

Dynamics of a class of nonautonomous semi-ratio-dependent predator–prey systems with functional responses[☆]

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Abstract

In this paper, we investigate the dynamics of a class of the so-called semi-ratio-dependent predator–prey interaction models with functional responses based on systems of nonautonomous differential equations with time-dependent parameters. The functional responses are classified into five types and typical examples of each type are provided. Then we establish sufficient criteria for the boundedness of solutions, the permanence of system, and the existence, uniqueness and globally asymptotic stability of positive periodic solution and positive almost periodic solution. Some conclusive discussion is presented at the end of this paper.

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1. Introduction

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [6]. At first sight, these problems may appear to be simple mathematically. However, in fact, they are often very challenging and complicated.

Recently, many authors have explored the dynamics of a class of the so-called semi-ratio-dependent predator–prey systems with functional responses

$$\begin{aligned}x' &= x[a - bx] - c(x)y, \\y' &= y\left[d - e\frac{y}{x}\right],\end{aligned}\tag{1.1}$$

where x and y stand for the population (or density) of the prey and the predator, respectively. $c(x)$ is the so-called predator functional response to prey.

In (1.1), it has been assumed that the prey grows logistically with growth rate a and carrying capacity a/b in the absence of predation. The predator consumes the prey according to the functional response $c(x)$ and grow logistically with growth rate d and carrying capacity $x(t)/e$ proportional to the population size of prey (or prey abundance). The parameter e is a measure of the food quality that the prey provides for conversion into predator birth.

The form of the predator equation in (1.1) was first proposed by Leslie [33]. In (1.1), the functional response $c(x)$ can be classified into five types.

When the functional response $c(x)$ is of type 1, i.e., $c(x) = mx$, then we have the following Leslie–Gower model [25,26,33]

$$\begin{aligned}x' &= x[a - bx] - mxy, \\y' &= y\left[d - e\frac{y}{x}\right],\end{aligned}\tag{1.2}$$

where the predation is assumed to be proportional to the population size of the prey.

When the functional response $c(x)$ is of type 2, in particular, $c(x) = mx/(A + x)$, then we have the following model of R.M. May also known as the so-called Holling–Tanner predator–prey model [3,5,6,11,13,18,20,23–27,35,37–40,45,46,48], which takes the form of

$$\begin{aligned}x' &= x[a - bx] - \frac{mx}{A + x}y, \\y' &= y\left[d - e\frac{y}{x}\right].\end{aligned}\tag{1.3}$$

The saturating functional response $mx/(A + x)$ is of Michaelis–Menten type in enzyme–substrate kinetics. The parameter m is the maximum specific rate of product formation, x is the substrate concentration, and A (the half-saturation constant) is the substrate concentration at which the rate of product formation is half maximal. The functional response $mx/(A + x)$ was also proposed by Holling [25] for “nonlearning” predators, which is also called a functional response of the predator of Holling type II. The label

nonlearning is a bit misleading because even predators capable of learning should exhibit this type of response when given only one type of prey for which to search. In predator–prey interaction, m is the maximal predator per capita consumption rate, i.e., the maximum number of prey that can be eaten by a predator in each time unit and A is the number of prey necessary to achieve one-half of the maximum rate m . For the derivation of the functional response $c(x)$ of type 2, one can refer to [25,27,39] and references cited therein for details. According to Hassell [22,23], type 2 functional response is the most common type of functional response among arthropod predators. The May model has been used by Wollkind et al. [48] to investigate numerically the dynamics of a predator–prey system for a pest in fruit-bearing trees, under the hypothesis that the parameters depend on the temperature.

When the functional response $c(x)$ is of type 3, in particular, $c(x) = mx^n/(A + x^n)$ ($n \geq 2$), then we have

$$\begin{aligned}x' &= x[a - bx] - \frac{mx^n}{A + x^n}y, \\y' &= y\left[d - e\frac{y}{x}\right].\end{aligned}\tag{1.4}$$

The functional response $c(x)$ of type 3 is sigmoid and it tends to an asymptotic value as the prey density increases. If we take into account the time a predator used in handling the prey it has captured, we find the predator has a functional response of type 3. The function $c(x) = x^2/(A + x^2)$ is also referred to as a function response of Holling type III, which was suggested by the biologist Holling [25]. The general form of function response of this type was introduced by Kazarinov and van den Driessche [31]. One can refer [23,25,30–32,42, 46,48] for related studies.

When the functional response $c(x)$ is of type 4, in particular, $c(x) = mx^2/((A + x) \times (B + x))$ [12,35,42,44,46,48], then we have

$$\begin{aligned}x' &= x[a - bx] - \frac{mx^2}{(A + x)(B + x)}y, \\y' &= y\left[d - e\frac{y}{x}\right].\end{aligned}\tag{1.5}$$

The function $c(x) = mx^2/((A + x)(B + x))$ is an S -shaped curve. The sigmoidal-type curves are indicative of predator which show some form of learning behavior in which, below a certain level of threshold density, the predator will not utilize the prey for food at any great intensity. However, above that density level, the predators increase their feeding rates until some saturation level is reached. Holling [25] reasoned that these animals tend both to learn slowly and to forget the value of a food unless they encounter it fairly often. Holling [25] gave some field evidence that an S -shaped functional response is typical for vertebrate predators with alternative prey available. One can refer to [12] for details of the derivation of $c(x) = mx^2/((A + x)(B + x))$. In fact, the domed functional response, which has been termed type 4, incorporates prey interference with predation in that the per capita predation rate increases with prey density to a maximum at a critical prey density beyond which it decreases. When the prey species is a spider mite, such as *T. mcdanieli*, an possible

source of interference is the webbing produced by these mites [12,13,42]. This webbing is known to interfere with predators by decreasing their walking speed and reducing their chances of contacting the prey [42]. In extreme cases predatory mites that are not adapted to walking on webbing can starve in the presence of spider mite prey.

When the functional response $c(x)$ is of type 5 (also Ivlev's functional response), in particular, $c(x) = m(1 - e^{-Ax})$ [8,17,26,27,29,33,38], then we have

$$\begin{aligned}x' &= x[a - bx] - m(1 - e^{-Ax})y, \\y' &= y\left[d - e\frac{y}{x}\right].\end{aligned}\tag{1.6}$$

For details of derivation, one can refer to [29].

Experimental results on the functional response of predators can be found, for example, in [1,2,12,14,20,25]. It should be pointed out that the expressions are used to define type 1–5 functional responses (e.g., see [12–14,23,32,46]), rather than they are used for their simplicity.

Although much progress has been seen in the predator–prey theories, such systems are not well studied in the sense that most results are autonomous cases related in which time t has not appeared explicitly in the equations. That is to say, in most of the predator–prey systems considered so far, it has been assumed that all biological and environmental parameters are constant in time. However, any biological or environmental parameters are naturally subject to fluctuation in time and if a model is desired which takes into account such fluctuation it must be nonautonomous, which is, of course, more difficult to study in general. One must of course ascribe some properties to the time dependence of the parameters in the models, for only then can the resulting dynamic be studied accordingly. One might assume they are periodic or almost periodic, etc.

To consider the fluctuative environmental factors in real populations, we will confine ourselves here to the case that time t appears explicitly in the biological and environmental parameters.

Although the autonomous case of (1.1) has been studied extensively in the literature [1–3,5,6,11–14,18,20–48], few works have been done on the nonautonomous predator–prey systems of type (1.1) with functional response of type 1–5 [7].

The principle aim of this paper is to perform systematic analysis on the dynamics of the nonautonomous semi-ratio-dependent predator–prey systems with functional responses of form (1.1).

For the sake of generality and conveniences in the following discussion, we prefer to study the following semi-ratio-dependent predator–prey system in a more general form

$$\begin{aligned}x' &= x[a(t) - b(t)x] - c(t, x)y, \\y' &= y\left[d(t) - e(t)\frac{y}{x}\right], \\x(t_0) &> 0, \quad y(t_0) > 0, \quad t_0 \in R.\end{aligned}\tag{1.7}$$

Specially, we will establish sufficient criteria for the boundedness of solutions, the permanence and globally asymptotic stability of systems and the uniqueness of positive periodic solution and almost periodic solution to be globally asymptotically stable.

The tree of this paper is the following:

1. Introduction.
2. General nonautonomous case: boundedness, permanence and globally asymptotic stability.
3. Periodic case: existence, uniqueness and globally asymptotic stability of positive periodic solutions.
4. Almost periodic case: existence, uniqueness and globally asymptotic stability of positive almost periodic solutions.
5. Conclusive discussion.
6. References.

2. General nonautonomous case

In this section, we shall consider the general nonautonomous case and present some preliminaries results, including the boundedness of solutions, the permanence and globally asymptotic stability of system (1.7). First, we shall introduce some notations that will be used throughout this paper.

Let $R_+^2 := \{(x, y)^T \in R^2 \mid x \geq 0, y \geq 0\}$ and $f(t)$ be a bounded continuous function on R . Define

$$f^u := \sup_{t \in R} f(t), \quad f^l := \inf_{t \in R} f(t).$$

Particularly, if $f(t), g(t, x)$ are ω periodic functions with respect to t , then

$$\bar{f} := \frac{1}{\omega} \int_0^\omega f(t) dt, \quad \bar{g}(x) := \frac{1}{\omega} \int_0^\omega g(t, x) dt.$$

Consider the nonautonomous predator–prey system (1.7) together with the following assumptions:

- (A₁) $a(t), b(t), d(t), e(t)$ are continuous on R and are bounded below and above by positive constants;
- (A₂) $c(t, x)$ is continuous with respect to the first variable and is differentiable with respect to the second variable, and $c(t, 0) = 0$, $(\partial c / \partial x)(t, x) > 0$ for any $t \in R$, $x > 0$, and $(\partial c / \partial x)(t, x)$ is bounded with respect to t ;
- (A₃) there exists a constant $C_0 > 0$, such that $c(t, x) \leq C_0 x$ for any $t \in R$, $x > 0$;
- (A₃) there exists a constant $\hat{C}_0 > 0$, such that $c(t, x) \leq \hat{C}_0$ for any $t \in R$, $x > 0$;
- (A₄) $a^l - C_0 M_2 > 0$;
- (A₄) $(a^l)^2 - 4b^u \hat{C}_0 \hat{M}_2 > 0$, where $m_i, M_i, \hat{m}_i, \hat{M}_i$, $i = 1, 2$, are positive constants such that

$$\begin{aligned} M_1 &> \frac{a^u}{b^l} := M_1^*, & M_2 &> \frac{d^u}{e^l} M_1 := M_2^*, \\ m_1 &< \frac{a^l - C_0 M_2}{b^u} := m_1^*, & m_2 &< \frac{d^l}{e^u} m_1 := m_2^*, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \hat{M}_1 &> \frac{a^u}{b^l}, & \hat{M}_2 &> \frac{d^u}{e^l} \hat{M}_1, \\ \hat{m}_1 &< \frac{a^l + \sqrt{(a^l)^2 - 4b^u \hat{C}_0 \hat{M}_2}}{2b^u}, & \hat{m}_2 &< \frac{d^l}{e^u} \hat{m}_1. \end{aligned} \quad (2.2)$$

