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Non-vanishing theta values of characters with special prime conductors

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ABSTRACT

Let p be a prime of the form $2\ell+1$ or $4\ell+1$, where ℓ is also a prime. We prove $\theta(\chi, i) \neq 0$ for all primitive Dirichlet characters χ with conductor p except for the quadratic and the quartic ones. Our results generalize a theorem of Bengoechea, which asserts $\theta(\chi, i) \neq 0$ for non-quadratic χ with "large" prime conductor $p = 2\ell + 1$, where ℓ is also a prime.

1. Introduction

Let χ be a primitive Dirichlet character of conductor N. The *theta function* associated to χ, denoted $\theta(\chi, \tau)$, is defined on the upper half plane as

$$\theta(\chi,\tau) = \begin{cases} \sum\limits_{n=1}^{\infty} \chi(n) e^{\frac{n^2}{N}\pi i \tau}, & \text{if } \chi \text{ is even,} \\ \sum\limits_{n=1}^{\infty} n\chi(n) e^{\frac{n^2}{N}\pi i \tau}, & \text{if } \chi \text{ is odd.} \end{cases} \tag{1.1}$$

If $\theta(\chi, i) \neq 0$ then the normalized Gauss sum can be expressed as

$$W(\chi) = \frac{\theta(\chi, i)}{\theta(\overline{\chi}, i)}. \tag{1.2}$$

One can use (1.1) to compute efficiently the numerical value of $W(\chi)$. So it is hoped that $\theta(\chi, i)$ does not vanish for "many" χ.

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Cohen and Zagier [2] showed that $\theta(\chi, i)$ vanishes for only two characters and their complex conjugates and for no other primitive characters of conductor ≤ 52100. Louboutin [3] proved that there is a positive constant c such that for prime p, at least $cp/\log(p)$ of the characters χ of conductor p have $\theta(\chi, i) \neq 0$. He raised the question whether $\theta(\chi, i)$ is always nonzero.

Bengoechea [1] calculated the Galois action on the special values of theta functions and proved:

Theorem 1.1 *([1], Theorem 4.7). There is a constant $c > 0$ such that for all non-quadratic χ with prime conductor $p = 2\ell + 1$, where ℓ is prime, satisfying $p > c$, we have $\theta(\chi, i) \neq 0$.*

Bengoechea's proof uses the following theorem, which is due to Louboutin for odd characters and Louboutin-Munsch for even characters.

Theorem 1.2. *There is a constant $c > 0$ such that $\theta(\chi, i) \neq 0$ for at least $cp/\log(p)$ characters of the $(p-1)/2$ odd characters with conductor p and of the $(p-1)/2$ even ones.*

For a fixed prime p, denote by X the group of Dirichlet characters modulo p, which is a cyclic group of order $p-1$. For any nonempty subset T of X, the second and fourth mean value of theta functions at i for characters ranging in T, are defined respectively to be

$$\begin{aligned} S_2(T) &:= \sum_{\chi \in T} |\theta(\chi, i)|^2, \\ S_4(T) &:= \sum_{\chi \in T} |\theta(\chi, i)|^4. \end{aligned} \tag{1.3}$$

Let $N(T)$ be the number of characters χ in T such that $\theta(\chi, i) \neq 0$. Then Cauchy-Schwarz inequality yields

$$N(T) \geqslant S_2^2(T)/S_4(T). \tag{1.4}$$

Louboutin and Munsch [4] gave the following asymptotic estimations for the set X^+ of the $\frac{p-1}{2}$ even characters as well as the set X^- of the $\frac{p-1}{2}$ odd ones:

$$\begin{aligned} S_2(X^+) &\sim \frac{p^{\frac{3}{2}}}{4\sqrt{2}}, \quad S_2(X^-) \sim \frac{p^{\frac{5}{2}}}{16\pi\sqrt{2}}, \\ S_4(X^+) &\sim \frac{3p^2 \log p}{16\pi}, \quad S_4(X^-) \sim \frac{3p^4 \log p}{512\pi^3}. \end{aligned} \tag{1.5}$$

Theorem 1.2 follows from (1.4) and (1.5).

