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Proper splittings and reduced solutions of matrix equations

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ABSTRACT

In this article we apply proper splittings of matrices to develop an iterative process to approximate solutions of matrix equations of the form $TX = W$. Moreover, by using the partial order induced by positive semidefinite matrices, we obtain equivalent conditions to the convergence of this process. We also include some speed comparison results of the convergence of this method. In addition, for all matrix T we propose a proper splitting based on the polar decomposition of T.

1. Introduction

The theory of splittings of matrices is a useful tool for the construction of iterative methods for solving systems of linear equations. Roughly speaking, a splitting of a matrix $T \in M^{m\times n}(\mathbb{C})$ is a partition of T of the form: $T = U - V$ with $U, V \in M^{m\times n}(\mathbb{C})$. A pioneering work regarding splitting of matrices is due to Varga [37] where the concept of regular splitting for invertible matrices $T \in M^n(\mathbb{R})$ was introduced to approximate the unique solution $T^{-1}w$ of the system $Tx = w$ by means of the following iterative process:

$$x^{i+1} = U^{-1}Vx^i + U^{-1}w. \tag{1}$$

The scheme (1) converges for every x^0 if and only if the spectral radius of $U^{-1}V$ is less than one (in symbols, $\rho(U^{-1}V) < 1$) and, in such case, it converges to $T^{-1}w$. Later, Berman and Plemmons [6] extended the notion of regular splitting for $T \in M^{m\times n}(\mathbb{R})$. They called a proper splitting of $T \in M^{m\times n}(\mathbb{R})$ to a partition $T = U - V$ where $U, V \in M^{m\times n}$, $\mathcal{R}(U) = \mathcal{R}(T)$ and $\mathcal{N}(U) = \mathcal{N}(T)$. Here, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote

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the range and nullspace of T, respectively. Based on this concept, Berman and Plemmons proved that the next iterative process

$$x^{i+1} = U^\dagger V x^i + U^\dagger w \tag{2}$$

converges to the best least square approximate solution $T^\dagger w$ of $Tx = w$ for every x^0 if and only if $\rho(U^\dagger V) < 1$.

In view of the above, during the last years several authors dealt with the problems of finding conditions on the matrices U, V (of the splitting) which ensure that $\rho(U^{-1}V) < 1$ (or, $\rho(U^\dagger V) < 1$) and of comparing the speed of convergence of these methods for different classes of splittings and matrices. Let us mention some of these classes: regular splitting [37], [17], nonnegative splitting with first and second type [34], [35], weak regular splitting with first and second type [38], [13], [19], proper splitting [29], [6], P-regular splitting [21], [22], P-proper splitting [23], weak nonnegative splitting of the first and the second type [14], T monotone [37], [38], [14], T positive definite [30], T positive semidefinite [23], among many others. All these articles are based on working with nonnegative matrices or positive definite matrices and the usual partial orders induced by them. In the case of considering nonnegative matrices, the results follow by the Perron-Frobenius theory. In [12], the authors unified the theory for these two partial orders and for the scheme (1) by using the partial order induced by positivity cone of matrices. However, the theory for the scheme (2) requires a deeper analysis. A first approach is given in [6], where under the idea of proper splitting and considering matrices that leave a cone invariant, many results as those of Varga [37], Ortega and Rheinboldt [33], Mangasarian [27] and Vandergraft [36] are extended to scheme (2). In fact, nonnegative matrices leave the cone $\mathbb{R}^n_+$ invariant and so applying the ideas of [6] many of the results about scheme (1) and nonnegative matrices are immediately extended for (2). However, these ideas can not be applied when the partial order induced by positive semidefinite matrices, also known as Löwner order, is considered. One of our goal in this article is to fill this gap. Moreover, we observe that proper splitting can also be applied to solve matrix equations of the form $TX = W$ and to get their so-called reduced solutions. This sort of solutions emerges as a generalization of the well-known Douglas' solutions [18] and is useful, for example, in many problems of engineering and physic in which it is needed to separate signals from noise (see [5], [8], among others). In this article, given a proper splitting $T = U - V$, we propose the iterative method (4) and we prove that it converges to a reduced solution of $TX = W$ if and only if $\rho(U^\dagger V) < 1$. In addition, equivalent conditions for (4) to converge to an Hermitian or positive definite reduced solution are provided. Motivated by all these facts, the remain of the paper is devoted to provide, for a proper splitting $T = U - V$, equivalent conditions to $\rho(U^\dagger V) < 1$ and some speed comparison results, considering the Löwner order. Furthermore, a proper splitting which is based on the polar decomposition of matrices is proposed.

We briefly describe the contents of the paper. In Section 2, we fix the notation and give some results that are needed in the following sections. In Section 3, we study some spectral properties of the so-called reduced solutions of matrix equations $TX = W$ and we analyse the existence of Hermitian and positive semidefinite reduced solutions. In Section 4, some properties of proper splitting that will be used in the next sections are studied. Section 5 is entirely devoted to explore the relationship between proper splitting and reduced solutions. In this sense, the iterative process (4) is proposed with the aim of approximating the reduced solutions of $TX = W$ by means of the use of a proper splitting of T. In addition, the convergence of this iterative process to Hermitian and positive definite solutions is studied. Here the main result is Theorem 5.1. In Section 6, the main results regarding convergence and speed comparison for the iterative method (4) considering the Löwner order are presented. The main results are Propositions 6.1 and 6.3 and Theorems 6.6 and 6.7. Finally, in Section 7, we present a particular proper splitting for $T \in \mathbb{M}^{m\times n}(\mathbb{C})$ based on the polar decomposition of T. In Proposition 7.3 the convergence of this particular splitting is analysed. In Theorem 7.4 and Proposition 7.6 this splitting is studied for the class of matrices $T \in M^{n\times n}(\mathbb{C})$ that can be factorized as the product of two orthogonal projections and for normal matrices.

2. Preliminaries

Along this article the set of n-uples of complex numbers is denoted by $\mathbb{C}^n$ and $M^{m\times n}(\mathbb{C})$ denotes the set of $m\times n$-matrices with coefficients in $\mathbb{C}$. If $m=n$ we abbreviate $M^n(\mathbb{C})$. Throughout, we shall consider the canonical inner product, $\langle\ ,\ \rangle$, on $\mathbb{C}^n$ and $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$ the norm induced by it. The spectral norm in $M^n(\mathbb{C})$ will be denoted by $\|\ \|$. Given $T\in M^{m\times n}(\mathbb{C})$ the symbols $\sigma(T)$ and $\rho(T)$ stand for the spectrum (set of eigenvalues) of T and spectral radius of T, respectively. In addition, given $T\in M^{m\times n}(\mathbb{C})$ we denote by $\mathcal{R}(T),\mathcal{N}(T),r(T),T^*$, the range, the nullspace, the rank and complex conjugate transpose of T, respectively. The Moore Penrose inverse of $T\in M^{m\times n}(\mathbb{C})$ is denoted by $T^\dagger$ and it is the unique matrix in $M^{n\times m}(\mathbb{C})$ which satisfies the four "Moore-Penrose equations"

$$TXT=T,\quad XTX=X,\quad XT=(XT)^*,\quad TX=(TX)^*. \tag{3}$$

The next classical result of Greville [20] related to the reverse order law for the Moore-Penrose inverse will be used along the article.

Theorem 2.1. *Let $T_1,T_2\in M^n(\mathbb{C})$. Then, $(T_1T_2)^\dagger=T_2^\dagger T_1^\dagger$ if and only if $\mathcal{R}(T_1^*T_1T_2)\subseteq\mathcal{R}(T_2)$ and $\mathcal{R}(T_2T_2^*T_1^*)\subseteq\mathcal{R}(T_1^*)$.*

Henceforth, we denote by $M_H^n(\mathbb{C})$ the set of Hermitian matrices of $M^n(\mathbb{C})$ and by $M_+^n(\mathbb{C})$ the set of positive semidefinite matrices, i.e., $T\in M_+^n(\mathbb{C})$ if $\langle Tx,x\rangle\geq 0$ for all $x\in\mathbb{C}^n$. Moreover, given $T_1,T_2\in M_H^n(\mathbb{C})$, we say that $T_1\leq T_2$ if $T_2-T_1\in M_+^n(\mathbb{C})$, i.e., $\leq$ denotes the Löwner order of matrices.