Define

$$\Gamma := \{(x, y)^T \in \mathbb{R}^2 \mid m_1 \leq x \leq M_1, m_2 \leq y \leq M_2\}, \quad (2.3)$$

$$\hat{\Gamma} := \{(x, y)^T \in \mathbb{R}^2 \mid \hat{m}_1 \leq x \leq \hat{M}_1, \hat{m}_2 \leq y \leq \hat{M}_2\}. \quad (2.4)$$

Theorem 2.1. *Both the nonnegative and positive cones of \mathbb{R}^2 are positively invariant with respect to system (1.7).*

Proof. Note that system (1.7) is equivalent to

$$\begin{aligned} x(t) &= x(t_0) \exp \left\{ \int_{t_0}^t \left[a(s) - b(s)x(s) - \frac{c(s, x(s))}{x(s)} y(s) \right] ds \right\}, \\ y(t) &= y(t_0) \exp \left\{ \int_{t_0}^t \left[d(s) - e(s) \frac{y(s)}{x(s)} \right] ds \right\}. \end{aligned}$$

The assertion of the lemma follows immediately for all $t \geq t_0$. The proof is complete. \square

Theorem 2.2. *Assume that (A_1) – (A_4) hold. Then the set Γ defined by (2.3) is positively invariant with respect to system (1.7).*

Proof. Let $(x(t), y(t))^T$ be the solution of (1.7) through $(x(t_0), y(t_0))^T$ with $m_1 \leq x(t_0) \leq M_1$ and $m_2 \leq y(t_0) \leq M_2$.

From the prey's equation of (1.7) and the positivity of the solution of (1.7), it follows that

$$x'(t) \leq x(t)[a^u - b^l x(t)] = b^l x(t)[M_1^* - x(t)] \leq b^l x(t)[M_1 - x(t)].$$

A standard comparison argument shows that

$$0 < x(t_0) \leq M_1 \quad \Rightarrow \quad x(t) \leq M_1, \quad t \geq t_0.$$

Similarly, by the predator's equation of (1.7), we have

$$y'(t) \leq y(t) \left[d^u - \frac{e^l}{M_1} y(t) \right] = \frac{e^l}{M_1} y(t) [M_2^* - y(t)] \leq \frac{e^l}{M_1} y(t) [M_2 - y(t)],$$

and hence,

$$0 < y(t_0) \leq M_2 \quad \Rightarrow \quad y(t) \leq M_2, \quad t \geq t_0.$$

The first equation of (1.7) and the above results together lead to

$$x'(t) \geq x(t)[a^l - b^u x(t) - C_0 M_2] = b^u x(t)[m_1^* - x(t)] \geq b^u x(t)[m_1 - x(t)],$$

then we have

$$x(t_0) \geq m_1 \Rightarrow x(t) \geq m_1, \quad t \geq t_0.$$

By the second equation of (1.7), we have

$$y'(t) \geq y(t) \left[d^l - \frac{e^u}{m_1} y(t) \right] = \frac{e^u}{m_1} y(t) [m_2^* - y(t)] \geq \frac{e^u}{m_1} y(t) [m_2 - y(t)],$$

which implies

$$y(t_0) \geq m_2 \Rightarrow y(t) \geq m_2, \quad t \geq t_0.$$

Now we can claim that Γ is positively invariant with respect to (1.7). The proof is complete. \square

Carrying out similar arguments as above, we can prove

Theorem 2.3. *If (A_1) , (A_2) , (\hat{A}_3) and (\hat{A}_4) hold, then the set $\hat{\Gamma}$ defined by (2.4) is positively invariant with respect to system (1.7).*

Definition 2.1. The solutions of system (1.7) are said to be ultimately bounded if there exist $B > 0$ and $T > 0$ such that every solution $(x(t), y(t))^T$ of (1.7) through $(x(t_0), y(t_0))^T \in R^2$ satisfies $\|(x(t), y(t))^T\| \leq B$, for all $t \geq t_0 + T$, where B is independent of particular solution while T may depend on each solution.

Definition 2.2 [29]. System (1.7) is said to be permanent if there exists a compact region $\Gamma \subset \text{Int } R_+^2$ such that for every solution $(x(t), y(t))^T$ of (1.7) with positive initial value $(x(t_0), y(t_0))^T$, there exists a $T > 0$ such that $(x(t), y(t))^T \in \Gamma$ for all $t \geq t_0 + T$.

Theorem 2.4. *If (A_1) – (A_4) hold, then the set Γ defined by (2.3) is an ultimately bounded region (or absorbing and positively invariant set) of system (1.7).*

Proof. Let $(x(t), y(t))^T$ be the solution of (1.7) with any positive initial value $(x(t_0), y(t_0))^T$.

If $x(t) > M_1$ for all $t \geq t_0$, then $x(t) - M_1^* > M_1 - M_1^* := \delta_1 > 0$ for all $t \geq t_0$, which, together with the first equation of (1.7), implies

$$x'(t) \leq x(t) [a^u - b^l x(t)] \leq x(t) [a^u - b^l (M_1^* + \delta_1)] = -b^l \delta_1 x(t), \quad t \geq t_0.$$

Thus, $x(t) \leq x(t_0) \exp\{-b^l \delta_1 (t - t_0)\} \rightarrow 0$ as $t \rightarrow +\infty$, which contradicts the fact $x(t) > M_1$ for all $t \geq t_0$. Hence, there must exist a $T_1 > 0$ such that $x(t) \leq M_1$ for all $t \geq t_0 + T_1$.

If $y(t) > M_2$ for all $t \geq t_0$, then $y(t) - M_2^* > M_2 - M_2^* := \delta_2 > 0$ for all $t \geq t_0$. By the second equation of (1.7), we have

$$\begin{aligned} y'(t) &\leq y(t) \left[d^u - \frac{e^l}{M_1} y(t) \right] \leq y(t) \left[d^u - \frac{e^l}{M_1} (M_2^* + \delta_2) \right] \\ &= -\frac{e^l}{M_1} \delta_2 y(t), \quad t \geq t_0 + T_1. \end{aligned}$$

Therefore,

$$y(t) \leq y(t_0 + T_1) \exp\left\{-\frac{e^l}{M_1} \delta_2(t - t_0 - T_1)\right\} \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

which contradicts the fact $y(t) > M_2$ for all $t \geq t_0$. Hence, there exists a $T_2 > T_1$ such that $y(t) \leq M_2$ for all $t \geq t_0 + T_2$.

If $x(t) < m_1$ for all $t \geq t_0$, then $m_1^* - x(t) > m_1^* - m_1 := \delta_3 > 0$ for all $t \geq t_0$. From the first equation of (1.7), it follows

$$\begin{aligned} x'(t) &\geq x(t)[a^l - b^u x(t) - C_0 M_2] \geq x(t)[a^l - C_0 M_2 - b^u(m_1^* - \delta_3)] \\ &= b^u \delta_3 x(t), \quad t \geq t_0 + T_2. \end{aligned}$$

Thus,

$$x(t) \geq x(t_0 + T_2) \exp\{b^u \delta_3(t - t_0 - T_2)\} \rightarrow +\infty \quad \text{as } t \rightarrow +\infty,$$

which contradicts the fact $x(t) < m_1$ for all $t \geq t_0$. Hence, there exists a $T_3 > T_2$ such that $x(t) \geq m_1$ for all $t \geq t_0 + T_3$.

If $y(t) < m_2$ for all $t \geq t_0$, then $m_2^* - y(t) > m_2^* - m_2 := \delta_4 > 0$ for all $t \geq t_0$. By the second equation of (1.7), we have

$$y'(t) \geq y(t)\left[d^l - \frac{e^u}{m_1} y(t)\right] \geq y(t)\left[d^l - \frac{e^l}{m_1}(m_2^* - \delta_4)\right] = \frac{e^l}{m_1} \delta_4 y(t), \quad t \geq t_0 + T_3.$$

Therefore,

$$y(t) \geq y(t_0 + T_3) \exp\left\{\frac{e^u}{m_1} \delta_4(t - t_0 - T_3)\right\} \rightarrow +\infty \quad \text{as } t \rightarrow +\infty,$$

which contradicts the fact $y(t) < m_2$ for all $t \geq t_0$. Hence, there exists a $T_4 > T_3$ such that $y(t) \geq m_2$ for all $t \geq t_0 + T_4$.

Hence, the above arguments imply that $(x(t), y(t))^T \in \Gamma$ for any $t \geq t_0 + T_4$. Therefore, Γ is an ultimately bounded region of system (1.7). The proof is complete. \square

By the similar arguments, we can establish the following result:

Theorem 2.5. Assume that (A_1) , (A_2) , (\hat{A}_3) and (\hat{A}_4) hold. Then the set $\hat{\Gamma}$ defined by (2.4) is an ultimately bounded region (or absorbing and positively invariant set) of system (1.7).

The above arguments show that

Theorem 2.6. If (A_1) – (A_4) or (A_1) , (A_2) , (\hat{A}_3) and (\hat{A}_4) hold, then system (1.7) is permanent.

Remark 2.1. Practical persistence [8–10], in which seems to have been some recent interest, refers to determining specific estimates in terms of model data for the asymptotic distance to the boundary of the feasible region for uniformly persistent population interaction models. In fact, the scenarios of the approach to Theorem 2.6 is a particular case of the so-called “practical persistence” approach to permanence.

Definition 2.3. System (1.7) is said to be globally asymptotically stable if any two solutions $(x_i(t), y_i(t))^T$, $i = 1, 2$, of (1.7) with positive initial values have the property

$$\lim_{t \rightarrow +\infty} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) = 0.$$

In order to explore the globally asymptotic stability, we introduce below a lemma due to Barbălat.

Lemma 2.1 [4]. Let f be a nonnegative function defined on $[0, +\infty)$ such that f is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$. Then $\lim_{t \rightarrow +\infty} f(t) = 0$.

Lemma 2.2. Let h be a real number and f be a nonnegative function defined on $[h, +\infty)$ such that f is integrable on $[h, +\infty)$ and is uniformly continuous on $[h, +\infty)$. Then $\lim_{t \rightarrow +\infty} f(t) = 0$.

Theorem 2.7. Assume (A_1) – (A_4) hold. Moreover, if

$$(A_5) \quad b^l - \frac{C_0 + C_1}{m_1} M_2 - \frac{e^u}{m_1^2} M_2 > 0, \quad \frac{e^l}{M_1} - C_0 > 0,$$

where m_i, M_i , $i = 1, 2$, are defined in (2.1) and

$$C_1 = \sup_{t \in [0, +\infty)} \left\{ \max_{x \in [m_1, M_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\} > 0,$$

then system (1.7) is globally asymptotically stable.

