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Existence results for some anisotropic Dirichlet problems

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ABSTRACT

This paper concerns with a class of elliptic anisotropic Dirichlet problems depending of one real parameter on bounded Euclidean domains. Our approach is based on variational and topological methods. More concretely, along the paper we show the existence of at least two weak solutions for the treated problem by using a direct consequence of the celebrated Pucci and Serrin theorem in addition to a local minimum result for differentiable functionals due to Ricceri. This abstract approach has been developed for equations on Carnot groups; see [15].

1. Introduction

It is well-known that a great attention in the last years has been focused by many authors on the study of anisotropic equations on bounded Euclidean domains. See, among others, the papers [2–6,9,11,13] as well as [14,17–19,22,23,28] and references therein.

Motivated by this large interest in the current literature, we study here the existence of weak solutions for the following anisotropic Dirichlet problem

$$(P_\lambda^f) \qquad \begin{cases} -\Delta_{p(x)} u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u := div(|\nabla^{p(x)-2}\nabla u)$ denotes the $p(\cdot)$-Laplace operator, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive real parameter, and $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function satisfying

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty. \tag{1}$$

Moreover, $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

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(f_1) there exist $a_1, a_2 > 0$ and $q \in C(\bar{\Omega})$ with $1 < p(x) < p^*$ for each $x \in \bar{\Omega}$, such that

$$|f(x,t)| \leq a_1 + a_2|t|^{q(x)-1},$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N \end{cases}.$$

Inspired by [1,15,24], we prove that, for small values of λ, problem (P_λ^f) admits at least two weak solutions [see Theorem 3.1] requiring that the continuous and subcritical nonlinear term f satisfies the celebrated Ambrosetti-Rabinowitz condition without the usual additional assumption at zero, that is

$$\lim_{t \to 0} \frac{f(x,t)}{t} = 0, \tag{2}$$

uniformly in $\bar{\Omega}$.

A special case of our result reads as follows.

Theorem 1.1. *Let Ω be a smooth and bounded domain of the Euclidean space $\mathbb{R}^N$, $p > 1$, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which*

(f_1') *there exist $a_1, a_2 > 0$ and $q \in \left(2, 2\left(\frac{N(p-1)+p}{N-p}\right)\right)$ such that*

$$|f(t)| \leq a_1 + a_2|t|^{q-1},$$

for every $t \in \mathbb{R}$;

(f_2') *there are $\mu > 2$ and $r > 0$ such that*

$$0 < \mu \int_0^t f(\tau)d\tau \leq tf(t),$$

for any $|t| \geq r$.

Then, there exists an open interval $\Lambda \subset (0, +\infty)$ such that, for every $\lambda \in \Lambda$, the following problem

$$\begin{cases} -\Delta p = \lambda f(u) & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

admits at least two (distinct) weak solutions in the Sobolev $W_0^{1,p}(\Omega)$.

The interval Λ in the above result can be explicitly determined. Set

$$c_s := \sup_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\|u\|_{L^s(\Omega)}}{\|u\|_{W_0^{1,p(x)}(\Omega)}}, \quad (\text{with } s \in \{1, q\})$$

Our abstract tool for proving the main result is the following abstract theorem that we recall here in a convenient form.

Theorem 1.2. *Let E be a reflexive real Banach space, and let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive. Further, assume that Ψ is sequentially weakly continuous. In addition, assume that, for each $\alpha > 0$, the functional $J_\alpha := \alpha\Phi - \Psi$ satisfies the classical compactness Palais-Smale (briefly (PS)) condition. Then, for each $\varrho > \inf\limits_E \Phi$ and each*

$$\alpha > \inf_{u\in\Phi^{-1}((-\infty,\varrho))} \frac{\sup\limits_{v\in\Phi^{-1}((-\infty,\varrho))} \Psi(v) - \Psi(u)}{\varrho - \Phi(u)},$$

the following alternative holds: either the functional J_α has a strict global minimum which lies in $\Phi^{-1}\big((-\infty, \varrho)\big)$, or J_α has at least two critical points one of which lies in $\Phi^{-1}\big((-\infty, \varrho)\big)$.