The following result will be frequently used in these notes. The proof can be found in [4, Corollary 2].

Theorem 2.2. *Let $T_1,T_2\in M_+^n(\mathbb{C})$ and consider the following conditions*

a) $T_1\leq T_2$;
b) $T_2^\dagger\leq T_1^\dagger$;
c_1) $r(T_1)=r(T_2)$;
c_2) $\mathcal{R}(T_2-T_1)\cap\mathcal{N}(T_2)=\mathcal{R}(T_1^\dagger-T_2^\dagger)\cap\mathcal{N}(T_1)=\{0\}$.

Then any of two the conditions a), b) and c_i) imply the third condition, $i=1,2$.

3. Reduced solutions of matrix equations

In this section we focus on the study of matrix equations of the form $TX=W$ for $T\in M^{m\times n}(\mathbb{C})$ and $W\in M^{m\times r}(\mathbb{C})$. More precisely, we are interested on the so-called "reduced solutions" of these equations introduced in the next result.

Theorem 3.1. *Let $T\in M^{m\times n}(\mathbb{C})$ and $W\in M^{m\times r}(\mathbb{C})$. There exists $X_0\in M^{n\times r}(\mathbb{C})$ such that $TX_0=W$ if and only if $\mathcal{R}(W)\subseteq\mathcal{R}(T)$. In such case, for each subspace $\mathcal{M}$ such that $\mathcal{M}\dot{+}\mathcal{N}(T)=\mathbb{C}^n$ there exists a unique $X_{\mathcal{M}}\in M^{n\times r}(\mathbb{C})$ such that $TX_{\mathcal{M}}=W$ and $\mathcal{R}(X_{\mathcal{M}})\subseteq\mathcal{M}$. Moreover, $\mathcal{N}(X_{\mathcal{M}})=\mathcal{N}(W)$. We called $X_{\mathcal{M}}$ the* **reduced solution** *for $\mathcal{M}$ of $TX=W$. In particular, it holds that $X_{\mathcal{R}(T^*)}=T^\dagger W$.*

Proof. See [1] and [18]. □

The previous result also holds for T,W and X_0 bounded linear operators on Hilbert spaces, see [18].

Remark 3.2. In this remark we comment some advantages of working with reduced solutions:

a) Reduced solutions and oblique projections are two concepts closely related. Indeed, the set of oblique projections with a fixed nullspace $\mathcal{T}$ coincides with the set of reduced solutions of the equation $(I - P_{\mathcal{T}})X = I - P_{\mathcal{T}}$, where $P_{\mathcal{T}}$ denotes the orthogonal projection onto $\mathcal{T}$. In a more general context, given $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and $\mathcal{M}, \mathcal{N}$ two algebraic complements of $\mathcal{N}(T)$, then it is easy to verify that $X_{\mathcal{N}} = Q_{\mathcal{N}//\mathcal{N}(T)} X_{\mathcal{M}}$ where $Q_{\mathcal{N}//\mathcal{N}(T)}$ is the projection with range $\mathcal{N}$ and nullspace $\mathcal{N}(T)$. This relationship between reduced solutions will be frequently used along the article.

   Oblique projections (and so reduced solutions) appear in several problems of engineering, physic, chemistry, among other areas. For instance, they are useful for solving a variety of signal processing problems where it is needed to separate signals from noise (see [5], [8], [7], among other sources). Furthermore, oblique projections are used in communication problems to remove intersymbol interference [5] and for estimation of directions-of-arrival [8].

b) Other class of matrix equations that frequently arises in many applications are those of the form $TXS = W$, with $T \in M^{m\times n}(\mathbb{C})$ and $S \in M^{r\times s}(\mathbb{C})$. Let us observe that this class of equations can be transformed into two-steps matrix equations of the previous form by means of reduced solutions. More precisely, if $Y_{\mathcal{M}}$ is a reduced solution of $TX = W$ then $\mathcal{N}(Y_{\mathcal{M}}) = \mathcal{N}(W)$ or, equivalently, $\mathcal{R}(Y_{\mathcal{M}}^*) = \mathcal{R}(W^*)$ and so $\mathcal{R}(Y_{\mathcal{M}}^*) \subseteq \mathcal{R}(S^*)$. Thus, there exists Y^* such that $S^*Y^* = Y_{\mathcal{M}}^*$. Hence, $TYS = TY_{\mathcal{M}} = W$, i.e., Y solves $TXS = W$. We shall return to this idea in Section 5.

c) Reduced solutions can also be applied to fully describe the generalized inverses of a matrix $T \in M^{m\times n}(\mathbb{C})$, see [1]. For instance, $T^\dagger$ coincides with the reduced solution of $TX = P_{R(T)}$ for $\mathcal{M} = R(T^*)$.

Lemma 3.3. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and $\mathcal{N}(T) \subseteq \mathcal{N}(W)$. Then the following assertions hold:*

a) $\sigma(X_{\mathcal{M}}) = \sigma(T^\dagger W)$ for all reduced solution $X_{\mathcal{M}}$ of $TX = W$.
b) $\rho(X_{\mathcal{M}}) = \rho(T^\dagger W)$ for all reduced solution $X_{\mathcal{M}}$ of $TX = W$.
c) Let $X_{\mathcal{M}}$ be the reduced solution for $\mathcal{M}$ of $TX = W$ and $\lambda \in \sigma(X_{\mathcal{M}})$. Hence, if v_λ is an eigenvector of $X_{\mathcal{M}}$ associated to λ then $Q_{\mathcal{N}//\mathcal{N}(T)} v_\lambda$ is an eigenvector of $X_{\mathcal{N}}$ associated to λ for all reduced solution $X_{\mathcal{N}}$ of $TX = W$.

Proof. *a)* Since $X_{\mathcal{M}} = Q_{\mathcal{M}//\mathcal{N}(T)} T^\dagger W$ we get that $\sigma(X_{\mathcal{M}}) = \sigma(Q_{\mathcal{M}//\mathcal{N}(T)} T^\dagger W) = \sigma(T^\dagger W Q_{\mathcal{M}//\mathcal{N}(T)}) = \sigma(T^\dagger W)$, since $\mathcal{N}(T) \subseteq \mathcal{N}(W)$.

b) It follows from the above item.

c) Again, it follows from the fact that $X_{\mathcal{N}} = Q_{\mathcal{N}//\mathcal{N}(T)} X_{\mathcal{M}}$. □

As we have already said, matrix equations of the form $TX = W$ arise in many problems of engineering, statistics, physics, among other areas. However, in many of these applications the only relevant solutions are those where the solution matrix is Hermitian or positive semidefinite. This occurs, for example, in many statistical problems (see [9], [28], just to mention a few). Evidently, not every solvable matrix equation $TX = W$ admits a positive semidefinite nor Hermitian solution. The following result due to Khatri and Mitra [24] provides equivalent conditions to the existence of such kind of solutions.

Theorem 3.4. *Let $T, W \in M^{m\times n}(\mathbb{C})$. The following assertions hold:*

a) If $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ then there exists $X_0 \in M_H^n(\mathbb{C})$ such that $TX_0 = W$ if and only if $TW^* \in M_H^m(\mathbb{C})$.
b) There exists $X_0 \in M_+^n(\mathbb{C})$ such that $TX_0 = W$ if and only if $TW^* \in M_+^m(\mathbb{C})$ and $r(TW^*) = r(W)$.

As previously mentioned, we are interested on the reduced solutions of the matrix equation $TX = W$. In the following results we study Hermitian and positive semidefinite reduced solutions:

Proposition 3.5. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$. The following conditions are equivalent:*

a) There exists an Hermitian reduced solution of $TX = W$.
b) $TW^* \in M_H^m(\mathbb{C})$ and $\mathcal{R}(W^*) \cap \mathcal{N}(T) = \{0\}$.

