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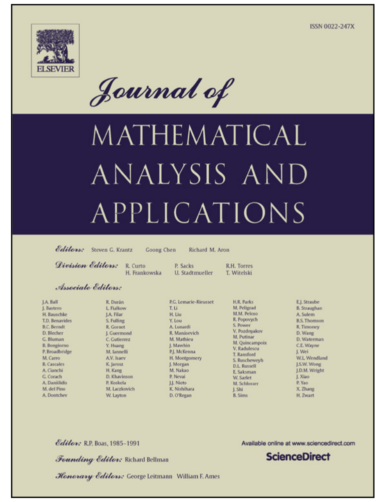

Explicit expressions for finite trigonometric sums

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EXPLICIT EXPRESSIONS FOR FINITE TRIGONOMETRIC SUMS

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ABSTRACT. In this paper, we perform a further investigation for finite trigonometric sums. We establish some connections between the higher-order trigonometric functions and Zeta functions. As applications, various known finite trigonometric sums are explicitly expressed as linear combinations of the Bernoulli and Euler polynomials and numbers.

1. INTRODUCTION

As is well known, the mathematical literature contains many evaluations of finite trigonometric sums of one sort such as (see, e.g., [10] and [33, p. 234])

$$\sum_{r=1}^{q-1}\cot^2\left(\frac{\pi r}{q}\right)=\frac{(q-1)(q-2)}{3}\quad(q\geq 2),\tag{1.1}$$

and

$$\sum_{r=1}^{q-1}\csc^2\left(\frac{\pi r}{q}\right)=\frac{(q-1)(q+1)}{3}\quad(q\geq 2).\tag{1.2}$$

It will become more difficult to find an explicit evaluation when we replace the power 2 on the left side of (1.1) and (1.2) by arbitrary positive even power, respectively. The earliest evaluation of finite trigonometric sums of another sort was posed by Eisenstein [22] in 1844 that for positive integers q, m with $1\leq m\leq q-1$,

$$\sum_{r=1}^{q-1}\sin\left(\frac{2\pi mr}{q}\right)\cot\left(\frac{\pi r}{q}\right)=q-2m,\tag{1.3}$$

which was first proved by Stern [32] in 1861. It is worth mentioning that Eisenstein [22] used (1.3) to give a proof of the law of quadratic reciprocity. Williams and Zhang [36] in 1994 used inductive hypotheses to generalize (1.3), and obtained that for positive integers q, m, n with $1\leq m\leq q-1$,

$$\begin{aligned}&\sum_{r=1}^{q-1}\sin\left(\frac{2\pi mr}{q}\right)\cot^{2n-1}\left(\frac{\pi r}{q}\right)\\&=\sum_{k=0}^{n-1}(-1)^{n-k}\frac{(2q)^{2n-2k-1}\widehat{A}(2n-1,2k)}{(2n-2k-1)!}B_{2n-2k-1}\left(\frac{m}{q}\right).\end{aligned}\tag{1.4}$$

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Here the rational numbers $\widehat{A}(n,k)$ are determined recursively for positive integer n and non-negative k with $0 \le k \le n-1$ by

$$\widehat{A}(n,k) = \frac{n-k-1}{n-1}\widehat{A}(n-1,k) - \widehat{A}(n-2,k-2) \quad (2 \le k \le n-2),$$

with the initial values $\widehat{A}(n,0) = 1$, $\widehat{A}(n,1) = 0$ and $\widehat{A}(n,n-1) = 0$ or $(-1)^{(n-1)/2}$ according to n is an even integer or n is an odd integer, and $B_n(x)$ are the Bernoulli polynomials given by the generating function (see, e.g., [1, 29])

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

In particular, the rational numbers $B_n = B_n(0)$ are called the Bernoulli numbers. The earliest evaluation known to us of a sum of the type (1.4) is attributed to Dowker, who first in detail examined and explicitly evaluated the sums

$$C_{2n}(q,m) := \sum_{r=1}^{q-1} \cos\left(\frac{2\pi mr}{q}\right)\csc^{2n}\left(\frac{\pi r}{q}\right), \tag{1.5}$$

where q is a positive integer with $q \ge 2$, and m is a non-negative integer such that $0 \le m \le q-1$. More precisely, the sums (1.5) appear in Dowker's [19, 20, 21] work in his theory of the Casimir effect, short-time expansion of the integrated heat kernel on locally flat generalized cones, and in connection with Verlinde's formula for the dimensions of vector bundles on moduli spaces.

Chu and Marini [14] in 1999 used generating functions to evaluate in closed form 24 different classes of finite trigonometric sums in terms of multiple sums of binomial coefficients. Berndt and Yeap [10] in 2002 used contour integration to establish some general theorems for cotangent sums and alternating cosecant sums, which include as special cases many explicit evaluations for finite trigonometric sums, and some reciprocity theorems for Dedekind sums stated in [9, 30]. For example, Berndt and Yeap [10, Corollaries 2.2 and 3.2] showed that for positive integers q, n with $q \ge 2$,

$$\frac{1}{q}\sum_{r=1}^{q-1}\cot^{2n}\left(\frac{\pi r}{q}\right) = (-1)^n - (-1)^n 2^{2n} \sum_{\substack{k_0+k_1+\cdots+k_{2n}=n \\ k_0,k_1,\ldots,k_{2n}\ge 0}} q^{2k_0-1}\prod_{j=0}^{2n}\frac{B_{2k_j}}{(2k_j)!}, \tag{1.6}$$

and

$$\begin{aligned}&\frac{1}{q}\sum_{r=1}^{q-1}(-1)^r\csc^{2n}\left(\frac{\pi r}{q}\right)\\ &= (-1)^n 2^{2n+1} \sum_{\substack{k_0+k_1+\cdots+k_{2n}=n \\ k_0,k_1,\ldots,k_{2n}\ge 0}} q^{2k_0-1}\prod_{j=0}^{2n}(2^{2k_j-1}-1)\frac{B_{2k_j}}{(2k_j)!}.\end{aligned} \tag{1.7}$$

Meanwhile, Berndt and Yeap [10, Theorem 4.1] used contour integration to study the sums on the left side of (1.4), and determined that for positive integers q, m, n

with $1 \leq m \leq q-1$,

$$\begin{aligned}&\sum_{r=1}^{q-1} \sin\left(\frac{2\pi mr}{q}\right)\cot^{2n-1}\left(\frac{\pi r}{q}\right)\\&= -2^{2n-1} \sum_{\substack{2k_1+\cdots+2k_{2n-1}+\mu+\upsilon=2n-1\\ k_1,\ldots,k_{2n-1},\mu,\upsilon\geq 0}} (-1)^{\frac{\mu+\upsilon-1}{2}} \frac{m^\mu}{\mu!}\frac{q^\upsilon B_\upsilon}{\upsilon!}\prod_{j=1}^{2n-1}\frac{B_{2k_j}}{(2k_j)!}. \quad (1.8)\end{aligned}$$

Two explicit evaluations for the sums (1.5) were also obtained by Berndt and Yeap [10] using contour integration. Cvijović and Srivastava [18] in 2012 used contour integration to further explore the sums (1.5), and derived the closed-form formulas for 12 different classes of finite trigonometric sums including the sums (1.5). All of Cvijović and Srivastava's [18] results involve the higher-order Bernoulli polynomials. Fonseca, Glasser and Kowalenko [24] in 2018 used integral approach to evaluate a trigonometric inverse power sum considered by Gardner [25] in 1969 and the case $m = 0$ in the sums (1.5) in terms of the symmetric polynomials over the set of quadratic powers up to $(n-1)^2$.

Motivated and inspired by the work of the above authors, in this paper we perform a further investigation for finite trigonometric sums. We establish some connections between the higher-order trigonometric functions and the period zeta function and the Lerch zeta function. As applications, various finite trigonometric sums studied by Williams and Zhang [36], Byrne and Smith [12], Chu and Marini [14], Berndt and Yeap [10], Cvijović and Srivastava [18], Kowalenko [24, 28], etc., are explicitly expressed as linear combinations of the Bernoulli and Euler polynomials and numbers. Moreover, we also depict that a finite trigonometric sum considered by Zhang and Lin [37] in 2018 is explicitly expressed by Dirichlet L-functions.

It is well known that the tangent function and the secant function can be converted by the cotangent function and the cosecant function, respectively. To avoid the repetition and tediousness of arguments, we confine our attention for finite trigonometric sums to only cotangent sums and cosecant sums, which are organized in the second section and in the third second, respectively.

2. Expressions for cotangent sums

For convenience, in the following we always denote by i the square root of -1 such that $\mathrm{i}^2 = -1$, and denote by $s(n,k)$ the Stirling numbers of the first kind given for non-negative integers n, k with $0 \leq k \leq n$ by the generating function (see, e.g., [16])

$$\frac{\big(\ln(1+t)\big)^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!}.$$

We also write, for positive integer n, $\alpha(n) = 0$ or $(-1)^{n/2}$ according to n is an odd integer or n is an even integer. We firstly state the following result.

