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# Three-Step Iterative Algorithms for Multivalued Quasi Variational Inclusions

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In this paper, we suggest and analyze some new classes of three-step iterative algorithms for solving multivalued quasi variational inclusions by using the resolvent equations technique. New iterative algorithms include the Ishikawa, Mann, and Noor iterations for solving variational inclusions (inequalities) and optimization problems as special cases. The results obtained in this paper represent an improvement and a significant refinement of previously known results. 

*Key Words:* variational inclusions; resolvent equations; algorithms; iterative methods; convergence.

## 1. INTRODUCTION

Multivalued quasi variational inclusion, which was introduced and studied by Noor [15, 20, 21, 23, 24], is a useful and important extension of the variational principles with a wide range of applications in industry, physical, regional, social, pure, and applied sciences. Some special cases have been studied by many authors including Ding [4], Noor [12–16], and Noor and Noor [25]. Quasi variational inclusions provide us with a unified, natural, novel, innovative, and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences. It is well known that the projection methods, Wiener–Hopf equation techniques, and auxiliary principle techniques cannot be extended and modified for solving variational inclusions. This fact motivated us to develop another technique, which involves the use of the resolvent operator associated with the maximal monotone operator. Using this technique, one shows



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that the variational inclusions are equivalent to the fixed point problem. This alternative formulation was used to develop numerical methods for solving various classes of variational inclusions and related problems; see [4, 9–26]. In recent years, three-step forward-backward splitting methods have been developed by Glowinski and Le Tallec [7] and Noor [13, 14, 22] for solving various classes of variational inequalities by using the Lagrangian multiplier, updating the solution and the auxiliary principle techniques. It has been shown in [7] that the three-step schemes give better numerical results than the two-step and one-step approximation iterations. For the applications of the splitting methods in partial differential equations, see Ames [1] and the references therein. Equally important is the area of the resolvent equations, which is mainly due to Noor [18]. Using the resolvent operator methods, it can be shown that the variational inclusions are equivalent to the resolvent equations. It has been shown in [9, 11–16, 20, 21, 25, 26] that the resolvent equations technique can be used effectively to develop some powerful iterative algorithms for various classes of variational inclusions (inequalities) as well as to study the sensitivity analysis for variational inclusions. It is well known that the resolvent equations include the Wiener–Hopf equations as a special case. The Wiener–Hopf equations were introduced and studied by Shi [30] and Robinson [29] in relations with classical variational inequalities. For the recent state-of-the-art, see, for example, [10] and the references therein. In this paper, we again use the resolvent equations technique to suggest and analyze a new class of three-step iterative schemes for solving multivalued quasi variational inclusions. Our results include the Ishikawa, Mann, and Noor [13, 14] iterations for solving variational inclusions (inequalities) as special cases. We also study the convergence criteria of these new methods. Since multivalued quasi variational inclusions include mixed quasi variational inequalities, complementarity problems, and nonconvex programming problems as special cases, our results continue to hold for these problems. Our results extend and generalize the previously known results.

## 2. PRELIMINARIES

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $C(H)$ be a family of all nonempty compact subsets of $H$. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators and $g : H \longrightarrow H$ be a single-valued operator. Let $A(\cdot, \cdot) : H \times H \longrightarrow H$ be a maximal monotone operator with respect to the first argument. For a given nonlinear operator $N(\cdot, \cdot) : H \times H \longrightarrow H$, consider the problem of finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w, y) + A\big(g(u), u\big), \tag{2.1}$$

which is called the multivalued quasi variational inclusions; see Noor [15, 20, 21, 23, 24]. Some special cases of (2.1) have been studied by Noor [14–22], Ding [4], Noor and Noor [25], and Noor *et al.* [26] recently. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of the multivalued quasi variational inclusions; see for example, [3, 26, 28].

*Special Cases*

(I) If $A(\cdot, u) = \partial\phi(\cdot, u) : H \times H \longrightarrow R \cup \{+\infty\}$, is the subdifferential of a convex, proper, and lower semi-continuous function $\phi(\cdot, u)$ with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w, y), g(v) - g(u)\rangle + \phi\big(g(v), g(u)\big) - \phi\big(g(u), g(u)\big) \geq 0, \qquad \text{for all } v \in H, \tag{2.2}$$

which is called the set-valued mixed quasi variational inequality. Problem (2.2) has been studied by Noor [11, 12] using the resolvent equations technique.

(II) If $A(g(u, v) \equiv A(g(u))$, for all $v \in H$, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w, y) + A\big(g(u)\big), \tag{2.3}$$

a problem considered and studied by Noor [16] using the resolvent equations technique. Some special cases have been studied by Robinson [28] and Uko [32].

