Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
J. Math. Anal. Appl. 289 (2004) 317–335
Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS
www.elsevier.com/locate/jmaa

Exponential dichotomies of linear dynamic equations on measure chains under slowly varying coefficients

Christian Pötzsche [1]

Department of Mathematics, University of Augsburg, D-86135 Augsburg, Germany

Received 22 April 2003
Submitted by A.C. Peterson

Abstract

Unifying ordinary differential and difference equations, we consider linear dynamic equations on measure chains or time scales, which possess an exponential dichotomy uniformly in a parameter, and show that this dichotomy is robust, if the mentioned parameter changes slowly in time. Here, the equations can be infinite dimensional and are not assumed to be invertible.

Keywords: Exponential dichotomy; Slowly varying; Time scale; Linear dynamic equation

1. Introduction and preliminaries

The well-known and established notion of an exponential dichotomy generalizes the concept of hyperbolicity from autonomous to nonautonomous linear equations, where the invariant subspaces are replaced by so-called invariant vector bundles and the stability properties of the solutions in these nontrivial invariant sets are uniform. The importance of exponential dichotomies in the theory of nonautonomous dynamical systems is due to the fact that they are a very useful tool to solve nonlinear problems as perturbations of linear ones, like in the persistence of integral manifolds (cf., e.g., [6,13,16]).

E-mail address: christian.poetzsche@math.uni-augsburg.de.
URL: http://www.math.uni-augsburg.de/~poetzsch.
[1] Research supported by the "Graduiertenkolleg: Nichtlineare Probleme in Analysis, Geometrie und Physik" (GRK 283) financed by the Deutsche Forschungsgemeinschaft and the State of Bavaria.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.09.063

Meanwhile dichotomies are widely used, and thorough introductions into the theory of exponentially dichotomous ordinary differential equations (ODEs) can be found in, e.g., the books [3,5]. For difference equations (OΔEs) the literature is slightly sparser, but [4] and [6, Section 7.6] pioneered here. Both concepts have been unified in [12,13] within the "calculus on measure chains," which goes back to [7]. This calculus allows a simultaneous treatment of ODEs, OΔEs and of equations on so-called inhomogeneous time scales, which allow applications in, for instance, discretization theory and population dynamics. To quote a reference about dynamic equations on measure chains or time scales we recommend [7] and the monograph [2].

In the present paper we prove an abstract perturbation result (Theorem 3.4) for parameter-dependent linear dynamic equations on measure chains in arbitrary Banach spaces. Such a result has two main applications:

- *Robustness of exponential dichotomies under slowly varying coefficients*: This means, if we consider for example a parameter-dependent linear ODE $\dot{x} = A(t,q)x$, which has an exponential dichotomy uniformly in a parameter q from, e.g., a metric space, then one can replace the constant value q by any function $q_*(t)$ which varies "slowly" in time, such that the equation $\dot{x} = A(t, q_*(t))x$ is also exponentially dichotomous. This, in turn, yields a sufficient condition for a dynamic equation to possess an exponential dichotomy in terms of the spectrum of their coefficient operator (see Remark 3.3(2)).
- *Construction of invariant fiber bundles*, which are the counterpart of integral manifolds in the theory of difference equations or general dynamic equations. Indeed, using Theorem 3.4 one is able to characterize invariant fiber bundles as fixed points of an abstract integral operator within a Lyapunov–Perron technique. Such applications are presented in, e.g., [6,16] for differential equations, while OΔEs are considered in [14] and the general case of dynamic equations on arbitrary measure chains will be published in a forthcoming paper.

The above mentioned result has its origins in [6] and [16]. Their approach has the advantage that, differing from [3, p. 50, Proposition 1], one can immediately apply it to infinite dimensional equations. Moreover, in the case of ODEs, and with an equivalent result (cf. [15]), it follows with Palmer [11] that our main Theorem 3.4 is more general than [17, p. 342, Theorem 6] in certain situations. In the case of difference equations, we do not know of any related results, and therefore the achievements of this paper (Theorem 3.4 and Corollary 3.6) seem to be new even in this setting.

To introduce our terminology, $\mathbb{N}$ are the positive integers, $\mathbb{Z}$ the integers, $\mathbb{R}$ is the real and $\mathbb{C}$ the complex field. In addition, for any real $h \geqslant 0$ we write $\mathbb{R}_h := \{x \in \mathbb{R}\colon\ 1 + hx > 0\}$. Now suppose for the following that $\mathcal{X}$ denotes a real or complex Banach space with the norm $\|\cdot\|$. $\mathcal{L}(\mathcal{X})$ stands for the linear space of continuous endomorphisms on $\mathcal{X}$ with the norm $\|T\| := \sup_{\|x\|=1} \|Tx\|$, and $\mathcal{GL}(\mathcal{X})$ for the group of toplinear isomorphisms on $\mathcal{X}$; $I_{\mathcal{X}}$ is the identity mapping on $\mathcal{X}$. We write $\mathcal{N}(T) := T^{-1}(\{0\})$ for the *kernel* and $\mathcal{R}(T) := T\mathcal{X}$ for the *range* of $T \in \mathcal{L}(\mathcal{X})$.

We also shortly introduce some notions, which are specific for the calculus on measure chains. $(\mathbb{T}, \preceq, \mu)$ denotes an arbitrary *measure chain* with order relation "$\preceq$" and *growth calibration* μ (cf. [7]). A *time scale* is a special case of a measure chain, where

$\mathbb{T}$ is a nonempty closed subset of the reals $\mathbb{R}$, "$\preceq$" is just the canonical ordering "$\leqslant$" and the growth calibration is given by $\mu(t,\tau) = t - \tau$. Differing from the usual notation, $\rho_+ : \mathbb{T} \to \mathbb{T}$, $\rho_+(t) := \inf\{s \in \mathbb{T}\colon\ t \prec s\}$ is the *forward jump operator* and we assume that the *graininess* $\mu^*(t) := \mu(\rho_+(t), t)$ is bounded throughout the paper. Measure chains with constant graininess are called *homogeneous*. A point $t \in \mathbb{T}$ is called *right-dense* if $\mu^*(t) = 0$ and otherwise *right-scattered*. In case $\sup\{s \in \mathbb{T}\colon\ s \prec t\} = t$ we speak of a *left-dense* point t. Besides, $(\mathbb{T}, \preceq, \mu)$ is assumed to be unbounded above and below, i.e., the set $\{\mu(t,\tau) \in \mathbb{R}\colon\ t \in J\}$ has the mentioned property for one $\tau \in \mathbb{T}$. A measure chain $(\tilde{\mathbb{T}}, \preceq, \tilde{\mu})$ is denoted as *discrete*, if $\tilde{\mathbb{T}} = \{t_k\}_{k \in \mathbb{Z}}$ and if there exist reals $h_0, h > 0$ such that

$$h_0 \leqslant \tilde{\mu}(t_{k+1}, t_k) \leqslant h \quad \text{for } k \in \mathbb{Z}. \tag{1.1}$$