Theorem 2.1. *Let n be a positive integer. Then, for real number a,*

$$\cot^n(a) = \alpha(n) + \sum_{j=-\infty}^{+\infty}\sum_{k=1}^{n}\frac{A(n,k)}{(a+j\pi)^k} \quad (a \neq 0, \pm\pi, \pm2\pi, \ldots), \tag{2.1}$$

and

$$\tan^n(a) = \alpha(n) + (-1)^n \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{n} \frac{A(n,k)}{\left(a + (j+\frac{1}{2})\pi\right)^k} \quad (a \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots), \tag{2.2}$$

where $A(n,k)$ is given for positive integers n,k with $1 \leq k \leq n$ by

$$A(n,k) = \mathrm{i}^{n-k} \frac{(k-1)!}{2^k} \sum_{l=k}^{n} \binom{n}{l} \frac{2^l s(l,k)}{(l-1)!}. \tag{2.3}$$

Proof. It is easily seen from Euler's formula $e^{\mathrm{i}a} = \cos(a) + \mathrm{i}\sin(a)$ that

$$\cot(a) = \frac{\cos(a)}{\sin(a)} = \mathrm{i}\frac{e^{\mathrm{i}a} + e^{-\mathrm{i}a}}{e^{\mathrm{i}a} - e^{-\mathrm{i}a}} = \mathrm{i}\left(\frac{2}{e^{2\mathrm{i}a} - 1} + 1\right), \tag{2.4}$$

and

$$\frac{1}{1 - e^{2\mathrm{i}a}} = \frac{\mathrm{i}}{2}\cot(a) + \frac{1}{2}. \tag{2.5}$$

It follows from (2.4) and the binomial theorem that for positive integer n,

$$\cot^n(a) = \mathrm{i}^n \sum_{l=0}^{n} \binom{n}{l} (-2)^l \frac{1}{(1 - e^{2\mathrm{i}a})^l}. \tag{2.6}$$

Let $\binom{\alpha}{l}$ be the binomial coefficients given for non-negative integer l and complex number α by

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{l} = \frac{\alpha(\alpha-1)\cdots(\alpha-l+1)}{l!} \quad (l \geq 1).$$

Since the binomial series is defined for complex number α by

$$(1+t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n,$$

so we obtain that for non-negative integer l,

$$\frac{1}{(1 - e^{2\mathrm{i}a})^l} = \sum_{j=0}^{\infty} \binom{-l}{j} (-e^{2\mathrm{i}a})^j = \sum_{j=0}^{\infty} \binom{j+l-1}{j} e^{2\mathrm{i}aj}, \tag{2.7}$$

and for positive integer k,

$$\frac{\partial^{k-1}}{\partial a^{k-1}}\left(\frac{1}{1 - e^{2\mathrm{i}a}}\right) = \sum_{j=0}^{\infty} (2\mathrm{i}j)^{k-1} e^{2\mathrm{i}aj}. \tag{2.8}$$

Notice that the Stirling numbers of the first kind can be characterized by the identity (see, e.g., [16, p. 213])

$$x(x-1)\cdots(x-l+1) = \sum_{k=0}^{l} s(l,k) x^k \quad (l \geq 1).$$

Hence, by taking $x = -j$ in above identity, in light of $s(l,0) = 0$ for positive integer l (see, e.g., [16, p. 214]), we get

$$(j+1)\cdots(j+l-1) = \sum_{k=1}^{l} (-1)^{l-k} s(l,k) j^{k-1} \quad (l \geq 1),$$

which can be rewritten as

$$\binom{j+l-1}{j} = \frac{1}{(l-1)!}\sum_{k=1}^{l}(-1)^{l-k}s(l,k)j^{k-1} \quad (l \geq 1). \tag{2.9}$$

If we apply (2.9) to (2.7), in view of (2.8), we discover that for positive integer l,

$$\frac{1}{(1-e^{2\mathrm{i}a})^l} = \frac{1}{(l-1)!}\sum_{k=1}^{l}(-1)^{l-k}s(l,k)\frac{1}{(2\mathrm{i})^{k-1}}\frac{\partial^{k-1}}{\partial a^{k-1}}\left(\frac{1}{1-e^{2\mathrm{i}a}}\right). \tag{2.10}$$

It follows from (2.5), (2.6) and (2.10) that

$$\begin{aligned}\cot^n(a) &= \mathrm{i}^n + \mathrm{i}^n\sum_{l=1}^{n}\binom{n}{l}\frac{2^l}{(l-1)!}\sum_{k=1}^{l}(-1)^k s(l,k)\frac{1}{(2\mathrm{i})^{k-1}} \\ &\quad \times \frac{\partial^{k-1}}{\partial a^{k-1}}\left(\frac{\mathrm{i}}{2}\cot(a)+\frac{1}{2}\right).\end{aligned}$$

Changing the order of the above summation gives

$$\begin{aligned}\cot^n(a) &= \mathrm{i}^n + \mathrm{i}^n\sum_{k=1}^{n}(-1)^k\frac{1}{(2\mathrm{i})^{k-1}}\frac{\partial^{k-1}}{\partial a^{k-1}}\left(\frac{\mathrm{i}}{2}\cot(a)+\frac{1}{2}\right) \\ &\quad \times \sum_{l=k}^{n}\binom{n}{l}\frac{2^l s(l,k)}{(l-1)!},\end{aligned}$$

which means that for positive integer $n \geq 2$,

$$\begin{aligned}\cot^n(a) &= \mathrm{i}^n - \mathrm{i}^n\left(\frac{\mathrm{i}}{2}\cot(a)+\frac{1}{2}\right)\sum_{l=1}^{n}\binom{n}{l}\frac{2^l s(l,1)}{(l-1)!} \\ &\quad + \mathrm{i}^n\sum_{k=2}^{n}(-1)^k\frac{1}{2^k\mathrm{i}^{k-2}}\frac{\partial^{k-1}}{\partial a^{k-1}}\big(\cot(a)\big)\sum_{l=k}^{n}\binom{n}{l}\frac{2^l s(l,k)}{(l-1)!}.\end{aligned} \tag{2.11}$$

Since $s(l,1) = (-1)^{l-1}(l-1)!$ for positive integer l (see, e.g., [16, p. 214]), so we have

$$\sum_{l=1}^{n}\binom{n}{l}\frac{2^l s(l,1)}{(l-1)!} = -\sum_{l=1}^{n}\binom{n}{l}(-2)^l = 1-(-1)^n \quad (n \geq 1), \tag{2.12}$$

which together with (2.11) yields that for positive integer $n \geq 2$,

$$\begin{aligned}\cot^n(a) &= \mathrm{i}^n - \mathrm{i}^n\left(\frac{\mathrm{i}}{2}\cot(a)+\frac{1}{2}\right)\big(1-(-1)^n\big) \\ &\quad + \mathrm{i}^n\sum_{k=2}^{n}(-1)^k\frac{1}{2^k\mathrm{i}^{k-2}}\frac{\partial^{k-1}}{\partial a^{k-1}}\big(\cot(a)\big)\sum_{l=k}^{n}\binom{n}{l}\frac{2^l s(l,k)}{(l-1)!}.\end{aligned} \tag{2.13}$$

It is well known that $\cot(a)$ has the following expression in partial fractions (see, e.g., [1, p. 75] or [31, p. 327])

$$\begin{aligned}\cot(a) &= \frac{1}{a} + 2a\sum_{j=1}^{\infty}\frac{1}{a^2-j^2\pi^2} \\ &= \sum_{j=-\infty}^{+\infty}\frac{1}{a+j\pi} \quad (a \neq 0, \pm\pi, \pm 2\pi, \ldots).\end{aligned} \tag{2.14}$$

Hence, from (2.14) we have

$$\begin{aligned}
&\mathrm{i}^n - \mathrm{i}^n\left(\frac{\mathrm{i}}{2}\cot(a) + \frac{1}{2}\right)\bigl(1-(-1)^n\bigr)\\
&= \mathrm{i}^n\frac{1+(-1)^n}{2} - \mathrm{i}^{n+1}\frac{1-(-1)^n}{2}\cot(a)\\
&= \mathrm{i}^n\frac{1+(-1)^n}{2} - \mathrm{i}^{n+1}\frac{1-(-1)^n}{2}\sum_{j=-\infty}^{+\infty}\frac{1}{a+j\pi}, \qquad (2.15)
\end{aligned}$$

and for positive integer k,

$$\frac{\partial^{k-1}}{\partial a^{k-1}}\bigl(\cot(a)\bigr) = (-1)^{k-1}(k-1)!\sum_{j=-\infty}^{+\infty}\frac{1}{(a+j\pi)^k}. \qquad (2.16)$$

Thus, by applying (2.15) and (2.16) to (2.13), in light of (2.12), we obtain (2.1). Clearly, $\cot(\pi/2+a) = -\tan(a)$ for $a \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$. It follows that replacing a by $\pi/2+a$ in (2.1) gives (2.2). This completes the proof of Theorem 2.1. $\square$

It becomes obvious from (2.1) and (2.3) that $A(n,k) = 0$ if $n \not\equiv k \pmod 2$. We next show some applications of Theorem 2.1. The following theorem can be regarded as the new development of the computational problem for a finite trigonometric sum considered by Zhang and Lin [37, Equation (1)].