In this paper we take a more direct approach to prove the non-vanishing of theta values. We consider the first moment of the theta values

$$S_1(T) := \sum_{\chi \in T} \theta(\chi, i). \tag{1.6}$$

Louboutin and Munsch [4] have studied $S_1(X^+)$ and $S_1(X^-)$. They showed that

$$S_1(X^+) \sim \frac{p}{2}, \quad S_1(X^-) \sim \frac{p}{2}. \tag{1.7}$$

We shall give lower and upper bounds of $S_1(T)$ for some specific subsets T. Since $N(T) = 0$ implies $S_1(T) = 0$, we can get $N(T) \neq 0$ as long as we prove $S_1(T) \neq 0$. In particular, when T is a Galois orbit and $N(T) \neq 0$, then a theorem of Bengoechea will force all the characters in T to have non-vanishing theta values.

The prime p in Theorem 1.1 is called a *safe prime*, and the prime ℓ is called a *Sophie Germain prime*. The heuristic estimate for the number of Sophie Germain primes less than x is

$$C\frac{x}{(\log x)^2}, \tag{1.8}$$

where

$$C = 2\prod_{p>2}\frac{p(p-2)}{(p-1)^2} \approx 1.32032.$$

We shall prove that Theorem 1.1 holds not just for large enough safe primes, it is actually valid for *all* safe primes:

Theorem 1.3. *$\theta(\chi, i) \neq 0$ for all non-quadratic χ with prime conductor $p = 2\ell + 1$, where ℓ is a prime.*

In this paper, we will also generalize Theorem 1.1 to the characters χ with prime conductor $p = 4\ell + 1$, where ℓ is a prime.

Theorem 1.4. *Let p be a prime of the form $4\ell + 1$, where ℓ is also a prime. If a primitive Dirichlet character χ with conductor p is neither quadratic nor quartic, then $\theta(\chi, i) \neq 0$.*

The heuristic estimate for the number of primes $\ell < x$ such that $p = 4\ell + 1$ is also prime is the same as (1.8). One can see Conjecture 5.24 and 5.25 of [5].

2. Main results

Let χ be a primitive character with odd conductor N and order m. Let $M = 24mN^2$. Consider the order $\mathcal{O} = \mathbb{Z}[iN]$ in $K = \mathbb{Q}(i)$. Let $H_{\mathcal{O}} = K(j(iN))$ be the ring class field of $\mathcal{O}$ and $H_{M,\mathcal{O}}$ be the ray class field with conductor M over $H_{\mathcal{O}}$. As in [1], we define

$$A_\chi(\tau) = \frac{\theta(\chi, \tau/N)}{\eta(\tau/N)^{1+2\epsilon}}, \quad B_\chi(\tau) = |A_\chi(\tau)|^2 = A_\chi(\tau)A_{\overline{\chi}}(\tau),$$

where

$$\epsilon = \begin{cases} 0, & \text{if } \chi \text{ is even;} \\ 1, & \text{if } \chi \text{ is odd;} \end{cases}$$

and η is the classical Dedekind η-function. In particular, if the conductor N is a prime, say p, we denote by $X(p, m)$ the set of characters with conductor p and order m up to complex conjugation. With these notations Bengoechea proved the following theorem.

Theorem 2.1 *([1], Theorem 4.2 (i)). The set*

$$\left\{B_\chi(ip)^2 \mid \chi \in X(p, m)\right\}$$

is an orbit for the action of the group $\mathrm{Gal}(H_{M,\mathcal{O}}/H_{\mathcal{O}})$.

This theorem leads to a straightforward result, which plays a crucial role in our argument:

Proposition 2.2. *Once there exists a $\chi \in X(p,m)$ such that $\theta(\chi,i) \neq 0$, it is so for each $\chi \in X(p,m)$.*

We will make a frequent use of a simple fact from elementary calculus: Let $f(x)$ be a nonnegative descending continuous real function on the interval $[n_0,+\infty)$, where n_0 is an integer. If the integral $\int_{n_0}^{+\infty} f(x)\,\mathrm{d}x$ converges, then it is an upper bound of the series $\sum_{n=n_0+1}^{\infty} f(n)$.

In what follows p is always an odd prime, X is the character group modulo p, X^+ is the subgroup of X consisting of all even characters modulo p, and χ_0 is the trivial character modulo p. For convenience we establish a lemma concerning S_1 of a general subgroup of X^+.