(III) If $A(g(u)) \equiv \partial\phi(g(u))$ is the subdifferential of a proper, convex, and lower, semicontinuous function $\phi : H \longrightarrow R \cup \{+\infty\}$, then problem (2.1) reduces to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w, y), g(v) - g(u)\rangle + \phi\big(g(v)\big) - \phi\big(g(u)\big) \geq 0. \tag{2.4}$$

Problem (2.4) is known as the set-valued mixed variational inequality and has been studied by Noor *et al.* [26].

(IV) If the function $\phi(\cdot, \cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in $H$, that is,

$$\phi(u, u) = K_{(u)}(u) = \begin{cases} 0, & \text{if } u \in K(u) \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.2) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$, $g(u) \in K(u)$ such that

$$\langle N(w, y), g(v) - g(u)\rangle \geq 0, \qquad \text{for all } v \in K(u), \tag{2.5}$$

a problem considered and studied by Noor [19], using the projection method and the implicit Wiener–Hopf equations technique.

(V) If $K^*(u) = \{u \in H, \langle u, v\rangle \geq 0, \text{ for all } v \in K(u)\}$ is a polar cone of the convex-valued cone $K(u)$ in $H$, then problem (2.5) is equivalent to finding $u \in, w \in T(u), y \in V(u)$ such that

$$g(u) \in K(u), \qquad N(w, y) \in K^*(u), \qquad \text{and} \qquad \langle N(w, y), g(u)\rangle = 0,$$

which is called the generalized multivalued implicit complementarity problem.

For special choices of the operators $T$, $N(\cdot, \cdot)$, $g$, and the convex set $K$, one can obtain a large number of variational inclusions (inequalities) and implicit (quasi) complementarity problems; see, for example, [2–28] and the references therein. We would like to mention that the problem of finding a zero of the sum of two maximal monotone operators [31], the location problem $\min_{u\in H}\{f(u) + g(u)\}$, where $f, g$ are both convex functions, various classes of variational inequalities, and the complementarity problems are very special cases of problem (2.1). Thus it is clear that problem (2.1) is a general and unifying one and has numerous applications in pure and applied sciences.

We now recall some basic concepts and results.

DEFINITION 2.1 [2]. If $T$ is a maximal monotone operator on $H$, then, for a constant $\rho > 0$, the resolvent operator associated with $T$ is defined by

$$J_T(u) = (I + \rho T)^{-1}(u), \qquad \text{for all } u \in H,$$

where $I$ is the identity operator. It is known that the monotone operator $T$ is maximal monotone if and only if the resolvent operator $J_T$ is defined everywhere on the space. Furthermore, the resolvent operator $J_T$ is single-valued and nonexpansive.

*Remark* 2.1. Since the operator $A(\cdot, \cdot)$ is a maximal monotone operator with respect to the first argument, for a constant $\rho > 0$, we denote by

$$J_{A(u)} \equiv \big(I + \rho A(u)\big)^{-1}(u), \qquad \text{for all } u \in H,$$

the resolvent operator associated with $A(\cdot, u) \equiv A(u)$. For example, if $A(\cdot, u) = \partial\phi(\cdot, u)$, for all $u \in H$, and $\phi(\cdot, \cdot) : H \times H \longrightarrow R \cup \{+\infty\}$ is proper, convex, and lower semicontinuous with respect to the first argument, then it is well known that $\partial\phi(\cdot, u)$ is a maximal monotone operator with respect to the first argument. In this case, the resolvent operator $J_{A(u)} = J_{\phi(u)}$ is

$$J_{\phi(u)} = \big(I + \rho\partial\phi(\cdot, u)\big)^{-1}(u) = \big(I + \rho\partial\phi(u)\big)^{-1}(u), \qquad \text{for all } u \in H,$$

which is defined everywhere on the space $H$, where $\partial\phi(u) \equiv \partial\phi(\cdot, u)$. For recent state-of-the-art of the nonconvex analysis, see Gao [5].

Let $R_{A(u)} = I - J_{A(u)}$, where $I$ is the identity operator and $J_{A(u)} = (I + \rho A(u))^{-1}$ is the resolvent operator. For given $T, V : H \longrightarrow C(H)$ and $N(\cdot,\cdot) : H \times H \longrightarrow H$, consider the problem of finding $z, u, \in H, w \in T(u), y \in V(u)$ such that

$$N(w, y) + \rho^{-1} R_{A(u)} z = 0, \tag{2.6}$$

where $\rho > 0$ is a constant. Equations (2.6) are called the implicit resolvent equations introduced and studied by Noor [21, 23]. In particular, if $A(g(u), u) \equiv A(u)$, then $J_{A(u)} = (I + \rho A)^{-1} = J_A$ and the implicit resolvent equations (2.6) are equivalent to finding $z, u, \in H, w \in T(u), y \in V(u)$ such that

$$N(w, y) + \rho^{-1} R_A z = 0, \tag{2.7}$$

which are called the resolvent equations; see Noor [16]. It has been shown in [16] that the problems (2.3) and (2.7) are equivalent by using the general duality principle. This equivalence was used to suggest and analyze some iterative methods for solving the generalized set-valued variational inclusions. For formulation and applications of the resolvent equations, see [11–13, 16, 17, 21–26].