The above critical point result comes out from a joint application of the classical Pucci-Serrin theorem (see [20]) and a local minimum result due to Ricceri (see [25]). For a proof of Theorem 1.2 see, for instance, [24, Theorem 6]. We refer the interested reader to [16,26,27] and references therein for recent applications of the Ricceri's variational principle.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework and preliminaries. Successively, in Section 3, Theorem 3.1 and some preparatory results concerning the compactness Palais-Smale condition (see Lemmas 3.2 and 3.3) are presented.

In the last section, Theorem 3.1 has been proved and a concrete example of an application is presented in Example 4.3.

2. Abstract framework

Here and in the sequel, we assume that $p \in C(\bar\Omega)$ verifies the previous condition and is globally log-Hölder continuous on Ω. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined

$$L^{p(x)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \rho_p(u) := \int_\Omega |u|^{p(x)} dx < +\infty\Big\}. \tag{3}$$

On $L^{p(x)}(\Omega)$ we consider the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u}{\lambda}\Big|^{p(x)} dx \le 1\Big\}.$$

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined by putting

$$W^{1,p(x)}(\Omega) := \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\}$$

and it is endowed with the following norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \|\,|\nabla u|\,\|_{L^{p(x)}(\Omega)}.$$

By $W_0^{1,p(x)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W_0^{1,p(x)}(\Omega)$. We recall that, since p is globally log-Hölder continuous on Ω, the Poincaré inequality is true.

On $W_0^{1,p(x)}(\Omega)$ we consider the norm

$$\|u\| := \|\,|\nabla u|\,\|_{L^{p(x)}(\Omega)}. \tag{4}$$

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It is well known that, in the view of (1), both $L^{p(x)}(\Omega)$ and $W^{1,p}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces.

The following result generalizes the well-known Sobolev embedding theorem.

Theorem 2.1. *Assume that $p \in C(\bar{\Omega})$ with $p(x) > 1$ for each $x \in \bar{\Omega}$. If $q \in C(\bar{\Omega})$ and $1 < q(x) < p^*(x)$ for all $x \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.*

Set

$$\Phi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is known that $\Phi \in C^1(W_0^{1,p(x)}, \mathbb{R})$, and

$$\langle \Phi'(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla v \rangle \, dx$$

for each $u, v \in W_0^{1,p(x)}(\Omega)$. Moreover, the functional Φ is convex, sequentially weakly lower semicontinuous and its derivative $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is a homeomorphism.

Proposition 2.2. *If we denote*

$$\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx \qquad \forall u \in L^{p(x)}(\Omega)$$

then

(i) $|u|^{p(x)} < 1 \ (= 1; > 1) \Leftrightarrow \rho(u) < 1 \ (= 1; > 1)$;
(ii) $|u|^{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \geq \rho(u) \geq |u|_{p(x)}^{p^+}$;
(iii) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $|u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

See Fan and Zhao [8] and Zhao et al. [31].

Proposition 2.3. *The following fact holds*

(i) *$W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces;*
(ii) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous;*
(iii) *There is a constant $C > 0$, such that*

$$|u|_{p(x)} \leq C \, |\nabla u|_{p(x)} \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Proposition 2.5, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

See Fan and Zhao [8].

Moreover, for $\alpha > 0$ and $h \in C(\Omega)$ with $1 < h^-$, we put

$$[\alpha]^h := \max\left\{\alpha^{h^-}, \alpha^{h^+}\right\}$$

and set

$$c_s := \sup_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^s(\Omega)}}{\|u\|_{W_0^{1,p(x)}(\Omega)}}, \quad (\text{with}\, s \in \{1, q\})$$

We cite the monograph [28] for a nice introduction on anisotropic spaces and [12] for related topics on variational methods used in this paper. See also [21].

3. The main result and some technical lemmas

The aim of this section is to prove that, under natural assumptions on the nonlinear term f, weak solutions to problem (P_λ^f) below do exist. With the above notations the main result reads as follows.