Proof. $a) \to b)$ Let $X_{\mathcal{M}}$ be an Hermitian reduced solution of $TX = W$. Hence, as $\mathcal{N}(X_{\mathcal{M}}) = \mathcal{N}(W)$, we get that $\mathcal{R}(X_{\mathcal{M}}) = \mathcal{R}(W^*)$ and so $\mathcal{R}(W^*) \cap \mathcal{N}(T) = \{0\}$. The proof is completed applying Theorem 3.4.

$b) \to a)$ By Theorem 3.4, there exists $X_0 \in M_H^n(\mathbb{C})$ such that $TX_0 = W$. On the other hand, since $\mathcal{R}(W^*) \cap \mathcal{N}(T) = \{0\}$, then $\mathcal{M} := \mathcal{R}(W^*) + (\mathcal{R}(W^*) + \mathcal{N}(T))^\perp$ is an algebraic complement of $\mathcal{N}(T)$ in $\mathbb{C}^n$. Therefore, the projection $Q_{\mathcal{M}//\mathcal{N}(T)}$ is well-defined and $TQ_{\mathcal{M}//\mathcal{N}(T)}X_0Q^*_{\mathcal{M}//\mathcal{N}(T)} = WQ^*_{\mathcal{M}//\mathcal{N}(T)} = W$ because $\mathcal{R}(W^*) \subseteq \mathcal{M}$. Thus, by the uniqueness of the reduced solution for $\mathcal{M}$, we get that $Q_{\mathcal{M}//\mathcal{N}(T)}X_0Q^*_{\mathcal{M}//\mathcal{N}(T)} \in M_H^n(\mathbb{C})$ is the reduced solution for $\mathcal{M}$ of $TX = W$. The proof is concluded. $\square$

If T and W are bounded operators defined on Hilbert spaces, then the condition "$\overline{\mathcal{R}(W^*)} \dotplus \mathcal{N}(T)$ is a closed subspace" should be added in item b) of Proposition 3.5 in order to get the result. This condition is equivalent to an angle condition between the subspaces $\overline{\mathcal{R}(W^*)}$ and $\mathcal{N}(T)$, see [2].

Proposition 3.6. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and let $\mathcal{M}$ be a subspace of $\mathbb{C}^n$ such that $\mathcal{M} \dotplus \mathcal{N}(T) = \mathbb{C}^n$. If $TW^* \in M_H^m(\mathbb{C})$ and $X_{\mathcal{M}}$ is the reduced solution for $\mathcal{M}$ of $TX = W$ then the following conditions are equivalent:*

a) $X_{\mathcal{M}}$ is Hermitian;
b) $X_{\mathcal{M}}$ is normal;
c) $\mathcal{R}(W^*) \subseteq \mathcal{M}$.

In particular, $T^\dagger W \in M_H^n(\mathbb{C})$ if and only if $T^\dagger W$ is normal if and only if $\mathcal{R}(W^*) \subseteq \mathcal{R}(T^*)$.

Proof. $a) \to b)$ It is immediate.

$b) \to c)$ If $X_{\mathcal{M}}X^*_{\mathcal{M}} = X^*_{\mathcal{M}}X_{\mathcal{M}}$ then $\mathcal{R}(X_{\mathcal{M}}) = \mathcal{R}(X^*_{\mathcal{M}})$. So that $\mathcal{R}(W^*) = \mathcal{N}(W)^\perp = \mathcal{N}(X_{\mathcal{M}})^\perp = \mathcal{R}(X_{\mathcal{M}}) \subseteq \mathcal{M}$.

$c) \to a)$ By Theorem 3.4, there exists $X_0 \in M_H^n(\mathbb{C})$ such that $TX_0 = W$. Therefore, $TQ_{\mathcal{M}//\mathcal{N}(T)}X_0Q^*_{\mathcal{M}//\mathcal{N}(T)} = WQ^*_{\mathcal{M}//\mathcal{N}(T)} = W$ because $\mathcal{R}(W^*) \subseteq \mathcal{M}$. Thus, by the uniqueness of the reduced solution for $\mathcal{M}$, we get that $X_{\mathcal{M}} = Q_{\mathcal{M}//\mathcal{N}(T)}X_0Q^*_{\mathcal{M}//\mathcal{N}(T)}$, i.e., $X_{\mathcal{M}} \in M_H^n(\mathbb{C})$. $\square$

Corollary 3.7. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$. If $T^\dagger W \in M_H^n(\mathbb{C})$ then $\rho(X_{\mathcal{M}}) = \rho(T^\dagger W)$ for all reduced solution for $\mathcal{M}$ of $TX = W$.*

Proof. It follows from Lemma 3.3 and Proposition 3.6. $\square$

The proofs of the next results concerning positive semidefinite reduced solutions of $TX = W$ follow proceeding as in the proofs of Propositions 3.5 and 3.6.

Proposition 3.8. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$. The following conditions are equivalent:*

a) There exists a positive semidefinite reduced solution of $TX = W$.
b) $TW^* \in M_+^m(\mathbb{C})$, $r(TW^*) = r(W)$ and $\mathcal{R}(W^*) \cap \mathcal{N}(T) = \{0\}$.

Proposition 3.9. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $TW^* \in M_+^n(\mathbb{C})$ and $r(TW^*) = r(W)$, and let $\mathcal{M}$ be a subspace of $\mathbb{C}^n$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$. If $X_\mathcal{M}$ is the reduced solution for $\mathcal{M}$ of $TX = W$ then the following conditions are equivalent:*

a) $X_\mathcal{M}$ *is positive semidefinite;*
b) $X_\mathcal{M}$ *is normal;*
c) $\mathcal{R}(W^*) \subseteq \mathcal{M}$.

In particular, $T^\dagger W \in M_+^n(\mathbb{C})$ if and only if $T^\dagger W$ is normal if and only if $\mathcal{R}(W^*) \subseteq \mathcal{R}(T^*)$.

4. Proper splitting of matrices

Given a matrix $T \in M^{m\times n}(\mathbb{C})$, a splitting of T is a partition of T in the following form: $T = U - V$ with $U, V \in M^{m\times n}(\mathbb{C})$. In [6], Berman and Plemmons introduced the concept of proper splitting for rectangular matrices with the aim of developing an iterative process that converges to the best least squares approximate solution of the system $Tx = w$. Let us formally introduce this concept:

Definition 4.1. Let $T \in M^{m\times n}(\mathbb{C})$. The decomposition

$$T = U - V,$$

where $U, V \in M^{m\times n}(\mathbb{C})$ is called a proper splitting of T if $\mathcal{R}(U) = \mathcal{R}(T)$ and $\mathcal{N}(U) = \mathcal{N}(T)$.

In the following result we gather some properties of proper splittings. Many of these properties can be found in [6].

Proposition 4.2. *If $T = U - V$ is a proper splitting of T then the following assertions hold:*

a) $\mathcal{N}(T) \subseteq \mathcal{N}(V)$ *and* $\mathcal{R}(V) \subseteq \mathcal{R}(T)$.
b) $T = U(I - U^\dagger V)$ *and* $I - U^\dagger V$ *is invertible.*
c) $T^\dagger = (I - U^\dagger V)^{-1}U^\dagger$.
d) $(T^\dagger U)^\dagger = U^\dagger T$.

Proof. Let $T = U - V$ be a proper splitting of T.

$a)$ If $x \in \mathcal{N}(T)$ then $Tx = 0$ and so $Ux = 0$. Then $Vx = 0$. Therefore $\mathcal{N}(T) \subseteq \mathcal{N}(V)$. In addition, since $V = U - T$ and $\mathcal{R}(U) = \mathcal{R}(T)$ then $\mathcal{R}(V) \subseteq \mathcal{R}(T)$.

$b), c)$ The proofs can be found in [6, Theorem 1].

$d)$ Since $\mathcal{R}(U) = \mathcal{R}(T)$ and $\mathcal{N}(U) = \mathcal{N}(T)$ the assertion follows from Theorem 2.1. □

The following results will be useful in the next section.

