, \quad \xi \notin \Gamma, \tag{3.11}$$

note that $\Upsilon(\xi)$ is analytic in Σ^+ and Σ^-, and $\Upsilon(\infty) = 0$. In general, $\Upsilon(\xi)$ has a singularity of logarithmic type at $\xi = 0$. We define the following function:

$$X(\xi) = \begin{cases} e^{\Gamma(\xi)}, & \xi \in \Sigma^+; \\ \xi^{-\mu} e^{\Gamma(\xi)}, & \xi \in \Sigma^-, \end{cases} \tag{3.12}$$

then, we have

$$X^+(\tau) = U(\tau) X^-(\tau), \quad \tau \in \Gamma. \tag{3.13}$$

To solve R-HP (3.5), we first consider its homogeneous problem given by the equality

$$F^+(\tau) = \frac{\Pi_2(\tau)}{\Pi_1(\tau)} U_0(\tau) F^-(\tau), \quad \tau \in \Gamma. \tag{3.14}$$

By using the generalized Liouville theorem [22], (3.14) has the following general solution:

$$F_*(\xi) = \begin{cases} \Pi_2(\xi) X(\xi) p_{\mu - N_1}(\xi), & \xi \in \Sigma^+; \\ \Pi_1(\xi) X(\xi) p_{\mu - N_1}(\xi), & \xi \in \Sigma^-. \end{cases} \tag{3.15}$$

In (3.15), when $\mu - N_1 \geq 0$, $p_{\mu-N_1}(\xi)$ is an arbitrary polynomial with degree $\mu - N_1$; and when $\mu - N_1 < 0$, $p_{\mu-N_1}(\xi) \equiv 0$, in this case, (3.14) has a unique solution (zero solution).

In the following, we shall solve R-HP (3.5). To do this, we define the following Cauchy principal value integral:

$$\Theta(\xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Pi_1(\tau) W(\tau)}{X^+(\tau)(\tau - \xi)} d\tau, \quad \xi \notin \Gamma. \tag{3.16}$$

It is clear that $\frac{\Pi_1(\tau)W(\tau)}{X^+(\tau)} \in H_0$, where H_0 denotes the class of Hölder continuous functions on any closed interval exterior to $\tau = 0$.

Since $\Omega(x) \in \{\{0\}\}$, $\Theta(\xi)$ is bounded and has no singularity at μ_j, v_k ($j \in \{1, 2, \ldots, s\}, k \in \{1, 2, \ldots, l\}$). We first discuss a particular solution to the problem (3.5), hence we need to introduce a Hermite interpolation polynomial $B_\rho(\xi)$ ($\rho = N_1 + N_2 - 1$), which has some zero-points μ_j ($j \in \{1, 2, \ldots, s\}$) and v_k ($k \in \{1, 2, \ldots, l\}$) with the orders α_j, β_k, respectively. It is easy to prove that $B_\rho(\xi)$ exists uniquely. Similar to the discussion in [16,25,26], we consider the following function

$$D(\xi) = \begin{cases} \frac{1}{\Pi_1(\xi)} X(\xi)(\Theta(\xi) - B_\rho(\xi)), & \xi \in \Sigma^+; \\ \frac{1}{\Pi_2(\xi)} X(\xi)(\Theta(\xi) - B_\rho(\xi)), & \xi \in \Sigma^- \end{cases} \tag{3.17}$$

and we easily prove that (3.17) is a particular solution to (3.5). By using the theory of linear algebra, we obtain a general solution of (3.5)

$$F(\xi) = D(\xi) + F_*(\xi), \quad \xi \in \Sigma^\pm,$$

that is,

$$F(\xi) = \begin{cases} \frac{1}{\Pi_1(\xi)} X(\xi)(\Theta(\xi) - B_\rho(\xi)) + X(\xi)\Pi_2(\xi) p_{\mu-N_1}(\xi), & \xi \in \Sigma^+; \\ \frac{1}{\Pi_2(\xi)} X(\xi)(\Theta(\xi) - B_\rho(\xi)) + X(\xi)\Pi_1(\xi) p_{\mu-N_1}(\xi), & \xi \in \Sigma^-. \end{cases} \tag{3.18}$$

Since $F(\xi) \in H$, thus $F(\xi)$ has no singularity at $\tau = \mu_j$, $\tau = v_k$ $(j \in \{1, 2, \ldots, s\}, k \in \{1, 2, \ldots, l\})$. Therefore, we also have the following Noether conditions of solvability

$$\begin{aligned} &\int_\Gamma \frac{\Pi_1(\tau)W(\tau)}{X^+(\tau)(\tau-\mu_j)^p} d\tau = 0; \\ &\int_\Gamma \frac{\Pi_1(\tau)W(\tau)}{X^+(\tau)(\tau-\nu_r)^q} d\tau = 0 \end{aligned} \tag{3.19}$$

for any $j \in \{1, 2, \ldots, s\}$, $p \in \{0, 1, 2, \ldots, \alpha_j\}$; $r \in \{1, 2, \ldots, l\}$, $q \in \{0, 1, 2, \ldots, \beta_r\}$.