Theorem 2.2. *Let q, n be positive integers with $q \geq 2$, and let χ be a non-principal Dirichlet character modulo q. Then*

$$\begin{aligned}
\sum_{r=1}^{q-1}\chi(r)\cot^n\left(\frac{\pi r}{q}\right) &= \sum_{k=1}^{n}\frac{q^k A(n,k)}{\pi^k}L(k,\chi)\\
&\quad + \chi(-1)\sum_{k=1}^{n}(-1)^k\frac{q^k A(n,k)}{\pi^k}L(k,\chi), \qquad (2.17)
\end{aligned}$$

where $A(n,k)$ is as in (2.3), and $L(s,\chi)$ is the Dirichlet L-function given for Dirichlet character χ modulo q and complex number $s = \sigma + it$ by (see, e.g., [6, 15])

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \quad (\sigma > 1).$$

Proof. Since χ is a non-principal Dirichlet character modulo q, so in this case $L(k,\chi)$ converges for positive integer k and

$$\sum_{r=1}^{q-1}\chi(r) = 0.$$

It follows that taking $a = \pi r/q$ in Theorem 2.1 and then making the operation $\sum_{r=1}^{q-1}\chi(r)$ on both sides of (2.1) gives

$$\begin{aligned}
\sum_{r=1}^{q-1}\chi(r)\cot^n\left(\frac{\pi r}{q}\right) &= \sum_{k=1}^{n}\frac{q^k A(n,k)}{\pi^k}\sum_{j=0}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(r)}{(r+qj)^k}\\
&\quad + \sum_{k=1}^{n}\frac{q^k A(n,k)}{\pi^k}\sum_{j=1}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(r)}{(r-qj)^k}. \qquad (2.18)
\end{aligned}$$

Observe that for positive integer k,

$$\sum_{j=0}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(r)}{(r+qj)^k}=\sum_{j=0}^{\infty}\sum_{r=1}^{q}\frac{\chi(r+qj)}{(r+qj)^k}=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^k}=L(k,\chi),$$

and

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(r)}{(r-qj)^k} &= \sum_{j=1}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(q-r)}{(q-r-qj)^k}\\ &= \chi(-1)(-1)^k\sum_{j=1}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(r)}{\big(r+q(j-1)\big)^k}\\ &= \chi(-1)(-1)^k\sum_{j=0}^{\infty}\sum_{r=1}^{q-1}\frac{\chi(r)}{(r+qj)^k}\\ &= \chi(-1)(-1)^kL(k,\chi).\end{aligned}$$

Thus, we know from (2.18) that Theorem 2.2 is complete. □

Corollary 2.3. *Let q,n be positive integers with $q\geq 2$, and let χ be a non-principal Dirichlet character modulo q. If $\chi(-1)=1$ then*

$$\sum_{r=1}^{q-1}\chi(r)\cot^{2n}\left(\frac{\pi r}{q}\right)=2\sum_{k=1}^{n}(-1)^{n-k}\frac{q^{2k}(2k-1)!\widetilde{A}(2n,2k)}{(2\pi)^{2k}}L(2k,\chi),\tag{2.19}$$

and if $\chi(-1)=-1$ then

$$\begin{aligned}&\sum_{r=1}^{q-1}\chi(r)\cot^{2n-1}\left(\frac{\pi r}{q}\right)\\ &=2\sum_{k=1}^{n}(-1)^{n-k}\frac{q^{2k-1}(2k-2)!\widetilde{A}(2n-1,2k-1)}{(2\pi)^{2k-1}}L(2k-1,\chi),\end{aligned}\tag{2.20}$$

where $\widetilde{A}(n,k)$ is given for positive integers n,k with $1\leq k\leq n$ by

$$\widetilde{A}(n,k)=\sum_{l=k}^{n}\binom{n}{l}\frac{2^ls(l,k)}{(l-1)!}.\tag{2.21}$$

Proof. We obtain from Theorem 2.2 that if $\chi(-1)=1$ then

$$\sum_{r=1}^{q-1}\chi(r)\cot^{n}\left(\frac{\pi r}{q}\right)=\sum_{k=1}^{n}\big(1+(-1)^k\big)\mathrm{i}^{n-k}\frac{q^k(k-1)!\widetilde{A}(n,k)}{2^k\pi^k}L(k,\chi),$$

and if $\chi(-1)=-1$ then

$$\sum_{r=1}^{q-1}\chi(r)\cot^{n}\left(\frac{\pi r}{q}\right)=\sum_{k=1}^{n}\big(1-(-1)^k\big)\mathrm{i}^{n-k}\frac{q^k(k-1)!\widetilde{A}(n,k)}{2^k\pi^k}L(k,\chi).$$

Since $A(n,k)=0$ when $n\not\equiv k \pmod 2$, which means $\widetilde{A}(n,k)=0$ when $n\not\equiv k$ (mod 2), so the desired results follow immediately. □

It is interesting to point out that one can find the corresponding relationships between cotangent sums and Gauss sums if we apply the special values of Dirichlet L-function at positive integers stated in [2, Theorem 1] to Corollary 2.3. For another finite trigonometric sums associated to Dirichlet character, one is referred to [7, 11]. We now use Theorem 2.1 to establish the following connections between the higher-order cotangent and tangent functions and the period zeta function.

Theorem 2.4. *Let q,n be positive integers with $q\geq 2$, and let θ_r be a real function defined on positive integer r. If $\theta_r\neq 0,\pm q,\pm 2q,\dots$ then*

$$\cot^n\left(\frac{\pi\theta_r}{q}\right) = \alpha(n)-\mathrm{i}^n\frac{1-(-1)^n}{2} +\mathrm{i}^n\sum_{k=1}^{n}(-1)^k\widetilde{A}(n,k)F(\theta_r/q,1-k), \tag{2.22}$$

and if $\theta_r\neq \pm\frac{q}{2},\pm\frac{3q}{2},\dots$ then

$$\tan^n\left(\frac{\pi\theta_r}{q}\right) = \alpha(n)+\mathrm{i}^n\frac{1-(-1)^n}{2} +(-\mathrm{i})^n\sum_{k=1}^{n}(-1)^k\widetilde{A}(n,k)F(1/2+\theta_r/q,1-k), \tag{2.23}$$

where $\widetilde{A}(n,k)$ is as in (2.21), and $F(a,s)$ is the period zeta function given for real number a and complex number $s=\sigma+\mathrm{i}t$ by (see, e.g., [6])

$$F(a,s)=\sum_{n=1}^{\infty}\frac{e^{2\pi\mathrm{i}na}}{n^s}\quad(\sigma>1).$$

Note that the series also converges for $\sigma>0$ when a is not an integer.

Proof. Clearly, from Theorem 2.1 we have

$$\cot^n\left(\frac{\pi\theta_r}{q}\right)=\alpha(n)+\sum_{k=1}^{n}A(n,k)\sum_{j=-\infty}^{+\infty}\frac{1}{\left(\frac{\pi\theta_r}{q}+j\pi\right)^k}\quad(\theta_r\neq 0,\pm q,\pm 2q,\dots). \tag{2.24}$$

It is easily seen from (2.16) that for positive integer k,

$$\begin{aligned}\sum_{j=-\infty}^{+\infty}\frac{1}{\left(\frac{\pi\theta_r}{q}+j\pi\right)^k} &= \frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial a^{k-1}}\big(\cot(a)\big)\bigg|_{a=\frac{\pi\theta_r}{q}}\\ &= \frac{(-1)^{k-1}}{\pi^{k-1}(k-1)!}\frac{\partial^{k-1}}{\partial a^{k-1}}\big(\cot(\pi a)\big)\bigg|_{a=\frac{\theta_r}{q}}.\end{aligned} \tag{2.25}$$

Since the period zeta function satisfies

$$F(a,s-1)=\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial a}\big(F(a,s)\big),$$

which holds true for all s by analytic continuation, so by using the above identity repeatedly, we discover

$$F\big(a,s-(k-1)\big)=\frac{1}{(2\pi\mathrm{i})^{k-1}}\frac{\partial^{k-1}}{\partial a^{k-1}}\big(F(a,s)\big)\quad(k\geq 1). \tag{2.26}$$

In particular, from $F(a,1) = -\ln(1-e^{2\pi i a})$ ($a \neq$ integer) and (2.5), we have

$$F(a,0) = \frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial a}\big(-\ln(1-e^{2\pi \mathrm{i}a})\big) = \frac{e^{2\pi \mathrm{i}a}}{1-e^{2\pi \mathrm{i}a}} = \frac{\mathrm{i}}{2}\cot(\pi a) - \frac{1}{2}. \tag{2.27}$$

By taking $s=0$ in (2.26), in light of (2.27), we get that for positive integer k,

$$\frac{\partial^{k-1}}{\partial a^{k-1}}\big(\cot(\pi a)\big)\bigg|_{a=\frac{\theta_r}{q}} = \frac{\delta_{1,k}}{\mathrm{i}} + 2^k\pi^{k-1}\mathrm{i}^{k-2}F(\theta_r/q, 1-k), \tag{2.28}$$

where $\delta_{1,k}$ is the Kronecker delta given by $\delta_{1,k}=1$ or 0 according to $k=1$ or $k\neq 1$. Hence, by combining (2.24), (2.25) and (2.28), in view of (2.3) and (2.12), we have

$$\begin{aligned}\cot^n\left(\frac{\pi\theta_r}{q}\right) &= \alpha(n) + \frac{1}{\mathrm{i}}A(n,1) + \sum_{k=1}^{n}(-1)^{k-1}\frac{2^k\mathrm{i}^{k-2}A(n,k)}{(k-1)!}F(\theta_r/q,1-k)\\ &= \alpha(n) - \mathrm{i}^n\frac{1-(-1)^n}{2} + \mathrm{i}^n\sum_{k=1}^{n}(-1)^k\widetilde{A}(n,k)F(\theta_r/q,1-k),\end{aligned}$$

as desired. If we substitute $q/2+\theta_r$ for θ_r in (2.22), we obtain (2.23). This completes the proof of Theorem 2.4. □

It follows that we improve Williams and Zhang's formula (1.4) and unify Berndt and Yeap's formulas (1.6) and (1.8), as follows.

Corollary 2.5. *Let q, n be positive integers with $q \geq 2$. Then, for non-negative integer m, if $1 \leq m \leq q-1$ then*

$$\begin{aligned}&\sum_{r=1}^{q-1}\sin\left(\frac{2\pi mr}{q}\right)\cot^{2n-1}\left(\frac{\pi r}{q}\right)\\ &= (-1)^n\sum_{k=1}^{n}\frac{q^{2k-1}\widetilde{A}(2n-1,2k-1)}{2k-1}B_{2k-1}\left(\frac{m}{q}\right),\end{aligned} \tag{2.29}$$

and if $0 \leq m \leq q-1$ then

$$\begin{aligned}&\sum_{r=1}^{q-1}\cos\left(\frac{2\pi mr}{q}\right)\cot^{2n}\left(\frac{\pi r}{q}\right)\\ &= (-1)^n(q\delta_{0,m}-1) - (-1)^n\sum_{k=1}^{n}\frac{\widetilde{A}(2n,2k)}{2k}\left\{q^{2k}B_{2k}\left(\frac{m}{q}\right) - B_{2k}\right\},\end{aligned} \tag{2.30}$$

where $\delta_{0,m}$ is the Kronecker delta given by $\delta_{0,m}=1$ or 0 according to $m=0$ or $m\neq 0$.