Proposition 4.3. *Let $T = U - V$ be a proper splitting of T. The following assertions are equivalent:*

a) $VT^* \in M_H^n(\mathbb{C})$;
b) $UT^* \in M_H^n(\mathbb{C})$;
c) $VU^* \in M_H^n(\mathbb{C})$;
d) $U^\dagger V \in M_H^n(\mathbb{C})$;
e) $T^\dagger U \in M_H^n(\mathbb{C})$;
f) $T^\dagger V \in M_H^n(\mathbb{C})$.

Proof. *a)* $\leftrightarrow$ *b)* It is obvious since $UT^* = TT^* + VT^*$.

b) $\leftrightarrow$ *c)* It is obvious since $VU^* = UU^* - TU^*$.

c) $\leftrightarrow$ *d)* If $VU^* \in M_H^n(\mathbb{C})$ then, by Theorem 3.4, the equation $UX = V$ admits an hermitian solution. Since $R(V^*) \subseteq R(U^*)$ we can apply Proposition 3.6, and we get that $U^\dagger V \in M_H^n(\mathbb{C})$. Conversely, if $U^\dagger V \in M_H^n(\mathbb{C})$ then the equation $UX = V$ admits an hermitian solution and, by Theorem 3.4, $VU^* \in M_H^n(\mathbb{C})$.

Similarly, the equivalences *b)* $\leftrightarrow$ *e)* and *a)* $\leftrightarrow$ *f)* follow from Proposition 3.6 and Theorem 3.4. $\square$

Proposition 4.4. *Let $T = U - V$ be a proper splitting of T. The following assertions are equivalent:*

a) $U^\dagger V \in M_+^n(\mathbb{C})$;
b) $VU^* \in M_+^n(\mathbb{C})$ *and* $r(VU^*) = r(V)$;
c) $U^\dagger T \in M_H^n(\mathbb{C})$ *and* $U^\dagger T \leq P_{T^*}$.

Proof. *a)* $\leftrightarrow$ *b)* It follows by Theorem 3.4 and Proposition 3.9.

a) $\leftrightarrow$ *c)* Suppose that $U^\dagger V \in M_+^n(\mathbb{C})$. Then $U^\dagger T = U^\dagger U - U^\dagger V \in M_H^n(\mathbb{C})$. In addition, $U^\dagger V = U^\dagger U - U^\dagger T \geq 0$. Therefore, $U^\dagger T \leq P_{T^*}$. Conversely, if $U^\dagger T \in M_H^n(\mathbb{C})$ and $U^\dagger T \leq P_{T^*} = U^\dagger U$ then $U^\dagger V = U^\dagger U - U^\dagger T \geq 0$. $\square$

Proposition 4.5. *Let $T = U - V$ be a proper splitting of T. The following assertions are equivalent:*

a) $U^\dagger T \in M_+^n(\mathbb{C})$;
b) $TU^* \in M_+^n(\mathbb{C})$ *and* $r(TU^*) = r(T)$;
c) $U^\dagger V \in M_H^n(\mathbb{C})$ *and* $U^\dagger V \leq P_{T^*}$.

Proof. *a)* $\leftrightarrow$ *b)* It follows by Theorem 3.4 and Proposition 3.9.

a) $\leftrightarrow$ *c)* If $U^\dagger T \in M_+^n(\mathbb{C})$ then $U^\dagger V = U^\dagger U - U^\dagger T \in M_H^n(\mathbb{C})$. Furthermore $U^\dagger T = U^\dagger U - U^\dagger V \geq 0$. Then $P_{T^*} \geq U^\dagger V$. Conversely, if $U^\dagger V \in M_H^n(\mathbb{C})$ and $P_{T^*} \geq U^\dagger V$ then $U^\dagger T = U^\dagger U - U^\dagger V \geq 0$. $\square$

5. Reduced solutions and proper splittings of matrices

In [6] the concept of proper splitting of a matrix T is applied to approximate the best minimum square solution of a system $Tx = w$. In this section, we shall observe that this class of splitting can also be used to get the reduced solution for $\mathcal{M}$ of a matrix equation $TX = W$. To be more specific, let $T \in M^{m\times n}(\mathbb{C})$ and $W \in M^{m\times r}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and let $\mathcal{M}$ be a subspace of $\mathbb{C}^n$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$. Let $T = U - V$ be a proper splitting of T. Let $Y_\mathcal{M} \in M^n(\mathbb{C})$ and $Z_\mathcal{M} \in M^{n\times r}(\mathbb{C})$ be the reduced solutions for $\mathcal{M}$ of $UY = V$ and $UZ = W$, respectively. Define the iterative process:

$$X^{i+1} = Y_\mathcal{M} X^i + Z_\mathcal{M}. \tag{4}$$

We call (4), *the iterative process for $\mathcal{M}$ of the proper splitting $T = U - V$ with respect to W.*

In the next theorem, we prove that whether the iteration (4) converges, then it converges to the reduced solution for $\mathcal{M}$ of $TX = W$. Furthermore, an equivalent condition for its convergence is provided.

Theorem 5.1. *Let $T \in M^{m\times n}(\mathbb{C})$ and $W \in M^{m\times r}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and let $\mathcal{M}$ be a subspace of $\mathbb{C}^n$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$. Consider the proper splitting $T = U - V$ of T. Then the iterative process (4) converges for all X^0 to $X_\mathcal{M}$, the reduced solution for $\mathcal{M}$ of the equation $TX = W$, if and only if $\rho(Y_\mathcal{M}) < 1$.*

Proof. It is clear that if the iterative process (4) converges to $X \in M^{n\times r}(\mathbb{C})$ then $TX = W$ and $\mathcal{R}(X) \subseteq \mathcal{M}$, i.e., $X = X_\mathcal{M}$. Now, $UX_\mathcal{M} = VX_\mathcal{M} + W$ and so $U(X^{i+1} - X_\mathcal{M}) = V(X^i - X_\mathcal{M}) = UY_\mathcal{M}(X^i - X_\mathcal{M})$. Then,

$X^{i+1} - X_{\mathcal{M}} = Y_{\mathcal{M}}(X^i - X_{\mathcal{M}}) = Y_{\mathcal{M}}^{i+1}(X^0 - X_{\mathcal{M}})$ since $\mathcal{M} \dot{+} \mathcal{N}(U) = \mathbb{C}^n$. Thus, the iterative process (4) is convergent for all X^0 if and only if $\rho(Y_{\mathcal{M}}) < 1$. $\square$

It should be mentioned that the iterative process (4) emerges as a generalization of the iteration introduced in [6] to obtain the best minimum square solution of $Tx = b$. Indeed, notice that choosing $W = P_T b$ and $\mathcal{M} = R(T^*)$ then the iteration (4) can be used to obtain the best minimum square error solution of $Tx = b$. Furthermore, we stress that the iteration (4) is practical when it is easier to solve the equations that involve U than the equation that involves T.

Remarks 5.2.

a) Taking into account Remark 3.2, we observe that the iterative process (4) can also be used for solving matrix equations of the form $TXS = W$. Evidently, in this case a proper splitting for T and a proper splitting for S are needed. We recommend [25] and [26] for other applications of splitting of matrices for solving matrix equations of the form $TXS = W$.
b) As it was pointed out in Remark 3.2, the generalized inverses of a matrix T can be described by means of reduced solutions of certain matrix equations. Therefore, the iterative process (4) can be applied to approximate generalized inverses of a matrix. We recommend [10] and [31] for other perspective of the use of splitting of matrices or operators to get generalized inverses.
c) Notice that our goal is to apply the iterative process (4) to approximate the reduced solution for $\mathcal{M}$ of the matrix equation $TX = W$. This leads to requiring that $\rho(Y_{\mathcal{M}}) < 1$ or, equivalently, that $\lim_{i\to\infty} Y_{\mathcal{M}}^i = 0$. If instead we were interested in approximating a solution of $TX = W$ by (4), then a different condition on $Y_{\mathcal{M}}$ is required. For more details, we recommend [32] and [11] where this problem is solved by considering $Y_{\mathcal{M}}$ semi-convergent, i.e., $\lim_{i\to\infty} Y_{\mathcal{M}}^i$ exists.