With given real numbers $h_0, h > 0$, and measure chain $\mathbb{T}$, we write $\mathbb{S}_{h_0}^h(\mathbb{T})$ for the set of all discrete measure chains $(\tilde{\mathbb{T}}, \preceq, \mu)$ with $\tilde{\mathbb{T}} \subseteq \mathbb{T}$ satisfying (1.1). Furthermore, we speak of a (h_0, h)-*measure chain* $(\mathbb{T}, \preceq, \mu)$, if for every point $t_0 \in \mathbb{T}$ there exist $t_k, t_{-k} \in \mathbb{T}$, $k \in \mathbb{N}$, such that $\{t_k\}_{k \in \mathbb{Z}} \in \mathbb{S}_{h_0}^h(\mathbb{T})$ holds. Any measure chain which is unbounded above and below, and with bounded graininess μ^*, is a (h_0, h)-measure chain for $h_0 > 0$ and $h \geqslant h_0 + \sup_{t \in \mathbb{T}} \mu^*(t)$ (cf. [13, p. 2, Lemma 1.1.7]). The following example should illuminate the above notions for readers who are primarily interested in ODEs or OΔEs.

Example 1.1. (1) For the reals $\mathbb{R}$ we have the identities $\rho_+(t) \equiv t$, $\mu^*(t) \equiv 0$ on $\mathbb{R}$ and each real number is a right- and left-dense point. Moreover, $\mathbb{R}$ is a (h_0, h)-time scale for any $0 < h_0 \leqslant h$.

(2) The discrete time scales $\bar{h}\mathbb{Z}$, $\bar{h} > 0$, and in particular the integers $\mathbb{Z}$, consist of right-scattered points. We have $\rho_+(t) \equiv t + \bar{h}$, $\mu^*(t) \equiv \bar{h}$ on $\bar{h}\mathbb{Z}$, and $\bar{h}\mathbb{Z}$ is a (h_0, h)-time scale for any $\bar{h} \leqslant h_0 \leqslant h$.

A mapping $\phi : \mathbb{T} \to \mathcal{X}$ is said to be *differentiable* (at $t_0 \in \mathbb{T}$), if there exists a unique *derivative* $\phi^\Delta(t_0) \in \mathcal{X}$, such that for every $\epsilon > 0$ the estimate

$$\bigl\|\phi\bigl(\rho_+(t_0)\bigr) - \phi(t) - \mu\bigl(\rho_+(t_0), t\bigr)\phi^\Delta(t_0)\bigr\| \leqslant \epsilon\bigl|\mu\bigl(\rho_+(t_0), t\bigr)\bigr| \quad \text{for } t \in U$$

holds in a neighborhood $U \subseteq \mathbb{T}$ of t_0 (see [7, Section 2.4]). As special cases we obtain in a time scale setting the usual derivative $\phi^\Delta(t) = \dot{\phi}(t)$ for $\mathbb{T} = \mathbb{R}$ and the forward difference operator $\phi^\Delta(t) = (\phi(t+h) - \phi(t))/h$ for $\mathbb{T} = h\mathbb{Z}$, $h > 0$.

Now let $(\mathcal{Q}, d)$ be a metric space. According to [7, Section 5.2], a mapping $f : \mathbb{T} \times \mathcal{Q} \to \mathcal{X}$ is said to be *rd-continuous*, if for every $q_0 \in \mathcal{Q}$ one has that f is continuous in (t_0, q_0) for every right-dense $t_0 \in \mathbb{T}$, and if for any left-dense $t_0 \in \mathbb{T}$ the limits $\lim_{q \to q_0} f(t_0, q)$, $\lim_{(t,q) \to (t_0, q_0),\, t \prec t_0} f(t,q)$ exist.

In addition, $\mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathcal{X})$ denotes the set of rd-continuous maps from $\mathbb{T}$ into $\mathcal{X}$ and

$$\mathcal{C}_{\mathrm{rd}}\mathcal{R}\bigl(\mathbb{T}, \mathcal{L}(\mathcal{X})\bigr) := \bigl\{A \in \mathcal{C}_{\mathrm{rd}}\bigl(\mathbb{T}, \mathcal{L}(\mathcal{X})\bigr)\colon\ I_{\mathcal{X}} + \mu^*(t)A(t) \in \mathcal{GL}(\mathcal{X}) \text{ for all } t \in \mathbb{T}\bigr\}$$

stands for the set of so-called *regressive* mappings. The *positively regressive group* is given by $\mathcal{C}_{\mathrm{rd}}^+\mathcal{R}(\mathbb{T}, \mathbb{R}) := \{a \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})\colon\ 1 + \mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}$ with the addition $(a \oplus b)(t) := a(t) + b(t) + \mu^*(t)a(t)b(t)$, and the subtraction $(a \ominus b)(t) := (a(t) - b(t))/(1 + \mu^*(t)b(t))$ for $t \in \mathbb{T}$. On the time scale $\mathbb{T} = \mathbb{R}$, rd-continuity means continuity, and the algebraic operations $\oplus$ or $\ominus$ reduce to the usual (pointwise) addition or subtraction of continuous real-valued mappings, respectively. On the other hand, for $\mathbb{T} = h\mathbb{Z}$, $h > 0$, any function is rd-continuous.