Lemma 2.3. *Let G be a subgroup of X^+ with order d. Then*

$$S_1(G) = d \sum_{\substack{n=1 \\ n^{\frac{p-1}{d}} \equiv 1 \pmod{p}}}^{\infty} e^{-\frac{\pi}{p}n^2},$$

which has the obvious lower bound $d\,e^{-\frac{\pi}{p}}$.

Proof. The orthogonality relation on G reads

$$\sum_{\chi\in G} \chi(n) = \begin{cases} d, & \text{if } n^{\frac{p-1}{d}} \equiv 1 \pmod{p}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} S_1(G) &= \sum_{\chi\in G} \theta(\chi,i) = \sum_{\chi\in G}\sum_{n=1}^{\infty} \chi(n)\, e^{-\frac{\pi}{p}n^2} = \sum_{n=1}^{\infty} \left(\sum_{\chi\in G}\chi(n)\right) e^{-\frac{\pi}{p}n^2} \\ &= d \sum_{\substack{n=1 \\ n^{\frac{p-1}{d}} \equiv 1 \pmod{p}}}^{\infty} e^{-\frac{\pi}{p}n^2} > d\,e^{-\frac{\pi}{p}}. \quad \square \end{aligned}$$

2.1. Even characters and odd characters

By definition,

$$\theta(\chi_0,i) = \sum_{\substack{n=1 \\ n\not\equiv 0 \pmod{p}}}^{\infty} e^{-\frac{\pi}{p}n^2} < \sum_{n=1}^{\infty} e^{-\frac{\pi}{p}n^2} \leqslant \int_0^{+\infty} e^{-\frac{\pi}{p}x^2}\,\mathrm{d}x = \frac{\sqrt{p}}{2}, \tag{2.1}$$

and by Lemma 2.3,

$$S_1(X^+) \geqslant \frac{p-1}{2}\, e^{-\frac{\pi}{p}}. \tag{2.2}$$

Thus

$$S_1(X^+\backslash\{\chi_0\}) = S_1(X^+) - \theta(\chi_0,i) \geqslant \frac{p-1}{2}\, e^{-\frac{\pi}{p}} - \frac{\sqrt{p}}{2}. \tag{2.3}$$

One sees the RHS of the above inequality stays positive when $p \geqslant 7$. Hence we obtain

Proposition 2.4. *For any prime $p \geqslant 7$ there exists a nontrivial $\chi \in X^+$ such that $\theta(\chi, i) \neq 0$.*

Then consider the set X^- of odd characters modulo p, that is, $X^- = X \backslash X^+$. Combining the orthogonality relations on X and on X^+, we have

$$\sum_{\chi \in X^-} \chi(n) = \begin{cases} (p-1)/2, & \text{if } n \equiv 1 \pmod p, \\ -(p-1)/2, & \text{if } n \equiv -1 \pmod p, \\ 0, & \text{if } n \not\equiv \pm 1 \pmod p. \end{cases}$$

Hence

$$\begin{aligned} S_1(X^-) &= \sum_{\chi \in X^-} \theta_\chi(i) = \sum_{\chi \in X^-} \sum_{n=1}^{\infty} n\,\chi(n)\,e^{-\frac{\pi}{p}n^2} = \sum_{n=1}^{\infty} \left(\sum_{\chi \in X^-} \chi(n) \right) n\,e^{-\frac{\pi}{p}n^2} \\ &= \frac{p-1}{2} \left(\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1 \\ n \equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} \right). \end{aligned} \tag{2.4}$$

It is clear that

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} > e^{-\frac{\pi}{p}} > 1 - \frac{\pi}{p}.$$

For the series $\sum_{n=1,\ n \equiv -1 \ (\mathrm{mod}\ p)}^{\infty} n\,e^{-\frac{\pi}{p}n^2}$, notice that the function $f(x) = x\,e^{-\frac{\pi}{p}x^2}$ descends on the interval $[\sqrt{\frac{p}{2\pi}}, +\infty)$, and particularly on $[p-2, +\infty)$, we have

$$\sum_{\substack{n=1 \\ n \equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} < \sum_{n=p-1}^{\infty} n\,e^{-\frac{\pi}{p}n^2} \leqslant \int_{p-2}^{+\infty} x\,e^{-\frac{\pi}{p}x^2}\,\mathrm{d}x = \frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}.$$

So

$$\sum_{\substack{n=1 \\ n \equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1 \\ n \equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} > 1 - \frac{\pi}{p} - \frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}, \tag{2.5}$$

and

$$S_1(X^-) > \frac{p-1}{2} \left(1 - \frac{\pi}{p} - \frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}} \right). \tag{2.6}$$

Note the RHS of the above inequality is positive when $p \geqslant 5$. Thus for any prime $p \geqslant 5$, there exists a $\chi \in X^-$ such that $\theta(\chi, i) \neq 0$.