Proof. Let $(x_i(t), y_i(t))^T$, $i = 1, 2$, be any two solutions of (1.7) with positive initial values $(x_i(t_0), y_i(t_0))^T$. Theorem 2.2 implies that there exists a $T_1 > 0$ such that $(x_i(t), y_i(t))^T \in \Gamma$, $i = 1, 2$, for all $t \geq t_0 + T_1$.

Consider a Lyapunov function defined by

$$V(t) = |\ln\{x_1(t)\} - \ln\{x_2(t)\}| + |\ln\{y_1(t)\} - \ln\{y_2(t)\}|, \quad t \geq t_0.$$

A direct calculation of the right derivative $D^+V(t)$ of $V(t)$ along the solutions of (1.7) leads to

$$\begin{aligned} D^+V(t) &= \left[-b(t)(x_1(t) - x_2(t)) - \left(\frac{c(t, x_1(t))}{x_1(t)} y_1(t) - \frac{c(t, x_2(t))}{x_2(t)} y_2(t) \right) \right] \\ &\quad \times \operatorname{sgn}(x_1(t) - x_2(t)) \\ &\quad + \left[-e(t) \left(\frac{y_1(t)}{x_1(t)} - \frac{y_2(t)}{x_2(t)} \right) \right] \operatorname{sgn}(y_1(t) - y_2(t)) \\ &= \left[-b(t)(x_1(t) - x_2(t)) - \left(\frac{c(t, x_1(t))}{x_1(t)} y_1(t) - \frac{c(t, x_1(t))}{x_1(t)} y_2(t) \right. \right. \\ &\quad \left. \left. + \frac{c(t, x_1(t))}{x_1(t)} y_2(t) - \frac{c(t, x_2(t))}{x_2(t)} y_2(t) \right) \right] \operatorname{sgn}(x_1(t) - x_2(t)) \\ &\quad + \left[-e(t) \left(\frac{y_1(t)}{x_1(t)} - \frac{y_2(t)}{x_1(t)} + \frac{y_2(t)}{x_1(t)} - \frac{y_2(t)}{x_2(t)} \right) \right] \operatorname{sgn}(y_1(t) - y_2(t)) \end{aligned}$$

$$\begin{aligned}
&\leq -b^l |x_1(t) - x_2(t)| + C_0 |y_1(t) - y_2(t)| \\
&\quad + M_2 \left| \frac{1}{\xi(t)} \frac{\partial c}{\partial x}(t, \xi(t)) - \frac{1}{(\xi(t))^2} c(t, \xi(t)) \right| |x_1(t) - x_2(t)| \\
&\quad - \frac{e^l}{M_1} |y_1(t) - y_2(t)| + \frac{e^u}{m_1^2} M_2 |x_1(t) - x_2(t)| \\
&\leq - \left[b^l - \frac{C_0 + C_1}{m_1} M_2 - \frac{e^u}{m_1^2} M_2 \right] |x_1(t) - x_2(t)| \\
&\quad - \left[\frac{e^l}{M_1} - C_0 \right] |y_1(t) - y_2(t)| \\
&\leq -\mu (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|), \\
&\quad t \geq \max\{t_0 + T_1, 0\} := T,
\end{aligned} \tag{2.5}$$

where $\xi(t)$ is between $x_1(t)$ and $x_2(t)$, and

$$\begin{aligned}
C_1 &= \sup_{t \in [0, +\infty)} \left\{ \max_{x \in [m_1, M_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\} > 0, \\
\mu &= \min \left\{ b^l - \frac{C_0 + C_1}{m_1} M_2 - \frac{e^u}{m_1^2} M_2, \frac{e^l}{M_1} - C_0 \right\} > 0.
\end{aligned}$$

Obviously,

$$V(T) = |\ln\{x_1(T)\} - \ln\{x_2(T)\}| + |\ln\{y_1(T)\} - \ln\{y_2(T)\}| < +\infty.$$

Integrating from T to t on both sides of (2.5) produces

$$V(t) + \mu \int_T^t (|x_1(s) - x_2(s)| + |y_1(s) - y_2(s)|) ds \leq V(T) < +\infty, \quad t \geq T.$$

Then

$$\int_T^{+\infty} (|x_1(s) - x_2(s)| + |y_1(s) - y_2(s)|) ds \leq \frac{V(T)}{\mu} < +\infty.$$

Hence, $|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| \in L^1([T, +\infty))$. By system (1.7) and Theorem 2.4, we get $x_i(t)$, $y_i(t)$, $i = 1, 2$, and their derivatives are bounded on $[T, +\infty)$, which implies that $|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|$ is uniformly continuous on $[T, +\infty)$. By Lemma 2.2, we reach

$$\lim_{t \rightarrow +\infty} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) = 0.$$

The proof is complete. \square

Theorem 2.8. Assume that (A_1) , (A_2) , (\hat{A}_3) and (\hat{A}_4) hold. If

$$(\hat{A}_5) \quad b^l - \frac{\hat{C}_0 + \hat{m}_1 \hat{C}_1}{\hat{m}_1^2} \hat{M}_2 - \frac{e^u}{\hat{m}_1^2} \hat{M}_2 > 0, \quad \frac{e^l}{\hat{M}_1} - \frac{\hat{C}_0}{\hat{m}_1} > 0,$$

where $\hat{m}_i, \hat{M}_i, i = 1, 2$, are defined in (2.2) and

$$\hat{C}_1 = \sup_{t \in [0, +\infty)} \left\{ \max_{x \in [\hat{m}_1, \hat{M}_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\},$$

then system (1.7) is globally asymptotically stable.

The proof is similar to that of Theorem 2.7, hence the details are omitted here.

3. Periodic case

In this section, we investigate the existence, uniqueness and stability of positive periodic solutions of (1.7) under the assumption that

(A₆) the parameters in system (1.7) are ω periodic with respect to t .

In addition to the assumptions in Section 2, it is clear that Theorems 2.2–2.8 remain valid for system (1.7) with the additional assumption (A₆).

Lemma 3.1 (Brouwer fixed point theorem). *Suppose that a continuous operator σ maps a closed, bounded, convex subset $\bar{\Omega} \subset R^n$ into itself. Then $\bar{\Omega}$ contains at least one fixed point of the operator σ , i.e., there exists an $x^* \in \bar{\Omega}$ such that $\sigma(x^*) = x^*$.*

Theorem 3.1. *If (A₁)–(A₄) and (A₆) hold, then system (1.7) has at least one positive ω periodic solution, say $(x^*(t), y^*(t))^T$, and $m_1 \leq x^*(t) \leq M_1$, $m_2 \leq y^*(t) \leq M_2$, where $m_i, M_i, i = 1, 2$, are defined in (2.1).*

Proof. First, we define a shift operator, which is also known as a Poincaré mapping $\sigma : R^2 \rightarrow R^2$ by

$$\sigma((x_0, y_0)^T) = (x(\omega, t_0, (x_0, y_0)^T), y(\omega, t_0, (x_0, y_0)^T))^T, \quad (x_0, y_0)^T \in R^2,$$

where $(x(t, t_0, (x_0, y_0)^T), y(t, t_0, (x_0, y_0)^T))^T$ denotes the solution of (1.7) through the point $(t_0, x_0, y_0)^T$. Theorem 2.2 tells us that the set Γ defined by (2.3) is positive invariant with respect to system (1.7), that is to say, the operator σ defined above maps Γ into itself, i.e., $\sigma(\Gamma) \subset \Gamma$. Since the solution of (1.7) is continuous with respect to the initial value, the operator σ is continuous. It is not difficult to show that Γ is a bounded, closed, convex set in R^2 . By Lemma 3.1, σ has at least one fixed point in Γ , i.e., there exists a $(x^*, y^*)^T \in \Gamma$ such that

$$(x^*, y^*)^T = (x(\omega, t_0, (x^*, y^*)^T), y(\omega, t_0, (x^*, y^*)^T))^T.$$

Hence, there exists at least one strictly positive ω periodic solution of (1.7) in Γ . The rest of the proof follows directly. \square

Similarly, we can easily prove that

Theorem 3.2. *If (A_1) , (A_2) , (\hat{A}_3) , (\hat{A}_4) and (A_6) hold, then system (1.7) has at least one positive ω periodic solution, say $(\hat{x}^*(t), \hat{y}^*(t))^T$, and $\hat{m}_1 \leq \hat{x}^*(t) \leq \hat{M}_1$, $\hat{m}_2 \leq \hat{y}^*(t) \leq \hat{M}_2$, where $\hat{m}_i, \hat{M}_i, i = 1, 2$, are defined in (2.2).*

Remark 3.1. It is fairly widely known that in an autonomous system of ODEs, permanence implies the existence of a componentwise positive equilibrium. Some authors have reported that, in a periodic setting, there are also results asserting that permanence implies the existence of a componentwise positive periodic orbit. Comparing Theorem 2.6 with Theorems 3.1 and 3.2, one can easily observe that our results fairly support the claim.

The conditions in Theorems 3.1 and 3.2 are given in terms of supremum and infimum of the parameters. Next, we will employ an alternative approach to establish some criteria for the same problem but in terms of the averages of the related parameters over an interval of the common period. That is a continuation theorem in coincidence degree theory, which have been successfully used to establish sufficient criteria for the existence of positive periodic solutions of Lotka–Volterra type multi-species competition systems and predator–prey systems with time delays; for example, one can consult [15–17,34] for details.

To this end, we shall first summarize below a few concepts and results from [19] borrowing notations and terminologies there.

Let X, Z be normed vector spaces, $L: \text{Dom } L \subset X \rightarrow Z$ be a linear mapping, $N: X \rightarrow Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim Im } L < +\infty$ and $\text{Im } L$ is closed in Z . If L is a Fredholm mapping of index zero and there exist continuous projectors $P: X \rightarrow X$ and $Q: Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$, $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$, it follows that $L|_{\text{Dom } L \cap \text{Ker } P}: (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of that map by K_P . If $\bar{\Omega}$ is an open bounded subset of X , the mapping N will be called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N: \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J: \text{Im } Q \rightarrow \text{Ker } L$.

Lemma 3.2 (Continuation theorem). *Let L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Suppose*

- (a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
- (b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \text{Ker } L$ and*

$$\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0.$$

Then the operator equation $Lx = Nx$ has at least one solution lying in $\text{Dom } L \cap \bar{\Omega}$.