We abbreviate $\lfloor a \rfloor := \inf_{t\in\mathbb{T}} a(t)$, $\lceil a \rceil := \sup_{t\in\mathbb{T}} a(t)$ and $a \lhd b :\Leftrightarrow 0 < \lfloor b - a \rfloor$ for functions $a, b : \mathbb{T} \to \mathbb{R}$. An element $a \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ is said to be *discretely bounded below*, if $\Gamma_-(a) := 1 + \lfloor \mu^* a \rfloor > 0$ holds. In addition, we say a is *discretely bounded above*, if $\Gamma_+(a) := 1 + \lceil \mu^* a \rceil < \infty$. For an arbitrary real $h \geqslant 0$ one easily verifies that the mappings $\xi_h : \mathbb{R}_h \to \mathbb{R}$, $\vartheta_h : \mathbb{R} \to \mathbb{R}_h$, given by

$$\xi_h(x) := \lim_{t\searrow h} \frac{\log(1 + tx)}{t}, \qquad \vartheta_h(x) := \lim_{t\searrow h} \frac{\exp(tx) - 1}{t},$$

are bijective and inverse to each other. Then the *real exponential function* $e_a(t, \tau) \in \mathbb{R}$, $t, \tau \in \mathbb{T}$, on $\mathbb{T}$, allows the representation

$$e_a(t, \tau) = \int_\tau^t \xi_{\mu^*(s)}\bigl(a(s)\bigr)\, \Delta s \tag{1.2}$$

and we have $e_{a\oplus b}(t, \tau) = e_a(t, \tau)e_b(t, \tau)$ for $t, \tau \in \mathbb{T}$ (cf. [7]). For homogeneous time scales and constant functions $a(t) \equiv \alpha$, one obtains explicitly

$$e_a(t, s) = e^{\alpha(t-s)} \quad \text{for } \mathbb{T} = \mathbb{R},$$
$$e_a(t, s) = (1 + h\alpha)^{(t-s)/h} \quad \text{for } \mathbb{T} = h\mathbb{Z},\ h > 0,$$

and formulas for the exponential function on various other time scales can be found in [2, pp. 69ff].

We close this section with two technical results on the real exponential function. The first one estimates the exponential function on bounded subsets of $\mathbb{T}$, while the second one relates real exponential functions on different measure chains.

Lemma 1.1. *Consider reals $0 < h_0 \leqslant h$ and functions $a, b \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$. Then the constants $E^-_a(h_0, h) := \inf_{h_0\leqslant\mu(t,s)\leqslant h} e_a(t, s)$, $E^+_b(h_0, h) := \sup_{h_0\leqslant\mu(t,s)\leqslant h} e_b(t, s)$ satisfy the following*:

(a) *If $0 \lhd a$, then for any $C \in \mathbb{R}$ there exist reals $0 < h_0 \leqslant h$, $\lceil \mu^* \rceil \leqslant h$ such that $C \leqslant E^-_a(h_0, h)$,*
(b) *if b is bounded above, we have $E^+_b(h_0, h) < \infty$.*

Proof. The easy proof can be found in [13, p. 115, Lemma 2.3.1]. □

Lemma 1.2. *Suppose $\tilde{\mathbb{T}} = \{t_k\}_{k\in\mathbb{Z}}$ is a discrete measure chain with $\tilde{\mathbb{T}} \subseteq \mathbb{T}$ and $\tilde c, \tilde d \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$. Then $c_0, d_0 : \mathbb{T} \to \mathbb{R}$,*

$$c_0(t) := \vartheta_{\mu^*(t)}\left(\sup_{k\in\mathbb{Z}} \frac{\ln(1 + \mu(t_{k+1}, t_k)\tilde c(t_k))}{\mu(t_{k+1}, t_k)}\right),$$
$$d_0(t) := \vartheta_{\mu^*(t)}\left(\inf_{k\in\mathbb{Z}} \frac{\ln(1 + \mu(t_{k+1}, t_k)\tilde d(t_k))}{\mu(t_{k+1}, t_k)}\right),$$

are positively regressive and satisfy

$$\tilde{e}_{\tilde{c}}(t_k, t_l) \leqslant e_{c_0}(t_k, t_l), \quad \tilde{e}_{\tilde{d}}(t_l, t_k) \leqslant e_{d_0}(t_l, t_k) \quad \textit{for } l \leqslant k, \tag{1.3}$$

where, from now on, $\tilde{e}_{\tilde{c}}$ denotes the real exponential function on $\tilde{\mathbb{T}}$.

Proof. See [13, p. 67, Lemma 1.3.32]. □

2. Bounded growth and exponential dichotomies

Consider an operator-valued mapping $A \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$. Differing from the existing literature on linear dynamic equations on measure chains we do not assume that the *coefficient operator A* is regressive and we can include noninvertible difference equations into our theory. Hence our standard reference for, e.g., existence and uniqueness results will be [13], instead of [2,7]. A *linear dynamic equation* (or a *linear system*) is an equation of the form

$$x^{\Delta} = A(t)x, \tag{2.1}$$

and a differentiable mapping $\lambda : I \to \mathcal{X}$ is said to solve (2.1) on a subset $I = \mathbb{T}$ or $I = \{t \in \mathbb{T}\colon \tau \preceq t\}$, $\tau \in \mathbb{T}$, if its derivative λ^{Δ} satisfies $\lambda^{\Delta}(t) \equiv A(t)\lambda(t)$ on I.

Example 2.1. On homogeneous time scales, the linear dynamic equation (2.1) describes ODEs and OΔEs. In fact, if $\mathbb{T} = \mathbb{R}$ we consider linear nonautonomous ODEs of the form $\dot{x} = A(t)x$. If $\mathbb{T} = h\mathbb{Z}$, then (2.1) reduces to the difference equation $(x(t+h) - x(t))/h = A(t)x(t)$ or equivalently $x(t+h) = [I_{\mathcal{X}} + hA(t)]x(t)$.

The linear dynamic equation (2.1) is said to have

- c^{+}*-bounded growth* (with constant C), if there exists a real number $C \geqslant 1$ and some $c \in \mathcal{C}_{\mathrm{rd}}^{+}\mathcal{R}(\mathbb{T}, \mathbb{R})$ bounded above, such that $\|\Phi_A(t, \tau)\| \leqslant Ce_c(t, \tau)$ for $\tau \preceq t$,
- (c, d)*-bounded growth* (with constant C), if it has c^{+}-bounded growth, one has $A \in \mathcal{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$ and if there exists some $d \in \mathcal{C}_{\mathrm{rd}}^{+}\mathcal{R}(\mathbb{T}, \mathbb{R})$ bounded below, such that $\|\Phi_A(t, \tau)\| \leqslant Ce_d(t, \tau)$ for $t \preceq \tau$,

where $\Phi_A(t, \tau) \in \mathcal{L}(\mathcal{X})$ is the *transition operator* of (2.1), i.e., the solution of the corresponding initial value problem $X^{\Delta} = A(t)X$, $X(\tau) = I_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X})$ for $\tau \preceq t$. It is easy to see that Φ_A has the properties

$$\Phi_A\big(\rho_{+}(t), t\big) = I_{\mathcal{X}} + \mu^{*}(t)A(t) \quad \text{for } t \in \mathbb{T}, \tag{2.2}$$

$$\Phi_A(t, \tau) = \Phi_A(t, s)\Phi_A(s, \tau) \quad \text{for } \tau \preceq s \preceq t \tag{2.3}$$

(cf. [13, p. 55, Satz 1.3.9]) and in case $A \in \mathcal{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$ one has the relation $\Phi_A(t, \tau) \in \mathcal{GL}(\mathcal{X})$ and the *linear cocycle property* (2.3) holds for all $\tau, s, t \in \mathbb{T}$.