Proof. We know from [6, Theorem 12.13] or [15, Corollary 9.6.10] that for non-negative integer k,

$$\zeta(-k,x) = -\frac{1}{k+1}B_{k+1}(x), \tag{2.31}$$

where $\zeta(s,x)$ is the Hurwitz zeta function given for real number $x>0$, and complex number $s=\sigma+it$ by

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s} \quad (\sigma>1).$$

It follows from (2.31) that for positive integers k, r,

$$F(r/q, 1-k) = \sum_{j=0}^{\infty}\sum_{l=1}^{q} \frac{e^{\frac{2\pi \mathrm{i}(qj+l)r}{q}}}{(qj+l)^{1-k}} = -\frac{1}{kq^{1-k}}\sum_{l=1}^{q} e^{\frac{2\pi \mathrm{i} lr}{q}} B_k\left(\frac{l}{q}\right). \tag{2.32}$$

By taking $\theta_r = r$ in Theorem 2.4 and then making the operation $\sum_{r=1}^{q-1} e^{\frac{2\pi \mathrm{i} mr}{q}}$ on both sides of (2.22), in view of (2.32) and the familiar geometric sum stated in [6, Theorem 8.1], we arrive at

$$\begin{aligned}
&\sum_{r=1}^{q-1} e^{\frac{2\pi \mathrm{i} mr}{q}} \cot^n\left(\frac{\pi r}{q}\right)\\
&= \left(\alpha(n) - \mathrm{i}^n \frac{1-(-1)^n}{2}\right)(q\delta_{0,m} - 1)\\
&\quad + \mathrm{i}^n \sum_{k=1}^{n} (-1)^{k-1} \frac{\widetilde{A}(n,k)}{kq^{1-k}} \sum_{l=1}^{q} B_k\left(\frac{l}{q}\right) \sum_{r=1}^{q-1} e^{\frac{2\pi \mathrm{i} r(m+l)}{q}}.
\end{aligned} \tag{2.33}$$

It is well known that the Bernoulli polynomials satisfy the symmetric relation

$$B_k(1-x) = (-1)^k B_k(x) \quad (k \geq 0), \tag{2.34}$$

and the multiplication formula

$$B_k(qx) = q^{k-1} \sum_{l=0}^{q-1} B_k\left(x + \frac{l}{q}\right) \quad (k \geq 0, q \geq 1), \tag{2.35}$$

which can be found in [1, p. 804] or [15, Proposition 9.1.3]. Hence, from (2.34) and (2.35) we have

$$\begin{aligned}
\sum_{l=1}^{q} B_k\left(\frac{l}{q}\right) \sum_{r=1}^{q-1} e^{\frac{2\pi \mathrm{i} r(m+l)}{q}} &= \sum_{l=0}^{q-1} B_k\left(1 - \frac{l}{q}\right) \sum_{r=1}^{q-1} e^{\frac{2\pi \mathrm{i} r(m+q-l)}{q}}\\
&= (-1)^k \sum_{l=0}^{q-1} B_k\left(\frac{l}{q}\right) \left\{\sum_{r=0}^{q-1} e^{\frac{2\pi \mathrm{i} r(m-l)}{q}} - 1\right\}\\
&= (-1)^k q B_k\left(\frac{m}{q}\right) - (-1)^k q^{1-k} B_k,
\end{aligned}$$

which together with (2.33) yields

$$\begin{aligned}
&\sum_{r=1}^{q-1} \left\{\cos\left(\frac{2\pi mr}{q}\right) + \mathrm{i} \sin\left(\frac{2\pi mr}{q}\right)\right\} \cot^n\left(\frac{\pi r}{q}\right)\\
&= \left(\alpha(n) - \mathrm{i}^n \frac{1-(-1)^n}{2}\right)(q\delta_{0,m} - 1)\\
&\quad - \mathrm{i}^n \sum_{k=1}^{n} \frac{\widetilde{A}(n,k)}{kq^{1-k}} \left\{q B_k\left(\frac{m}{q}\right) - q^{1-k} B_k\right\}.
\end{aligned} \tag{2.36}$$

Since $B_1 = -1/2$ and $B_{2k-1} = 0$ for positive integer $k \geq 2$ (see, e.g., [6, pp. 265-266]), so by (2.12), (2.36) and $\widetilde{A}(n,k) = 0$ when $n \not\equiv k \pmod 2$, we complete the proof of Corollary 2.5. $\square$

It is easy from $s(2,2)=1$ and $B_2=1/6$ to check the formula (1.1) when taking $m=0$ and $n=1$ in (2.30). An alternative version of (2.30) on condition that $1 \le m \le q-1$ was obtained by Williams and Zhang [36], where they expressed the left side of (2.30) by the rational numbers $\widehat{A}(n,k)$ appearing in (1.4) and the Bernoulli polynomials. We also mention that different proofs and formulations of the case $m=0$ in (2.30) have been given by Chu and Marini [14, pp. 137-139] and Cvijović and Klinowski [17]. For an alternative version of Corollary 2.5, see [18, Equations (2.1) and (2.2)] for details. We next present the explicit expressions for alternating cotangent sums.

Corollary 2.6. *Let q,n be positive integers with $q \ge 2$. If q is an even integer then*

$$\begin{aligned}&\sum_{r=1}^{q-1}(-1)^r \cot^{2n}\left(\frac{\pi r}{q}\right)\\ &= -(-1)^n-(-1)^n\sum_{k=1}^{n}\frac{\left(q^{2k}(2^{1-2k}-1)-1\right)\widetilde{A}(2n,2k)}{2k}B_{2k},\end{aligned} \tag{2.37}$$

and if q is an odd integer then

$$\begin{aligned}&\sum_{r=1}^{q-1}(-1)^r \cot^{2n-1}\left(\frac{\pi r}{q}\right)\\ &= -(-1)^n\sum_{k=1}^{n}\frac{q^{2k-2}\widetilde{A}(2n-1,2k-1)}{(2k-1)}\sum_{l=1}^{q-1}\tan\left(\frac{\pi l}{q}\right)B_{2k-1}\left(\frac{l}{q}\right).\end{aligned} \tag{2.38}$$

Proof. In a similar consideration to (2.33), we have

$$\begin{aligned}&\sum_{r=1}^{q-1}(-1)^r \cot^{n}\left(\frac{\pi r}{q}\right)\\ &= -\left(\alpha(n)-\mathrm{i}^n\frac{1-(-1)^n}{2}\right)\frac{1+(-1)^q}{2}\\ &\quad+\mathrm{i}^n\sum_{k=1}^{n}(-1)^{k-1}\frac{\widetilde{A}(n,k)}{kq^{1-k}}\sum_{l=1}^{q}B_k\left(\frac{l}{q}\right)\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi i l r}{q}}.\end{aligned} \tag{2.39}$$

It is easily seen that for positive integer l with $1 \le l \le q$ and $l \ne \frac{q}{2}$,

$$\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi i l r}{q}} = -\frac{1-(-1)^{q-1}e^{-\frac{2\pi i l}{q}}}{1+e^{-\frac{2\pi i l}{q}}} = \begin{cases}-1, & 2 \mid q,\\ -\mathrm{i}\tan\left(\frac{\pi l}{q}\right), & 2 \nmid q,\end{cases}$$

and the case $q=2$ and $x=0$ in (2.35) implies

$$B_k\left(\frac{1}{2}\right) = (2^{1-k}-1)B_k \quad (k \ge 0). \tag{2.40}$$

It follows from (2.34), (2.35) and (2.40) that if q is an even integer then

$$\begin{aligned}\sum_{l=1}^{q} B_k\left(\frac{l}{q}\right)\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi i l r}{q}} &= qB_k\left(\frac{1}{2}\right)-\sum_{l=1}^{q} B_k\left(\frac{l}{q}\right)\\ &= qB_k\left(\frac{1}{2}\right)-\sum_{l=0}^{q-1} B_k\left(1-\frac{l}{q}\right)\\ &= \left\{(2^{1-k}-1)q-(-1)^k q^{1-k}\right\}B_k,\end{aligned}$$

and if q is an odd integer then

$$\sum_{l=1}^{q} B_k\left(\frac{l}{q}\right)\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi i l r}{q}} = -\mathrm{i}\sum_{l=1}^{q}\tan\left(\frac{\pi l}{q}\right)B_k\left(\frac{l}{q}\right).$$

Thus, by applying the above two identities to (2.39), in view of $\widetilde{A}(n,k)=0$ when $n\not\equiv k \pmod 2$, we get the desired results immediately. □

It is trivially seen that the sum on the left side of (2.37) is equal to zero when q is an odd integer, and the sum on the left side of (2.38) is equal to zero when q is an even integer. For an alternative version of (2.37), see [14, pp. 155-156] for details.