Theorem 3.1. *Let $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(f_1) *there exist $a_1, a_2 > 0$ and $q \in C(\bar{\Omega})$ with $1 < q(x) < p^*$ for each $x \in \bar{\Omega}$, such that*

$$|f(x,t)| \le a_1 + a_2|t|^{q(x)-1},$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \ge N \end{cases};$$

(f_2) *there are $\mu > p^+$ and $r > 0$ such that*

$$0 < \mu \int_0^t f(x,\tau)d\tau \le tf(x,t),$$

for any $x \in \bar{\Omega}$, and $|t| \ge r$.

Then, for every $\varrho > 0$ and each

$$0 < \lambda < \frac{\varrho}{a_1 c_1 (p^+)^{\frac{1}{p^-}} [\varrho]^{\frac{1}{p}} + \frac{a_2}{q^-}[c_q]^q (p^+)^{\frac{q^+}{p^-}} [[\varrho]^{\frac{1}{p}}]^q}, \tag{5}$$

problem (P_λ^f) admits at least two weak solutions one of which lies in

$$\mathbb{B}_\varrho := \left\{ u \in W_0^{1,p(x)}(\Omega) : \|u\| < \min\left\{ (p^+\varrho)^{\frac{1}{p^-}}, (p^+\varrho)^{\frac{1}{p^+}} \right\} \right\}.$$

We recall that a *weak solution* for the problem (P_λ^f), is a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\langle \nabla u, \nabla v\rangle\, dx = \lambda \int_\Omega f(x,u)v dx, \quad \forall\, v \in W_0^{1,p(x)}(\Omega) \\ u \in W_0^{1,p(x)}(\Omega). \end{cases}$$

Now, for the sake of completeness, we recall that a C^1-functional $J : E \to \mathbb{R}$, where E is a real Banach space with topological dual E^*, satisfies the *Palais-Smale condition at level* $\zeta \in \mathbb{R}$, (abbreviated $(\mathrm{PS})_\zeta$) when:

$(\mathrm{PS})_\zeta$ *Every sequence* $\{u_n\}$ *in* E *such that*

$$J(u_n) \to \zeta, \quad \text{and} \quad \|J'(u_n)\|_{E^*} \to 0,$$

as $n \to +\infty$, *possesses a convergent subsequence.*

We say that J satisfies the *Palais-Smale condition* (abbreviated (PS)) if $(\mathrm{PS})_\zeta$ holds for every $\zeta \in \mathbb{R}$.

For our goal, in the next two lemmas we shall verify the compactness (PS) condition for the functional $\mathcal{J}_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}_\lambda(u) := \frac{1}{\lambda}\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \int_\Omega F(x,u)dx, \quad \forall\, u \in W_0^{1,p(x)}(\Omega) \tag{6}$$

where $\lambda > 0$ and, as usual, we set $F(x,t) := \int_0^t f(x,\tau)d\tau$.

Note that the functional $\mathcal{J}_\lambda \in C^1(W_0^{1,p(x)}(\Omega))$ and its derivative at $u \in W_0^{1,p(x)}(\Omega)$ is given by

$$\langle \mathcal{J}_\lambda'(u), v\rangle = \frac{1}{\lambda}\int_\Omega |\nabla u|^{p(x)-2}\langle \nabla u, \nabla v\rangle\, dx - \int_\Omega f(x,u)v dx,$$

for every $v \in W_0^{1,p(x)}(\Omega)$.

Lemma 3.2. *Assume that conditions* (f_1) *and* (f_2) *are verified. Then, every Palais-Smale sequence for the functional* $\mathcal{J}_\lambda$ *is bounded in the Sobolev space* $W_0^{1,p(x)}(\Omega)$.