2.2. Quadratic character

Let χ_1 be the unique quadratic character in X, which is even when $p \equiv 1 \pmod 4$ and odd when $p \equiv 3 \pmod 4$. Note $\chi_1(n) = \pm 1$ for any $n \geqslant 1$.

Case $p \equiv 1 \pmod 4$**.** The theta value of χ_1 is

$$\theta(\chi_1, i) = \sum_{\substack{n=1\\ n\equiv\pm1 \pmod p}}^{\infty} e^{-\frac{\pi}{p}n^2} + \sum_{\substack{n=1\\ n\not\equiv 0,\pm1 \pmod p}}^{\infty} \chi_1(n)\, e^{-\frac{\pi}{p}n^2}, \tag{2.7}$$

hence

$$\theta(\chi_1, i) < \sum_{n=1}^{\infty} e^{-\frac{\pi}{p}n^2} \leqslant \int_0^{+\infty} e^{-\frac{\pi}{p}x^2}\, \mathrm{d}x = \frac{\sqrt{p}}{2}. \tag{2.8}$$

Hence by (2.6) and (2.8),

$$S_1(X^-\backslash\{\chi_1\}) > \frac{p-1}{2}\left(1 - \frac{\pi}{p} - \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}}\right) - \frac{\sqrt{p}}{2} \tag{2.9}$$

and the RHS of the above inequality is positive when $p \geqslant 5$.

Case $p \equiv 3 \pmod 4$**.** The theta value of χ_1 is

$$\theta(\chi_1, i) = \sum_{\substack{n=1\\ n\equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1\\ n\equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} + \sum_{\substack{n=1\\ n\not\equiv 0,\pm1 \pmod p}}^{\infty} n\,\chi_1(n)\, e^{-\frac{\pi}{p}n^2}. \tag{2.10}$$

Note that

$$\sum_{\substack{n=1\\ n\not\equiv 0,\pm1 \pmod p}}^{\infty} n\,\chi_1(n)\, e^{-\frac{\pi}{p}n^2} < \sum_{\substack{n=1\\ n\not\equiv 0,\pm1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} < \sum_{n=1}^{\infty} n\, e^{-\frac{\pi}{p}n^2}.$$

Since the function $f(x) = x\, e^{-\frac{\pi}{p}x^2}$ ascends on the interval $[0, \sqrt{\frac{p}{2\pi}}]$ and descends on the interval $[\sqrt{\frac{p}{2\pi}}, +\infty)$, it takes its maximum value $\sqrt{\frac{p}{2\pi}}\, e^{-\frac{1}{2}}$ at $x = \sqrt{\frac{p}{2\pi}}$. Therefore,

$$\sum_{n=1}^{\infty} n\, e^{-\frac{\pi}{p}n^2} < \sqrt{\frac{p}{2\pi}} \cdot \sqrt{\frac{p}{2\pi}}\, e^{-\frac{1}{2}} + \int_{\sqrt{\frac{p}{2\pi}}}^{+\infty} x\, e^{-\frac{\pi}{p}x^2}\, \mathrm{d}x = \frac{p}{\pi}\, e^{-\frac{1}{2}}.$$

So

$$\begin{aligned} & S_1(X^-\backslash\{\chi_1\}) = S_1(X^-) - \theta(\chi_1, i) \\ =& \frac{p-3}{2}\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1\\ n\equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p}n^2}\right) - \sum_{\substack{n=1\\ n\not\equiv \pm1 \pmod p}}^{\infty} n\,\chi_1(n)\, e^{-\frac{\pi}{p}n^2} \\ >& \frac{p-3}{2}\left(1 - \frac{\pi}{p} - \frac{p}{2\pi}\, e^{-\frac{\pi(p-2)^2}{p}}\right) - \frac{p}{\pi}\, e^{-\frac{1}{2}}. \end{aligned} \tag{2.11}$$

One sees the RHS of the above inequality (2.11) is positive when $p \geqslant 11$. By (2.9) and (2.11), we have proved the following proposition.