We now turn to another cotangent sums. Byrne and Smith [12, Theorems 1 and 2] in 1997 used the Lagrange formula for polynomial interpolation based on the zeros of the Chebyshev polynomial of the first kind to discover that for positive integers q, n,

$$\sum_{r=1}^{q}(-1)^{r-1}\cot^{2n-1}\left(\frac{\pi(2r-1)}{4q}\right)=\sum_{k=1}^{n}a_{n,k}q^{2k-1}, \tag{2.41}$$

and

$$\sum_{r=1}^{q}\cot^{2n}\left(\frac{\pi(2r-1)}{4q}\right)=(-1)^n q+\sum_{k=1}^{n}b_{n,k}q^{2k}, \tag{2.42}$$

where $a_{n,k}$ and $b_{n,k}$ can be determined recursively for positive integers k,n with $1\le k\le n$ by

$$a_{n,k}=\frac{1}{2^{2(n-k)}-1}\sum_{j=1}^{n-k}\binom{2n-1}{j}(-1)^j a_{n-j,k}\quad (1\le k\le n-1)$$

with the special values $a_{n,1}=(-1)^{n-1}$ and $a_{n,1}+\cdots+a_{n,n}=1$, and

$$b_{n,k}=\frac{1}{2^{2(n-k)}-1}\sum_{j=1}^{n-k}\binom{2n}{j}(-1)^j b_{n-j,k}\quad (1\le k\le n-1)$$

with the special values $b_{n,1}+\cdots+b_{n,n}=1+(-1)^{n-1}$, respectively. To illustrate the advantage of the results and methods presented here, we improve Byrne and Smith's formulas (2.41) and (2.42), as follows.

Corollary 2.7. *Let q, n be positive integers. Then*

$$\sum_{r=1}^{q}(-1)^{r-1}\cot^{2n-1}\left(\frac{\pi(2r-1)}{4q}\right)$$
$$= -\frac{(-1)^n}{2}\sum_{k=1}^{n} q^{2k-1}\widetilde{A}(2n-1,2k-1)E_{2k-2}, \tag{2.43}$$

and

$$\sum_{r=1}^{q}\cot^{2n}\left(\frac{\pi(2r-1)}{4q}\right)$$
$$= (-1)^n q - (-1)^n\sum_{k=1}^{n}\frac{q^{2k}2^{2k}(2^{2k}-1)\widetilde{A}(2n,2k)}{4k}B_{2k}, \tag{2.44}$$

where $E_n(x)$ are the Euler polynomials given by the generating function (see, e.g., [1, 29])

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

In particular, the integers $E_n = 2^nE_n(1/2)$ are called the Euler numbers.

Proof. It is clear from Theorem 2.4 that for positive integer r with $1 \le r \le q$,

$$\cot^n\left(\frac{\pi(2r-1)}{4q}\right) = \alpha(n) - \mathrm{i}^n\frac{1-(-1)^n}{2} + \mathrm{i}^n\sum_{k=1}^{n}(-1)^k\widetilde{A}(n,k)\sum_{j=1}^{\infty}\frac{e^{\frac{2\pi \mathrm{i} j(2r-1)}{4q}}}{j^{1-k}}. \tag{2.45}$$

Notice that from (2.31) we have

$$\sum_{j=1}^{\infty}\frac{e^{\frac{2\pi \mathrm{i} j(2r-1)}{4q}}}{j^{1-k}} = \sum_{m=0}^{\infty}\sum_{l=1}^{4q}\frac{e^{\frac{2\pi \mathrm{i}(4qm+l)(2r-1)}{4q}}}{(4qm+l)^{1-k}} = -\frac{1}{k(4q)^{1-k}}\sum_{l=1}^{4q}e^{\frac{2\pi \mathrm{i} l(2r-1)}{4q}}B_k\left(\frac{l}{4q}\right).$$

It follows from (2.45) that

$$\sum_{r=1}^{q}(-1)^{r-1}\cot^{n}\left(\frac{\pi(2r-1)}{4q}\right)$$
$$= \left(\alpha(n) - \mathrm{i}^n\frac{1-(-1)^n}{2}\right)\frac{1-(-1)^q}{2}$$
$$-\mathrm{i}^n\sum_{k=1}^{n}(-1)^k\frac{\widetilde{A}(n,k)}{k(4q)^{1-k}}\sum_{l=1}^{4q}B_k\left(\frac{l}{4q}\right)\sum_{r=1}^{q}(-1)^{r-1}e^{\frac{2\pi \mathrm{i} l(2r-1)}{4q}}, \tag{2.46}$$

and

$$\begin{aligned}&\sum_{r=1}^{q}\cot^n\left(\frac{\pi(2r-1)}{4q}\right)\\&=\alpha(n)q-\mathrm{i}^n\frac{(1-(-1)^n)q}{2}\\&\quad-\mathrm{i}^n\sum_{k=1}^{n}(-1)^k\frac{\widetilde{A}(n,k)}{k(4q)^{1-k}}\sum_{l=1}^{4q}B_k\left(\frac{l}{4q}\right)\sum_{r=1}^{q}e^{\frac{2\pi\mathrm{i}l(2r-1)}{4q}}.\end{aligned}\tag{2.47}$$

By a simple calculation, we obtain that for positive integer l such that $1\leq l\leq 4q$ and $l\neq q,3q$,

$$\sum_{r=1}^{q}(-1)^{r-1}e^{\frac{2\pi\mathrm{i}l(2r-1)}{4q}}=e^{\frac{\pi\mathrm{i}l}{2q}}\frac{1-(-1)^{q+l}}{1+e^{\frac{\pi\mathrm{i}l}{q}}}=\frac{1-(-1)^{q+l}}{2\cos(\frac{\pi l}{2q})},$$

and for positive integer l such that $1\leq l\leq 4q$ and $l\neq 2q,4q$,

$$\sum_{r=1}^{q}e^{\frac{2\pi\mathrm{i}l(2r-1)}{4q}}=e^{\frac{\pi\mathrm{i}l}{2q}}\frac{1-(-1)^{l}}{1-e^{\frac{\pi\mathrm{i}l}{q}}}=\frac{(-1)^l-1}{2\mathrm{i}\sin(\frac{\pi l}{2q})}.$$

Hence, we get from (2.34), (2.46), (2.47) and $\widetilde{A}(n,k)=0$ when $n\not\equiv k \pmod 2$ that

$$\begin{aligned}&\sum_{r=1}^{q}(-1)^{r-1}\cot^{2n-1}\left(\frac{\pi(2r-1)}{4q}\right)\\&=-(-1)^n\sum_{k=1}^{2n-1}(-1)^k\frac{\widetilde{A}(2n-1,k)}{k(4q)^{1-k}}\left\{qB_k\left(\frac{1}{4}\right)-qB_k\left(\frac{3}{4}\right)\right\}\\&=-(-1)^n\sum_{k=1}^{n}\frac{q\widetilde{A}(2n-1,2k-1)}{(2k-1)(4q)^{2-2k}}\left\{B_{2k-1}\left(\frac{3}{4}\right)-B_{2k-1}\left(\frac{1}{4}\right)\right\},\end{aligned}\tag{2.48}$$

and

$$\begin{aligned}&\sum_{r=1}^{q}\cot^{2n}\left(\frac{\pi(2r-1)}{4q}\right)\\&=(-1)^nq-(-1)^n\sum_{k=1}^{2n}(-1)^k\frac{\widetilde{A}(2n,k)}{k(4q)^{1-k}}\left\{-qB_k\left(\frac{1}{2}\right)+qB_k(1)\right\}\\&=(-1)^nq-(-1)^n\sum_{k=1}^{n}\frac{q\widetilde{A}(2n,2k)}{2k(4q)^{1-2k}}\left\{B_{2k}-B_{2k}\left(\frac{1}{2}\right)\right\}.\end{aligned}\tag{2.49}$$

Since the Euler polynomials can be expressed by the Bernoulli polynomials in the following way (see, e.g., [1, p. 806])

$$E_k(x)=\frac{2^{k+1}}{k+1}\left\{B_{k+1}\left(\frac{x+1}{2}\right)-B_{k+1}\left(\frac{x}{2}\right)\right\}\quad(k\geq 0),\tag{2.50}$$

which can be easily proved by using the generating functions of the Bernoulli and Euler polynomials, so by taking $x=1/2$ in (2.50), we have

$$B_{k+1}\left(\frac{3}{4}\right)-B_{k+1}\left(\frac{1}{4}\right)=\frac{k+1}{2^{2k+1}}E_k\quad(k\geq 0).\tag{2.51}$$

Thus, by applying (2.40) and (2.51) to (2.48) and (2.49), we complete the proof of Corollary 2.7. □

In fact, from the proof of Corollary 2.7, the corresponding expression can be easily obtained when the odd power $2n-1$ on the left side of (2.43) is replaced by an even power $2n$ or the even power $2n$ on the left side of (2.44) is replaced by an odd power $2n-1$. By almost identically the same argument, one can improve the results for cotangent sums stated in [3, Theorems 3.2 and 3.3] and [27, Theorem 4.3]. In addition, some explicit expressions for tangent sums can be easily established by using the identity (see, e.g., [13, Corollary 3])

$$\eta(-k,x)=\frac{1}{2}E_k(x)\quad (k\geq 0), \tag{2.52}$$

where $\eta(s,x)$ is the alternating Hurwitz zeta function (also called Hurwitz Euler-zeta function) given for real number $x>0$ and complex number $s=\sigma+\mathrm{i}t$ by

$$\eta(s,x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+x)^s}\quad (\sigma>0).$$

For example, let q,n be positive integers with $q\geq 2$ and let m be a non-negative integer with $0\leq m\leq q-1$. We obtain from (2.31) and (2.52) that for positive integers k,r,

$$\begin{aligned}F(1/2+r/q,1-k) &= \sum_{j=0}^{\infty}\sum_{l=1}^{q}(-1)^{qj+l}\frac{e^{\frac{2\pi\mathrm{i}(qj+l)r}{q}}}{(qj+l)^{1-k}}\\ &= \begin{cases}-\frac{1}{kq^{1-k}}\sum_{l=1}^{q}(-1)^l e^{\frac{2\pi\mathrm{i}lr}{q}}B_k\left(\frac{l}{q}\right), & 2\mid q,\\ \frac{1}{2q^{1-k}}\sum_{l=1}^{q}(-1)^l e^{\frac{2\pi\mathrm{i}lr}{q}}E_{k-1}\left(\frac{l}{q}\right), & 2\nmid q.\end{cases}\end{aligned}$$

By applying the above identity to Theorem 2.4 , with the help of the geometric sum and some basic properties of the Bernoulli and Euler polynomials described in [1, pp. 804-806], one can evaluate

$$\sum_{\substack{r=1\\(r\neq\frac{q}{2},q\text{ is even})}}^{q-1}e^{\frac{2\pi\mathrm{i}(m+\epsilon)r}{q}}\tan^n\left(\frac{\pi r}{q}\right)\quad\text{and}\quad\sum_{\substack{r=1\\(q\text{ is odd})}}^{q-1}e^{\frac{2\pi\mathrm{i}(m+\epsilon)r}{q}}\tan^n\left(\frac{\pi r}{q}\right)$$

in terms of linear combinations of the Bernoulli and Euler polynomials and numbers, where $\epsilon\in\{0,q/2\}$. In particular, one can improve the results presented in [18, Equations (2.7), (2.8), (2.10) and (2.11)].