Proof. Suppose that $\{u_n\} \subset X$, $\{\varphi(u_n)\}$ is bounded and $\|\varphi'(u_n)\| \to 0$. Then

$$\begin{aligned}
C \geq \varphi(u_n) &= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx - \int_\Omega F(x,u)\, dx \\
&\geq \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx - \int_\Omega \frac{u_n}{\mu} f(x,u_n)\, dx - c \\
&\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right)|\nabla u_n|^{p(x)}\, dx + \int_\Omega \frac{1}{\mu}\left(|\nabla u_n|^{p(x)} - u_n f(x,u_n)\right) dx - c \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\mu}\right)|\nabla u_n|_{p(x)}^{p^-} - \frac{1}{\mu}\|\varphi'(u_n)\|\,\|u_n\| - c.
\end{aligned}$$

Hence, the sequence $\{\|u_n\|\}$ is bounded. □

We say that a map $L : E \to E^*$ satisfies the (S_+) condition if: $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty}\langle L(u_n) - L(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in E.

Next lemma is quite standard and directly follows by [7]. However, for the sake of completeness, we give direct proof of it.

Lemma 3.3. *Assume that conditions* (f_1) *and* (f_2) *are verified. Then, the functional* $\mathcal{J}_\lambda$ *satisfies the compactness* (PS) *condition.*

Proof. Consider the following functional:

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \qquad u \in X := W_0^{1,p(x)}(\Omega).$$

Standard computations improve that the functional $J \in C^1(X, R)$; see [7]. Moreover, $p(x)$-Laplace operator is the derivative operator of J in the weak sense. Let us consider $\Phi' : X \to X^*$, the functional given by

$$\langle \Phi'(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \qquad \forall v, u \in X.$$

(i) The functional Φ' is continuous and bounded. Indeed, for any $\xi, \eta \in \mathbb{R}^N$, we have the following relations (see [7])

$$\frac{\left[\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta)\right]}{\left(|\xi|^p + |\eta|^p\right)^{(p-2)/p}} \geq (p-1)|\xi-\eta|^p, \tag{7}$$

wherever $1 < p < 2$

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta) \geq \left(\frac{1}{2}\right)^p |\xi-\eta|^p, \quad p \geq 2. \tag{8}$$

The above inequalities ensure that Φ' is strictly monotone.

(ii) By put (i), if $u_n \to u$ and $\limsup_{n\to\infty} \langle \Phi(u_n) - \Phi(u), u_n - u \rangle \leq 0$ it follows that

$$\limsup_{n\to\infty} \langle \Phi(u_n) - \Phi(u), u_n - u \rangle = 0.$$

By of (1) and (2), ∇u_n converges in measure to ∇u in Ω, so we get a subsequence (which we still denote by ∇u_n) satisfying $\nabla u_n(x) \to \nabla u(x)$, a.e. $x \in \Omega$. The Fatou's Lemma gives

$$\liminf_{n\to\infty} \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \geq \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx. \tag{9}$$

Since $u_n \to u$ it follows that

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u \rangle = \lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle = 0. \tag{10}$$

On the other hand

$$\begin{aligned}\left\langle \Phi'(u_n), u_n - u\right\rangle &= \int_\Omega |\nabla u_n|^{p(x)}\, dx - \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u \nabla v dx \\ &\geq \int_\Omega |\nabla u_n|^{p(x)}\, dx - \int_\Omega |\nabla u_n|^{p(x)-1} |\nabla u|\, dx \\ &\geq \int_\Omega |\nabla u_n|^{p(x)}\, dx - \int_\Omega (\frac{p(x)-1}{p(x)} |\nabla u_n|^{p(x)}\, dx \\ &- \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx \\ &\geq \int_\Omega \frac{1}{p\,(x)} |\nabla u_n|^{p(x)}\, dx - \int_\Omega \frac{1}{p\,(x)} |\nabla u|^{p(x)}\, dx,\end{aligned}$$

for $x \to \infty$.