We take the boundary values for $F(\xi)$ in (3.18) and obtain

$$\begin{aligned} F^+(\tau) &= \frac{1}{2}W(\tau) + \frac{1}{\Pi_1(\tau)}X^+(\tau)(\Theta(\tau) - B_\rho(\tau)) + X^+(\tau)\Pi_2(\tau)p_{\mu-N_1}(\tau); \\ F^-(\tau) &= -\frac{\Pi_1(\tau)W(\tau)}{2\Pi_2(\tau)P_0(\tau)} + \frac{1}{\Pi_2(\tau)}X^-(\tau)(\Theta(\tau) - B_\rho(\tau)) + X^-(\tau)\Pi_1(\tau)p_{\mu-N_1}(\tau) \end{aligned} \tag{3.20}$$

for any $\tau \in \Gamma$.

Thus, we get the explicit solutions of (3.5) as follows

$$F(\tau) = F^+(\tau) - F^-(\tau), \quad \tau \in \Gamma. \tag{3.21}$$

By (3.18) and (3.20), we know that if $N_1 - \mu - 1 > 0$, then $D(\xi)$ has a pole point with the order $N_1 - \mu - 1$ at ∞. In order that $F(\xi)$ is bounded at $\xi = \infty$, if we assume that $B_\rho(\xi)$ given by the equality

$$B_\rho(\xi) = e_0\xi^\rho + e_1\xi^{\rho-1} + \ldots + e_\rho,$$

then one must have

$$e_0 = e_1 = \ldots = e_{N_1-\mu-2} = 0, \tag{3.22}$$

where e_j are constants. Therefore, we have the following conclusions:

(1) when $\rho - (N_1 - \mu - 1) = N_2 + \mu > -1$, (3.5) has always a solution;

(2) when $N_2 + \mu \leq -1$ and the following conditions of solvability

$$\int_\Gamma \frac{\Pi_1(\tau)W(\tau)}{X^+(\tau)} \tau^{k-1} d\tau = 0, \quad k = 1, 2, \ldots, -N_2 - \mu \tag{3.23}$$

are fulfilled, (3.5) has a solution.

Therefore, we have

Theorem 3.1. *Assume that (3.6) and $a_j - b_j \neq 0, a_j + b_j \neq 0$ $(j \in \{0, 1, 2, \ldots, n\})$ are fulfilled, the general solution of problem (3.5) is given by formula (3.21), where $X^\pm(\xi)$ are expressed by (3.12), and $p_{\mu-N_1}(\xi)$ is an arbitrary polynomial with degree $\mu - N_1$. If $\mu - N_1 \geq 0$, (3.21) contains $\mu - N_1$ arbitrary constants; if $\mu - N_1 < 0$, then $p_{\mu-N_1}(\xi) \equiv 0$, and (3.5) has a unique solution. Once $F(\xi)$ is obtained, we can determine $\Omega(s)$, therefore Eq. (1.1) has a solution given by $\omega(t) = \mathbb{F}^{-1}\Omega(s)$.*

3.2. Solution of R-HP (3.5) under conditions (3.8)

Under conditions (3.8), we can easily verify that (3.5) is a R-HP with discontinuous coefficients and nodal points $\tau = 0, -i$.

At node $\tau = -i$, it is easy to know that $U(-i+0) = 1$. But, for $U(-i-0)$ (see [20] for the definitions of $U(-i \pm 0)$), we have

(I) $U(-i-0) = C$ $(C \neq 0, \infty)$.
(II) $U(-i-0) = 0$.
(III) $U(-i-0) = \infty$.