Remark 2.1. (1) Without the condition that c is bounded above, it would be possible to show that every system (2.1) has c^{+}-bounded growth (cf. [1]).

(2) On discrete measure chains, the system (2.1) has c^+-bounded growth for a certain c, if and only if A is bounded (cf. [13, p. 71, Satz 1.3.42]).

The following two lemmas can be shown using Gronwall's inequality on measure chains (see [2, p. 256, Theorem 6.4]).

Lemma 2.1. *Assume $c \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ is discretely bounded below. Consider the linear systems* (2.1) *and*

$$x^\Delta = B(t)x \tag{2.4}$$

with $B \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$. If there exists a real number $C \geqslant 1$ and a bounded function $\epsilon \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ satisfying $\|\Phi_A(t,\tau)\| \leqslant Ce_c(t,\tau)$ for $\tau \preceq t$ and $\|A(t) - B(t)\| \leqslant \epsilon(t)$ for $t \in \mathbb{T}$, then

$$\big\|\Phi_B(t,\tau) - \Phi_A(t,\tau)\big\| \leqslant \frac{C^2\lceil\epsilon\rceil}{\Gamma_-(c + C\epsilon)}\mu(t,\tau)e_{c+C\epsilon}(t,\tau) \quad \textit{for } \tau \preceq t.$$

Proof. See [13, p. 73, Korollar 1.3.45(a)]. □

Lemma 2.2. *Let $C_1, C_2 \geqslant 1$ be reals and $c \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$. If the linear systems* (2.1) *and* (2.4) *have c^+-bounded growth with constants C_1 and C_2, respectively, then*

$$\big\|\Phi_B(t,\tau) - \Phi_A(t,\tau)\big\| \leqslant C_1C_2e_c(t,\tau)\int_\tau^t \frac{\|B(s) - A(s)\|}{1 + \mu^*(s)c(s)}\,\Delta s \quad \textit{for } \tau \preceq t.$$

Proof. See [13, p. 74, Korollar 1.3.46(a)]. □

A mapping of projections $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is called an *invariant projector* of the linear system (2.1), if $P(t)\Phi_A(t,\tau) = \Phi_A(t,\tau)P(\tau)$ for $\tau \preceq t$ holds, and in case

$$\big[I_{\mathcal{X}} + \mu^*(t)A(t)\big]\big|_{\mathcal{N}(P(t))} : \mathcal{N}\big(P(t)\big) \to \mathcal{N}\big(P\big(\rho_+(t)\big)\big) \tag{2.5}$$

is bijective for all right-scattered $t \in \mathbb{T}$, we speak of a *regular invariant projector*. Then one can show that the restriction

$$\bar\Phi_A(t,\tau) := \Phi_A(t,\tau)|_{\mathcal{N}(P(\tau))} : \mathcal{N}\big(P(\tau)\big) \to \big(\mathcal{N}\big(P(t)\big)\big) \quad \text{for } \tau \preceq t$$

is a well-defined isomorphism, and we denote its inverse by $\bar\Phi_A(\tau,t)$ (cf. [12, Proposition 2.3]). The linear system (2.1) is said to possess an *exponential dichotomy* (ED for short) with a, b, K_1, K_2, if there exists a regular invariant projector $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ of (2.1) satisfying

$$\big\|\Phi_A(t,\tau)P(\tau)\big\| \leqslant K_1e_a(t,\tau) \quad \text{for } \tau \preceq t, \tag{2.6}$$

$$\big\|\bar\Phi_A(t,\tau)\big[I_{\mathcal{X}} - P(\tau)\big]\big\| \leqslant K_2e_b(t,\tau) \quad \text{for } t \preceq \tau, \tag{2.7}$$

with real constants $K_1, K_2 \geqslant 1$ and $a, b \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \lhd b$. Note that on the time scale $\mathbb{T} = \mathbb{R}$ any invariant projector is regular.

Example 2.2. Let $\alpha, \beta, h \geqslant 0$ be reals with $\alpha < \beta$. On homogeneous measure chains with $\mu^*(t) \equiv h$ on $\mathbb{T}$ and for constant coefficient operators $A(t) \equiv A$ on $\mathbb{T}$, one has the following situation:

(1) In case $h = 0$ (ODEs), the linear dynamic equation (2.1) has an ED with α, β, if the spectrum $\sigma(A) \subseteq \mathbb{C}$ is disjoint from the vertical strip $\{\lambda \in \mathbb{C}\colon \alpha \leqslant \Re\lambda \leqslant \beta\}$ in the complex plane. The corresponding invariant projector is given by the spectral projection related to the spectral set $\{\lambda \in \sigma(A)\colon \Re\lambda < \alpha\}$ (cf. [5, p. 72ff]).

(2) Analogously, in case $h > 0$ (OΔEs), the system (2.1) possesses an ED with α, β, if $\sigma(I_{\mathcal{X}} + hA)$ is disjoint from the annulus $\{\lambda \in \mathbb{C}\colon \alpha \leqslant |\alpha| \leqslant \beta\}$, and the invariant projector is given by the spectral projection related to $\{\lambda \in \mathbb{C}\colon |\lambda| < \alpha\}$.

Remark 2.2. In our definition of an exponential dichotomy, the growth functions a, b are not assumed to be constants. For ODEs this generalization dates back to [10]. A second feature of our definition is that we do not insist on a hyperbolicity condition like $a \lhd 0 \lhd b$. Thus, one can speak of a *pseudo-hyperbolic dichotomy*, which makes the above notion more flexible. Eventually, we point out again that Eq. (2.1) does not have to be regressive. For OΔEs this has its origins in [6, p. 229, Definition 7.6.4] and with a different, but equivalent definition in [9].

The proof of the next lemma is too excessive to be presented here. It is based on the fact that certain spaces of exponentially bounded functions are admissible for Eq. (2.1) (cf. [13, p. 106, Satz 2.2.7]).