Taking into account (14) and (15) one has

$$\lim_{n\to\infty} \int_\Omega \frac{1}{p\,(x)} |\nabla u_n|^{p(x)}\, dx = \int_\Omega \frac{1}{p\,(x)} |\nabla u|^{p(x)}\, dx. \tag{11}$$

By (6) it follows that the integrals of the functions family

$$\left\{(1/p\,(x))\, |\nabla u_n|^{p(x)}\right\}$$

possess absolutely equicontinuity on Ω. Since

$$\frac{|\nabla u_n(x) - \nabla u(x)|^{p(x)}}{p(x)} \leq C\left(\frac{1}{p\,(x)} |\nabla u_n\,(x)|^{p(x)} + \frac{1}{p\,(x)} |\nabla u\,(x)|^{p(x)}\right), \tag{12}$$

the integrals of the family $\left\{(1/p\,(x))\, |\nabla u_n\,(x) - \nabla u\,(x)|^{p(x)}\right\}$ are also absolutely equicontinuous on Ω (see [30, Chapter 6, Section 3]) and therefore

$$\lim_{n\to\infty} \int_\Omega \frac{1}{p\,(x)} |\nabla u_n\,(x) - \nabla u\,(x)|^{p(x)}\, dx = 0. \tag{13}$$

By (8)

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n\,(x) - \nabla u\,(x)|^{p(x)}\, dx = 0. \tag{14}$$

By Proposition 2.4 and (9) it follows that $u_n \to u$, i.e. Φ' is of type (S_+).

(iii) By the strictly monotonicity, Φ' is an injection. Since

$$\lim_{\|u\|\to\infty} \frac{\left\langle \Phi' u, u\right\rangle}{\|u\|} = \lim_{\|u\|\to\infty} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\|u\|} = \infty,$$

the functional Φ' is coercive, thus Φ' is a surjection in view of Minty-Browder Theorem (see [32, Theorem 26A]). Hence Φ' has an inverse mapping $\Phi^{-1} : X^* \to X$. Therefore, the continuity of Φ^{-1} is sufficient to ensure Φ' to be a homeomorphism.

If $f_n, f \in X^*$, $f_n \to f$, let $u_n = \Phi^{-1}(f_n)$, $u = \Phi^{-1}(f)$, then $\Phi'(u_n) = f_n$, $\Phi'(u) = f$.

So $\{u_n\}$ is bounded in X. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$.

Since $f_n \to f$, then

$$\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle = \lim_{n\to\infty} \langle f_n, u_n - u_0 \rangle = 0. \tag{15}$$

Since Φ' is of type (S_+), $u_n \rightharpoonup u_0$, we conclude that $u_n \to u_0$, so Φ^{-1} is continuous.

Indent, let $\Psi(u) = \int_\Omega F(x,u)dx$, and $\Psi' : X \to X^*$ its derivative, without loss of generality, we assume that $u_n \rightharpoonup u_0$, then $\Psi'(u_n) \to \Psi'(x_0)$.

Since $\mathcal{J}'_\lambda(u_n) = \Phi'(u_n) - \Psi'(u_n) \to 0$, $\Phi'(u_n) \to \Psi'(u_0)$. Since Φ' is a homeomorphism, $u_n \to u_0$, and so the functional $\mathcal{J}_\lambda$ satisfies (PS) condition. □

4. Proof of Theorem 3.1

For the proof of our result, we observe that problem (P_λ^f) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_\lambda$. Note that the functional $\mathcal{J}_\lambda$ is continuously Gâteaux differentiable in $u \in W_0^{1,p(x)}(\Omega)$ and one has

$$\langle \mathcal{J}'_\lambda(u), v \rangle = \frac{1}{\lambda} \int_\Omega |\nabla u|^{p(x)-2} \langle \nabla u, \nabla v \rangle \, dx \; - \int_\Omega f(x,u) v dx$$

for every $v \in W_0^{1,p(x)}(\Omega)$. Thus, the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions to problem (P_λ^f). Let $\varrho > 0$ and set $\alpha := 1/\lambda$, with λ as in the statement.

Hence, let us apply Theorem 1.2, taking $E := W_0^{1,p(x)}(\Omega)$, endowed by the norm (3), $J_\alpha := \mathcal{J}_\lambda$, and setting

$$\Phi(u) := \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx,$$

as well as

$$\Psi(u) := \int_\Omega F(x,u) dx,$$

for every $u \in W_0^{1,p(x)}(\Omega)$.

Now, Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly continuous thanks to the Rellich-Kondrachov theorem. Hence, the regularity assumptions on the functional J_α are verified.