By Lemma 3.3, Proposition 4.2 and Theorem 5.1 we get the next result:

Corollary 5.3. *Let $T = U - V$ be a proper splitting of $T \in M^{m\times n}(\mathbb{C})$ and let $W \in M^{m\times r}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$. Then, the iterative process of the proper splitting $T = U - V$ converges for some $\mathcal{M}$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$ if and only if the iterative process of the proper splitting $T = U - V$ converges for all $\mathcal{M}$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$. In particular, the iterative process (4) is convergent if and only if $\rho(U^\dagger V) < 1$.*

The following result follows from Propositions 3.6, 3.9 and Theorem 5.1.

Corollary 5.4. *Let $T, W \in M^{m\times n}(\mathbb{C})$ such that $\mathcal{R}(W) \subseteq \mathcal{R}(T)$ and let $\mathcal{M}$ be a subspace of $\mathbb{C}^n$ such that $\mathcal{M} \dot{+} \mathcal{N}(T) = \mathbb{C}^n$. Consider the iterative process for $\mathcal{M}$ of the proper splitting $T = U - V$ with respect to W, given by (4). Then the following assertions hold:*

a) If $WT^* \in M_H^m(\mathbb{C})$ and $\rho(Y_{\mathcal{M}}) < 1$ then the iterative process (4) converges to an Hermitian reduced solution of $TX = W$ if and only if $\mathcal{R}(W^*) \subseteq \mathcal{M}$.
b) If $WT^* \in M_H^m(\mathbb{C})$, $r(WT^*) = r(W)$ and $\rho(Y_{\mathcal{M}}) < 1$ then the iterative process (4) converges to a positive semidefinite reduced solution of $TX = W$ if and only if $\mathcal{R}(W^*) \subseteq \mathcal{M}$.

6. Convergence of proper splittings

In the previous section, the iterative process (4) was defined on the basis of a proper splitting $T = U - V$ and it was proven that this iterative process is convergent (in which case it converges to the reduced solution for $\mathcal{M}$ of $TX = W$) if and only if $\rho(U^\dagger V) < 1$. Accordingly, this section is devoted to provide conditions that guarantee $\rho(U^\dagger V) < 1$. Most of the results in this section are inspired by [6], [14], [34]. In order to compare our results with those of the previously mentioned articles, in what follows:

- by K we denote a full cone in $\mathbb{R}^n$ and given $A \in M^{m\times n}(\mathbb{R})$, by $A \in \mathcal{I}_K$ we mean that A leaves the full cone K invariant, i.e., $AK \subset K$;
- by $A \in M_+^n(\mathbb{C})$ we mean $A \geq 0$ where $\geq$ denotes the Löwner order in $M^n(\mathbb{C})$;
- by $A \in M_{\succ 0}^n(\mathbb{R})$ we mean that A has nonnegative entries and by $\succ$ we denote the usual order in $M^{m\times n}(\mathbb{R})$ induced by nonnegative matrices.

Evidently, $M_{\succ 0}^n(\mathbb{R}) = \mathcal{I}_K$ with $K = \mathbb{R}_+^n$. In this section, we work with the Löwner order. Theorems 2.1 and 2.2 play a key role for proving the main results of this section.

Proposition 6.1. *Let $T = U - V$ be a proper splitting of T. Then the following assertions are equivalent:*

a) $T^\dagger V \in M_+^n(\mathbb{C})$;
b) $TV^* \in M_+^n(\mathbb{C})$ *and* $r(TV^*) = r(V)$;
c) $0 \leq U^\dagger V \leq P_{V^*}$;
d) $U^\dagger V \in M_+^n(\mathbb{C})$ *and* $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger V)}{1+\rho(T^\dagger V)} < 1$.

Proof. *a)* $\leftrightarrow$ *b)* It follows from Theorem 3.4 and Proposition 3.9.

a) $\leftrightarrow$ *c)* Let $T = U - V$ be a proper splitting of T. Since $T^\dagger = (I - U^\dagger V)^{-1}U^\dagger$ then

$$\begin{aligned} 0 \leq T^\dagger V &\leftrightarrow 0 \leq (I - U^\dagger V)^{-1}U^\dagger V \leftrightarrow 0 \leq \left((I - U^\dagger V)^{-1}U^\dagger V\right)^\dagger \\ &\leftrightarrow 0 \leq \left(U^\dagger V\right)^\dagger (I - U^\dagger V) \leftrightarrow 0 \leq \left(U^\dagger V\right)^\dagger - P_{\mathcal{N}(U^\dagger V)^\perp} \\ &\leftrightarrow 0 \leq P_{V^*} \leq \left(U^\dagger V\right)^\dagger. \end{aligned} \tag{5}$$

Now, if $0 \leq P_{V^*} \leq \left(U^\dagger V\right)^\dagger$ then $\mathcal{R}(P_{V^*}) \subseteq \mathcal{R}((U^\dagger V)^\dagger)$. So, $\mathcal{R}((U^\dagger V)^\dagger - P_{V^*}) \cap \mathcal{N}((U^\dagger V)^\dagger) \subseteq \mathcal{R}((U^\dagger V)^\dagger) \cap \mathcal{N}((U^\dagger V)^\dagger) = \{0\}$. On the other hand, if $0 \leq P_{V^*} \leq \left(U^\dagger V\right)^\dagger$ then $\mathcal{R}(P_{V^*}) \subseteq \mathcal{R}((U^\dagger V)^\dagger) = \mathcal{R}(U^\dagger V)$. Now, $\mathcal{R}(P_{V^*} - U^\dagger V) \cap \mathcal{N}(P_{V^*}) \subseteq \mathcal{R}(U^\dagger V) \cap \mathcal{N}(V) = \mathcal{R}(V^*(U^*)^\dagger) \cap \mathcal{N}(V) \subseteq \mathcal{R}(V^*) \cap \mathcal{N}(V) = \{0\}$. So that, by Theorem 2.2, we get $0 \leq U^\dagger V \leq P_{V^*}$. Conversely, if $0 \leq U^\dagger V \leq P_{V^*}$ then $\mathcal{R}((U^\dagger V)^\dagger) = \mathcal{R}(U^\dagger V) \subseteq \mathcal{R}(V^*)$. In addition, as $\mathcal{N}(U^\dagger V) = \mathcal{N}(V)$ then it holds that $\mathcal{R}(P_{V^*} - U^\dagger V) \cap \mathcal{N}(P_{V^*}) = \{0\}$ and $\mathcal{R}((U^\dagger V)^\dagger - P_{V^*}) \cap \mathcal{N}(U^\dagger V) = \{0\}$. Therefore, applying again Theorem 2.2, $0 \leq P_{V^*} \leq \left(U^\dagger V\right)^\dagger$. Then, by the equivalences (5), the assertion follows.

a) $\leftrightarrow$ *d)* First, let us prove that $\lambda \in \sigma(T^\dagger V)$ if and only if $\mu = \frac{\lambda}{1+\lambda} \in \sigma(U^\dagger V)$. In fact,

$$\begin{aligned} T^\dagger V x = \lambda x &\leftrightarrow (I - U^\dagger V)^{-1}U^\dagger V x = \lambda x \leftrightarrow U^\dagger V x = (I - U^\dagger V)\lambda x \\ &\leftrightarrow U^\dagger V(1+\lambda)x = \lambda x \leftrightarrow U^\dagger V x = \frac{\lambda}{1+\lambda}x. \end{aligned}$$

Assume that $T^\dagger V \in M_+^n(\mathbb{C})$. Then $\rho(T^\dagger V) \in \sigma(T^\dagger V)$ and $\mu = \dfrac{\lambda}{1+\lambda}$ achieves its maximum when $\lambda = \rho(T^\dagger V)$. Therefore $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger V)}{1+\rho(T^\dagger V)} < 1$. Conversely, if $U^\dagger V \in M_+^n(\mathbb{C})$ and $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger V)}{1+\rho(T^\dagger V)} < 1$ then $\|U^\dagger V\| = \rho(U^\dagger V) < 1$. Then $(I - U^\dagger V)^{-1} = \sum\limits_{k=0}^{\infty}(U^\dagger V)^k$. So that $T^\dagger V = (I - U^\dagger V)^{-1}U^\dagger V = \sum\limits_{k=1}^{\infty}(U^\dagger V)^k$. Thus, $T^\dagger V \in M_+^n(\mathbb{C})$. $\square$

Remark 6.2. Similar results to Proposition 6.1 are disseminated in the literature in different contexts. For example, in [6, Theorem 2], Berman and Plemmons proved that:

If $T = U - V$ is a proper splitting of $T \in M^{m\times n}(\mathbb{R})$ with $U^\dagger V \in \mathcal{I}_K$ then, $T^\dagger V \in \mathcal{I}_K$ if and only if $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger V)}{1+\rho(T^\dagger V)} < 1$.