Lemma 2.3. *Let $K_1, K_2, L_1, L_2 \geqslant 1$, $\epsilon \geqslant 0$ be reals and $a, b, c, d \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ such that $a \lhd c \lhd d \lhd b$. Then under the assumptions*

(i) *the linear system* (2.1) *possesses an ED with a, b, K_1, K_2 and P,*
(ii) *the linear system* (2.4) *possesses an ED with c, d, L_1, L_2 and Q,*
(iii) $\|A(t) - B(t)\| \leqslant \epsilon$ *for all* $t \in \mathbb{T}$,

the invariant projectors satisfy

$$\left\| P(t) - Q(t) \right\| \leqslant \epsilon \max\{L_1, L_2\} C_{a,b}(c, d) \quad \text{for } t \in \mathbb{T},$$

with

$$C_{a,b}(c, d) := \frac{K_1}{\lfloor d - a \rfloor} + \frac{K_2}{\lfloor c - a \rfloor} + \max\left\{ \frac{K_1}{\lfloor c - a \rfloor}, \frac{K_2}{\lfloor b - d \rfloor} \right\}.$$

Proof. See [13, p. 108, Korollar 2.2.9]. □

One of the main properties of an exponential dichotomy is its roughness. At the end of this section we present a roughness theorem for exponential dichotomies under $\mathcal{L}^\infty$-perturbations of dynamic equations on discrete measure chains, which is sufficient for our purposes.

Theorem 2.4. *Let $(\tilde{\mathbb{T}}, \preceq, \tilde{\mu})$ be a discrete measure chain and consider a mapping $\tilde{A} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$. The linear dynamic equation $x^\Delta = \tilde{A}(t)x$ on $\tilde{\mathbb{T}}$ is assumed to possess an ED with $\tilde{a}, \tilde{b}, K_1, K_2$ and an invariant projector $\tilde{P}$, where $\tilde{b}$ is bounded above. Moreover, let $\tilde{c}, \tilde{d} \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$ with $\tilde{a} \lhd \tilde{c} \lhd \tilde{d} \lhd \tilde{b}$, and suppose the mapping $\tilde{B} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$ satisfies $\|\tilde{B}(t) - \tilde{A}(t)\| \leqslant \epsilon$ for $t \in \tilde{\mathbb{T}}$ with a real number $\epsilon \geqslant 0$ such that $\epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d}) < 1$. Then $x^\Delta = \tilde{B}(t)x$ has an ED with $\tilde{c}, \tilde{d}$,*

$$L_1 := \left(\frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})}\right)^2, \qquad L_2 := \left(1 + \frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})}\right)\frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})}$$

and an invariant projector $\tilde{Q} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$ satisfying

$$\|\tilde{Q}(t) - \tilde{P}(t)\| \leqslant \epsilon\left(1 + \frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})}\right)\frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})^2\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})} \quad \text{for } t \in \tilde{\mathbb{T}}.$$

Proof. See [13, pp. 113–114, Satz 2.2.14]. However, the proof is very similar to the difference equations case presented in [9, p. 45, Satz 3.2.1]. □

3. Uniform exponential dichotomies

In this section we are confronted with exponential dichotomies on three different "time scales," namely $\mathbb{Z}$, discrete and general measure chains. The subsequent lemma clarifies to what extend the dichotomy notion for difference equations from [9, p. 7, Definition 2.1.2] carries over to dynamic equations on discrete measure chains.

Lemma 3.1. *Consider reals $K_1, K_2, M_1, M_2 \geqslant 1$, a discrete measure chain $(\tilde{\mathbb{T}}, \preceq, \tilde{\mu})$ with $\tilde{\mathbb{T}} = \{t_k\}_{k\in\mathbb{Z}}$, functions $\tilde{a}, \tilde{b} \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$, $\tilde{a} \lhd \tilde{b}$, a sequence $\hat{A} : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ and*

$$\Psi_{\hat{A}}(k, l) := \begin{cases} I_{\mathcal{X}} & \text{for } l = k, \\ \hat{A}(k-1)\dots\hat{A}(l) & \text{for } l < k. \end{cases} \tag{3.1}$$

If $\hat{P} : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ is a sequence of projections such that

$$\|\Psi_{\hat{A}}(k, l)\hat{P}(l)x\| \leqslant K_1\tilde{e}_{\tilde{a}}(t_k, t_l)\|\hat{P}(l)x\| \quad \text{for } l \leqslant k, \tag{3.2}$$

$$\|\Psi_{\hat{A}}(k, l)[I_{\mathcal{X}} - \hat{P}(l)]x\| \geqslant K_2^{-1}\tilde{e}_{\tilde{b}}(t_k, t_l)\|[I_{\mathcal{X}} - \hat{P}(l)]x\| \quad \text{for } l \leqslant k, \tag{3.3}$$

and $x \in \mathcal{X}$, and if

$$\hat{P}(k+1)\hat{A}(k) = \hat{A}(k)\hat{P}(k), \qquad \mathcal{N}(\hat{P}(k+1)) \subseteq \mathcal{R}(\hat{A}(k)), \tag{3.4}$$

$$\|\hat{P}(k)\| \leqslant M_1, \qquad \|I_{\mathcal{X}} - \hat{P}(k)\| \leqslant M_2 \tag{3.5}$$

for $k \in \mathbb{Z}$ holds, then the linear system

$$x^\Delta = \tilde{A}(t)x, \qquad \tilde{A}(t_k) := \frac{1}{\tilde{\mu}^*(t_k)}\big(\hat{A}(k) - I_{\mathcal{X}}\big) \quad \text{for } k \in \mathbb{Z} \tag{3.6}$$

on $\tilde{\mathbb{T}}$ possesses an ED with $\tilde{a}, \tilde{b}$, constants M_1K_1, M_2K_2 and the invariant projector $\tilde{P} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$, $\tilde{P}(t_k) := \hat{P}(k)$.