If $A(\cdot,\cdot) \equiv \phi(\cdot,\cdot)$ is the indicator function of a closed convex set $K(u)$ in $H$, then the resolvent operator $J_{A(u)} \equiv P_K(u)$, the projection of $H$ onto $K(u)$. Consequently, problem (2.6) is equivalent to finding $z, u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$N(w, y) + \rho Q_{K(u)} z = 0, \tag{2.8}$$

where $Q_{K(u)} = I - P_{K(u)}$ and $I$ is the indentity operator. The equations of the type (2.8) are called the implicit Wiener–Hopf equations introduced and studied by Noor [19]. For recent applications and numerical methods of the Wiener–Hopf equations, see [10, 18, 19, 27, 29, 30] and the references therein.

DEFINITION 2.2. For all $u_1, u_2 \in H$, the operator $N(\cdot,\cdot)$ is said to be strongly monotone and Lipschitz continuous with respect to the first argument, if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\langle N(w_1, \cdot) - N(w_2, \cdot), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2,$$

$$\text{for all } w_1 \in T(u_1), w_2 \in T(u_2)$$

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|.$$

In a similar way, we can define strong monotonicity and Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument.

DEFINITION 2.3. The set-valued operator $V : H \longrightarrow C(H)$ is said to be $M$-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$M\big(V(u), V(v)\big) \leq \xi\|u - v\|, \qquad \text{for all } u, v \in H,$$

where $M(\cdot, \cdot)$ is the Hausdorff metric on $C(H)$.

We also need the following condition.

*Assumption* 2.1. For all $u, v, w \in H$, the resolvent operator $J_{A(u)}$ satisfies the condition

$$\|J_{A(u)}w - J_{A(v)}w\| \leq \nu\|u - v\|,$$

where $\nu > 0$ is a constant.

Assumption 2.1 is satisfied when the operator $A$ is monotone jointly with respect to two arguments. In particular, this implies that $A$ is monotone with respect to the first argument.

## 3. MAIN RESULTS

In this section, we use the resolvent operator technique to establish the equivalence between the multivalued quasi variational inclusions and the implicit resolvent fixed points. This equivalence is used to suggest an iterative method for solving the quasi variational inclusions. For this purpose, we need the following well known result; see Noor [20, 21]. However, we include its proof for the sake of completeness.

LEMMA 3.1 $(u, w, y)$ is a solution of (2.1) if and only if $(u, w, y)$ satisfies the relation

$$g(u) = J_{A(u)}[g(u) - \rho N(w, y)], \tag{3.1}$$

where $\rho > 0$ is a constant and $J_{A(u)} = (I + \rho A(u))^{-1}$ is the resolvent operator.

*Proof.* Let $u \in H$, $w \in T(u)$, $y \in V(u)$ be a solution of (2.1). Then, for a constant $\rho > 0$,

$$\begin{aligned}
(2.1) &\iff 0 \in \rho N(w, y) + \rho A\big(g(u), u\big) \\
&\iff 0 \in -\big(g(u) - \rho N(w, y)\big) + \big(I + \rho A(u)\big)g(u) \\
&\iff g(u) = J_{A(u)}\big[g(u) - \rho N(w, y)\big],
\end{aligned}$$

the required result. ∎

From Lemma 3.1, we conclude that the multivalued quasi variational inclusions (2.1) are equivalent to the implicit fixed-point problem (3.1). This alternative formulation is very useful from both the theoretical and numerical analysis points of view. We use this equivalence to propose some three-step iterative algorithms for solving multivalued quasi variational inclusions (2.1) and related optimization problems.

The relation (3.1) can be written as

$$u = u - g(u) + J_{A(u)}\big[g(u) - \rho N(w, u)\big], \tag{3.2}$$

where $\rho > 0$ is a constant.

This fixed-point formulation allows us to suggest the following unified three-step iterative algorithm.