Evidently, the result holds if $\mathcal{I}_K$ is replaced by $M^n_{\succ 0}(\mathbb{R})$. The particular case $\mathcal{I}_K = M^n_{\succ 0}(\mathbb{R})$ together with other equivalent conditions can be found in [34, Lemma 2.6] and [29, Lemma 3.5].

Later, Climent and Perea proved that Berman and Plemmons' result holds if $\mathcal{I}_K$ is replaced by $M^n_+(\mathbb{C})$ and T, U are non singular matrices (see [14, Theorem 2.5]). Finally, in [12, Theorem 3], Climent et al. unified [14, Theorem 2.5] and [34, Lemma 2.6] by considering a partial order induced by a positivity cone of matrices.

Nevertheless, all these results include the general hypotheses $U^\dagger V \in \mathcal{I}_K$ or $U^{-1}V \in M^n_+(\mathbb{C})$. Notice that in Proposition 6.1 we relaxed this general assumption. However, let us observe that this condition can not be omitted in the context of nonnegative matrices. In fact, $T^{-1}V \succ 0$ does not imply $U^{-1}V \succ 0$, in general. For example, consider the proper splitting of $T = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$ given by $T = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$. Then, $T^{-1}V = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix} \succ 0$, but $U^{-1}V = \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix} \not\succ 0$.

Proposition 6.3. *Let $T = U - V$ be proper splitting of T. Then the following assertions are equivalent:*

a) $0 \le U^\dagger T \le P_{T^*}$*;*

b) $U^\dagger V \in M^n_+(\mathbb{C})$ *and* $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger U) - 1}{\rho(T^\dagger U)} < 1$.

Proof. Suppose that $0 \le U^\dagger T \le P_{T^*} = U^\dagger U$. Then, $0 \le U^\dagger(U - T) = U^\dagger V$ and so $\rho(U^\dagger V) \in \sigma(U^\dagger V)$. Then there exists $0 \ne x \in \mathbb{C}^n$ such that $U^\dagger V x = \rho(U^\dagger V)x$. So that $x \in \mathcal{R}(U^*)$ and then $x = U^\dagger U x$. Therefore we get that $T^\dagger U x = (I - U^\dagger V)^{-1} U^\dagger U x = (I - U^\dagger V)^{-1} x = \dfrac{1}{1-\rho(U^\dagger V)} x$. Then $0 \le \dfrac{1}{1-\rho(U^\dagger V)} \le \rho(T^\dagger U)$ and so $\rho(U^\dagger V) \le \dfrac{\rho(T^\dagger U) - 1}{\rho(T^\dagger U)}$. On the other hand, since $T^\dagger U \ge 0$ then $\rho(T^\dagger U) \in \sigma(T^\dagger U)$. Hence, there exists $0 \ne y \in \mathbb{C}^n$ such that $T^\dagger U y = \rho(T^\dagger U)y$. In consequence, $y \in \mathcal{R}(T^*)$ and so $y = U^\dagger U y$. Therefore, $\rho(T^\dagger U)y = T^\dagger U y = (I - U^\dagger V)^{-1} U^\dagger U y = (I - U^\dagger V)^{-1} y$. After some computations we get $U^\dagger V y = \dfrac{\rho(T^\dagger U) - 1}{\rho(T^\dagger U)} y$. Then $\dfrac{\rho(T^\dagger U) - 1}{\rho(T^\dagger U)} \le \rho(U^\dagger V)$. Therefore, $\rho(U^\dagger V) = \dfrac{\rho(T^\dagger U) - 1}{\rho(T^\dagger U)} < 1$. Conversely, if item b) holds then $\|U^\dagger V\| = \rho(U^\dagger V) < 1$. Then $I - U^\dagger V$ is invertible. So that $(I - U^\dagger V)^{-1} = \sum_{k=0}^{\infty} (U^\dagger V)^k$. Then, by Proposition 4.2, we get that $T^\dagger U = (I - U^\dagger V)^{-1} U^\dagger U = \sum_{k=0}^{\infty} (U^\dagger V)^k U^\dagger U = P_{U^*} + \sum_{k=1}^{\infty} (U^\dagger V)^k \ge 0$, or equivalently, $U^\dagger T = (T^\dagger U)^\dagger \ge 0$. Moreover, $0 \le U^\dagger V = U^\dagger(U - T) = P_{T^*} - U^\dagger T$. The proof is complete. $\square$

Remark 6.4. We recommend [34], [29] and [14] for similar results to Proposition 6.3 but for the partial order $\succ$. On the other hand, a similar result to the previous one but considering $T \in M^n(\mathbb{C})$ non singular and under the general hypothesis that $U^{-1}V > 0$, can be found in [14, Theorem 2.5].

Let us observe that in the context of nonnegative matrices if $T = U - V$ is a proper splitting of T then $U^\dagger T \succ 0$ does not imply $U^\dagger V \succ 0$, in general. For example, consider the proper splitting of $T = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ given by $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. Then, $U^\dagger T = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \succ 0$ but $U^\dagger V = \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} \not\succ 0$.

Proposition 6.5. *Let $T = U - V$ be a proper splitting of T. If there exists $\tilde{X} \in M_H^n(\mathbb{C})$ such that $U\tilde{X} = V$ and $\rho(\tilde{X}) < 1$ then $\rho(U^\dagger V) \leq \rho(\tilde{X}) < 1$.*

Proof. Let $\tilde{X} \in M_H^n(\mathbb{C})$ such that $U\tilde{X} = V$ and $\rho(\tilde{X}) < 1$. Then, by Theorem 3.4 and Proposition 4.3, it holds that $U^\dagger V \in M_H^n(\mathbb{C})$ and so $\rho(U^\dagger V) = \|U^\dagger V\|$. Now, since $\tilde{X} = U^\dagger V + Z$ for some $Z \in M^n(\mathbb{C})$ with $\mathcal{R}(Z) \subseteq \mathcal{N}(U)$, we get that $\rho(U^\dagger V) = \|U^\dagger V\| \leq \|\tilde{X}\| = \rho(\tilde{X}) < 1$. □

Once the converge of the iterative process is guaranteed, then the second problem to be addressed is to improve the speed of convergence of the method. To this end, we provide two comparison results.