Proof. Above all, we remark that the transition operators $\Phi_{\tilde{A}}$ of (3.6) and $\Psi_{\hat{A}}$ satisfy $\Phi_{\tilde{A}}(t_k, t_l) = \Psi_{\hat{A}}(k, l)$ (cf. (2.2), (2.3)) for all $l \leqslant k$. Inductively one can see from (3.4) that $\tilde{P} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$ is an invariant projector of (3.6) and to show that $\tilde{P}$ is regular, we verify that

$$[I_{\mathcal{X}} + \tilde{\mu}^*(t)\tilde{A}(t)]\big|_{\mathcal{N}(\tilde{P}(t))} : \mathcal{N}(\tilde{P}(t)) \to \mathcal{N}(\tilde{P}(\tilde{\rho}_+(t))) \tag{3.7}$$

is bijective for all $t \in \tilde{\mathbb{T}}$. For an arbitrary $t \in \tilde{\mathbb{T}}$ we choose $\xi_0 \in \mathcal{N}(\tilde{P}(t))$ such that $[I_{\mathcal{X}} + \tilde{\mu}^*(t)\tilde{A}(t)]\xi_0 = 0$ and the estimate

$$\begin{aligned} &K_2^{-1}\tilde{e}_{\tilde{b}}(\tilde{\rho}_+(t), t)\|\xi_0\| = K_2^{-1}\tilde{e}_{\tilde{b}}(\tilde{\rho}_+(t), t)\big\|[I_{\mathcal{X}} - \tilde{P}(t)]\xi_0\big\| \\ &\overset{(3.3)}{\leqslant} \big\|\Phi_{\tilde{A}}(\tilde{\rho}_+(t), t)[I_{\mathcal{X}} - \tilde{P}(t)]\xi_0\big\| \overset{(2.2)}{=} \big\|[I_{\mathcal{X}} + \tilde{\mu}^*(t)\tilde{A}(t)]\xi_0\big\| = 0 \end{aligned}$$

yields $\xi_0 = 0$. Therefore, the linear operator (3.7) is one-to-one. Due to the inclusion (3.4) we know that for every $\xi \in \mathcal{N}(\tilde{P}(\tilde{\rho}_+(t)))$ there exists a $\xi_0 \in \mathcal{X}$ with $[I_{\mathcal{X}} + \tilde{\mu}^*(t)\tilde{A}(t)]\xi_0 = \xi$. Hence, $\xi = [I_{\mathcal{X}} - \tilde{P}(\tilde{\rho}_+(t))]\xi = [I_{\mathcal{X}} - \tilde{P}(\tilde{\rho}_+(t))][I_{\mathcal{X}} + \tilde{\mu}^*(t)\tilde{A}(t)]\xi_0$ and because the two expressions in brackets on the right-hand side commute due to (3.4), the operator (3.7) is onto. It remains to prove that (3.6) satisfies the claimed dichotomy estimates w.r.t. the invariant projector $\tilde{P}$. Passing over to the least upper bound for $x \in \mathcal{X}$, $\|x\| = 1$, in (3.2) immediately gives us

$$\big\|\Phi_{\tilde{A}}(t,\tau)\tilde{P}(\tau)\big\| \overset{(3.2)}{\leqslant} K_1\tilde{e}_{\tilde{a}}(t,\tau)\big\|\tilde{P}(\tau)\big\| \overset{(3.5)}{\leqslant} K_1M_1\tilde{e}_{\tilde{a}}(t,\tau) \quad \text{for } \tau \preceq t.$$

On the other side, since the operator (3.7) is bijective, we know that the extended transition operator $\bar{\Phi}_{\tilde{A}}(t,\tau) : \mathcal{N}(\tilde{P}(\tau)) \to \mathcal{N}(\tilde{P}(t))$, $t \preceq \tau$, is well-defined (cf. [12, Proposition 2.3]) and for any $x \in \mathcal{X}$ we have

$$\begin{aligned} &K_2^{-1}\tilde{e}_{\tilde{b}}(\tau, t)\big\|\bar{\Phi}_{\tilde{A}}(t,\tau)[I_{\mathcal{X}} - \tilde{P}(\tau)]x\big\| \\ &\overset{(3.3)}{\leqslant} \big\|\Phi_{\tilde{A}}(\tau,t)[I_{\mathcal{X}} - \tilde{P}(t)]\bar{\Phi}_{\tilde{A}}(t,\tau)[I_{\mathcal{X}} - \tilde{P}(\tau)]x\big\| = \big\|[I_{\mathcal{X}} - \tilde{P}(\tau)]x\big\| \end{aligned}$$

for $t \preceq \tau$. Passing over to the least upper bound over $x \in \mathcal{X}$, $\|x\| = 1$, finally gives

$$\big\|\bar{\Phi}_{\tilde{A}}(t,\tau)[I_{\mathcal{X}} - \tilde{P}(\tau)]\big\| \leqslant K_2\tilde{e}_{\tilde{b}}(t,\tau)\big\|I_{\mathcal{X}} - \tilde{P}(\tau)\big\| \overset{(3.5)}{\leqslant} K_2M_2\tilde{e}_{\tilde{b}}(t,\tau) \quad \text{for } t \preceq \tau,$$

and the proof is finished. □

The following result can be considered as a perturbation result, as well as a sufficient condition for an exponential dichotomy on discrete measure chains. For difference equations it goes back to [6, p. 234, Theorem 7.6.8] and [16, Theorem 4].

Lemma 3.2. *Consider a discrete measure chain* $(\tilde{\mathbb{T}}, \preceq, \tilde{\mu})$, $\tilde{\mathbb{T}} = \{t_k\}_{k\in\mathbb{Z}}$, *real numbers* $0 < \theta_1 < 1 < \theta_2$, $K_1, K_2 \geqslant 1$, $N_0 \geqslant 0$, *functions* $\tilde{a}, \tilde{b} \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$, $\tilde{a} \lhd \tilde{b}$, *where* $\tilde{b}$ *is bounded above, sequences* $\hat{A}, \hat{B} : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$, *and sequences of projections* $\hat{P}_1, \hat{P}_2 : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ *such that*

$$\big\|\hat{A}(k)\eta\big\| \leqslant \theta_1\big(1 + \tilde{\mu}^*(t_k)\tilde{a}(t_k)\big)\|\eta\| \quad \textit{for } \eta \in \mathcal{R}(\hat{P}_1(k)), \tag{3.8}$$

$$\big\|\hat{A}(k)\xi\big\| \geqslant \theta_2\big(1 + \tilde{\mu}^*(t_k)\tilde{b}(t_k)\big)\|\xi\| \quad \textit{for } \xi \in \mathcal{N}(\hat{P}_1(k)) \tag{3.9}$$

and $\|I_{\mathcal{X}} - \hat{P}_2(k)\| \leqslant K_2$, $\|\hat{A}(k)\| \leqslant N_0$,

$$\hat{P}_2(k+1)\hat{A}(k) = \hat{A}(k)\hat{P}_1(k), \qquad \mathcal{N}\big(\hat{P}_2(k+1)\big) \subseteq \mathcal{R}\big(\hat{A}(k)\big), \tag{3.10}$$