Now, we observe that there exists $u_0 \in W_0^{1,p(x)}(\Omega)$ such that

$$J_\alpha(tu_0) \to -\infty, \tag{16}$$

as $t \to +\infty$.

Indeed by (f_1) it follows that

$$F(x,t) \geq C\,|t|^\mu \quad \forall x \in \overline{\Omega},\ |t| \geq M.$$

For $u_0 \in X \backslash \{0\}$ and $t > 1$, we have

$$\varphi(tu_0) = \int_\Omega \frac{1}{p(x)} |t\nabla u_0|^{p(x)} dx - \int_\Omega F(x, tu_0) dx$$
$$\leq t^{p^+} \int_\Omega \frac{1}{p(x)} |\nabla u_0|^{p(x)} - ct^\mu \int_\Omega |u_0|^\mu dx - C_1,$$

which implies $\varphi(tu_0) \to -\infty$ as $t_{\to+\infty}$.

Hence, the functional J_μ is unbounded from below and, by Lemmas 3.2 and 3.3, the compactness (PS) condition is verified.

We claim that

$$\mu > \chi(\varrho) := \inf_{u\in\Phi^{-1}((-\infty,\varrho))} \frac{\sup_{v\in\Phi^{-1}((-\infty,\varrho))} \int_\Omega F(\xi, v)dx - \int_\Omega F(\xi, u(x))dx}{\varrho - \|u\|^2_{W_0^{1,p(x)}(\Omega)}}, \tag{17}$$

for every $\varrho > 0$. For our goal, let us fix $\varrho > 0$. Since $0 \in \Phi^{-1}((-\infty, \varrho))$, it follows that

$$\chi(\varrho) \leq \frac{\sup_{v\in\Phi^{-1}((-\infty,\varrho))} \int_\Omega F(x, v)dx}{\varrho}. \tag{18}$$

On the other hand, one has

$$\frac{1}{\varrho} \sup_{\Phi(u)\leq\varrho} \Psi(u) \leq \frac{1}{\varrho}\left\{a_1c_1(p^+)^{\frac{1}{p^-}}[\varrho]^{\frac{1}{p}} + \frac{a_2}{q^-}[c_q]^q(p^+)^{\frac{q^+}{p^-}}[[\varrho]^{\frac{1}{p}}]^q\right\}. \tag{19}$$

See, for instance, [3].

Indeed by Theorem 1.3 of [14] and from the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, we have

$$\int_\Omega |u|^{q(x)}dx = \varrho_q(u) \leq [\|u\|_{L^{q(x)}(\Omega)}]^q \leq [c_q\|u\|]^q$$

for each $u \in X$.

For each $u \in \Phi^{-1}((-\infty, \varrho))$, thanks to Proposition 2.2, one has

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p}} < [p^+\varrho]^{\frac{1}{p}} \leq (p^+)^{\frac{1}{p^-}}[\varrho]^{\frac{1}{p}}.$$

So, the compact embedding $X \hookrightarrow L^1(\Omega)$, (f_1) and (23) imply that, for each $u \in \Phi^{-1}((-\infty, \varrho))$, we have

$$\Psi(u) \leq a_1 \int_\Omega |u|dx + \frac{a_2}{q^-} \int_\Omega |u|^{q(x)}dx$$
$$\leq a_1c_1\|u\| + \frac{a_2}{q^-}[c_q\|u\|]^q$$
$$\leq a_1c_1\|u\| + \frac{a_2}{q^-}[c_q]^q[\|u\|]^q$$
$$< a_1c_1(p^+)^{\frac{1}{p^-}}[\varrho]^{\frac{1}{p}} + \frac{a_2}{q^-}[c_q]^q(p^+)^{\frac{q^+}{p^-}}[[\varrho]^{\frac{1}{p}}]^q.$$

Then, thanks to

$$\sup_{v\in\Phi^{-1}((-\infty,\varrho))}\int_{\Omega} F(x,v)dx \leq a_1c_1(p^+)^{\frac{1}{p^-}}[\varrho]^{\frac{1}{p}} + \frac{a_2}{q^-}[c_q]^q(p^+)^{\frac{q^+}{p^-}}[[\varrho]^{\frac{1}{p}}]^q,$$

inequality (19) immediately holds.