At node $\tau = 0$, without loss of generality, we assume that $E_{0,1}(0) \neq 0, a_0 + b_0 + E_{0,1}(0) \neq 0$. Therefore, we need to consider the following three cases (T_1), (T_2), (T_3):

(T_1) $a_0 + b_0 + E_{0,2}(0) \neq 0$, $E_{0,2}(0) \neq 0$, that is, $U_0(+0) \neq 0$, $U_0(-0) \neq 0$.
(T_2) $E_{0,2}(0) = 0$, $a_0 + b_0 + E_{0,2}(0) \neq 0$, that is, $U_0(+0) \neq 0$, $U_0(-0) = 0$.
(T_3) $a_0 + b_0 + E_{0,2}(0) = 0$, $E_{0,2}(0) \neq 0$, that is, $U_0(+0) = 0$, $U_0(-0) \neq 0$.

Denote $C_1 = -i, C_2 = 0$. Since C_1, C_2 are the nodes of (3.5), according to the method used in [7,17], we can obtain that, near $C_k (k = 1, 2)$,

$$U_0(\tau) = U_k(\tau)(\tau - C_k)^{\alpha_k}, \tag{3.24}$$

where

$$\alpha_k = \begin{cases} \beta_k^{(1)}, & \tau \in \Gamma_1, \\ \beta_k^{(2)}, & \tau \in \Gamma_2; \end{cases} \qquad U_k(\tau) = \begin{cases} U_k^{(1)}(\tau), & \tau \in \Gamma_1, \\ U_k^{(2)}(\tau), & \tau \in \Gamma_2, \end{cases}$$

$U_k(\tau) \neq 0$ $(k = 1, 2)$ on Γ and $\beta_k^{(j)}$ $(k, j = 1, 2)$ are real numbers.

Since $U_k(\tau) \neq 0$ $(k = 1, 2)$, we can take a single-valued continuous branch of $\log U_0(\tau)$ on Γ and introduce the following piece-wise holomorphic function:

$$\Lambda(\xi) = \frac{1}{2\pi i} \int_\Gamma \frac{\log U_0(\tau)}{\tau - \xi} d\tau, \quad \xi \notin \Gamma. \tag{3.25}$$

Note that, $\Lambda(\xi)$ has a singularity of logarithmic type at $\xi = C_1, C_2$. By using Sokhotski-Plemelj formula [3] for $\Lambda(\xi)$ in (3.25), we have, near C_k $(k = 1, 2)$,

$$\begin{aligned} \Lambda^+(\xi) &= \Psi_k^+(\xi) \log(\xi - C_k) + \Delta_k^+(\xi), \quad \xi \in \Sigma^+; \\ \Lambda^-(\xi) &= \Psi_k^-(\xi) \log(\xi - C_k) + \Delta_k^-(\xi), \quad \xi \in \Sigma^-, \end{aligned} \tag{3.26}$$

where

$$\Psi_k^\pm(\xi) = \frac{\theta_k(\xi)(\beta_k^{(1)} - \beta_k^{(2)})}{2\pi} + \frac{\arg U_k^{(2)}(C_k + 0) - \arg U_k^{(1)}(C_k - 0)}{2\pi} + \frac{\beta_k^{(2)} \pm \beta_k^{(1)}}{2},$$
$$\theta_k(\xi) = \arg(\xi - C_k),\ \xi \in D_k^\pm \setminus \{C_k\},\ D_k^+ = \Sigma^+ \cap G_k,\ D_k^- = \Sigma^- \cap G_k,\ k = 1, 2,$$

and $G_k (k = 1, 2)$ are sufficiently small neighborhood of C_k. And $|\mathrm{Re}\{\Delta_k^\pm(\xi)\}| \leq M$ $(M \in \mathbb{R}^+)$, $\xi \in D_k^\pm$. Suppose that the tangential direction of Γ at C_k is the same as the forward direction of Γ.

It is readily seen that, when z (near C_k) moves along the positive direction around C_k, according to the values of $\Psi_k^{\pm}(C_k \pm 0)$, we can define the index of (3.5). Let

$$A_k = \min\{\Psi_k^+(C_k+0), \Psi_k^+(C_k-0), \Psi_k^-(C_k+0), \Psi_k^-(C_k-0)\}\ (k=1,2),$$

where the definitions of $\Psi_k^{\pm}(C_k \pm 0)$ are the same as $U(-i \pm 0)$. When A_k is an integer, we call C_k a special node and take $\sigma_k = -A_k$, otherwise, C_k is called ordinary node and take $\sigma_k = -[A_k]$. Moreover, we call

$$\mu = \sum_{j=1}^{2}(-\sigma_j) \tag{3.27}$$

the index of (3.5). We define the following function

$$X(\xi) = (\xi - C_1)^{\sigma_1}(\xi - C_2)^{\sigma_2} e^{\Gamma(\xi)}, \quad \xi \notin \Gamma. \tag{3.28}$$

It is not difficult to verify that (3.28) is the canonical function of R-HP (3.5).