$$\big\|\hat{P}_1(k)\big\| \leqslant K_1, \qquad \big\|\hat{P}_2(k)\big\| \leqslant K_1, \tag{3.11}$$

for $k \in \mathbb{Z}$. *For fixed functions* $\tilde{c}, \tilde{d} \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$ *with* $\tilde{a} \lhd \tilde{c} \lhd \tilde{d} \lhd \tilde{b}$ *we assume*

$$\big\|\hat{A}(k) - \hat{B}(k)\big\| \leqslant \epsilon_1, \qquad \big\|\hat{P}_2(k) - \hat{P}_1(k)\big\| \leqslant \epsilon_2 \quad \text{for } k \in \mathbb{Z}, \tag{3.12}$$

where the reals $\epsilon_0, \epsilon_1 \geqslant 0$ *may satisfy*

$$2\epsilon_2 K_1 \leqslant \min\{1-\theta_1, \theta_2 - 1\}, \qquad \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d}) < 1 \tag{3.13}$$

with the abbreviation

$$\epsilon := \frac{1}{\lfloor \tilde{\mu}^* \rfloor}\left(\epsilon_1 + \frac{2\epsilon_2 K_1 N_0}{1 - 2\epsilon_2 K_1}\right).$$

Then the linear dynamic equation

$$x^{\Delta} = \tilde{B}(t)x, \qquad \tilde{B}(t_k) := \frac{1}{\tilde{\mu}^*(t_k)}\big(\hat{B}(k) - I_{\mathcal{X}}\big), \quad k \in \mathbb{Z},$$

on $\tilde{\mathbb{T}}$ *possesses an ED with* $\tilde{c}, \tilde{d}, L_1, L_2$ *given in Theorem* 2.4, *and an invariant projector* $\tilde{Q} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$ *such that*

$$\big\|\tilde{Q}(t_k) - \hat{P}_2(k)\big\| \leqslant \epsilon\left(1 + \frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})}\right)\frac{C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})^2\Gamma_+(\tilde{d})}{1 - \epsilon C_{\tilde{a},\tilde{b}}(\tilde{c}, \tilde{d})} \quad \text{for } k \in \mathbb{Z}.$$

Proof. The crucial object in our considerations is the operator sequence $\Gamma : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$, $\Gamma(k) := \hat{P}_2(k)\hat{P}_1(k) + [I_{\mathcal{X}} - \hat{P}_2(k)][I_{\mathcal{X}} - \hat{P}_1(k)]$, which satisfies

$$\begin{aligned}\hat{P}_2(k)\Gamma(k) &\equiv \hat{P}_2(k)^2\hat{P}_1(k) + \big[\hat{P}_2(k) - \hat{P}_2(k)^2\big]\big[I_{\mathcal{X}} - \hat{P}_1(k)\big] \\ &\equiv \hat{P}_2(k)\hat{P}_1(k)^2 + \big[I_{\mathcal{X}} - \hat{P}_2(k)\big]\big[\hat{P}_1(k) - \hat{P}_1(k)^2\big] \equiv \Gamma(k)\hat{P}_1(k)\end{aligned} \tag{3.14}$$

on $\mathbb{Z}$. Moreover, one has

$$\begin{aligned}\big\|I_{\mathcal{X}} - \Gamma(k)\big\| &\leqslant \big\|\hat{P}_1(k) - \hat{P}_2(k)\big\|\big\|\hat{P}_1(k)\big\| + \big\|\hat{P}_2(k)\big\|\big\|\hat{P}_2(k) - \hat{P}_1(k)\big\| \\ &\overset{(3.11)}{\leqslant} 2K_1\big\|\hat{P}_2(k) - \hat{P}_1(k)\big\| \overset{(3.12)}{\leqslant} 2\epsilon_2 K_1 \quad \text{for } k \in \mathbb{Z},\end{aligned} \tag{3.15}$$

and consequently the linear operator $\Gamma(k) \in \mathcal{L}(\mathcal{X})$ is invertible due to (3.13) and the Neumann series. This guarantees

$$\big\|\Gamma(k)\big\| \leqslant 1 + 2\epsilon_2 K_1, \qquad \big\|\Gamma(k)^{-1}\big\| \leqslant [1 - 2\epsilon_2 K_1]^{-1} \quad \text{for } k \in \mathbb{Z}, \tag{3.16}$$

and the identity (3.14) gives us $\Gamma(k)^{-1}\hat{P}_2(k) \equiv \hat{P}_1(k)\Gamma(k)^{-1}$ on $\mathbb{Z}$. For the mapping $\hat{C} : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$, $\hat{C}(k) := \hat{A}(k)\Gamma(k)^{-1}$ we have

$$\hat{P}_2(k+1)\hat{C}(k) \overset{(3.10)}{\equiv} \hat{A}(k)\hat{P}_1(k)\Gamma(k)^{-1} \equiv \hat{C}(k)\hat{P}_2(k) \quad \text{on } \mathbb{Z},$$

and the definition of $\hat{C}(k) \in \mathcal{L}(\mathcal{X})$ leads to $\mathcal{R}(\hat{C}(k)) = \mathcal{R}(\hat{A}(k))$. Additionally, (3.10) implies $\mathcal{N}(\hat{P}_2(k+1)) \subseteq \mathcal{R}(\hat{C}(k))$ for all $k \in \mathbb{Z}$. With arbitrary $\eta \in \mathcal{R}(\hat{P}_2(k))$ we get

$\hat{P}_1(k)\Gamma(k)^{-1}\eta \equiv \Gamma(k)^{-1}\hat{P}_2(k)\eta \equiv \Gamma(k)^{-1}\eta$ on $\mathbb{Z}$, hence $\Gamma(k)^{-1}\eta \in \mathcal{R}(\hat{P}_1(k))$ and therefore due to (2.2), (2.3), applied to $\tilde{e}_{\tilde{a}}$, the estimate

$$\begin{aligned}\|\hat{C}(k)\eta\| &= \|\hat{A}(k)\Gamma(k)^{-1}\eta\| \overset{(3.8)}{\leqslant} \theta_1\tilde{e}_{\tilde{a}}(t_{k+1},t_k)\|\Gamma(k)^{-1}\eta\| \\ &\overset{(3.16)}{\leqslant} \frac{\theta_1}{1-2\epsilon_2K_1}\tilde{e}_{\tilde{a}}(t_{k+1},t_k)\|\eta\| \overset{(3.13)}{\leqslant} \tilde{e}_{\tilde{a}}(t_{k+1},t_k)\|\eta\| \quad \text{for } k\in\mathbb{Z}.\end{aligned}$$