Theorem 3.3. *Assume (A_1) – (A_3) and (A_6) hold. Moreover, if*

$$(A_7) \quad \frac{C_0 \bar{d}}{\bar{b} \bar{e}} \exp\{2(\bar{a} + \bar{d})\omega\} < 1,$$

then system (1.7) has at least one positive ω periodic solution, say $(x^(t), y^*(t))^T$, and there exist positive constants $\alpha_i^*, \beta_i^*, i = 1, 2$, such that $\alpha_1^* \leq x^*(t) \leq \beta_1^*, \alpha_2^* \leq y^*(t) \leq \beta_2^*$.*

Proof. Making the change of variables

$$x(t) = \exp\{\tilde{x}(t)\}, \quad y(t) = \exp\{\tilde{y}(t)\},$$

system (1.7) is reformulated as

$$\begin{aligned} \tilde{x}'(t) &= a(t) - b(t) \exp\{\tilde{x}(t)\} - c(t, \exp\{\tilde{x}(t)\}) \exp\{\tilde{y}(t) - \tilde{x}(t)\}, \\ \tilde{y}'(t) &= d(t) - e(t) \exp\{\tilde{y}(t) - \tilde{x}(t)\}. \end{aligned} \quad (3.1)$$

Let

$$\begin{aligned} X = Z &= \{(\tilde{x}, \tilde{y})^T \in C(R, R^2) \mid \tilde{x}(t + \omega) = \tilde{x}(t), \tilde{y}(t + \omega) = \tilde{y}(t)\}, \\ \|(\tilde{x}, \tilde{y})^T\| &= \max_{t \in [0, \omega]} |\tilde{x}(t)| + \max_{t \in [0, \omega]} |\tilde{y}(t)|, \quad (\tilde{x}, \tilde{y})^T \in X \text{ (or } Z). \end{aligned}$$

Then X, Z are both Banach spaces when they are endowed with the above norm $\|\cdot\|$.

Let

$$\begin{aligned} N \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} &= \begin{bmatrix} N_1(t) \\ N_2(t) \end{bmatrix} = \begin{bmatrix} a(t) - b(t) \exp\{\tilde{x}(t)\} - c(t, \exp\{\tilde{x}(t)\}) \exp\{\tilde{y}(t) - \tilde{x}(t)\} \\ d(t) - e(t) \exp\{\tilde{y}(t) - \tilde{x}(t)\} \end{bmatrix}, \\ L \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} &= \begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix}, \quad P \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = Q \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \tilde{x}(t) dt \\ \frac{1}{\omega} \int_0^\omega \tilde{y}(t) dt \end{bmatrix}, \quad \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \in X. \end{aligned}$$

Then

$$\begin{aligned} \text{Ker } L &= \{(\tilde{x}, \tilde{y})^T \in X \mid (\tilde{x}, \tilde{y})^T = (h_1, h_2)^T \in R^2\}, \\ \text{Im } L &= \left\{ (\tilde{x}, \tilde{y})^T \in Z \mid \int_0^\omega \tilde{x}(t) dt = 0, \int_0^\omega \tilde{y}(t) dt = 0 \right\}, \end{aligned}$$

and

$$\dim \text{Ker } L = 2 = \text{codim Im } L.$$

Since $\text{Im } L$ is closed in Z , L is a Fredholm mapping of index zero. It is easy to show that P, Q are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Im } L = \text{Ker } Q = \text{Im}(I - Q).$$

Furthermore, the generalized inverse (to L) $K_P : \text{Im } L \rightarrow \text{Dom } L \cap \text{Ker } P$ exists and is given by

$$K_P \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \int_0^t \tilde{x}(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t \tilde{x}(s) ds dt \\ \int_0^t \tilde{y}(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t \tilde{y}(s) ds dt \end{bmatrix}.$$

Thus

$$\begin{aligned} QN \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} &= \begin{bmatrix} \frac{1}{\omega} \int_0^\omega (a(s) - b(s) \exp\{\tilde{x}(s)\} - c(s, \exp\{\tilde{x}(s)\}) \exp\{\tilde{y}(s) - \tilde{x}(s)\}) ds \\ \frac{1}{\omega} \int_0^\omega (d(s) - e(s) \exp\{\tilde{y}(s) - \tilde{x}(s)\}) ds \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
& K_P(I - Q)N \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \\
&= \begin{bmatrix} \int_0^t N_1(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t N_1(s) ds dt - \left(\frac{t}{\omega} - \frac{1}{2}\right) \int_0^\omega N_1(t) dt \\ \int_0^t N_2(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t N_2(s) ds dt - \left(\frac{t}{\omega} - \frac{1}{2}\right) \int_0^\omega N_2(t) dt \end{bmatrix}.
\end{aligned}$$

Obviously, QN and $K_P(I - Q)N$ are continuous. Using the Arzela–Ascoli theorem, it is not difficult to show that $K_P(I - Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L -compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $L\tilde{x} = \lambda N\tilde{x}$, $\lambda \in (0, 1)$, we have

$$\begin{aligned}
\tilde{x}'(t) &= \lambda[a(t) - b(t) \exp\{\tilde{x}(t)\} - c(t, \exp\{\tilde{x}(t)\}) \exp\{\tilde{y}(t) - \tilde{x}(t)\}], \\
\tilde{y}'(t) &= \lambda[d(t) - e(t) \exp\{\tilde{y}(t) - \tilde{x}(t)\}].
\end{aligned} \tag{3.2}$$

Suppose that $(\tilde{x}, \tilde{y})^T \in X$ is a solution of system (3.2) for a certain $\lambda \in (0, 1)$. Integrating on both sides of (3.2) from 0 to ω , we obtain

$$\begin{aligned}
\bar{a}\omega &= \int_0^\omega b(t) \exp\{\tilde{x}(t)\} dt + \int_0^\omega c(t, \exp\{\tilde{x}(t)\}) \exp\{\tilde{y}(t) - \tilde{x}(t)\} dt, \\
\bar{d}\omega &= \int_0^\omega e(t) \exp\{\tilde{y}(t) - \tilde{x}(t)\} dt.
\end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}
\int_0^\omega |\tilde{x}'(t)| dt &\leq \lambda \left[\int_0^\omega a(t) dt + \int_0^\omega b(t) \exp\{\tilde{x}(t)\} dt \right. \\
&\quad \left. + \int_0^\omega c(t, \exp\{\tilde{x}(t)\}) \exp\{\tilde{y}(t) - \tilde{x}(t)\} dt \right] < 2\bar{a}\omega, \\
\int_0^\omega |\tilde{y}'(t)| dt &\leq \lambda \left[\int_0^\omega d(t) dt + \int_0^\omega e(t) \exp\{\tilde{y}(t) - \tilde{x}(t)\} dt \right] < 2\bar{d}\omega.
\end{aligned} \tag{3.4}$$

Since $(\tilde{x}, \tilde{y})^T \in X$, there exist $\xi_i, \eta_i \in [0, \omega]$, $i = 1, 2$, such that

$$\begin{aligned}
\tilde{x}(\xi_1) &= \min_{t \in [0, \omega]} \tilde{x}(t), & \tilde{x}(\eta_1) &= \max_{t \in [0, \omega]} \tilde{x}(t), \\
\tilde{y}(\xi_2) &= \min_{t \in [0, \omega]} \tilde{y}(t), & \tilde{y}(\eta_2) &= \max_{t \in [0, \omega]} \tilde{y}(t).
\end{aligned} \tag{3.5}$$

From (3.3) and (3.5), we obtain

$$\begin{aligned}\bar{a}\omega &\geq \int_0^\omega b(t) \exp\{\tilde{x}(\xi_1)\} dt = \bar{b}\omega \exp\{\tilde{x}(\xi_1)\}, \\ \bar{d}\omega &\geq \int_0^\omega e(t) \exp\{\tilde{y}(\xi_2) - \tilde{x}(\eta_1)\} dt = \bar{e}\omega \exp\{\tilde{y}(\xi_2) - \tilde{x}(\eta_1)\},\end{aligned}$$

and hence,

$$\tilde{x}(\xi_1) \leq \ln\left\{\frac{\bar{a}}{\bar{b}}\right\}, \quad \tilde{y}(\xi_2) \leq \ln\left\{\frac{\bar{d}}{\bar{e}}\right\} + \tilde{x}(\eta_1). \quad (3.6)$$

From (3.4) and (3.6), we obtain

$$\begin{aligned}\tilde{x}(t) &\leq \tilde{x}(\xi_1) + \int_0^\omega |\tilde{x}'(t)| dt < \ln\left\{\frac{\bar{a}}{\bar{b}}\right\} + 2\bar{a}\omega := H_1, \\ \tilde{y}(t) &\leq \tilde{y}(\xi_2) + \int_0^\omega |\tilde{y}'(t)| dt < \ln\left\{\frac{\bar{d}}{\bar{e}}\right\} + H_1 + 2\bar{d}\omega := H_2.\end{aligned} \quad (3.7)$$

On the other hand, by (3.3) and (3.5), we also have

$$\begin{aligned}\bar{a}\omega &\leq \int_0^\omega b(t) \exp\{\tilde{x}(\eta_1)\} dt + \int_0^\omega C_0 \exp\{\tilde{y}(\eta_2)\} dt \\ &= \bar{b}\omega \exp\{\tilde{x}(\eta_1)\} + C_0\omega \exp\{\tilde{y}(\eta_2)\}, \\ \bar{d}\omega &\leq \int_0^\omega e(t) \exp\{\tilde{y}(\eta_2) - \tilde{x}(\xi_1)\} dt = \bar{e}\omega \exp\{\tilde{y}(\eta_2) - \tilde{x}(\xi_1)\},\end{aligned}$$

and hence,

$$\begin{aligned}\tilde{x}(\eta_1) &\geq \ln\left\{\frac{1}{\bar{b}}[\bar{a} - C_0 \exp\{\tilde{y}(\eta_2)\}]\right\} > \ln\left\{\frac{1}{\bar{b}}[\bar{a} - C_0 \exp\{H_2\}]\right\} \\ &= \ln\left\{\frac{\bar{a}}{\bar{b}}\left[1 - \frac{C_0\bar{d}}{\bar{b}\bar{e}} \exp\{2(\bar{a} + \bar{d})\omega\}\right]\right\} := H_3, \\ \tilde{y}(\eta_2) &\geq \ln\left\{\frac{\bar{d}}{\bar{e}}\right\} + \tilde{x}(\xi_1).\end{aligned} \quad (3.8)$$

From (3.4) and (3.8), we have

$$\begin{aligned}\tilde{x}(t) &\geq \tilde{x}(\eta_1) - \int_0^\omega |\tilde{x}'(t)| dt > H_3 - 2\bar{a}\omega, \\ \tilde{y}(t) &\geq \tilde{y}(\eta_2) - \int_0^\omega |\tilde{y}'(t)| dt > \ln\frac{\bar{d}}{\bar{e}} + H_3 - 2(\bar{a} + \bar{d})\omega := H_4,\end{aligned} \quad (3.9)$$

which, together with (3.7), implies

$$\begin{aligned}\max_{t \in [0, \omega]} |\tilde{x}(t)| &< \max\{|H_1|, |H_3 - 2\bar{a}\omega|\} := H_5, \\ \max_{t \in [0, \omega]} |\tilde{y}(t)| &< \max\{|H_2|, |H_4|\} := H_6.\end{aligned}$$

Clearly, H_5 and H_6 are independent of λ .