By using the principle of analytic continuation, we take the boundary values for $X(\xi)$ in (3.28) and obtain

$$X^{\pm}(\tau) = (\tau - C_k)^{-\mu+\Psi_k^{\pm}(\tau)} M(\tau), \quad k=1,2, \tag{3.29}$$

where $M(\tau)$ and $\frac{1}{M(\tau)}$ are the bounded functions in the neighborhood of C_k $(k=1,2)$.

Under conditions (3.8), the problem to find solutions of R-HP (3.5) is similar to the discussion in Subsection 3.1, and further discussion is omitted here.

Therefore, we have

Theorem 3.2. *If (3.6) and $a_j - b_j = 0, a_j + b_j \neq 0$ $(j \in \{0,1,2,\dots,n\})$ are fulfilled, then (3.5) is solvable in class $\{0\}$, and all the conditions of solvability and its solution are similar to these in Theorem 3.1, that is, the solution of (3.5) is also expressed by (3.21), but $X^{\pm}(\xi)$ are given by (3.29).*

4. The properties of the solution at nodes 0, $-i$

In Section 3, we discuss the general solution and the conditions of solvability to problem (3.5). In this section we shall consider the situations of the solution at nodes $0, -i$.

4.1. The behaviors of the solution at $\tau = 0$

At node $\tau = 0$, according to our previous assumption, that is,

$$E_{0,1}(0) \neq 0, \quad a_0 + b_0 + E_{0,1}(0) \neq 0. \tag{4.1}$$

Therefore, we need to discuss three cases mentioned above: the cases $(T_1), (T_2), (T_3)$.

Now we set

$$R_2 = \frac{\arg U_2^{(2)}(-0) - \arg U_2^{(2)}(+0)}{2\pi}. \tag{4.2}$$

For case (T_1), we have

$$\Psi_2^+(+0) = \Psi_2^-(-0) = \Psi_2^+(-0) = \Psi_2^-(+0) = R_2.$$

For case (T_2), we have

$$\Psi_2^+(+0) = R_2 + \frac{3}{2}\beta_2^{(1)}, \quad \Psi_2^+(-0) = R_2 + \beta_2^{(1)};$$
$$\Psi_2^-(+0) = R_2 - \frac{1}{2}\beta_2^{(1)}, \quad \Psi_2^-(-0) = R_2 + \beta_2^{(1)},$$

and we also obtain the relation between the index and the node: $A_2 = \Psi_2^-(+0)$.

For case (T_3), we have

$$\Psi_2^+(+0) = R_2 - \frac{1}{2}\beta_2^{(2)}, \quad \Psi_2^+(-0) = R_2;$$
$$\Psi_2^-(+0) = R_2 + \frac{1}{2}\beta_2^{(2)}, \quad \Psi_2^-(-0) = R_2 - \beta_2^{(2)},$$

in this case, $A_2 = \Psi_2^-(-0)$.

We denote by ∂H the order of zero-point of the following function

$$H(\xi) = C(\xi) + (2\pi)^{-\frac{1}{2}} \sum_{j=1}^{n} \sum_{m=0}^{j-1} (-\frac{\xi}{\xi + i})^m (E_{j,1}(\xi) - E_{j,2}(\xi)) \tag{4.3}$$

at $\xi = 0$.

Since $\Omega(x) \in \{\{0\}\}$, so $\Omega(0) = 0$ and hence $F(0) = 0$, by (3.6) and (3.29), we have the following condition of solvability

$$\partial H \geq \max\{\sigma_2 + \Psi_2^+(+0),\ \sigma_2 + \Psi_2^+(-0)\}. \tag{4.4}$$

Let $\tau = 0$ be an ordinary nodal point. For cases (T_1), (T_2), (T_3), if the following condition

$$C(0) + (2\pi)^{-\frac{1}{2}} \sum_{j=1}^{n} (E_{j,1}(0) - E_{j,2}(0)) = 0 \tag{4.5}$$

is fulfilled, (3.5) is always solvable.