3. Expressions for cosecant sums

Before stating the connections between the higher-order cosecant and secant functions and the Lerch zeta function, we firstly give the following results.

Theorem 3.1. *Let n be a positive integer. Then, for real number a,*

$$\csc^n(a)=\sum_{j=-\infty}^{+\infty}(-1)^j\sum_{k=0}^{n-1}\frac{B(n,k)}{(a+j\pi)^{k+1}}\quad (a\neq 0,\pm\pi,\pm 2\pi,\ldots), \tag{3.1}$$

and

$$\sec^n(a)=\sum_{j=-\infty}^{+\infty}(-1)^j\sum_{k=0}^{n-1}\frac{B(n,k)}{\left(a+(j+\frac{1}{2})\pi\right)^{k+1}}\quad (a\neq\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\ldots), \tag{3.2}$$

where $B(n,k)$ is given for positive integer n and non-negative integer k with $0 \leq k \leq n-1$ by

$$B(n,k) = -\mathrm{i}^{n+1-k}\frac{2^n k! e^{\mathrm{i}a(n-1)}}{(n-1)!}\sum_{l=k+1}^{n}\binom{l-1}{k}\frac{s(n,l)}{2^l}. \tag{3.3}$$

Proof. Clearly, the following identities are complete

$$\csc(a) = \frac{1}{\sin(a)} = \frac{2\mathrm{i}}{e^{\mathrm{i}a} - e^{-\mathrm{i}a}} = -2\mathrm{i}e^{\mathrm{i}a}\frac{1}{1-e^{2\mathrm{i}a}}, \tag{3.4}$$

and

$$\frac{1}{1-e^{2\mathrm{i}a}} = -\frac{\csc(a)}{2\mathrm{i}e^{\mathrm{i}a}} = \frac{\mathrm{i}}{2}e^{-\mathrm{i}a}\csc(a). \tag{3.5}$$

It follows from (2.10), (3.4) and (3.5) that for positive integer n,

$$\begin{aligned}\csc^n(a) &= \frac{(-2\mathrm{i}e^{\mathrm{i}a})^n}{(n-1)!}\sum_{l=1}^{n}s(n,l)(-1)^{n-l}\frac{1}{(2\mathrm{i})^{l-1}}\frac{\partial^{l-1}}{\partial a^{l-1}}\left(\frac{1}{1-e^{2\mathrm{i}a}}\right)\\ &= \frac{(-2\mathrm{i}e^{\mathrm{i}a})^n}{(n-1)!}\sum_{l=1}^{n}s(n,l)(-1)^{n-l}\frac{1}{2^l\mathrm{i}^{l-2}}\frac{\partial^{l-1}}{\partial a^{l-1}}\big(e^{-\mathrm{i}a}\csc(a)\big). \end{aligned}\tag{3.6}$$

It is well known that $\csc(a)$ has the following expression in partial fractions (see, e.g., [1, p. 75] or [31, p. 329])

$$\begin{aligned}\csc(a) &= \frac{1}{a} + 2a\sum_{j=1}^{\infty}\frac{(-1)^j}{a^2-j^2\pi^2}\\ &= \sum_{j=-\infty}^{+\infty}(-1)^j\frac{1}{a+j\pi} \quad (a \neq 0, \pm\pi, \pm 2\pi, \ldots),\end{aligned}\tag{3.7}$$

which implies that for non-negative integer k,

$$\frac{\partial^k}{\partial a^k}\big(\csc(a)\big) = (-1)^k k! \sum_{j=-\infty}^{+\infty}(-1)^j\frac{1}{(a+j\pi)^{k+1}}. \tag{3.8}$$

With the help of the familiar Leibniz rule, we obtain that for positive integer l,

$$\frac{\partial^{l-1}}{\partial a^{l-1}}\big(e^{-\mathrm{i}a}\csc(a)\big) = \sum_{k=0}^{l-1}\binom{l-1}{k}\frac{\partial^{l-1-k}}{\partial a^{l-1-k}}\big(e^{-\mathrm{i}a}\big)\frac{\partial^k}{\partial a^k}\big(\csc(a)\big). \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned}\frac{\partial^{l-1}}{\partial a^{l-1}}\big(e^{-\mathrm{i}a}\csc(a)\big) &= e^{-\mathrm{i}a}\sum_{k=0}^{l-1}\binom{l-1}{k}(-\mathrm{i})^{l-1-k}(-1)^k k!\\ &\quad\times\sum_{j=-\infty}^{+\infty}(-1)^j\frac{1}{(a+j\pi)^{k+1}}.\end{aligned}\tag{3.10}$$

Applying (3.10) to (3.6) gives

$$\csc^n(a) = -\frac{2^n e^{\mathrm{i}a(n-1)}}{(n-1)!}\sum_{l=1}^{n} s(n,l)\frac{1}{2^l}\sum_{k=0}^{l-1}\binom{l-1}{k}\mathrm{i}^{n+1-k}k! \times \sum_{j=-\infty}^{+\infty}(-1)^j\frac{1}{(a+j\pi)^{k+1}}. \tag{3.11}$$

Thus, by changing the order of the summation in (3.11), we obtain (3.1). Since $\csc(\pi/2+a)=\sec(a)$ for $a \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$, so by replacing a by $\pi/2+a$ in (3.1), we get (3.2). This completes the proof of Theorem 3.1. □

We now use Theorem 3.1 to establish the following connections between the higher-order cosecant and secant functions and the Lerch zeta function.

Theorem 3.2. *Let q, n be positive integers, and let θ_r be a real function defined on positive integer r. If $\theta_r \neq 0, \pm q, \pm 2q, \ldots$ then*

$$\csc^n\left(\frac{\pi\theta_r}{q}\right) = -\mathrm{i}^n\frac{2^{n+1}e^{\frac{n\pi\mathrm{i}\theta_r}{q}}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\widetilde{B}(n,k)\phi(\theta_r/q, 1/2, -k), \tag{3.12}$$

and if $\theta_r \neq \pm\frac{q}{2}, \pm\frac{3q}{2}, \ldots$ then

$$\sec^n\left(\frac{\pi\theta_r}{q}\right) = -(-1)^n\frac{2^{n+1}e^{\frac{n\pi\mathrm{i}\theta_r}{q}}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\widetilde{B}(n,k)\phi(1/2+\theta_r/q, 1/2, -k), \tag{3.13}$$

where $\widetilde{B}(n,k)$ is given for positive integer n and non-negative integer k with $0 \leq k \leq n-1$ by

$$\widetilde{B}(n,k) = \sum_{l=k+1}^{n}\binom{l-1}{k}\frac{s(n,l)}{2^{l-k}}, \tag{3.14}$$

and $\phi(a,x,s)$ is the Lerch zeta function given for real number a, $x \neq$ negative integer or zero, and complex number $s = \sigma + \mathrm{i}t$ by (see, e.g., [5, 8])

$$\phi(a,x,s) = \sum_{n=0}^{\infty}\frac{e^{2\pi\mathrm{i}na}}{(n+x)^s} \quad (\sigma > 1).$$

Note that the series is an entire function of s when a is not an integer.