By assumption (A_2) , it is easy to show that

$$QN \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \bar{a} - \bar{b} \exp\{\tilde{x}\} - \bar{c}(\exp\{\tilde{x}\}) \exp\{\tilde{y} - \tilde{x}\} \\ \bar{d} - \bar{e} \exp\{\tilde{y} - \tilde{x}\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.10)$$

has a unique solution $(\tilde{x}^*, \tilde{y}^*)^T$ in $\text{Int } R^2$. Set $H = H_5 + H_6 + C$, which is taken sufficiently large such that the unique solution of (3.10) satisfies $\|(\tilde{x}^*, \tilde{y}^*)^T\| = |\tilde{x}^*| + |\tilde{y}^*| < H$.

Let $\Omega = \{(\tilde{x}, \tilde{y})^T \in X \mid \|(\tilde{x}, \tilde{y})^T\| < H\}$, then it is clear that Ω verifies the requirement (a) of Lemma 3.2. When $(\tilde{x}, \tilde{y})^T \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap R^2$, $(\tilde{x}, \tilde{y})^T$ is a constant vector in R^2 with $\|(\tilde{x}, \tilde{y})^T\| = |\tilde{x}| + |\tilde{y}| = H$. Then

$$QN \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \bar{a} - \bar{b} \exp\{\tilde{x}\} - \bar{c}(\exp\{\tilde{x}\}) \exp\{\tilde{y} - \tilde{x}\} \\ \bar{d} - \bar{e} \exp\{\tilde{y} - \tilde{x}\} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In view of Theorem 3.3, direct calculation produces

$$\begin{aligned}\deg(JQN, \Omega \cap \text{Ker } L, 0) &= \text{sgn} \begin{vmatrix} -\bar{b} \exp\{\tilde{x}^*\} - \frac{\partial \bar{c}}{\partial x}(\exp\{\tilde{x}^*\}) \exp\{\tilde{y}^*\} & -\bar{c}(\exp\{\tilde{x}^*\}) \exp\{\tilde{y}^* - \tilde{x}^*\} \\ \bar{e} \exp\{\tilde{y}^* - \tilde{x}^*\} & -\bar{e} \exp\{\tilde{y}^* - \tilde{x}^*\} \end{vmatrix} \\ &= \text{sgn} \left\{ \bar{d} \left[\bar{b} \exp\{\tilde{x}^*\} + \frac{\partial \bar{c}}{\partial x}(\exp\{\tilde{x}^*\}) \exp\{\tilde{y}^*\} - \bar{c}(\exp\{\tilde{x}^*\}) \exp\{\tilde{y}^* - \tilde{x}^*\} \right. \right. \\ &\quad \left. \left. + \bar{c}(\exp\{\tilde{x}^*\}) \exp\{\tilde{y}^* - \tilde{x}^*\} \right] \right\} \\ &= \text{sgn} \left\{ \bar{d} \left[\bar{b} \exp\{\tilde{x}^*\} + \frac{\partial \bar{c}}{\partial x}(\exp\{\tilde{x}^*\}) \exp\{\tilde{y}^*\} \right] \right\} > 0, \end{aligned} \quad (3.11)$$

where the degree is Brouwer degree, and the isomorphism J of $\text{Im } Q$ onto $\text{Ker } L$ can be chosen to be the identity mapping, since $\text{Im } Q = \text{Ker } L$. By now we have proved that Ω verifies all requirements of Lemma 3.2, then

$$L \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = N \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

has at least one solution in $\text{Dom } L \cap \bar{\Omega}$, i.e., (3.1) has at least one ω periodic solution in $\text{Dom } L \cap \bar{\Omega}$, say $(\tilde{x}^*(t), \tilde{y}^*(t))^T$. Set $x^*(t) = \exp\{\tilde{x}^*(t)\}$, $y^*(t) = \exp\{\tilde{y}^*(t)\}$, then $(x^*(t), y^*(t))^T$ is one positive ω periodic solution of system (1.7). The existence of positive constants $\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*$ are obvious. The proof is complete. \square

Carrying out similar arguments, we have

Theorem 3.4. Assume (A_1) , (A_2) , (\hat{A}_3) and (A_6) hold. Moreover, if

$$(\hat{A}_7) \quad \bar{a} - \frac{\bar{d}}{e^l} \hat{C}_0 > 0,$$

then system (1.7) has at least one positive ω periodic solution, say $(\hat{x}^*(t), \hat{y}^*(t))^T$, and there exist positive constants $\hat{\alpha}_i^*, \hat{\beta}_i^*, i = 1, 2$, such that $\hat{\alpha}_1^* \leq \hat{x}^*(t) \leq \hat{\beta}_1^*, \hat{\alpha}_2^* \leq \hat{y}^*(t) \leq \hat{\beta}_2^*$.

Definition 3.1. Let $(x^*(t), y^*(t))^T, i = 1, 2$, be a positive ω periodic solution of system (1.7) with positive initial value. We say that $(x^*(t), y^*(t))^T$ is globally asymptotically stable if any other solution $(x(t), y(t))^T$ of (1.7) has the property

$$\lim_{t \rightarrow +\infty} (|x(t) - x^*(t)| + |y(t) - y^*(t)|) = 0.$$

It is immediate that if $(x^*(t), y^*(t))^T$ is globally asymptotically stable, then $(x^*(t), y^*(t))^T$ is in fact unique.

From Theorems 2.7 and 3.1, it follows that

Theorem 3.5. If (A_1) – (A_5) and (A_6) hold, then system (1.7) has a unique positive ω periodic solution in Γ which is globally asymptotically stable.

Theorem 3.6. Assume (A_1) – (A_4) , (A_6) and (A_7) hold. Moreover, if

$$b^l - \frac{C_0 + C_2}{m_1} \beta_2 - \frac{e^u}{m_1 \alpha_1} \beta_2 > 0, \quad \frac{e^l}{M_1} - C_0 > 0,$$

or

$$b^l - \frac{C_0 + C_2}{m_1} \beta_2 - \frac{e^u}{m_1 \alpha_1} M_2 > 0, \quad \frac{e^l}{\beta_1} - C_0 > 0,$$

where $\alpha_i = \max\{\alpha_i^*, m_i\}$, $\beta_i = \min\{\beta_i^*, M_i\}$, $m_i, M_i, i = 1, 2$, are defined in (2.1), $\alpha_i^*, \beta_i^*, i = 1, 2$, are defined in Theorem 3.3 and

$$C_2 = \max_{t \in [0, \omega]} \left\{ \max_{x \in [m_1, \beta_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\}, \max_{x \in [\alpha_1, M_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\} > 0,$$

then system (1.7) has a unique positive ω periodic solution, say $(x^*(t), y^*(t))^T$, which is globally asymptotically stable and $\alpha_1 \leq x^*(t) \leq \beta_1, \alpha_2 \leq y^*(t) \leq \beta_2$.

Proof. Theorem 3.3 implies that system (1.7) has at least one positive ω periodic solution, say $(x^*(t), y^*(t))^T$, and there exist positive constants $\alpha_i^*, \beta_i^*, i = 1, 2$, such that $\alpha_1^* \leq x^*(t) \leq \beta_1^*, \alpha_2^* \leq y^*(t) \leq \beta_2^*$. In addition, since Γ is an ultimately bounded region of (1.7) and $(x^*(t), y^*(t))^T$ is a periodic solution, it follows that $\alpha_1 \leq x^*(t) \leq \beta_1, \alpha_2 \leq y^*(t) \leq \beta_2$. To complete the proof, we only need to show that $(x^*(t), y^*(t))^T$ is globally asymptotically stable.

Let $(x(t), y(t))^T$ be any other solution of (1.7) with initial value $(x(t_0), y(t_0))^T$. By Theorem 2.2, we have that there exists a $T_1 > 0$ such that $m_1 \leq x(t) \leq M_1, m_2 \leq y(t) \leq M_2$, for all $t \geq t_0 + T_1$, where $m_i, M_i, i = 1, 2$, are defined in (2.1). We denote $T := \max\{t_0 + T_1, 0\}$.

Consider a Lyapunov function defined by

$$V(t) = |\ln\{x(t)\} - \ln\{x^*(t)\}| + |\ln\{y(t)\} - \ln\{y^*(t)\}|.$$

Obviously,

$$V(T) = |\ln\{x(T)\} - \ln\{x^*(T)\}| + |\ln\{y(T)\} - \ln\{y^*(T)\}| < +\infty.$$

A direct calculation of the right derivative $D^+V(t)$ of $V(t)$ along the solutions of (1.7) leads to

$$\begin{aligned} D^+V(t) = & \left[-b(t)(x(t) - x^*(t)) - \left(\frac{c(t, x(t))}{x(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t) \right) \right] \\ & \times \operatorname{sgn}(x(t) - x^*(t)) \\ & + \left[-e(t) \left(\frac{y(t)}{x(t)} - \frac{y^*(t)}{x^*(t)} \right) \right] \operatorname{sgn}(y(t) - y^*(t)). \end{aligned} \quad (3.12)$$

Just because the different intersections will lead to different estimations, we will discuss $D^+V(t)$ in the following four cases.