Let $\tau = 0$ be a special nodal point. For case (T_1), in addition to (4.5), the constant term of $p_{\mu - N_1}(\xi)$ should take the value

$$p_{\mu - N_1}(0) = (\Pi_1(0)\Pi_2(0))^{-1}(B_\rho(0) - F(0)). \tag{4.6}$$

For case (T_2), we have

$$\partial H \geq \sigma_2 + \Psi_2^+(+0). \tag{4.7}$$

For case (T_3), we get

$$\partial H \geq \sigma_2 + \Psi_2^+(-0). \tag{4.8}$$

Moreover, in cases (T_2) and (T_3), besides (4.7) and (4.8), (4.6) is also fulfilled.

4.2. The behaviors of the solution at $\tau = -i$

Now, we discuss the case of the solution at $\tau = -i$. Note that when $U(-i-0) = 1$, $\tau = -i$ is not a nodal point; when $U(-i-0) \neq 1$, $\tau = -i$ is a nodal point, in this case, we consider the following three cases.

(I) If $U(-i-0) = C$ ($\neq 0, 1, \infty$), then we have

$$\Psi_1^+(-i+0) = \Psi_1^-(-i-0) = \Psi_1^+(-i-0) = \Psi_1^-(-i+0) = R_1,$$

where R_1 is defined by

$$R_1 = \frac{\arg U_1^{(2)}(-i-0) - \arg U_1^{(2)}(-i+0)}{2\pi}.$$

(II) If $U(-i-0) = 0$, then we get

$$\Psi_1^+(-i+0) = R_1 + \frac{1}{2}\beta_1^{(1)}, \quad \Psi_1^-(-i+0) = R_1 + \frac{1}{2}\beta_1^{(1)};$$
$$\Psi_1^+(-i-0) = R_1 + \beta_1^{(1)}, \qquad \Psi_1^-(-i-0) = R_1 - \beta_1^{(1)}.$$

Therefore, we have $A_1 = \Psi_1^-(-i-0)$.

(III) If $U(-i-0) = \infty$, then

$$\Psi_1^+(-i+0) = R_1 + \frac{1}{2}\beta_1^{(1)}, \quad \Psi_1^-(-i+0) = R_1 + \frac{1}{2}\beta_1^{(1)};$$
$$\Psi_1^+(-i-0) = R_1 + \beta_1^{(1)}, \qquad \Psi_1^-(-i-0) = R_1 - \beta_1^{(1)},$$

and $A_1 = \Psi_1^+(-i-0)$.

In order to discuss the solvable conditions for (3.5), we need to check the behavior of $W(\tau)$ at $\tau = -i$ and denote

$$W(\tau) = \begin{cases} W_1(\tau), & \tau \in \Gamma_1, \\ W_2(\tau), & \tau \in \Gamma_2. \end{cases} \tag{4.9}$$

Under conditions $a_j - b_j = 0,\ a_j + b_j \neq 0$ $(j = 0, 1, \ldots n)$, owing to $\delta(-i-0) = 1$, we denote by $V_1(\tau)$ and $V_2(\tau)$ the numerator and the denominator of $W(\tau)$ multiplied by $(-\frac{\tau+i}{\tau})^{n-1}$ respectively, namely,

$$V_1(\tau) = C(\tau)(-\frac{\tau+i}{\tau})^{n-1} + (2\pi)^{-\frac{1}{2}} \sum_{j=0}^{n-1} \sum_{m=j+1}^{n} (-\frac{\tau+i}{\tau})^{n-m+j} (E_{j,1}(\tau) - E_{j,2}(\tau)), \tag{4.10}$$

$$V_2(\tau) = \sum_{j=0}^{n-1} (-\frac{\tau+i}{\tau})^{n-1-j} E_{j,1}(\tau) - (\frac{\tau}{\tau+i}) E_{n,1}(\tau). \tag{4.11}$$

We denote by $\partial W_1, \partial W_2, \partial V_1, \partial V_2$ the orders of zero-points of $W_1(\tau)$, $W_2(\tau)$, $V_1(\tau)$, $V_2(\tau)$ at $\tau = -i$, respectively. Because $W(\tau)$ is bounded on Γ, it is easy to obtain that

$$\partial W_2 \geq \Psi_1^+(-i+0) + \sigma_1, \quad \partial V_1 \geq \partial V_2 + \Psi_1^+(-i-0) + \sigma_1. \tag{4.12}$$

Following the method used in [9,13,18], we can deduce from (3.14) that

$$F(\tau) = \frac{(-1)^k}{2\pi i} \lim_{\tau \to -i} \frac{\Pi_1(\tau) W_k(\tau) \log(\tau + i)}{X^+(\tau)} + F_k^*(\tau) \tag{4.13}$$

for any $\tau \in \Gamma_k$ with $k = 1, 2$, where $F_k^*(\tau) \in H$ near $\tau = -i$ and its limit exists as $\tau \to -i$.