ALGORITHM 3.1 [24]. Assume that $T, V : H \longrightarrow C(H)$, $g : H \longrightarrow H$, and $N(\cdot,\cdot)$, $A(\cdot,\cdot) : H \times H \longrightarrow H$ are operators. For a given $u_0 \in H$, compute the sequences $\{v_n\}$, $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ , $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ by the iterative schemes

$$\begin{aligned}
w_n &\in T(u_n) : \|w_{n+1} - w_n\| \le M\big(T(u_{n+1}), T(u_n)\big)\\
y_n &\in V(u_n) : \|y_{n+1} - y_n\| \le M\big(V(u_{n+1}), V(u_n)\big)\\
\overline{w_n} &\in T(x_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \le M\big(T(x_{n+1}), T(x_n)\big)\\
\overline{y_n} &\in V(x_n) : \|\overline{y_{n+1}} - \overline{y_n}\| \le M\big(V(x_{n+1}), V(x_n)\big)\\
\eta &\in T(v_n) : \|\eta_{n+1} - \eta_n\| \le M\big(T(v_{n+1}), T(v_n)\big)\\
\xi_n &\in V(v_n) : \|\xi_{n+1} - \xi_n\| \le M\big(V(v_{n+1}), V(v_n)\big)\\
x_n &= (1-\gamma_n)u_n + \gamma_n\Big\{u_n - g(u_n) + J_{A(u_n)}\big[g(u_n) - \rho N(w_n, y_n)\big]\Big\}\\
v_n &= (1-\beta_n)u_n + \beta_n\Big\{x_n - g(x_n) + J_{A(x_n)}\big[g(x_n) - \rho N(\overline{w_n}, \overline{y_n})\big]\Big\}\\
u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\Big\{v_n - g(v_n) + J_{A(v_n)}\big[g(v_n) - \rho N(\eta_n, \xi_n)\big]\Big\},\\
&\qquad n = 0, 1, 2, \ldots,
\end{aligned}$$

where $0 \le \alpha_n, \beta_n, \gamma_n \le 1$; for all $n \ge 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges. For $\gamma_n = 0$, Algorithm 3.1 is the Ishikawa iterative scheme for solving multivalued quasi variational inclusions; see Noor [15]. For $\beta_n = 0 = \gamma_n$ and $\alpha_n = \lambda$, Algorithm 3.1 has been studied by Noor [20, 21].

If $A(\cdot, v) \equiv \phi(\cdot, v)$, for all $v \in H$, is an indicator function of a closed convex-valued set $K(u)$ in $H$, then $J_{A(u)} \equiv P_{K(u)}$, the projection of $H$ onto the convex-valued set $K(u)$ in $H$. Consequently, Algorithm 3.1

collapses to:

ALGORITHM 3.2. For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, $g(u_0) \in K(u_0)$, compute the sequences $\{v_n\}$, $\{x_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\eta_n\}$, and $\{\xi_n\}$ from the iterative schemes

$$
\begin{aligned}
w_n &\in T(u_n) : \|w_{n+1} - w_n\| \le M\big(T(u_{n+1}), T(u_n)\big)\\
y_n &\in V(u_n) : \|y_{n+1} - y_n\| \le M\big(V(u_{n+1}), V(u_n)\big)\\
\overline{w_n} &\in T(x_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \le M\big(T(x_{n+1}), T(x_n)\big)\\
\overline{y_n} &\in V(x_n) : \|\overline{y_{n+1}} - \overline{y_n}\| \le M\big(V(x_{n+1}), V(x_n)\big)\\
\eta_n &\in T(v_n) : \|\eta_{n+1} - \eta_n\| \le M\big(T(v_{n+1}), T(v_n)\big)\\
\xi_n &\in V(v_n) : \|\xi_{n+1} - \xi_n\| \le M\big(V(v_{n+1}), V(v_n)\big)\\
x_n &= (1-\gamma_n)u_n + \gamma_n\Big\{u_n - g(u_n) + P_{K(u_n)}\big[g(u_n) - \rho N(w_n, y_n)\big]\Big\}\\
v_n &= (1-\beta_n)u_n + \beta_n\Big\{x_n - g(x_n) + P_{K(x_n)}\big[g(x_n) - \rho N(\overline{w_n}, \overline{y_n})\big]\Big\}\\
u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\Big\{v_n - g(v_n) + P_{K(v_n)}\big[g(v_n) - \rho N(\eta_n, \xi_n)\big]\Big\},\\
&\qquad n = 0, 1, 2, \ldots,
\end{aligned}
$$

where $0 < \alpha_n, \beta_n, \gamma_n < 1$ for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges. Algorithm 3.2 appears to be a new one for multivalued variational inequalities (2.5).

Let $\{A^n(., u)\}_{n \in N}$ be a sequence of maximal monotone operators with respect to the first argument, which approximates $A(, .u) \equiv A(u)$ on $H \times H$. We now suggest and analyze some perturbed type algorithms for multivalued quasi variational inclusions (2.1).