Case 1:

$$\begin{aligned} D^+V(t) = & \left[-b(t)(x(t) - x^*(t)) - \left(\frac{c(t, x(t))}{x(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t) \right) \right. \\ & \left. + \frac{c(t, x(t))}{x(t)}y^*(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t) \right] \operatorname{sgn}(x(t) - x^*(t)) \\ & + \left[-e(t) \left(\frac{y(t)}{x(t)} - \frac{y^*(t)}{x(t)} + \frac{y^*(t)}{x(t)} - \frac{y^*(t)}{x^*(t)} \right) \right] \operatorname{sgn}(y(t) - y^*(t)) \\ \leq & -b^l|x(t) - x^*(t)| + C_0|y(t) - y^*(t)| \\ & + \beta_2 \left| \frac{1}{\xi(t)} \frac{\partial c}{\partial x}(t, \xi(t)) - \frac{1}{(\xi(t))^2} c(t, \xi(t)) \right| |x(t) - x^*(t)| \\ & - \frac{e^l}{M_1}|y(t) - y^*(t)| + \frac{e^u}{m_1\alpha_1}\beta_2|x(t) - x^*(t)| \\ \leq & -b^l|x(t) - x^*(t)| + C_0|y(t) - y^*(t)| + \frac{C_0 + C_2}{m_1}\beta_2|x(t) - x^*(t)| \\ & - \frac{e^l}{M_1}|y(t) - y^*(t)| + \frac{e^u}{m_1\alpha_1}\beta_2|x(t) - x^*(t)| \\ \leq & \left[b^l - \frac{C_0 + C_2}{m_1}\beta_2 - \frac{e^u}{m_1\alpha_1}\beta_2 \right] |x(t) - x^*(t)| \\ & - \left[\frac{e^l}{M_1} - C_0 \right] |y(t) - y^*(t)|, \quad t \geq T, \end{aligned} \quad (3.13)$$

where $\xi(t)$ is between $x(t)$ and $x^*(t)$, and

$$C_2 = \max_{t \in [0, \omega]} \left\{ \max_{x \in [m_1, \beta_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\}, \max_{x \in [\alpha_1, M_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\} > 0.$$

Case 2:

$$\begin{aligned}
 D^+V(t) &= \left[-b(t)(x(t) - x^*(t)) - \left(\frac{c(t, x(t))}{x(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t) \right. \right. \\
 &\quad \left. \left. + \frac{c(t, x(t))}{x(t)}y^*(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t) \right) \right] \operatorname{sgn}(x(t) - x^*(t)) \\
 &\quad + \left[-e(t) \left(\frac{y(t)}{x(t)} - \frac{y(t)}{x^*(t)} + \frac{y(t)}{x^*(t)} - \frac{y^*(t)}{x^*(t)} \right) \right] \operatorname{sgn}(y(t) - y^*(t)) \\
 &\leq -b^l |x(t) - x^*(t)| + C_0 |y(t) - y^*(t)| \\
 &\quad + \beta_2 \left| \frac{1}{\xi(t)} \frac{\partial c}{\partial x}(t, \xi(t)) - \frac{1}{(\xi(t))^2} c(t, \xi(t)) \right| |x(t) - x^*(t)| \\
 &\quad - \frac{e^l}{\beta_1} |y(t) - y^*(t)| + \frac{e^u}{m_1 \alpha_1} M_2 |x(t) - x^*(t)| \\
 &\leq -b^l |x(t) - x^*(t)| + C_0 |y(t) - y^*(t)| + \frac{C_0 + C_2}{m_1} \beta_2 |x(t) - x^*(t)| \\
 &\quad - \frac{e^l}{\beta_1} |y(t) - y^*(t)| + \frac{e^u}{m_1 \alpha_1} M_2 |x(t) - x^*(t)| \\
 &\leq - \left[b^l - \frac{C_0 + C_2}{m_1} \beta_2 - \frac{e^u}{m_1 \alpha_1} M_2 \right] |x(t) - x^*(t)| \\
 &\quad - \left[\frac{e^l}{\beta_1} - C_0 \right] |y(t) - y^*(t)|, \quad t \geq T. \tag{3.14}
 \end{aligned}$$

Case 3:

$$\begin{aligned}
 D^+V(t) &= \left[-b(t)(u(t) - u^*(t)) - \left(\frac{c(t, x(t))}{x(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y(t) \right. \right. \\
 &\quad \left. \left. + \frac{c(t, x^*(t))}{x^*(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t) \right) \right] \operatorname{sgn}(x(t) - x^*(t)) \\
 &\quad + \left[-e(t) \left(\frac{y(t)}{x(t)} - \frac{y^*(t)}{x(t)} + \frac{y^*(t)}{x(t)} - \frac{y^*(t)}{x^*(t)} \right) \right] \operatorname{sgn}(y(t) - y^*(t)) \\
 &\leq -b^l |x(t) - x^*(t)| + C_0 |y(t) - y^*(t)| \\
 &\quad + M_2 \left| \frac{1}{\xi(t)} \frac{\partial c}{\partial x}(t, \xi(t)) - \frac{1}{(\xi(t))^2} c(t, \xi(t)) \right| |x(t) - x^*(t)| \\
 &\quad - \frac{e^l}{M_1} |y(t) - y^*(t)| + \frac{e^u}{m_1 \alpha_1} \beta_2 |x(t) - x^*(t)| \\
 &\leq -b^l |x(t) - x^*(t)| + C_0 |y(t) - y^*(t)| + \frac{C_0 + C_2}{m_1} M_2 |x(t) - x^*(t)| \\
 &\quad - \frac{e^l}{M_1} |y(t) - y^*(t)| + \frac{e^u}{m_1 \alpha_1} \beta_2 |x(t) - x^*(t)|
 \end{aligned}$$

$$\begin{aligned} &\leq -\left[b^l - \frac{C_0 + C_2}{m_1}M_2 - \frac{e^u}{m_1\alpha_1}\beta_2\right]|x(t) - x^*(t)| \\ &\quad - \left[\frac{e^l}{M_1} - C_0\right]|y(t) - y^*(t)|, \quad t \geq T. \end{aligned} \quad (3.15)$$

Case 4:

$$\begin{aligned} D^+V(t) &= \left[-b(t)(x(t) - x^*(t)) - \left(\frac{c(t, x(t))}{x(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y(t) \right.\right. \\ &\quad \left.+\frac{c(t, x^*(t))}{x^*(t)}y(t) - \frac{c(t, x^*(t))}{x^*(t)}y^*(t)\right)\operatorname{sgn}(x(t) - x^*(t)) \\ &\quad + \left[-e(t)\left(\frac{y(t)}{x(t)} - \frac{y(t)}{x^*(t)} + \frac{y(t)}{x^*(t)} - \frac{y^*(t)}{x^*(t)}\right)\right]\operatorname{sgn}(y(t) - y^*(t)) \\ &\leq -b^l|x(t) - x^*(t)| + C_0|y(t) - y^*(t)| \\ &\quad + M_2\left|\frac{1}{\xi(t)}\frac{\partial c}{\partial x}(t, \xi(t)) - \frac{1}{(\xi(t))^2}c(t, \xi(t))\right||x(t) - x^*(t)| \\ &\quad - \frac{e^l}{\beta_1}|y(t) - y^*(t)| + \frac{e^u}{m_1\alpha_1}M_2|x(t) - x^*(t)| \\ &\leq -b^l|x(t) - x^*(t)| + C_0|y(t) - y^*(t)| + \frac{C_0 + C_2}{m_1}M_2|x(t) - x^*(t)| \\ &\quad - \frac{e^l}{\beta_1}|y(t) - y^*(t)| + \frac{e^u}{m_1\alpha_1}M_2|x(t) - x^*(t)| \\ &\leq -\left[b^l - \frac{C_0 + C_2}{m_1}M_2 - \frac{e^u}{m_1\alpha_1}M_2\right]|x(t) - x^*(t)| \\ &\quad - \left[\frac{e^l}{\beta_1} - C_0\right]|y(t) - y^*(t)|, \quad t \geq T. \end{aligned} \quad (3.16)$$

It is easy to know that cases 1 and 2 give weaker conditions. And by the assumption (A_8) , we have

$$\begin{aligned} D^+V(t) &\leq -\mu_1(|x(t) - x^*(t)| + |y(t) - y^*(t)|), \quad t \geq T, \quad \text{or} \\ D^+V(t) &\leq -\mu_2(|x(t) - x^*(t)| + |y(t) - y^*(t)|), \quad t \geq T, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} \mu_1 &= \min\left\{b^l - \frac{C_0 + C_0}{m_1}\beta_2 - \frac{e^u}{m_1\alpha_1}\beta_2, \frac{e^l}{M_1} - C_0\right\} > 0, \\ \mu_2 &= \min\left\{b^l - \frac{C_0 + C_2}{m_1}\beta_2 - \frac{e^u}{m_1\alpha_1}M_2, \frac{e^l}{\beta_1} - C_0\right\} > 0. \end{aligned}$$

Integrating on both sides of (3.17) from T to t produces

$$V(t) + \mu_i \int_T^t (|x(s) - x^*(s)| + |y(s) - y^*(s)|) ds \leq V(T) < +\infty, \quad t \geq T.$$

Then

$$\int_T^t (|x(s) - x^*(s)| + |y(s) - y^*(s)|) ds \leq \frac{V(T)}{\mu_i} < +\infty, \quad t \geq T,$$

and hence, $|x(t) - x^*(t)| + |y(t) - y^*(t)| \in L^1([T, +\infty))$. By (3.12) and (3.17), we obtain

$$\begin{aligned} |\ln\{x(t)\} - \ln\{x^*(t)\}| &\leq V(T) < +\infty, \quad t \geq T, \\ |\ln\{y(t)\} - \ln\{y^*(t)\}| &\leq V(T) < +\infty, \quad t \geq T. \end{aligned}$$

Therefore,

$$\begin{aligned} \min_{t \in [0, \omega]} \{x^*(t)\} \exp\{-V(T)\} &\leq x(t) \leq \max_{t \in [0, \omega]} \{x^*(t)\} \exp\{V(T)\} < +\infty, \quad t \geq T, \\ \min_{t \in [0, \omega]} \{y^*(t)\} \exp\{-V(T)\} &\leq y(t) \leq \max_{t \in [0, \omega]} \{y^*(t)\} \exp\{V(T)\} < +\infty, \quad t \geq T. \end{aligned}$$

The boundedness of $x^*(t)$, $y^*(t)$ implies that $x(t)$, $y(t)$ are bounded above and below by positive constants for all $t \geq T$. Since $x(t)$, $y(t)$, $x^*(t)$, $y^*(t)$ are bounded with bounded derivatives (from the equations satisfied by them), it will follow that $|x(t) - x^*(t)| + |y(t) - y^*(t)|$ is uniformly continuous on $[T, +\infty)$. By Lemma 2.2, we get

$$\lim_{t \rightarrow +\infty} (|x(t) - x^*(t)| + |y(t) - y^*(t)|) = 0.$$

Now the proof is complete. \square

Combining Theorem 2.8 with Theorem 3.2, we conclude:

Theorem 3.7. Assume that (A_1) , (A_2) , (\hat{A}_3) , (\hat{A}_4) , (\hat{A}_5) and (A_6) hold. Then system (1.7) has a unique positive ω periodic solution in $\hat{\Gamma}$, which is globally asymptotically stable.

Similarly, we can prove the following theorem.