Theorem 6.6. *Let $T = U_1 - V_1 = U_2 - V_2$ be two proper splittings of T such that $0 \leq T^\dagger V_1 \leq T^\dagger V_2$ and $\mathcal{N}(V_1) = \mathcal{N}(V_2)$. Then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Proof. Since $0 \leq T^\dagger V_1 \leq T^\dagger V_2$ and $\mathcal{N}(V_1) = \mathcal{N}(V_2)$ we get that $\mathcal{N}(T^\dagger V_1) = \mathcal{N}(V_1) = \mathcal{N}(V_2) = \mathcal{N}(T^\dagger V_2)$ and $\mathcal{R}(T^\dagger V_1) = \mathcal{R}(T^\dagger V_2)$. Now, observe that $\mathcal{R}(T^\dagger V_2 - T^\dagger V_1) \cap \mathcal{N}(T^\dagger V_2) \subseteq \mathcal{R}(T^\dagger V_2) \cap \mathcal{N}(T^\dagger V_2) = \{0\}$. On the other hand, $\mathcal{R}((T^\dagger V_1)^\dagger - (T^\dagger V_2)^\dagger) \cap \mathcal{N}(T^\dagger V_1) \subseteq \mathcal{N}(V_1)^\perp \cap \mathcal{N}(T^\dagger V_1) = \mathcal{N}(V_1)^\perp \cap \mathcal{N}(V_1) = \{0\}$. So that, by Theorem 2.2 it holds that $0 \leq (T^\dagger V_2)^\dagger \leq (T^\dagger V_1)^\dagger$. Now, since $T^\dagger = (I - U_1^\dagger V_1)^{-1} U_1^\dagger = (I - U_2^\dagger V_2)^{-1} U_2^\dagger$ it holds that

$$
\begin{aligned}
0 \leq \left(T^\dagger V_2\right)^\dagger \leq \left(T^\dagger V_1\right)^\dagger &\leftrightarrow \left((I - U_2^\dagger V_2)^{-1} U_2^\dagger V_2\right)^\dagger \leq \left((I - U_1^\dagger V_1)^{-1} U_1^\dagger V_1\right)^\dagger \\
&\leftrightarrow 0 \leq \left(U_2^\dagger V_2\right)^\dagger \left(I - U_2^\dagger V_2\right) \leq (U_1^\dagger V_1)^\dagger (I - U_1^\dagger V_1) \\
&\leftrightarrow 0 \leq \left(U_2^\dagger V_2\right)^\dagger - P_{\mathcal{N}(U_2^\dagger V_2)^\perp} \leq \left(U_1^\dagger V_1\right)^\dagger - P_{\mathcal{N}\left(U_1^\dagger V_1\right)^\perp} \\
&\leftrightarrow 0 \leq \left(U_2^\dagger V_2\right)^\dagger - P_{V_1^*} \leq \left(U_1^\dagger V_1\right)^\dagger - P_{V_1^*} \\
&\leftrightarrow 0 \leq \left(U_2^\dagger V_2\right)^\dagger \leq \left(U_1^\dagger V_1\right)^\dagger \\
&\leftrightarrow 0 \leq U_1^\dagger V_1 \leq U_2^\dagger V_2.
\end{aligned}
$$

Then, by Proposition 6.1, we get that $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. □

Theorem 6.7. *Let $T = U_1 - V_1 = U_2 - V_2$ be two proper splittings of T. If $0 \leq T^\dagger U_1 \leq T^\dagger U_2 \leq P_{T^*}$ then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.*

Proof. By Proposition 6.3, it holds that $\rho(U_1^\dagger V_1) < 1$ and $\rho(U_2^\dagger V_2) < 1$. Now, observe that

$$
\begin{aligned}
0 \leq T^\dagger U_1 \leq T^\dagger U_2 \leq P_{T^*} &\leftrightarrow 0 \leq (I - U_1^\dagger V_1)^{-1} U_1^\dagger U_1 \leq (I - U_2^\dagger V_2)^{-1} U_2^\dagger U_2 \leq P_{T^*} \\
&\leftrightarrow 0 \leq (I - U_1^\dagger V_1)^{-1} P_{T^*} \leq (I - U_2^\dagger V_2)^{-1} P_{T^*} \leq P_{T^*} \\
&\leftrightarrow 0 \leq \left((I - U_2^\dagger V_2)^{-1} P_{T^*}\right)^\dagger \leq \left((I - U_1^\dagger V_1)^{-1} P_{T^*}\right)^\dagger \leq P_{T^*} \\
&\leftrightarrow 0 \leq P_{T^*}(I - U_2^\dagger V_2) \leq P_{T^*}(I - U_1^\dagger V_1) \leq P_{T^*} \\
&\leftrightarrow 0 \leq P_{T^*} - U_2^\dagger V_2 \leq P_{T^*} - U_1^\dagger V_1 \leq P_{T^*} \\
&\leftrightarrow 0 \leq U_1^\dagger V_1 \leq U_2^\dagger V_2 \leq P_{T^*},
\end{aligned}
$$

where the third equivalence holds because $r((I - U_1^\dagger V_1)^{-1} P_{T^*}) = r(P_{T^*})$ and $r((I - U_2^\dagger V_2)^{-1} P_{T^*}) = r(P_{T^*})$ (see Theorem 2.2). Therefore, $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$. □

Remark 6.8. Comparison results similar to those above but for the partial order $\succ$, can be found in [34], [29]. When contrasting our results with those for the partial order $\succ$, it can be observed that we require extra conditions on the matrices of the splittings in order to get the conclusions. More precisely, we require that $\mathcal{N}(V_1) = \mathcal{N}(V_2)$ in Theorem 6.6 and that $T^\dagger U_1 \leq T^\dagger U_2 \leq P_{T^*}$ in Theorem 6.7. We leave as an open problem to determine if these conditions can be relaxed.

7. Some examples of proper splittings

Recall that every $T \in M^{m\times n}(\mathbb{C})$ can be factorized as $T = U|T|$, where $U \in M^{m\times n}(\mathbb{C})$ is a partial isometry and $|T| = (T^*T)^{1/2}$. Such a factorization is called a polar decomposition of T and the matrix U is unique if $\mathcal{R}(U) = \mathcal{R}(T)$. In what follows, we call **the polar decomposition** of T to the unique factorization $T = U_T|T|$ where $U_T \in M^{m\times n}(\mathbb{C})$ is a partial isometry with $\mathcal{R}(U_T) = \mathcal{R}(T)$. Notice that it also holds that $\mathcal{N}(U_T) = \mathcal{N}(T)$. Therefore, the partial isometry U_T can be used to define a proper splitting of T, $T = U_T - V$. This proper splitting has the advantage that $U_T^\dagger = U_T^*$. The aim in this section is to study this particular proper splitting. We study equivalent conditions to guarantee the convergence of this particular proper splitting and we compare it with some proper splittings defined for certain classes of matrices.

Proposition 7.1. *Let $T \in M^{m\times n}(\mathbb{C})$. Then $T = U_T - V$ is a proper splitting of T such that $U_T^*V \in M_H^n(\mathbb{C})$.*

Proof. Since $\mathcal{R}(U_T) = \mathcal{R}(T)$ and $\mathcal{N}(U_T) = \mathcal{N}(T)$ then $T = U_T - V$ is a proper splitting. Now, as $U_T^*V = U_T^*(U_T - T) = P_{T^*} - |T|$ then U_T^*V is Hermitian. □

The next result due to Baksalary, Liski and Trenkler [3] will be used to prove Proposition 7.3. We present an alternative proof.

Lemma 7.2. *Let $A, B \in M_+^n(\mathbb{C})$. Then, $A \leq B$ if and only if $\rho(B^\dagger A) \leq 1$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

Proof. Let $A, B \in M_+^n(\mathbb{C})$. If $A \leq B$ then, by Douglas' theorem [18, Theorem 1], $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\|(B^{1/2})^\dagger A^{1/2}\| = \inf\{\lambda : A \leq \lambda B\}$. Thus, $\|(B^{1/2})^\dagger A^{1/2}\| \leq 1$. Therefore, $\rho(B^\dagger A) = \rho(A^{1/2}B^\dagger A^{1/2}) = \|A^{1/2}B^\dagger A^{1/2}\| = \|(B^{1/2})^\dagger A^{1/2}\|^2 \leq 1$.

Conversely, suppose that $\rho(B^\dagger A) \leq 1$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. Hence, since $\sigma(A^{1/2}B^\dagger A^{1/2}) = \sigma(B^\dagger A)$ we get that $\|(B^{1/2})^\dagger A^{1/2}\|^2 = \|A^{1/2}B^\dagger A^{1/2}\| = \rho(A^{1/2}B^\dagger A^{1/2}) = \rho(B^\dagger A) \leq 1$. Then, applying again [18, Theorem 1], $\inf\{\lambda : A \leq \lambda B\} = \|(B^{1/2})^\dagger A^{1/2}\| \leq 1$. Therefore $A \leq B$. □

In the next proposition we provide sufficient conditions for the convergence of the proper splitting $T = U_T - V$.