Proposition 2.5. *For any prime $p \geqslant 11$, there exists a non-quadratic $\chi \in X^-$ such that $\theta(\chi, i) \neq 0$.*

Table 1
Table of primitive characters with conductor $p = 2\ell + 1$.

order d	$\#X(p,d)$	parity
1	1	even
2	1	odd
ℓ	$\ell - 1$	even
2ℓ	$\ell - 1$	odd

2.3. Quartic characters

Let χ_2 and $\overline{\chi_2}$ be the two quartic characters in X, which are even when $p \equiv 1 \pmod 8$ and odd otherwise. To deduce our main result we only need to provide an upper bound for $\theta(\chi_2, i) + \theta(\overline{\chi_2}, i)$. Note that $\chi_2(n) + \overline{\chi_2}(n) = 0, \pm 2$ for any $n \geqslant 1$.

Case $p \equiv 1 \pmod 8$. By definition, $\theta(\chi_2, i) + \theta(\overline{\chi_2}, i)$ is

$$\sum_{\substack{n=1 \\ n \not\equiv 0 \pmod p}}^{\infty} (\chi_2(n) + \overline{\chi_2}(n))\, e^{-\frac{\pi}{p} n^2},$$

which is less than $2 \sum\limits_{n=1}^{\infty} e^{-\frac{\pi}{p} n^2}$. So

$$\theta(\chi_2, i) + \theta(\overline{\chi_2}, i) < 2 \sum_{n=1}^{\infty} e^{-\frac{\pi}{p} n^2} \leqslant 2 \int_0^{+\infty} e^{-\frac{\pi}{p} x^2}\, \mathrm{d}x = \sqrt{p}.$$

Case $p \not\equiv 1 \pmod 8$. In this case $\theta(\chi_2, i) + \theta(\overline{\chi_2}, i)$ is

$$2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} - 2 \sum_{\substack{n=1 \\ n \equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} + \sum_{\substack{n=1 \\ n \not\equiv 0, \pm 1 \pmod p}}^{\infty} n\, (\chi_2(n) + \overline{\chi_2}(n))\; e^{-\frac{\pi}{p} n^2}, \tag{2.12}$$

wherein

$$\sum_{\substack{n=1 \\ n \not\equiv 0, \pm 1 \pmod p}}^{\infty} n\, (\chi_2(n) + \overline{\chi_2}(n))\, e^{-\frac{\pi}{p} n^2} < 2 \sum_{n=1}^{\infty} n\, e^{-\frac{\pi}{p} n^2} = \frac{2p}{\pi}\, e^{-\frac{1}{2}}.$$

Therefore,

$$\theta(\chi_2, i) + \theta(\overline{\chi_2}, i) < 2 \left(\sum_{\substack{n=1 \\ n \equiv 1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} - \sum_{\substack{n=1 \\ n \equiv -1 \pmod p}}^{\infty} n\, e^{-\frac{\pi}{p} n^2} + \frac{p}{\pi}\, e^{-\frac{1}{2}} \right). \tag{2.13}$$

2.4. Primes of type $2\ell + 1$

Let $p = 2\ell + 1$ be a prime, where ℓ is also a prime. Then the character group X is a cyclic group of order 2ℓ. The sizes and parity of each orbit $X(p, d)$ for $d \,|\, 2\ell$ are listed in Table 1.