ALGORITHM 3.3 [24]. For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{v_n\}$, $\{x_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ from the iterative schemes

$$
\begin{aligned}
w_n &\in T(u_n) : \|w_{n+1} - w_n\| \le M\big(T(u_{n+1}), T(u_n)\big)\\
y_n &\in V(u_n) : \|y_{n+1} - y_n\| \le M\big(V(u_{n+1}), V(u_n)\big)\\
\overline{w_n} &\in T(x_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \le M\big(T(x_{n+1}), T(x_n)\big)\\
\overline{y_n} &\in V(x_n) : \|\overline{y_{n+1}} - \overline{y_n}\| \le M\big(V(x_{n+1}), V(x_n)\big)\\
\eta_n &\in T(v_n) : \|\eta_{n+1} - \eta_n\| \le M\big(T(v_{n+1}), T(v_n)\big)\\
\xi_n &\in V(v_n) : \|\xi_{n+1} - \xi_n\| \le M\big(V(v_{n+1}), V(v_n)\big)\\
x_n &= (1-\gamma_n)u_n + \gamma_n\Big\{u_n - g(u_n) + J_{A^n_{(u_n)}}\big[g(u_n) - \rho N(w_n, y_n)\big]\Big\} + \gamma_n h_n
\end{aligned}
$$

$$v_n = (1-\beta_n)u_n + \beta_n\Big\{x_n - g(x_n) + J_{A^n_{(x_n)}}\big[g(x_n) - \rho N(\overline{w_n}, \overline{y_n})\big]\Big\} + \beta_n f_n$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\Big\{v_n - g(v_n) + J_{A^n_{(v_n)}}\big[g(v_n) - \rho N(\eta_n, \xi_n)\big]\Big\} + \alpha_n e_n,$$

$$n = 0, 1, 2, \ldots,$$

where $0 \le \alpha_n, \beta_n, \gamma_n \le 1$; for all $n \ge 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges and $\rho > 0$ is a constant. Here $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are sequences of the elements of $H$ to take into account possible inexact computations.

For $\gamma_n = 0$, Algorithm 3.3 is two-step perturbed iterative method for solving multivalued quasi variational inclusions, which appears to be a new one. For $e_n = f_n = h_n = 0$ and $A^n(u) \equiv A(u)$, Algorithm 3.3 is exactly the Algorithm 3.1. which has been studied by Noor [24].

We now suggest and analyze another class of three-step iterative schemes using the resolvent equations technique. For this purpose, we need the following result, which is due to Noor [21].

LEMMA 3.2. *The multivalued quasi variational inclusion* (2.1) *has a solution* $u \in H$, $w \in T(u)$, $y \in V(u)$ *if and only if* $z, u, \in H$, $w \in T(u)$, $y \in V(u)$ *is a solution of the implicit resolvent equations* (2.6), *where*

$$g(u) = J_{A(u)}z \tag{3.3}$$

$$z = g(u) - \rho N(w, y), \tag{3.4}$$

*and* $\rho > 0$ *is a constant*.

Lemma 3.2 implies that the problems (2.1) and (2.6) are equivalent. This equivalent interplay between these problems plays an important and crucial role in suggesting and analyzing various iterative methods for solving multivalued quasi variational inclusions and related optimization problems. By a suitable and appropriate rearrangement of the implicit resolvent equations (2.6), we suggest and analyze a class of three-step iterative methods for the multivalued quasi variational inclusions (2.1).

Equations (2.6) can be written as

$$R_{A(u)}z = -\rho N(w, y),$$

which implies that

$$z = J_{A(u)}z - \rho N(w, y) = g(u) - \rho N(w, y), \qquad \text{using (3.3).}$$

We use this fixed-point formulation to suggest the following three-step iterative scheme for solving multivalued quasi variational inclusions (2.1).

ALGORITHM 3.4. For given $z_o$, $u_o \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ by the iterative schemes

$$g(u_n) = J_{A(u_n)} z_n \tag{3.5}$$

$$g(x_n) = J_{A(x_n)} x_n \tag{3.6}$$

$$g(v_n) = J_{A(v_n)} v_n \tag{3.7}$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \le M\big(T(u_{n+1}), T(u_n)\big) \tag{3.8}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \le M\big(V(u_{n+1}), V(u_n)\big) \tag{3.9}$$

$$\overline{w_n} \in T(x_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \le M\big(T(x_{n+1}), T(x_n)\big) \tag{3.10}$$

$$\overline{y_n} \in V(x_n) : \|\overline{y_{n+1}} - \overline{y_n}\| \le M\big(V(x_{n+1}), V(x_n)\big) \tag{3.11}$$