Proof. It is easily seen from Theorem 3.1 and (3.8) that

$$\begin{aligned}\csc^n\left(\frac{\pi\theta_r}{q}\right) &= -\frac{2^n e^{\frac{(n-1)\pi\mathrm{i}\theta_r}{q}}}{(n-1)!}\sum_{k=0}^{n-1}\frac{\mathrm{i}^{n+1-k}k!\widetilde{B}(n,k)}{2^k}\sum_{j=-\infty}^{+\infty}\frac{(-1)^j}{(\frac{\pi\theta_r}{q}+j\pi)^{k+1}} \\ &= -\frac{2^n e^{\frac{(n-1)\pi\mathrm{i}\theta_r}{q}}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\frac{\mathrm{i}^{n+1-k}\widetilde{B}(n,k)}{2^k\pi^k} \\ &\quad \times\frac{\partial^k}{\partial a^k}\left(\csc(\pi a)\right)\Big|_{a=\frac{\theta_r}{q}} \quad (\theta_r \neq 0, \pm q, \pm 2q, \ldots).\end{aligned} \tag{3.15}$$

Clearly, the Lerch zeta function satisfies

$$\phi(a,x,s-1) - x\phi(a,x,s) = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial a}\big(\phi(a,x,s)\big), \tag{3.16}$$

which holds true for all s by analytic continuation. Hence, by taking $x = 1/2$ in (3.16), we discover

$$e^{\pi \mathrm{i} a}\phi(a,1/2,s-1) = \frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial a}\big(e^{\pi \mathrm{i} a}\phi(a,1/2,s)\big).$$

Applying the above identity repeatedly gives

$$e^{\pi \mathrm{i} a}\phi(a,1/2,s-k) = \frac{1}{(2\pi \mathrm{i})^k}\frac{\partial^k}{\partial a^k}\big(e^{\pi \mathrm{i} a}\phi(a,1/2,s)\big) \quad (k \geq 0). \tag{3.17}$$

Since

$$e^{\pi \mathrm{i} a}\phi(a,1/2,1) = \ln(1+e^{\pi \mathrm{i} a}) - \ln(1-e^{\pi \mathrm{i} a}),$$

so by (3.5) and (3.17), we have

$$e^{\pi \mathrm{i} a}\phi(a,1/2,0) = \frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial a}\big(\ln(1+e^{\pi \mathrm{i} a}) - \ln(1-e^{\pi \mathrm{i} a})\big) = \frac{\mathrm{i}}{2}\csc(\pi a). \tag{3.18}$$

By taking $s = 0$ in (3.17), in view of (3.18), we obtain

$$\frac{\partial^k}{\partial a^k}\big(\csc(\pi a)\big)\Big|_{a=\frac{\theta_r}{q}} = 2^{k+1}\pi^k \mathrm{i}^{k-1} e^{\frac{\pi \mathrm{i}\theta_r}{q}}\phi(\theta_r/q,1/2,-k). \tag{3.19}$$

Thus, by combining (3.15) and (3.19), we obtain (3.12). If we substitute $q/2+\theta_r$ for θ_r in (3.12), we get (3.13). This completes the proof of Theorem 3.2. □

We next show some applications of Theorem 3.2. We firstly give the following explicit expressions for the sums (1.5).

Corollary 3.3. *Let q, n be positive integers with $q \geq 2$, and let m be a non-negative integer with $0 \leq m \leq q-1$. Assume that $n \equiv p \pmod q$ with $1 \leq p \leq q$. If $1 \leq m+p \leq q$ then*

$$\begin{aligned}&\sum_{r=1}^{q-1}\cos\left(\frac{2\pi mr}{q}\right)\csc^{2n}\left(\frac{\pi r}{q}\right)\\ &= -(-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}\frac{\widetilde{B}(2n,k)}{k+1}\\ &\times\left\{q^{k+1}B_{k+1}\left(\frac{2(m+p)-1}{2q}\right)-B_{k+1}\left(\frac{1}{2}\right)\right\},\end{aligned} \tag{3.20}$$

and if $q+1 \leq m+p \leq 2q-1$ then

$$\begin{aligned}&\sum_{r=1}^{q-1}\cos\left(\frac{2\pi mr}{q}\right)\csc^{2n}\left(\frac{\pi r}{q}\right)\\ &= -(-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}\frac{\widetilde{B}(2n,k)}{k+1}\\ &\times\left\{q^{k+1}B_{k+1}\left(\frac{2(m+p-q)-1}{2q}\right)-B_{k+1}\left(\frac{1}{2}\right)\right\}.\end{aligned} \tag{3.21}$$

Proof. By substituting $2n$ for n and r for θ_r in Theorem 3.2, we obtain that for positive integer r with $1 \leq r \leq q-1$,

$$\csc^{2n}\left(\frac{\pi r}{q}\right) = -(-1)^n\frac{2^{2n+1}e^{\frac{2\pi \mathrm{i}pr}{q}}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k\widetilde{B}(2n,k)\sum_{j=0}^{\infty}\frac{e^{\frac{2\pi \mathrm{i}jr}{q}}}{(j+\frac{1}{2})^{-k}}. \tag{3.22}$$

It is easily seen from (2.31) that

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{e^{\frac{2\pi \mathrm{i} jr}{q}}}{(j+\frac{1}{2})^{-k}} &= \sum_{l=0}^{q-1}\sum_{m=0}^{\infty}\frac{e^{\frac{2\pi \mathrm{i}(qm+l)r}{q}}}{(qm+l+\frac{1}{2})^{-k}}\\ &= -\frac{q^k}{k+1}\sum_{l=0}^{q-1}e^{\frac{2\pi \mathrm{i} lr}{q}}B_{k+1}\left(\frac{2l+1}{2q}\right).\end{aligned}$$

It follows from (3.22) that for positive integer r with $1\le r\le q-1$,

$$\begin{aligned}\csc^{2n}\left(\frac{\pi r}{q}\right) &= (-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k\frac{q^k\widetilde{B}(2n,k)}{k+1}\\ &\quad\times\sum_{l=0}^{q-1}B_{k+1}\left(\frac{2l+1}{2q}\right)e^{\frac{2\pi \mathrm{i} r(p+l)}{q}}.\end{aligned}\tag{3.23}$$

Hence, making the operation $\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i} mr}{q}}$ on both sides of (3.23) gives

$$\begin{aligned}&\sum_{r=1}^{q-1}\left\{\cos\left(\frac{2\pi mr}{q}\right)+\mathrm{i}\sin\left(\frac{2\pi mr}{q}\right)\right\}\csc^{2n}\left(\frac{\pi r}{q}\right)\\ &= (-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k\frac{q^k\widetilde{B}(2n,k)}{k+1}\\ &\quad\times\sum_{l=0}^{q-1}B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i} r(m+p+l)}{q}}.\end{aligned}\tag{3.24}$$

Notice that from (2.34) we have

$$\begin{aligned}&\sum_{l=0}^{q-1}B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i} r(m+p+l)}{q}}\\ &= \sum_{l=1}^{q}B_{k+1}\left(1-\frac{2l-1}{2q}\right)\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i} r(m+p+q-l)}{q}}\\ &= (-1)^{k+1}\sum_{l=1}^{q}B_{k+1}\left(\frac{2l-1}{2q}\right)\left\{\sum_{r=0}^{q-1}e^{\frac{2\pi \mathrm{i} r(m+p-l)}{q}}-1\right\}\\ &= (-1)^{k+1}\sum_{l=1}^{q}B_{k+1}\left(\frac{2l-1}{2q}\right)\sum_{r=0}^{q-1}e^{\frac{2\pi \mathrm{i} r(m+p-l)}{q}}\\ &\quad-(-1)^{k+1}\sum_{l=0}^{q-1}B_{k+1}\left(\frac{1}{2q}+\frac{l}{q}\right).\end{aligned}$$

Hence, we get from (2.35) that if $1\le m+p\le q$ then

$$\begin{aligned}&\sum_{l=0}^{q-1}B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i} r(m+p+l)}{q}}\\ &= (-1)^{k+1}qB_{k+1}\left(\frac{2(m+p)-1}{2q}\right)-(-1)^{k+1}q^{-k}B_{k+1}\left(\frac{1}{2}\right),\end{aligned}\tag{3.25}$$

and if $q+1 \leq m+p \leq 2q-1$ then

$$\begin{aligned}&\sum_{l=0}^{q-1} B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1} e^{\frac{2\pi i r(m+p+l)}{q}}\\&=(-1)^{k+1} q B_{k+1}\left(\frac{2(m+p-q)-1}{2q}\right)-(-1)^{k+1} q^{-k} B_{k+1}\left(\frac{1}{2}\right).\qquad(3.26)\end{aligned}$$

Thus, by applying (3.25) and (3.26) to (3.24), we complete the proof of Corollary 3.3. □

The different formulations of Corollary 3.3 have been derived by Berndt and Yeap [10, Theorems 5.1 and 5.3] and Cvijović and Srivastava [18, Equation (2.3)]. For another expressions of the case $m=0$ on the left side of (3.20), see [14, pp. 126-128] and [24, Theorem 4.2] for details. We are now in the position to present an alternative version of Berndt and Yeap's formula (1.7).

Corollary 3.4. *Let q, n be positive integers with q being an even integer. Assume that $n \equiv p \pmod q$ such that $1 \leq p \leq q$. If $1 \leq p \leq q/2$ then*

$$\begin{aligned}&\sum_{r=1}^{q-1}(-1)^r \csc^{2n}\left(\frac{\pi r}{q}\right)\\&=(-1)^n \frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k \frac{\widetilde{B}(2n,k)}{k+1}\\&\quad\times\left\{q^{k+1} B_{k+1}\left(\frac{q+1-2p}{2q}\right)-B_{k+1}\left(\frac{1}{2}\right)\right\},\qquad(3.27)\end{aligned}$$

and if $q/2+1 \leq p \leq q$ then

$$\begin{aligned}&\sum_{r=1}^{q-1}(-1)^r \csc^{2n}\left(\frac{\pi r}{q}\right)\\&=(-1)^n \frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k \frac{\widetilde{B}(2n,k)}{k+1}\\&\quad\times\left\{q^{k+1} B_{k+1}\left(\frac{3q+1-2p}{2q}\right)-B_{k+1}\left(\frac{1}{2}\right)\right\}.\qquad(3.28)\end{aligned}$$

Note that the sum on the left side of (1.7) is trivially equal to zero when q is an odd integer.