Theorem 2.6. *$\theta(\chi, i) \neq 0$ for all non-quadratic characters χ with prime conductor $p = 2\ell + 1$, where ℓ is also a prime.*

Table 2
Table of primitive characters with conductor $p = 4\ell + 1$.

order d	$\#X(p,d)$	parity
1	1	even
2	1	even
4	2	odd
ℓ	$\ell - 1$	even
2ℓ	$\ell - 1$	even
4ℓ	$2(\ell - 1)$	odd

Proof. Since $X^+ = X(p,\ell) \cup \{\chi_0\}$, it follows from Proposition 2.4 that, when $p \geqslant 7$ there exists a $\chi \in X(p,\ell)$ such that $\theta(\chi,i) \neq 0$, and then by Proposition 2.2, $\theta(\chi,i) \neq 0$ for all $\chi \in X(p,\ell)$. Similarly, $X^- = X(p,2\ell) \cup \{\chi_1\}$ and it follows from Propositions 2.2 and 2.5 that $\theta(\chi,i) \neq 0$ for all $\chi \in X(p,2\ell)$ when $p \geqslant 11$. Since Cohen and Zagier proved $\theta(\chi,i) = 0$ for only two characters of conductors 300 and 600 and their complex conjugates and for no other primitive characters of conductor ≤ 52100, the theorem holds also for $p < 11$. □

2.5. Primes of type $4\ell + 1$

Now let $p = 4\ell + 1$, where ℓ is also a prime. The least such prime is 13. In this situation the sizes and parity of each orbit $X(p,d)$ for $d \,|\, 4\ell$ are listed in Table 2.

Note

$$X^+ = \{\chi_0\} \cup \{\chi_1\} \cup X(p,\ell) \cup X(p,2\ell), \quad X^- = \{\chi_2, \overline{\chi_2}\} \cup X(p,4\ell).$$

Lemma 2.7. *$S_1(X(p,\ell)) > 0$ for $p \geqslant 13$, and $S_1(X(p,2\ell)) > 0$ for $p \geqslant 29$.*

Proof. We assume that $p \geqslant 13$. Since $X(p,\ell)$ together with χ_0 constitutes a subgroup of X^+, it follows from Lemma 2.3 that $S_1(X(p,\ell) \cup \{\chi_0\}) > \ell\, e^{-\frac{\pi}{p}}$. Combining this with (2.1) yields

$$S_1(X(p,\ell)) = S_1(X(p,\ell) \cup \{\chi_0\}) - \theta(\chi_0,i) > \ell\, e^{-\frac{\pi}{p}} - \frac{\sqrt{p}}{2},$$

wherein $\ell\, e^{-\frac{\pi}{p}} - \frac{\sqrt{p}}{2}$ is positive for any $p \geqslant 13$.

To show $S_1(X(p,2\ell))$ is also positive an upper bound of $S_1(X(p,\ell) \cup \{\chi_0\})$ is needed. In view of Lemma 2.3,

$$S_1(X(p,\ell) \cup \{\chi_0\}) = \ell \sum_{\substack{n=1 \\ n^4 \equiv 1 \pmod{p}}}^{\infty} e^{-\frac{\pi}{p} n^2}.$$

Let b be the solution of the congruence equation $x^2 \equiv -1 \pmod{p}$ with $2 \leqslant b \leqslant \frac{p-1}{2}$. Since $b^2 \equiv -1 \pmod{p}$ and the least positive integer satisfying this congruence condition is $p-1$, we have

$$b^2 \geqslant p - 1. \tag{2.14}$$

Similarly,

$$(p-b)^2 \geqslant 2p - 1. \tag{2.15}$$

Note that $1, b, p-b$ and $p-1$ are the four solutions of the congruence equation $x^4 \equiv 1 \pmod{p}$ between 1 and $p-1$.

Thus

$$\sum_{\substack{n=1\\ n^4\equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} e^{-\frac{\pi}{p}n^2} = \sum_{k=0}^{\infty}\left(e^{-\frac{\pi(kp+1)^2}{p}} + e^{-\frac{\pi(kp+b)^2}{p}} + e^{-\frac{\pi(kp+p-b)^2}{p}} + e^{-\frac{\pi(kp+p-1)^2}{p}}\right)$$
$$< e^{-\frac{\pi}{p}} + e^{-\frac{\pi b^2}{p}} + e^{-\frac{\pi(p-b)^2}{p}} + e^{-\frac{\pi(p-1)^2}{p}} + 4\sum_{k=1}^{\infty} e^{-\frac{\pi(kp+1)^2}{p}}.$$

By (2.14) and (2.15),

$$e^{-\frac{\pi b^2}{p}} + e^{-\frac{\pi(p-b)^2}{p}} \leqslant 2\,e^{-\frac{\pi(p-1)}{p}} < 2\,e^{-\frac{12\pi}{13}} < 0.12.$$