By (3.20) and (4.11), we have the following results.

For case (I), if

$$\lim_{\tau \to -i-0} \frac{E_{n,1}(\tau)}{\tau + i} = c, \tag{4.14}$$

then we have

$$p_{\mu - N_1}(-i) = (\Pi_1(-i)\Pi_2(-i))^{-1}(B_\rho(-i) - F(-i)), \tag{4.15}$$

where $c \neq 0, \infty$. If

$$\lim_{\tau \to -i-0} \frac{E_{n,1}(\tau)}{\tau + i} = 0, \tag{4.16}$$

owing to the boundedness of $W(\tau)$, we have $\partial V_1 > \partial V_2$ and (4.15).

For case (II), if

$$\lim_{\tau \to -i-0} \frac{E_{n,1}(\tau)}{\tau + i} = c, \tag{4.17}$$

similar to the case (I), we have the following solvable conditions (4.18) and (4.19)

$$\partial W_1 > \Psi_1^+(-i - 0) + \sigma_1, \quad \partial W_2 > \Psi_1^+(-i + 0) + \sigma_1, \tag{4.18}$$

and

$$p_{\mu - N_1}(-i) = (\Pi_1(-i)\Pi_2(-i))^{-1}(B_\rho(-i) - F_1^*(-i)), \tag{4.19}$$

where c is as the above. If

$$\lim_{\tau \to -i-0} \frac{E_{n,1}(\tau)}{\tau + i} = 0, \tag{4.20}$$

then the solvable conditions are (4.19) and the following (4.21)

$$\partial V_1 \geq \partial V_2 + \Psi_1^+(-i - 0) + \sigma_1, \quad \partial W_2 \geq \Psi_1^+(-i + 0) + \sigma_1. \tag{4.21}$$

For case (III), if

$$\lim_{\tau \to -i-0} \frac{E_{n,1}(\tau)}{\tau + i} = c, \tag{4.22}$$

then, (4.19) and the following (4.23) are fulfilled

$$\partial W_2 \geq \Psi_1^+(-i + 0) + \sigma_1, \tag{4.23}$$

where c is as the above. If

$$\lim_{\tau\to -i-0}\frac{E_{n,1}(\tau)}{\tau+i}=0, \tag{4.24}$$

then, (4.19) and the following (4.25) are fulfilled

$$\partial W_2 \geq \Psi_1^+(-i+0)+\sigma_1, \quad \partial V_1 > \partial V_2. \tag{4.25}$$

On the other hand, when $\delta(-i+0)=-1$, we let

$$\lim_{\tau\to -i+0}\frac{a_n+b_n+E_{n,1}(\tau)}{\tau+i}=c, \tag{4.26}$$

where $c\neq\infty$. Similar to the case $\lim_{\tau\to -i-0}\frac{E_{n,1}(\tau)}{\tau+i}$, we need to consider two cases: $c\neq 0$ and $c=0$ in (4.26). The remaining discussion is the same as the above.

From our previous discussions, we can state the main results of this paper in following form.

Theorem 4.1. *Under conditions (3.7) and (3.8), the necessary conditions of solvability to the Eq. (1.1) are (3.6) and (4.4). Assume that this is fulfilled. If $\tau=0$ is an ordinary nodal point of (3.5), then (4.5) is true. If $\tau=0$ is a special nodal point of (3.5), then (4.5) and (4.6) are true. If $\tau=-i$ is an ordinary nodal point of (3.5), then (4.12) is true. If $\tau=-i$ is a special nodal point of (3.5), then (4.5) is true. Assume that (4.5), (4.6) and (4.12) are fulfilled, Eq. (3.5) has the solutions and its solutions are given by (3.21). Once $F(x)$ is obtained, $\Omega(x)$ also is obtained and so the solution of Eq. (1.1) is of the form $\omega(t)=\mathbb{F}^{-1}[\Omega(x)]$.*

In this paper, we have solved Eq. (1.1) in class $\{0\}$. Indeed, it is possible to study the above mentioned equation in Clifford analysis, which is similar to that in [5,6,14,19,23]. Further discussion is omitted here.

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