Proof. It is easily seen from (3.23) that

$$\begin{aligned}\sum_{r=1}^{q-1}(-1)^r \csc^{2n}\left(\frac{\pi r}{q}\right) &= (-1)^n \frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k \frac{q^k \widetilde{B}(2n,k)}{k+1}\\&\quad\times\sum_{l=0}^{q-1} B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi i r(p+l)}{q}}.\qquad(3.29)\end{aligned}$$

Since $1 \le p+l \le 2q-1$ for non-negative integer l with $0 \le l \le q-1$, so if $p+l \ne q/2, 3q/2$ then

$$\sum_{r=0}^{q-1}(-1)^r e^{\frac{2\pi \mathrm{i} r(p+l)}{q}} = \frac{1-(-1)^q}{1+e^{\frac{2\pi \mathrm{i}(p+l)}{q}}} = 0.$$

It follows from (2.34) and (2.35) that if $1 \le p \le q/2$ then

$$\begin{aligned}
&\sum_{l=0}^{q-1} B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi \mathrm{i} r(p+l)}{q}} \\
&= \sum_{l=0}^{q-1} B_{k+1}\left(\frac{2l+1}{2q}\right)\left\{\sum_{r=0}^{q-1}(-1)^r e^{\frac{2\pi \mathrm{i} r(p+l)}{q}} - 1\right\} \\
&= qB_{k+1}\left(\frac{q+1-2p}{2q}\right) - \sum_{l=0}^{q-1} B_{k+1}\left(\frac{1}{2q}+\frac{l}{q}\right) \\
&= qB_{k+1}\left(\frac{q+1-2p}{2q}\right) - q^{-k}B_{k+1}\left(\frac{1}{2}\right),
\end{aligned} \tag{3.30}$$

and if $q/2+1 \le p \le q$ then

$$\begin{aligned}
&\sum_{l=0}^{q-1} B_{k+1}\left(\frac{2l+1}{2q}\right)\sum_{r=1}^{q-1}(-1)^r e^{\frac{2\pi \mathrm{i} r(p+l)}{q}} \\
&= \sum_{l=0}^{q-1} B_{k+1}\left(\frac{2l+1}{2q}\right)\left\{\sum_{r=0}^{q-1}(-1)^r e^{\frac{2\pi \mathrm{i} r(p+l)}{q}} - 1\right\} \\
&= qB_{k+1}\left(\frac{3q+1-2p}{2q}\right) - \sum_{l=0}^{q-1} B_{k+1}\left(\frac{1}{2q}+\frac{l}{q}\right) \\
&= qB_{k+1}\left(\frac{3q+1-2p}{2q}\right) - q^{-k}B_{k+1}\left(\frac{1}{2}\right).
\end{aligned} \tag{3.31}$$

Thus, applying (3.30) and (3.31) to (3.29) gives the desired results. □

It is worth noticing that the alternating cosecant sums of Corollary 3.4 appear in another formula for vector bundles due to Thaddeus [34], and was also evaluated by Chu and Marini [14, pp. 149-150].

To conclude this paper, we consider a trigonometric inverse power sum. Gardner [25] in 1969 asked whether it was possible to obtain a simple closed-form expression for $S_{n,2}(q)$ given for positive integers q, n with $q \ge 2$ by

$$S_{n,2}(q) := \frac{\pi^{2n}}{(2q)^{2n}}\sum_{r=1}^{q-1}\cos^{-2n}\left(\frac{\pi r}{2q}\right). \tag{3.32}$$

Fisher [23] in 1971 studied in an equivalent form the sum

$$S_{n,2}(q) := \frac{\pi^{2n}}{(2q)^{2n}}\sum_{r=1}^{q-1}\sin^{-2n}\left(\frac{\pi r}{2q}\right), \tag{3.33}$$

and used generating function to derive the identities

$$S_{1,2}(q) = \frac{\pi^2}{6}\left(1-\frac{1}{q^2}\right), \quad S_{2,2}(q) = \frac{\pi^4}{90}\left(1+\frac{5}{2q^2}-\frac{7}{2q^4}\right),$$

and the asymptotic formula for enough large q,

$$S_{n,2}(q)=\zeta(2n)+\frac{n}{12q^2}\zeta(2n-2)+O(q^{-4}),$$

where $\zeta(\cdot)$ is the Riemann zeta function. Kowalenko [28] in 2011 used empirical method to give some series expansions of $S_{n,2}(q)$, and tabulated the first ten values of $S_{n,2}(q)$. Fonseca, Glasser and Kowalenko [24, Theorem 3.1] in 2018 used integral approach to discover

$$S_{n,2}(q)=\frac{1}{(2n-1)!}\sum_{k=0}^{n-1}\frac{\pi^{2k}}{q^{2k}}\Gamma(2n-2k)\zeta(2n-2k)\widetilde{s}(n,k)\left(1-\frac{1}{q^{2n-2k}}\right),$$

where $\Gamma(\cdot)$ is the Gamma function, and $\widetilde{s}(n,k)$ represents the k-th elementary symmetric polynomial obtained by summing over the entire sequence of quadratic powers or squares of integers from 1^2 to $(n-1)^2$. We here give the explicit expression of the cosecant sums appearing in (3.33) in terms of the Bernoulli polynomials.

Corollary 3.5. *Let q,n be positive integers with $q\geq 2$. Assume that $n\equiv p$ (mod $2q$) such that $1\leq p\leq 2q$. Then*

$$\begin{aligned}\sum_{r=1}^{q-1}\csc^{2n}\left(\frac{\pi r}{2q}\right) &= -(-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}\frac{\widetilde{B}(2n,k)}{k+1}\\ &\quad\times\left\{2^k q^{k+1}B_{k+1}\left(\frac{2p-1}{4q}\right)-B_{k+1}\left(\frac{1}{2}\right)\right\}.\end{aligned}\tag{3.34}$$

Proof. By substituting $2n$ for n and $r/2$ for θ_r in Theorem 3.2 and then making the operation $\sum_{r=1}^{q-1}$ on both sides of (3.12), we obtain

$$\begin{aligned}\sum_{r=1}^{q-1}\csc^{2n}\left(\frac{\pi r}{2q}\right) &= -(-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k\widetilde{B}(2n,k)\\ &\quad\times\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i}pr}{2q}}\sum_{j=0}^{\infty}\frac{e^{\frac{2\pi \mathrm{i}jr}{2q}}}{(j+\frac{1}{2})^{-k}}.\end{aligned}\tag{3.35}$$

It is easily seen from (2.31) that

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{e^{\frac{2\pi \mathrm{i}jr}{2q}}}{(j+\frac{1}{2})^{-k}} &= \sum_{l=0}^{2q-1}\sum_{m=0}^{\infty}\frac{e^{\frac{2\pi\mathrm{i}(2qm+l)r}{2q}}}{(2qm+l+\frac{1}{2})^{-k}}\\ &= -\frac{(2q)^k}{k+1}\sum_{l=0}^{2q-1}e^{\frac{2\pi \mathrm{i}lr}{2q}}B_{k+1}\left(\frac{2l+1}{4q}\right),\end{aligned}$$

which together with (3.35) yields

$$\begin{aligned}\sum_{r=1}^{q-1}\csc^{2n}\left(\frac{\pi r}{2q}\right) &= (-1)^n\frac{2^{2n+1}}{(2n-1)!}\sum_{k=0}^{2n-1}(-1)^k\frac{(2q)^k\widetilde{B}(2n,k)}{k+1}\\ &\quad\times\sum_{l=0}^{2q-1}B_{k+1}\left(\frac{2l+1}{4q}\right)\sum_{r=1}^{q-1}e^{\frac{2\pi \mathrm{i}r(p+l)}{2q}}.\end{aligned}\tag{3.36}$$

Clearly, from (2.34) and (2.35) we have

$$\begin{aligned}
&\sum_{l=0}^{2q-1} B_{k+1}\left(\frac{2l+1}{4q}\right)\sum_{r=1}^{q-1} e^{\frac{2\pi i r(p+l)}{2q}} \\
&= \sum_{l=1}^{2q} B_{k+1}\left(1-\frac{2l-1}{4q}\right)\left\{\sum_{r=0}^{q-1} e^{\frac{2\pi i r(p+2q-l)}{2q}} - 1\right\} \\
&= (-1)^{k+1}\sum_{l=1}^{2q} B_{k+1}\left(\frac{2l-1}{4q}\right)\sum_{r=0}^{q-1} e^{\frac{2\pi i r(p-l)}{2q}} \\
&\quad -(-1)^{k+1}\sum_{l=0}^{2q-1} B_{k+1}\left(\frac{1}{4q}+\frac{l}{2q}\right) \\
&= (-1)^{k+1} q B_{k+1}\left(\frac{2p-1}{4q}\right) - (-1)^{k+1}(2q)^{-k} B_{k+1}\left(\frac{1}{2}\right). \qquad (3.37)
\end{aligned}$$

Thus, by applying (3.37) to (3.36), we conclude the proof of Corollary 3.5. □

It is obvious that multiplying on both sides of (3.34) by $\pi^{2n}/(2q)^{2n}$ gives the explicit expression of $S_{n,2}(q)$. It is worth noticing that one can use Theorem 3.2 to evaluate the secant sums

$$\sum_{\substack{r=1 \\ (r\neq \frac{q}{2}, q \text{ is even})}}^{q-1} e^{\frac{2\pi i(m+\epsilon)r}{q}} \sec^{2n}\left(\frac{\pi r}{q}\right) \quad \text{and} \quad \sum_{\substack{r=1 \\ (q \text{ is odd})}}^{q-1} e^{\frac{2\pi i(m+\epsilon)r}{q}} \sec^{2n}\left(\frac{\pi r}{q}\right),$$

where q, n are positive integers with $q \geq 2$, m is a non-negative integer with $0 \leq m \leq q-1$, and $\epsilon \in \{0, q/2\}$. It turns out that the results showed in [18, Equations (2.9) and (2.12)] can be improved. Moreover, one can also use Theorems 2.4 and 3.2 to evaluate some finite trigonometric sums dealt in [4, 26, 27, 35] in terms of linear combinations of the Bernoulli and Euler polynomials and numbers. For another applications of Theorems 2.1, 2.4, 3.1 and 3.2, we leave them to the interested read for further exploration.

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