For the item $e^{-\frac{\pi(p-1)^2}{p}}$, it is no greater than $e^{-\frac{12^2\pi}{13}} < 0.01$. Meanwhile, the sum $\sum_{k=1}^{\infty} e^{-\frac{\pi(kp+1)^2}{p}}$ has an upper bound

$$\int_0^{+\infty} e^{-\frac{\pi(px+1)^2}{p}}\,\mathrm{d}x < \frac{1}{\sqrt{\pi p}}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t = \frac{1}{2\sqrt{p}}.$$

Summing up all the above estimates, we obtain

$$S_1(X(p,\ell)\cup\{\chi_0\}) = \ell \sum_{\substack{n=1\\ n^4\equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} e^{-\frac{\pi}{p}n^2} < \ell\left(e^{-\frac{\pi}{p}} + 0.13 + \frac{2}{\sqrt{p}}\right). \tag{2.16}$$

Finally, from (2.2), (2.8) and (2.16) it follows that

$$\begin{aligned} S_1(X(p,2\ell)) &= S_1(X^+) - S_1(X(p,\ell)\cup\{\chi_0\}) - \theta(\chi_1, i)\\ &> 2\ell\, e^{-\frac{\pi}{p}} - \ell\left(e^{-\frac{\pi}{p}} + 0.13 + \frac{2}{\sqrt{p}}\right) - \frac{\sqrt{p}}{2}\\ &= \ell\left(e^{-\frac{\pi}{p}} - 0.13 - \frac{2}{\sqrt{p}}\right) - \frac{\sqrt{p}}{2}. \end{aligned} \tag{2.17}$$

One checks that $\ell\left(e^{-\frac{\pi}{p}} - 0.13 - \frac{2}{\sqrt{p}}\right) - \frac{\sqrt{p}}{2} > 0$ for any $p \geqslant 29$. $\square$

Lemma 2.8. *$S_1(X(p,4\ell))$ is positive for $p \geqslant 53$.*

Proof. By (2.4) and (2.13),

$$\begin{aligned} S_1(X(p,4\ell)) =& S_1(X^-) - \theta(\chi_2, i) - \theta(\overline{\chi_2}, i)\\ >& 2\ell\left(\sum_{\substack{n=1\\ n\equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1\\ n\equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2}\right)\\ &- 2\left(\sum_{\substack{n=1\\ n\equiv 1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} - \sum_{\substack{n=1\\ n\equiv -1 \ (\mathrm{mod}\ p)}}^{\infty} n\,e^{-\frac{\pi}{p}n^2} + \frac{p}{\pi}\,e^{-\frac{1}{2}}\right) \end{aligned}$$

$$=(2\ell-2)\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod p}}^{\infty} n\,e^{-\frac{\pi}{p}n^2}-\sum_{\substack{n=1\\ n\equiv -1 \pmod p}}^{\infty} n\,e^{-\frac{\pi}{p}n^2}\right)-\frac{2p}{\pi}\,e^{-\frac{1}{2}}.$$

In view of (2.5), we have

$$S_1(X(p,4\ell))>(2\ell-2)\left(1-\frac{\pi}{p}-\frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}\right)-\frac{2p}{\pi}\,e^{-\frac{1}{2}}.$$

It suffices to show

$$\frac{p-5}{4}\left(1-\frac{\pi}{p}-\frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}\right)-\frac{p}{\pi}\,e^{-\frac{1}{2}}>0,$$

or equivalently,

$$\left(1-\frac{5}{p}\right)\left(1-\frac{\pi}{p}-\frac{p}{2\pi}\,e^{-\frac{\pi(p-2)^2}{p}}\right)-\frac{4}{\pi\sqrt{e}}>0.$$

One easily checks that the above inequality holds for $p\geqslant 53$. □

Theorem 2.9. *$\theta(\chi,i)\neq 0$ for all non-quadratic, non-quartic characters χ with prime conductor $p=4\ell+1$, where ℓ is also a prime.*

Proof. This theorem follows from Proposition 2.2, Lemma 2.7, Lemma 2.8 and the fact that Cohen and Zagier proved $\theta(\chi,i)=0$ for only two characters of conductors 300 and 600 and their complex conjugates and for no other primitive characters of conductor ≤ 52100. □

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