Proposition 7.3. *Let $T \in M^{m\times n}(\mathbb{C})$ and consider the proper splitting $T = U_T - V$ of T. The following conditions are equivalent:*

a) $T^\dagger V \in M_+^n(\mathbb{C})$;
b) $U_T^*V \in M_+^n(\mathbb{C})$;
c) $\|T\| \leq 1$.

If any of the above conditions holds then the proper splitting $T = U_T - V$ is convergent.

Proof. $a) \leftrightarrow b)$ Suppose that $T^\dagger V \in M_+^n(\mathbb{C})$. Since $T^\dagger V = |T|^\dagger - P_{T^*} \geq 0$ then, by Theorem 2.2, it holds $|T| \leq P_{T^*}$. Therefore, $U_T^*V = U_T^*(U_T - T) = P_{T^*} - |T| \in M_+^n(\mathbb{C})$. The converse is similar.

b) $\leftrightarrow$ *c)* If $U_T^*V = P_{T^*} - |T| \geq 0$ then $|T| \leq P_{T^*}$ and so $\|T\| = \||T|\| \leq 1$. Conversely, if $\|T\| \leq 1$ we get that $\|P_{T^*}|T|\| = \||T|\| = \|T\| \leq 1$. Thus, $\rho(P_{T^*}|T|) \leq 1$. So that, as $\mathcal{R}(|T|) = \mathcal{R}(P_{T^*})$, by Lemma 7.2 we get that $|T| \leq P_{T^*}$. Therefore, $U_T^*V \in M_+^n(\mathbb{C})$.

The last part follows from Proposition 6.1. $\square$

The remaining of this section is devoted to study proper splittings for normal matrices and for matrices that can be factorized as the product of two orthogonal projections.

To this end, we denote by $\mathcal{P} \cdot \mathcal{P} = \{T \in M^n(\mathbb{C}) : T = P_1P_2 \text{ with } P_1, P_2 \in \mathcal{P}\}$, where $\mathcal{P} = \{Q \in M^n(\mathbb{C}) : Q = Q^2 = Q^*\}$. This set was studied by Corach and Maestripieri in [16] in the more general context of bounded linear operators defined on a Hilbert space. In the finite dimensional case, they proved that if $T \in \mathcal{P} \cdot \mathcal{P}$ then $T = P_TP_{T^*}$ and $\mathcal{R}(T)\dot{+}\mathcal{N}(T) = \mathbb{C}^n$ (see [16, Theorem 3.2]). In such case, notice that the projection $Q := Q_{\mathcal{R}(T)//\mathcal{N}(T)}$ is well-defined and it can be used to define a proper splitting of T, $T = Q - V_1$. In the next proposition we study the convergence of this proper splitting and we compare it with $T = U_T - V$.

Proposition 7.4. *Let $T \in \mathcal{P} \cdot \mathcal{P}$ and $Q = Q_{\mathcal{R}(T)//\mathcal{N}(T)}$. Then the following assertions hold:*

a) The proper splitting $T = Q - V_1$ is convergent.
b) The proper splitting $T = U_T - V$ is convergent.
c) $\rho(U_T^*V) \leq \rho(Q^\dagger V_1) < 1$.

Proof. *a*) By Proposition 6.1, it suffices to prove that $T^\dagger V_1 \in M_+^n(\mathbb{C})$. Now, by [15, Theorem 4.1], it holds that $T^\dagger = Q^*$. Then $T^\dagger V_1 = T^\dagger(Q - T) = Q^*Q - P_{T^*} \geq 0$. Thus, $T^\dagger V_1 \in M_+^n(\mathbb{C})$ and the splitting $T = Q - V_1$ is convergent.

b) Since $T \in \mathcal{P} \cdot \mathcal{P}$ then $T^* \in \mathcal{P} \cdot \mathcal{P}$. So that $0 \leq T^*T = P_{T^*}P_TP_{T^*} = P_{T^*}T \leq P_{T^*}$. Therefore, $|T| \leq P_{T^*}$ and so, $U_T^*V = P_{T^*} - |T| \geq 0$. Therefore, by Proposition 7.3 we get that the proper splitting $T = U_T - V$ is convergent.

c) By the above items, the proper splittings $T = Q - V_1$ and $T = U_T - V$ are convergent. As $Q^\dagger = P_{T^*}P_T$ (see [15, Theorem 4.1]) we get that

$$\begin{aligned} Q^\dagger V_1 &= P_{T^*}P_TV_1 = P_{T^*}P_T(Q - T) \\ &= P_{T^*}(Q - T) = P_{T^*} - P_{T^*}P_TP_{T^*}, \end{aligned}$$

and

$$\begin{aligned} U_T^\dagger V &= U_T^\dagger(U_T - T) = P_{T^*} - |T| \\ &= P_{T^*} - |P_TP_{T^*}|. \end{aligned}$$

Since $0 \leq P_{T^*}P_TP_{T^*} \leq I$ then $0 \leq P_{T^*}P_TP_{T^*} \leq (P_{T^*}P_TP_{T^*})^{1/2} \leq I$ and so $P_{T^*}P_TP_{T^*} \leq |P_TP_{T^*}|$. Hence, $0 \leq U_T^\dagger V \leq Q^\dagger V_1$, where $0 \leq U_T^\dagger V$ holds by the proof of item b). Therefore, $\rho(U_T^*V) \leq \rho(Q^\dagger V_1) < 1$. $\square$

Example 7.5. Observe that the inequality $\rho(U_T^*V) \leq \rho(Q^\dagger V_1)$ can be strict. Take $T = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{P}$.

Consider the proper splitting $T = U_T - V = \begin{pmatrix} \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & 0 \end{pmatrix}$. Now, since $U_T^* V = \begin{pmatrix} \frac{2-\sqrt{2}}{2} & 0 \\ 0 & 0 \end{pmatrix}$ then $\rho(U_T^* V) = \frac{2-\sqrt{2}}{2}$. On the other hand, consider the proper splitting $T = Q - V_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} - V_1$. Then, $Q^\dagger V_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$ and so $\rho(Q^\dagger V_1) = \frac{1}{2}$. Therefore $\rho(U_T^* V) < \rho(Q^\dagger V_1) < 1$.

We end this section by studying proper splittings of normal matrices. More precisely, given a normal matrix T we study the proper splitting $T = P_T - V_2$ and we compare it with $T = U_T - V$.

Proposition 7.6. *Let $T \in M^n(\mathbb{C})$ be a normal matrix. Then, the following assertions hold:*

a) The proper splitting $T = P_T - V_2$ is convergent if and only if $\|P_T - T\| < 1$.
b) If $T \in M_H^n(\mathbb{C})$, $P_T - |T| \in M_+^n(\mathbb{C})$ and $\|P_T - T\| < 1$ then the proper splittings $T = P_T - V_2$ and $T = U_T - V$ are convergent and $\rho(U_T^* V) \le \rho(V_2) < 1$.

Proof. *a*) Observe that if T is normal then $V_2 = P_T - T$ is also normal. Therefore, $\rho(P_T V_2) = \rho(V) = \|P_T - T\|$. Thus, $T = P_T - V$ is convergent if and only if $\|P_T - T\| < 1$.

b) Let $T \in M_H^n(\mathbb{C})$ and consider the proper splittings $T = P_T - V = U_T - V_1$. Since $\|P_T - T\| < 1$ and $U_T^* V = P_T - |T| \in M_+^n(\mathbb{C})$ then both splittings are convergent. Moreover, as $T \in M_H^n(\mathbb{C})$ then $T \le |T|$. Thus, $0 \le U_T^* V = U_T^*(U_T - T) = P_T - |T| \le P_T - T = V_2 = P_T V_2$. Therefore, $\rho(U_T^* V) = \|P_T - |T|\| \le \|P_T - T\| = \rho(V_2) < 1$. $\square$

Given a normal matrix $T \in M^n(\mathbb{C})$, then a natural proper splitting to be considered is: $T = |T| - V_3$. However, it can be proved that if $\rho(|T|^\dagger V_3) < 1$ then $T \in M_+^n(\mathbb{C})$. Thus, $T = |T|$ and the proposed splitting has not any sense.

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