$$\eta_n \in T(v_n) : \|\eta_{n+1} - \eta_n\| \le M\big(T(v_{n+1}), T(v_n)\big) \tag{3.12}$$

$$\xi_n \in V(v_n) : \|\xi_{n+1} - \xi_n\| \le M\big(V(v_{n+1}), V(v_n)\big) \tag{3.13}$$

$$x_n = (1 - \gamma_n) z_n + \gamma_n \big\{g(u_n) - \rho N(w_n, y_n)\big\} \tag{3.14}$$

$$v_n = (1 - \beta_n) z_n + \beta_n \big\{g(x_n) - \rho N(\overline{w_n}, \overline{y_n})\big\} \tag{3.15}$$

$$z_{n+1} = (1 - \alpha_n) z_n + \alpha_n \big\{g(v_n) - \rho N(\eta_n . \xi_n)\big\}, \quad n = 0, 1, 2, \ldots, \tag{3.16}$$

where $0 < \alpha_n, \beta_n, \gamma_n < 1$; for all $n \ge 1$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

For $\lambda_n = 0$, Algorithms 3.4 is known as the two-step iterative method for solving multivalued quasi variational inequalities (2.1); see Noor [23]. In brief, for suitable and appropriate choice of the operators $T, V, g$, and the spaces $H, K$, one can obtain a number of new and previously known algorithms for solving variational inclusions (inequalities) and related optimization problems.

We now study the convergence criteria of Algorithm 3.4. using the method of Noor [23, 24].

THEOREM 3.1. *Let the operator $N(\cdot, \cdot)$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. Let $g : H \longrightarrow H$ be strongly monotone with constant $\sigma > 0$ and be Lipschitz continuous with constant $\delta > 0$. Assume that the operator $N(\cdot, \cdot)$ is Lipschitz continuous with constant $\lambda > 0$ with respect to the second argument and $V$ is M-Lipschitz continuous with constant $\zeta > 0$. Let $T : H \longrightarrow C(H)$ be a M-Lipschitz continuous with constant $\mu > 0$. If Assumption* 2.1 *holds*

*and*

$$\left|\rho - \frac{\alpha - (1-k)\lambda\zeta}{\beta^2\mu^2 - \lambda^2\zeta^2}\right| < \frac{\sqrt{\left[\alpha - (1-k)\lambda\zeta\right]^2 - k\left(\beta^2\mu^2 - \lambda^2\zeta^2\right)(2-k)}}{\beta^2\mu^2 - \lambda^2\zeta^2} \tag{3.17}$$

$$\alpha > (1-k)\lambda\zeta + \sqrt{k\left(\beta^2\mu^2 - \lambda^2\zeta^2\right)(2-k)} \tag{3.18}$$

$$\rho\lambda\zeta < 1 - k \tag{3.19}$$

$$k = 2\left(\sqrt{1 - 2\sigma + \delta^2}\right) + \nu, \tag{3.20}$$

*then there exist* $z, u \in H$, $w \in T(u)$, $y \in V(u)$ *satisfying the implicit resolvent equations* (2.6) *and the sequences* $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, *and* $\{\xi_n\}$, *generated by Algorithm* 3.1 *converge to* $u, w, y, \bar{w}, \bar{y}, \eta,$ *and* $\xi$ *strongly in* $H$, *respectively.*

*Proof.* If Assumption 2.1 and the conditions (3.17)–(3.19) hold, then it has been shown in [20, Theorem 3.1, p. 106] that there exists a solution $u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the multivalued quasi variational inclusion (2.1). Let $u \in H$ be the solution of (2.1). Then from Lemma 3.2, it follows that $z, u \in H$ is also a solution of the resolvent equations (2.6) and

$$g(u) = J_{A(u)}z \tag{3.21}$$

$$z = (1-\alpha_n)z + \alpha_n\left\{g(u) - \rho N(w, y)\right\} \tag{3.22}$$

$$= (1-\beta_n)z + \beta_n\left\{g(u) - \rho N(w, y)\right\} \tag{3.23}$$

$$= (1-\gamma_n)z + \gamma_n\left\{g(u) - \rho N(w, y)\right\}, \tag{3.24}$$

where $0 < \alpha_n, \beta_n, \gamma_n < 1$ are constants.