Theorem 3.8. Assume that (A_1) , (A_2) , (\hat{A}_3) , (\hat{A}_4) , (A_6) and (\hat{A}_7) hold. Moreover, if one of the following conditions holds

$$\begin{aligned} b^l - \frac{\hat{C}_0 + \hat{m}_1 \hat{C}_2}{\hat{m}_1^2} \hat{\beta}_2 - \frac{e^u}{\hat{m}_1 \hat{\alpha}_1} \hat{\beta}_2 &> 0, & \frac{e^l}{\hat{M}_1} - \frac{\hat{C}_0}{\hat{m}_1} &> 0, \\ b^l - \frac{\hat{C}_0 + \hat{m}_1 \hat{C}_2}{\hat{m}_1^2} \hat{\beta}_2 - \frac{e^u}{\hat{m}_1 \hat{\alpha}_1} \hat{M}_2 &> 0, & \frac{e^l}{\hat{\beta}_1} - \frac{\hat{C}_0}{\hat{m}_1} &> 0, \\ b^l - \frac{\hat{C}_0 + \hat{m}_1 \hat{C}_2}{\hat{m}_1^2} \hat{M}_2 - \frac{e^u}{\hat{m}_1 \hat{\alpha}_1} \hat{M}_2 &> 0, & \frac{e^l}{\hat{\beta}_1} - \frac{\hat{C}_0}{\hat{\alpha}_1} &> 0, \\ b^l - \frac{\hat{C}_0 + \hat{m}_1 \hat{C}_2}{\hat{m}_1^2} \hat{M}_2 - \frac{e^u}{\hat{m}_1 \hat{\alpha}_1} \hat{\beta}_2 &> 0, & \frac{e^l}{\hat{M}_1} - \frac{\hat{C}_0}{\hat{\alpha}_1} &> 0, \end{aligned}$$

where $\hat{\alpha}_i = \max\{\hat{\alpha}_i^*, \hat{m}_i\}$, $\hat{\beta}_i = \min\{\hat{\beta}_i^*, \hat{M}_i\}$, $i = 1, 2$, \hat{m}_i, \hat{M}_i , $i = 1, 2$, are defined in (2.2), $\hat{\alpha}_i^*, \hat{\beta}_i^*$, $i = 1, 2$, are defined in Theorem 3.4 and

$$\hat{C}_2 = \max_{t \in [0, \omega]} \left\{ \max_{x \in [\hat{m}_1, \hat{\beta}_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\}, \max_{x \in [\hat{\alpha}_1, \hat{M}_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\} > 0,$$

then system (1.7) has a unique positive ω periodic solution, say $(\hat{x}^*(t), \hat{y}^*(t))^T$, which is globally asymptotically stable and $\hat{\alpha}_1 \leq \hat{x}^*(t) \leq \hat{\beta}_1$, $\hat{\alpha}_2 \leq \hat{y}^*(t) \leq \hat{\beta}_2$.

4. Almost periodic case

In this section, we devote ourselves to the existence, uniqueness and stability of positive almost periodic solution of (1.7) under the assumption that

(A₈) $a(t), b(t), d(t), e(t)$ are almost periodic functions, $c(t, x)$ is almost periodic in t uniformly with respect to $x \in [0, +\infty)$.

In addition to the assumptions in Section 2, it is clear that Theorems 2.2–2.8 remain valid for system (1.7) with assumption (A₈).

Let

$$x(t) = \exp\{\tilde{x}(t)\}, \quad y(t) = \exp\{\tilde{y}(t)\}.$$

Then system (1.7) becomes

$$\begin{aligned} \tilde{x}'(t) &= a(t) - b(t) \exp\{\tilde{x}(t)\} - c(t, \exp\{\tilde{x}(t)\}) \exp\{\tilde{y}(t) - \tilde{x}(t)\}, \\ \tilde{y}'(t) &= d(t) - e(t) \exp\{\tilde{y}(t) - \tilde{x}(t)\}. \end{aligned} \quad (4.1)$$

By Theorems 2.2–2.5, it is not difficult to show that

Theorem 4.1. *If (A₁)–(A₄) hold, then the set $\Gamma^* := \{(x, y)^T \in \mathbb{R}^2 \mid \ln\{m_1\} \leq x \leq \ln\{M_1\}, \ln\{m_2\} \leq y \leq \ln\{M_2\}\}$ is the positively invariant and ultimately bounded region of system (4.1), where m_i, M_i , $i = 1, 2$, are defined in (2.1).*

Theorem 4.2. *If (A₁), (A₂), (\hat{A}_3), and (\hat{A}_4) hold, then the set $\hat{\Gamma}^* := \{(x, y)^T \in \mathbb{R}^2 \mid \ln\{\hat{m}_1\} \leq x \leq \ln\{\hat{M}_1\}, \ln\{\hat{m}_2\} \leq y \leq \ln\{\hat{M}_2\}\}$ is the positively invariant and ultimately bounded region of system (4.1), where \hat{m}_i, \hat{M}_i , $i = 1, 2$, are defined in (2.2).*

In order to prove the main result of this section, we shall first make some preparation.

Consider

$$x' = f(t, x), \quad f(t, x) \in C(R \times D, \mathbb{R}^n), \quad (4.2)$$

where D is an open set in \mathbb{R}^n , $f(t, x)$ is almost periodic in t uniformly with respect to $x \in D$.

To discuss the existence of an almost periodic solution of (4.2), we investigate the product system of (4.2)

$$x' = f(t, x), \quad y' = f(t, y). \quad (4.3)$$

Lemma 4.1 [49]. Suppose that there exists a Lyapunov function $V(t, x, y)$ defined on $[0, +\infty) \times D \times D$ which satisfies the following conditions:

- (i) $a(\|x - y\|) \leq V(t, x, y) \leq b(\|x - y\|)$, where $a(\gamma), b(\gamma)$ are continuous, increasing and positive definite;
- (ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq K\{\|x_1 - x_2\| + \|y_1 - y_2\|\}$, where $K > 0$ is a constant;
- (iii) $V'_{(2)}(t, x, y) \leq -cV(t, x, y)$, where $c > 0$ is a constant.

Moreover, suppose that system (4.2) has a solution in a compact set S for all $t \geq t_0 \geq 0$, $S \subset D$. Then system (4.2) has a unique almost periodic solution in S , say $p(t)$, which is uniformly asymptotically stable in D . Furthermore, $\text{mod}(p) \subset \text{mod}(f)$.

Theorem 4.3. If $(A_1) - (A_5)$ and (A_8) hold, then system (1.7) has a unique positive almost periodic solution which is uniformly asymptotically stable in Γ and is globally asymptotically stable.

Proof. For $(x, y)^T \in \text{Int } R_+^2$, we define $\|(x, y)^T\| = x + y$. In order to prove that system (1.7) has a unique positive almost periodic solution, which is uniformly asymptotically stable in Γ , it is equivalent to show that system (4.1) has a unique almost periodic solution to be uniformly asymptotically stable in Γ^* .

Consider the product system of (4.1)

$$\begin{aligned}\tilde{x}'_1(t) &= a(t) - b(t) \exp\{\tilde{x}_1(t)\} - c(t, \exp\{\tilde{x}_1(t)\}) \exp\{\tilde{y}_1(t) - \tilde{x}_1(t)\}, \\ \tilde{y}'_1(t) &= d(t) - e(t) \exp\{\tilde{y}_1(t) - \tilde{x}_1(t)\}, \\ \tilde{x}'_2(t) &= a(t) - b(t) \exp\{\tilde{x}_2(t)\} - c(t, \exp\{\tilde{x}_2(t)\}) \exp\{\tilde{y}_2(t) - \tilde{x}_2(t)\}, \\ \tilde{y}'_2(t) &= d(t) - e(t) \exp\{\tilde{y}_2(t) - \tilde{x}_2(t)\}.\end{aligned}\quad (4.4)$$

Now we define a Lyapunov function on $[0, +\infty) \times \Gamma^* \times \Gamma^*$ as

$$V(t, \tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2) = |\tilde{x}_1(t) - \tilde{x}_2(t)| + |\tilde{y}_1(t) - \tilde{y}_2(t)|.$$

Set

$$\begin{aligned}a(\|(\tilde{x}_1, \tilde{y}_1)^T - (\tilde{x}_2, \tilde{y}_2)^T\|) &= b(\|(\tilde{x}_1, \tilde{y}_1)^T - (\tilde{x}_2, \tilde{y}_2)^T\|) \\ &= \|(\tilde{x}_1, \tilde{y}_1)^T - (\tilde{x}_2, \tilde{y}_2)^T\| = |\tilde{x}_1 - \tilde{x}_2| + |\tilde{y}_1 - \tilde{y}_2|.\end{aligned}$$

It is clear that the condition (i) of Lemma 4.1 is satisfied. Moreover,

$$\begin{aligned}&|V(t, \tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2) - V(t, \tilde{x}_3, \tilde{y}_3, \tilde{x}_4, \tilde{y}_4)| \\ &= \left| (|\tilde{x}_1(t) - \tilde{x}_2(t)| + |\tilde{y}_1(t) - \tilde{y}_2(t)|) - (|\tilde{x}_3(t) - \tilde{x}_4(t)| + |\tilde{y}_3(t) - \tilde{y}_4(t)|) \right| \\ &\leq |\tilde{x}_1(t) - \tilde{x}_3(t)| + |\tilde{y}_1(t) - \tilde{y}_3(t)| + |\tilde{x}_2(t) - \tilde{x}_4(t)| + |\tilde{y}_2(t) - \tilde{y}_4(t)|,\end{aligned}\quad (4.5)$$

which shows that the condition (ii) of Lemma 4.1 is satisfied.

Let $(\tilde{x}_i(t), \tilde{y}_i(t))^T$, $i = 1, 2$, be any two solutions of (4.1) defined on $[0, +\infty) \times \Gamma^* \times \Gamma^*$.