Since (5) holds, conditions (18) and (19) immediately yield

$$\chi(\varrho) \leq \frac{1}{\varrho}\left(a_1c_1(p^+)^{\frac{1}{p^-}}[\varrho]^{\frac{1}{p}} + \frac{a_2}{q^-}[c_q]^q(p^+)^{\frac{q^+}{p^-}}[[\varrho]^{\frac{1}{p}}]^q\right) < \frac{1}{\lambda} = \alpha.$$

Thus, inequality (17) is proved.

Then, owing to Theorem 1.2, problem (P_λ^f) admits at least two weak solutions one of which lies in $\Phi^{-1}((-\infty,\varrho))$. This completes the proof.

Remark 4.1. Theorem 3.1 can be viewed as a subelliptic counterpart of [24, Theorem 4].

Remark 4.2. We emphasize that Theorem 3.1 ensures the existence of at least two weak solutions whenever

$$\lambda \in \Lambda := \left(0, \max_{\varrho>0} h(\varrho)\right),$$

where $h : [0,+\infty) \to [0,+\infty)$ is the continuous function given by

$$h(\varrho) := \frac{\varrho}{a_1c_1(p^+)^{\frac{1}{p^-}}[\varrho]^{\frac{1}{p}} + \frac{a_2}{q^-}[c_q]^q(p^+)^{\frac{q^+}{p^-}}[[\varrho]^{\frac{1}{p}}]^q}.$$

Note that $\max_{\varrho>0} h(\varrho) < +\infty$, since $q > 2$.

Hence, Proposition 1.1 in Introduction is an immediate consequence of Theorem 3.1 taking into account of Section 2. Moreover, we also point out that, in Theorem 1.1, due to the presence of the parameter λ, on the contrary of [10, Theorem 3.1], no conditions at zero on the nonlinear term f is requested.

In conclusion, we present a direct application of the main result.

Example 4.3. Let Ω be a bounded domain with smooth boundary $\partial\Omega$, and $p, q \in C(\bar{\Omega})$ such that $1 < p^+ < q^- \leq q(x) < p^*(x)$ for each $x \in \bar{\Omega}$. Then, owing to Theorem 3.1, there exists an open interval $\Lambda \subset (0,+\infty)$ such that, for every $\lambda \in \Lambda$, the following problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(x,t) = \begin{cases} 1 + q(x)t^{q(x)-1} & x \in \bar{\Omega}, t \geq 0 \\ 1 - q(x)(-t)^{q(x)-1} & x \in \bar{\Omega}, t < 0, \end{cases}$$

admits at least two distinct and non-trivial weak solutions in $W_0^{1,p(x)}(\Omega)$. Note that in our setting, condition (2) is clearly not verified.

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Remark 4.4. Taking into account the results contained in [3] a concrete upper bound for the constants c_1 and c_q in Theorem 3.1 can be done. More precisely, one has

$$c_1 \leq k_{(p^-)^*}|\Omega|\frac{(p^-)^*-1}{(p^-)^*}$$

and

$$c_q \leq (|\Omega|+1)^2 k_{(p^-)^*}|\Omega|\frac{(p^-)^*-q^+}{(p^-)^* q^+},$$

where $q \in C^0(\bar{\Omega})$, and $1 < q(x) \leq q^+ \leq (p^-)^* \leq p^*(x)$ for each $x \in \bar{\Omega}$. Here, we recall that, by Talenti's result [29], if $1 < p^- < N$

$$k_{p^*} = \frac{1}{\sqrt{\pi}}\frac{1}{N^{\frac{1}{p^-}}}\left(\frac{p^- - 1}{N-p^-}\right)^{1-\frac{1}{p}}\left[\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p^-}\right)\Gamma(1+N-\frac{N}{p^-})}\right]^{\frac{1}{N}}.$$

Finally, we notice that Theorem 3.1 (and its consequences) represents a more precise version of Theorem 4.1 in [3].

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