From (3.14) and (3.24), we have

$$\begin{aligned}\|x_n - z\| &\le (1-\gamma_n)\|z_n - z\| \\ &\quad + \gamma_n\|g(u_n) - g(u) - \rho\left\{N(w_n, y_n) - N(w, y)\right\}\| \\ &\le (1-\gamma_n)\|z_n - z\| + \gamma_n\|u_n - u - \left(g(u_n) - g(u)\right)\| \\ &\quad + \gamma_n\|u_n - u - \rho\left(N(w_n, y_n) - N(w, y_n)\right)\| \\ &\quad + \rho\gamma_n\|N(w, y_n) - N(w, y)\|. \end{aligned} \tag{3.25}$$

Since $g$ is a strongly monotone and Lipschitz continuous operator with constants $\sigma > 0$, $\delta > 0$, it follows that

$$\begin{aligned}
&\|u_n - u - \big(g(u_n) - g(u)\big)\|^2 \\
&\quad = \|u_n - u\|^2 - 2\langle u_n - u, g(u_n) - g(u)\rangle + \|g(u_n) - g(u)\|^2 \\
&\quad \le (1 - 2\sigma + \delta^2)\|u_n - u\|^2 \\
&\quad = \left(\frac{k-\nu}{2}\right)\|u_n - u\|, \qquad \text{using (3.20).}
\end{aligned} \tag{3.26}$$

Using the strong monotonicity and Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the first argument, we have

$$\begin{aligned}
&\|u_n - u - \rho\big(N(w_n, y_n) - N(w, y_n)\big)\|^2 \\
&\quad = \|u_n - u\|^2 - 2\rho\langle N(w_n, y_n) - N(w, y_n), u_n - u\rangle \\
&\qquad + \rho^2\|N(w_n, y_n) - N(w, y_n)\|^2 \\
&\quad \le (1 - 2\rho\alpha + \rho^2\beta^2\mu^2)\|u_n - u\|^2.
\end{aligned} \tag{3.27}$$

From the Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument and the $M$-Lipschitz continuity of $V$, we have

$$\begin{aligned}
\|N(w, y_n) - N(w, y)\| &\le \lambda\|y_n - y\| \\
&\le \lambda M\big(V(u_n), V(u)\big) \\
&\le \lambda\zeta\|u_n - u\|.
\end{aligned} \tag{3.28}$$

Combining (3.25)–(3.28), we obtain

$$\|x_n - z\| \le (1 - \gamma_n)\|z_n - z\| + \gamma_n\left\{\frac{k-\nu}{2} + \rho\lambda\zeta + t(\rho)\right\}\|u_n - u\|, \tag{3.29}$$

where

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}. \tag{3.30}$$

From (3.5) and (3.3), we have

$$\begin{aligned}
\|u_n - u\| &\le \|u_n - u - \big(g(u_n) - g(u)\big)\| + \|J_{A(u_n)}z_n - J_{A(u)}z\| \\
&\le \|u_n - u - \big(g(u_n) - g(u)\big)\| + \|J_{A(u_n)}z_n - J_{A(u_n)}z\| \\
&\qquad + \|J_{A(u_n)}z - J_{A(u)}z\| \\
&\le \left(\frac{k-\nu}{2}\right)\|u_n - u\| + \nu\|u_n - u\| + \|z_n - z\|
\end{aligned}$$

which implies that

$$\|u_n - u\| \leq \left(\frac{1}{1-(k+\nu)/2}\right)\|z_n - z\|. \tag{3.31}$$

Combining (3.29) and (3.31), we obtain

$$\begin{aligned}\|x_n - z\| &\leq (1-\gamma_n)\|z_n - z\| + \gamma_n\left\{\frac{(k-\nu)/2 + \rho\lambda\zeta + t(\rho)}{1-(k+\nu)/2}\right\}\|z_n - z\| \\ &\leq (1-\gamma_n)\|z_n - z\| + \gamma_n\theta\|z_n - z\| \\ &= \big(1-\gamma_n(1-\theta)\big)\|z_n - z\| \leq \|z_n - z\|, \end{aligned}\tag{3.32}$$

where

$$\theta = \frac{(k-\nu)/2 + \rho\lambda\zeta + t(\rho)}{1-(k+\nu)/2}. \tag{3.33}$$

In a similar way, from (3.6) and (3.3), we obtain

$$\begin{aligned}\|x_n - u\| &\leq \|x_n - u - \big(g(x_n) - g(u)\big)\| + \|J_{A(x_n)}x_n - J_{A(x_n)}z\| \\ &\quad + \|J_{A(x_n)}z - J_{A(u)}z\| \\ &\leq \left(\frac{k-\nu}{2}\right)\|x_n - u\| + \nu\|x_n - u\| + \|x_n - z\|, \end{aligned}$$

which implies that

$$\|x_n - u\| \leq \frac{1}{1-(k+\nu)/2}\|x_n - z\|. \tag{3.34}$$

Also from (3.15), (3.23), (3.32), (3.33), and (3.34), we obtain

$$\begin{aligned}\|v_n - z\| &\leq (1-\beta_n)\|z_n - z\| + \beta_n\left\{\frac{k-\nu}{2} + \rho\nu\xi + t(\rho)\right\}\|x_n - u\| \\ &\leq (1-\beta_n)\|z_n - z\| + \beta_n\theta\|z_n - z\| \\ &= \big(1-\beta_n(1-\theta)\big)\|z_n - z\| \leq \|z_n - z\|. \end{aligned}\tag{3.35}$$