Calculating the right derivative $D^+V(t)$ of $V(t)$ along the solutions of (4.1), we have

$$\begin{aligned}
D^+V(t) &= \left[-b(t)(\exp\{\tilde{x}_1(t)\} - \exp\{\tilde{x}_2(t)\}) - (c(t, \exp\{\tilde{x}_1(t)\}) \exp\{\tilde{y}_1(t) - \tilde{x}_1(t)\} \right. \\
&\quad \left. - c(t, \exp\{\tilde{x}_2(t)\}) \exp\{\tilde{y}_2(t) - \tilde{x}_2(t)\}) \right] \operatorname{sgn}(\tilde{x}_1(t) - \tilde{x}_2(t)) \\
&\quad + \left[-e(t)(\exp\{\tilde{y}_1(t) - \tilde{x}_1(t)\} - \exp\{\tilde{y}_2(t) - \tilde{x}_2(t)\}) \right] \operatorname{sgn}(\tilde{y}_1(t) - \tilde{y}_2(t)) \\
&\leq - \left[b^l - \frac{C_0 + C_1}{m_1} M_2 - \frac{e^u}{m_1^2} M_2 \right] |\exp\{\tilde{x}_1(t)\} - \exp\{\tilde{x}_2(t)\}| \\
&\quad - \left[\frac{e^l}{M_1} - C_0 \right] |\exp\{\tilde{y}_1(t)\} - \exp\{\tilde{y}_2(t)\}|, \tag{4.6}
\end{aligned}$$

where

$$C_1 = \sup_{t \in [0, +\infty)} \left\{ \max_{x \in [m_1, M_1]} \left\{ \frac{\partial c}{\partial x}(t, x) \right\} \right\} > 0.$$

By Theorem 4.1 and

$$\begin{aligned}
|\exp\{\tilde{x}_1(t)\} - \exp\{\tilde{x}_2(t)\}| &\leq \exp\{\xi(t)\} |\tilde{x}_1(t) - \tilde{x}_2(t)|, \\
|\exp\{\tilde{y}_1(t)\} - \exp\{\tilde{y}_2(t)\}| &\leq \exp\{\eta(t)\} |\tilde{y}_1(t) - \tilde{y}_2(t)|, \tag{4.7}
\end{aligned}$$

where $\xi(t)$ is between $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, $\eta(t)$ is between $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$, we have

$$\begin{aligned}
D^+V(t) &\leq - \left[b^l - \frac{C_0 + C_1}{m_1} M_2 - \frac{e^u}{m_1^2} M_2 \right] m_1 |\tilde{x}_1(t) - \tilde{x}_2(t)| \\
&\quad - \left[\frac{e^l}{M_1} - C_0 \right] m_2 |\tilde{y}_1(t) - \tilde{y}_2(t)| \\
&= -\mu (|\tilde{x}_1(t) - \tilde{x}_2(t)| + |\tilde{y}_1(t) - \tilde{y}_2(t)|), \tag{4.8}
\end{aligned}$$

where

$$\mu = \min \left\{ \left[b^l - \frac{C_0 + C_1}{m_1} M_2 - \frac{e^u}{m_1^2} M_2 \right] m_1, \left[\frac{e^l}{M_1} - C_0 \right] m_2 \right\} > 0.$$

Hence, the condition (iii) of Lemma 4.1 is satisfied.

Therefore, from Theorem 4.1 and Lemma 4.1, it follows that system (4.1) has a unique almost periodic solution in Γ^* , say $(\tilde{x}^*(t), \tilde{y}^*(t))^T$, which is uniformly asymptotically stable in Γ^* . Hence, system (1.7) has a unique positive almost periodic solution $(x^*(t), y^*(t))^T$ in Γ , which is uniformly asymptotically stable in Γ . By Theorem 2.7, one can easily show that $(x^*(t), y^*(t))^T$ is globally asymptotically stable. The proof is complete. \square

By similar arguments, we also have

Theorem 4.4. *If (A_1) , (A_2) , (\hat{A}_3) , (\hat{A}_4) , (\hat{A}_5) and (A_8) hold, then system (1.7) has a unique positive almost periodic solution which is uniformly asymptotically stable in $\hat{\Gamma}$ and is globally asymptotically stable.*

Table 1
Applicability of general theorems to nonautonomous systems of form (1.2)–(1.6)

$c(t, x)$	PI	UB	P	GAS	EPS	GAS of PS	APS
$m(t)x$	Th2.1, Th2.2	Th2.4	Th2.6	Th2.7	Th3.1, Th3.3	Th3.5, Th3.6	Th4.3
$\frac{m(t)x}{A+x}$	Th2.1–2.3	Th2.4, Th2.5	Th2.6	Th2.7, Th2.8	Th3.1–3.4	Th3.5–3.8	Th4.3, Th4.4
$\frac{m(t)x^n}{A+x^n}$	Th2.1, Th2.3	Th2.5	Th2.6	Th2.8	Th3.2, Th3.4	Th3.7, Th3.8	Th4.4
$\frac{m(t)x^2}{(A+x)(B+x)}$	Th2.1–2.3	Th2.4, Th2.5	Th2.6	Th2.7, Th2.8	Th3.1–3.4	Th3.5–3.8	Th4.3, Th4.4
$m(t)(1 - e^{-Ax})$	Th2.1, Th2.3	Th2.5	Th2.6	Th2.8	Th3.2, Th3.4	Th3.7, Th3.8	Th4.4

P: permanence, PI: positive invariance, GAS: globally asymptotic stability, APS: almost periodic solutions, Th: theorem, UB: ultimate boundedness, PS: periodic solutions, EPS: existence of periodic solutions.

5. Conclusive discussion

In this paper, we have investigated the dynamical behavior of a class of nonautonomous semi-ratio-dependent predator–prey systems, which incorporates a number of possible terms for the predator’s functional responses to the prey. In order to enhance the applicability of the general results established previously, we shall go back to some of the particular forms for the functional responses and interpret the general results in some of the particular cases. One can easily see that it is very trivial to apply the general results to nonautonomous predator–prey systems of form (1.2)–(1.6). So we prefer to illustrate in Table 1 the applicability of such general theorems to systems of form (1.2)–(1.6).

From Table 1, one can easily observe that, for a given predator’s functional response to prey, different sufficient criteria are established for certain dynamical behavior of such systems. For example, both Theorem 3.1 and Theorem 3.3 assert the existence of componentwise positive ω periodic solutions of system (1.7) when the functional response is of type 1, 2 and 4.

Naturally, it is interesting to know how these corresponding theorem actually compare. Without loss of generality, as an example, we will talk about this topic based on Theorems 3.1 and 3.3.

Exploring (A_4) (from Theorem 3.1) versus (A_7) (from Theorem 3.3) is clearly the heart of the matter, since these are the only hypothesis that vary from Theorem 3.1 to Theorem 3.3. By (2.1), we can take $M_2 = a^u d^u / b^l e^l + \varepsilon$, where ε is taken sufficient small. From (A_4) , one can easily derive that

$$C_0 < \frac{a^l}{M_2} = \frac{a^l b^l e^l}{a^u d^u + \varepsilon b^l e^l} < \frac{a^l}{a^u} \times \frac{b^l e^l}{d^u}, \quad (5.1)$$

while (A_7) can be rewritten as

$$C_0 < \frac{\bar{b}\bar{e}}{\bar{d}} \exp\{-2(\bar{a} + \bar{d})\}. \quad (5.2)$$

It is trivial to show that (5.1) implies (5.2) if and only if

$$a^l / a^u \leq \exp\{-2(\bar{a} + \bar{d})\},$$

since

$$\frac{b^l e^l}{d^u} \leq \frac{\bar{b}\bar{e}}{\bar{d}}.$$

Generally speaking, assumptions (A_4) and (A_7) cannot contain each other as special case. That is to say, Theorems 3.1 and 3.3 do provide different sufficient criteria for the existence of componentwise positive periodic solutions of system (1.7). For example, consider the following predator–prey system of form (1.2):

$$\begin{aligned}\dot{x} &= x(0.3 - x) - (0.5 \sin 2\pi t + \delta)xy, \\ \dot{y} &= y \left[0.2 - (\cos 2\pi t + 2) \frac{y}{x} \right].\end{aligned}\tag{5.3}$$

In system (5.3),

$$\begin{aligned}a(t) &\equiv 0.3, & b(t) &\equiv 1, & d(t) &\equiv 0.2, & e(t) &= \cos 2\pi t + 2, \\ c(t, x) &= (0.5 \sin 2\pi t + \delta)x.\end{aligned}$$

Then direct calculation shows that

$$\begin{aligned}c(t, x) &\leq 0.5 + \delta \leq C_0, \\ \frac{a^l}{M_2} &= \frac{a^l b^l e^l}{a^u d^u + \varepsilon b^l e^l} = \frac{30}{6 + 100\varepsilon}, \\ \frac{\bar{b}\bar{e}}{\bar{d}} \exp\{-2(\bar{a} + \bar{d})\} &= 6 \exp\{-1\} \approx 2.21,\end{aligned}$$

where $\varepsilon > 0$ can be taken sufficient small.

Take $\delta = 3$, $C_0 = 3.5$ and ε sufficient small; then we have

$$C_0 < \frac{a^l}{M_2},$$

which shows that for system (5.3) Theorem 3.1 applies. However, for any $C_0 \geq 3.5$, we always have

$$C_0 > \frac{\bar{b}\bar{e}}{\bar{d}} \exp\{-2(\bar{a} + \bar{d})\},$$

so we can conclude that Theorem 3.3 fails.

Take $\delta = 1.5$, $C_0 = 2$ and ε sufficient small; then we have

$$C_0 < \frac{a^l}{M_2}, \quad C_0 < \frac{\bar{b}\bar{e}}{\bar{d}} \exp\{-2(\bar{a} + \bar{d})\},$$

therefore, both Theorem 3.1 and Theorem 3.3 apply.

Take $\delta = 6$; then $C_0 \geq 6.5$, hence

$$C_0 > \frac{a^l}{M_2}, \quad C_0 > \frac{\bar{b}\bar{e}}{\bar{d}} \exp\{-2(\bar{a} + \bar{d})\},$$

which implies neither Theorem 3.1 nor Theorem 3.3 applies. In this case, from the criteria established in this paper, we learn nothing about the existence of positive periodic

solutions. Stronger and more effective criteria should be established by using other methods.

Also, the above discussion and Table 1 tell us that generally there are no forms of functional responses for which Theorem 3.1 applies but Theorem 3.3 does not for vice versa. However, for some concrete predator–prey systems, the answer is completely different. For system (5.3) with $\delta = 3$, which is of form (1.2) and the functional response is of type 1, we have proved that Theorem 3.1 applies while Theorem 3.3 does not.

Now let us consider a predator–prey system of form (1.3), where the functional response is of type 2,

$$\begin{aligned}\dot{x} &= x(0.4 - 0.5x) - \frac{0.7x}{1+x}y, \\ \dot{y} &= y\left(0.1 - 0.3\frac{y}{x}\right);\end{aligned}\tag{5.4}$$

here

$$\begin{aligned}a(t) &\equiv 0.4, & b(t) &\equiv 0.5, & d(t) &\equiv 0.1, & e(t) &= 0.3 \sin 2\pi t + 0.4, \\ c(t, x) &= \frac{0.7x}{1+x} \leq 0.7 \leq C_0.\end{aligned}$$

Then for any $C_0 \geq 0.7$, we have

$$\frac{a^l}{M_2} = \frac{a^l b^l e^l}{a^u d^u + \varepsilon b^l e^l} < \frac{a^l b^l e^l}{a^u d^u} < \frac{b^l e^l}{d^u} = 0.5 < C_0,$$

that is Theorem 3.1 does not apply. However, for $C_0 = 0.71$,

$$\frac{\bar{b}\bar{e}}{\bar{d}} \exp\{-2(\bar{a} + \bar{d})\} = 2 \exp\{-1\} \approx 0.74 > C_0,$$

which implies Theorem 3.3 applies.

Finally, in view of the above discussion, we would like to mention that some results in Sections 3 and 4 have room for further improvement. However, significant improvement appears to be difficult unless new approaches can be found. The methods used here are very powerful and effective and can be used to attack other problems. It also seems interesting but more challenging to derive sufficient and necessary criteria for the dynamics of systems of form (1.7).

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