Similarly, from (3.37), (3.3), and (3.35), we can have

$$\|v_n - u\| \leq \frac{1}{1-(k+\nu)/2}\|v_n - z\| \leq \frac{1}{1-(k+\nu)/2}\|z_n - z\|. \tag{3.36}$$

From (3.22), (3.16), (3.26)–(3.28), and (3.30), we have

$$\begin{aligned}
\|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| \\
&\quad + \alpha_n \|g(v_n) - g(u) - \rho\big(N(\eta_n, \xi_n) - N(w, y)\big)\| \\
&\leq (1-\alpha_n)\|z_n - z\| + \alpha_n \|v_n - u - \big(g(v_n) - g(u)\big)\| \\
&\quad + \rho\alpha_n \|N(w, \xi_n) - N(w, y)\| \\
&\quad + \alpha_n \|v_n - u - \rho\big(N(\eta_n, \xi_n) - N(w, \xi_n)\big)\| \\
&\leq (1-\alpha_n)\|z_n - z\| + \alpha_n \left\{ \frac{k+\nu}{2} + \rho\gamma\zeta + t(\rho) \right\} \|v_n - u\|, \\
&\leq (1-\alpha_n)\|z_n - z\| + \alpha_n \theta \|z_n - z\|, \\
&\qquad \text{using (3.33), (3.35), and (3.36),} \\
&= \prod_{i=0}^{\infty} \big\{1 - (1-\theta)\alpha_i\big\} \|z_0 - z\|. \qquad (3.37)
\end{aligned}$$

From (3.17), (3.18), and (3.19), it follows that $\theta < 1$. Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\sum_{i=0}^{\infty} \{1 - (1-\theta)\alpha_i\} = 0$. Hence the sequence $\{z_n\}$ converges strongly to $z$. Also from (3.32) and (3.31), we see that the sequences $\{x_n\}$ and $\{u_n\}$ converge to $z$ and $u$ strongly, respectively. Using the technique of Noor [15], one can easily show that the sequences $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ converge strongly to $w$, $y$, $\bar{w}$, $\bar{y}$, $\eta$, and $\xi$, respectively. Now by using the continuity of the operators $T$, $V$, $g$, $J_{A(u)}$, and Lemma 3.2, we have

$$z = g(u) - \rho N(w, y) = J_{A(u)} z - \rho N(w, y) \in H.$$

We now show that $w \in T(u)$, $y \in V(u)$, $\bar{w} \in T(x)$, $\bar{y} \in V(x)$, $\eta \in T(v)$, $\xi \in V(v)$. In fact,

$$\begin{aligned}
d\big(w, T(u)\big) &\leq \|w - w_n\| + d\big(w_n, T(u)\big) \\
&\leq \|w - w_n\| + M\big(T(u_n), T(u)\big) \\
&\leq \|w - w_n\| + \mu \|u_n - u\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty,
\end{aligned}$$

where $d\big(w, T(u)\big) = inf\{\|w - z\| : z \in T(u)\}$. Since the sequences $\{w_n\}$ and $\{u_n\}$ are the Cauchy sequences, it follows that $d\big(w, T(u)\big) = 0$. This implies that $w \in T(u)$. In a similar way, one show that $y \in V(u)$, $\bar{w} \in T(x)$, $\bar{y} \in V(x)$, $\eta \in T(v)$, and $\xi \in V(v)$. By invoking Lemma 3.2, we have $z, u \in H$, $w \in T(u)$, $y \in V(u)$, $y \in V(u)$, which satisfy the implicit resolvent equations (2.6) and the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w}_n\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ converge strongly to $z$, $u$, $w$, $y$, $\bar{w}$, $\bar{y}$, $\eta$, and $\xi$ in $H$, respectively, the required result. ∎

*Remark* 3.1. It is worth mentioning that Assumption 2.1 and the conditions (3.9)–(3.11), which play an important part in the derivation of the main result, that is, Theorem 3.1, are very convenient and reasonably easy to verify in practical problems; see Noor [14, 20, 21]. For a different choice of the operators $T$, $V$, $g$, $N(.,.)$, and $A(.,.)$, these conditions are well known and have been already used in the existence of solutions of variational inequalities and inclusions. It is worth mentioning that the concept of fuzzy mappings can be extended fro multivalued quasi variational inclusions (2.1) by using the technique of Noor [33] and Noor and Al-Said [34].

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