Mathematical induction over $k \geqslant l$ implies

$$\|\Psi_{\hat{C}}(k,l)\hat{P}_2(l)x\| \overset{(2.3)}{\leqslant} \tilde{e}_{\tilde{a}}(t_k,t_l)\|\hat{P}_2(l)x\| \quad \text{for } l\leqslant k,\ x\in\mathcal{X},$$

with the operator $\Psi_{\hat{C}}(k,l)\in\mathcal{L}(\mathcal{X})$ given by (3.1), and similarly one derives

$$\|\Psi_{\hat{C}}(k,l)[I_{\mathcal{X}}-\hat{P}_2(l)]x\| \geqslant \tilde{e}_{\tilde{b}}(t_k,t_l)\|[I_{\mathcal{X}}-\hat{P}_2(l)]x\| \quad \text{for } l\leqslant k.$$

Thus, the assumptions of Lemma 3.1 with $M_1 = K_1$, $M_2 = K_2$ are satisfied for the sequences $\hat{C}, \hat{P}_2 : \mathbb{Z}\to\mathcal{L}(\mathcal{X})$, and the linear dynamic equation

$$x^{\Delta} = \tilde{C}(t)x, \qquad \tilde{C}(t_k) := \frac{1}{\tilde{\mu}^*(t_k)}\big(\hat{C}(k)-I_{\mathcal{X}}\big) \quad \text{for } k\in\mathbb{Z} \tag{3.17}$$

on $\tilde{\mathbb{T}}$ consequently possesses an ED with $\tilde{a}$, $\tilde{b}$, K_1, K_2 and the invariant projector $\tilde{P}(t_k) := \hat{P}_2(k)$, $k\in\mathbb{Z}$. Due to the estimate

$$\begin{aligned}\tilde{\mu}^*(t_k)\|\tilde{C}(t_k)-\tilde{B}(t_k)\| &\overset{(3.17)}{\leqslant} \|\hat{A}(k)\Gamma(k)^{-1}-\hat{A}(k)\| + \|\hat{A}(k)-\hat{B}(k)\| \\ &\overset{(3.12)}{\leqslant} \|\hat{A}(k)\|\|\Gamma(k)^{-1}\|\|I_{\mathcal{X}}-\Gamma(k)\| + \epsilon_1 \\ &\overset{(3.15)}{\leqslant} 2\epsilon_2K_1\|\hat{A}(k)\|\|\Gamma(k)^{-1}\| + \epsilon_1 \overset{(3.16)}{\leqslant} \frac{2\epsilon_2K_1N_0}{1-2\epsilon_2K_1} + \epsilon_1 \quad \text{for } k\in\mathbb{Z}\end{aligned}$$

and the inequality (3.13), one can finally apply Theorem 2.4 to (3.17). □

Our last preparation concerning discrete measure chains provides another sufficient condition for an exponential dichotomy on quite general measure chains.

Lemma 3.3. *Consider reals $0 < h_0 \leqslant h$, $\lceil\mu^*\rceil \leqslant h$, such that $(\mathbb{T},\preceq,\mu)$ is a (h_0,h)-measure chain, a real $C_2 \geqslant 1$ and functions $c, c_2, d, d_2 \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T},\mathbb{R})$, d bounded above, d_2 discretely bounded below and $c \lhd d$, $\sup_{s\in\mathbb{T}}\xi_{\mu^*(s)}(c(s)) < \inf_{s\in\mathbb{T}}\xi_{\mu^*(s)}(d(s))$, as well as a linear system*

$$x^{\Delta} = B(t)x \tag{3.18}$$

on $\mathbb{T}$ with $B\in\mathcal{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T},\mathcal{L}(\mathcal{X}))$. Under the assumptions

(i) *the system* (3.18) *has (c_2,d_2)-bounded growth with constant C_2,*
(ii) *there exist real numbers $L_1, L_2 \geqslant 1$, such that for any discrete measure chain $\tilde{\mathbb{T}} = \{t_k\}_{k\in\mathbb{Z}} \in \mathbb{S}^h_{h_0}(\mathbb{T})$ the equation*

$$x^{\Delta} = \tilde{B}(t)x, \qquad \tilde{B}(t_k) := \frac{1}{\mu(t_{k+1},t_k)}\big(\Phi_B(t_{k+1},t_k)-I_{\mathcal{X}}\big), \quad k\in\mathbb{Z},$$

on $\tilde{\mathbb{T}}$ *has an ED with* $\tilde{c}, \tilde{d} : \tilde{\mathbb{T}} \to \mathbb{R}$,

$$\tilde{c}(t_k) := \frac{e_c(t_{k+1}, t_k) - 1}{\mu(t_{k+1}, t_k)}, \qquad \tilde{d}(t_k) := \frac{e_d(t_{k+1}, t_k) - 1}{\mu(t_{k+1}, t_k)},$$

L_1, L_2 *and an invariant projector* $\tilde{Q}_{t_0} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$,

the system (3.18) *possesses an ED with* $\bar{c}, \bar{d} : \mathbb{T} \to \mathbb{R}$,

$$\bar{c}(t) := \vartheta_{\mu^*(t)}\Big(\sup_{s\in\mathbb{T}} \xi_{\mu^*(s)}\big(c(s)\big)\Big), \qquad \bar{d}(t) := \vartheta_{\mu^*(t)}\Big(\inf_{s\in\mathbb{T}} \xi_{\mu^*(s)}\big(d(s)\big)\Big),$$
$$\bar{L}_1 := L_1 C_2 E^+_{c_2 \ominus \bar{c}}(h_0, h), \qquad \bar{L}_2 := L_2 C_2 E^+_{\bar{d}\ominus d_2}(h_0, h) \tag{3.19}$$

and the invariant projector $Q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ *given by* $Q(t) := \tilde{Q}_t(t)$.

Proof. Since the function d is bounded above, and since d_2 is discretely bounded below, it is not difficult to verify that $c_2 \ominus \bar{c}$, $\bar{d} \ominus d_2$ are bounded above. Therefore, using Lemma 1.1(b) we obtain $E^+_{c_2\ominus\bar{c}}(h_0, h), E^+_{\bar{d}\ominus d_2}(h_0, h) < \infty$. Now let $t_0 \in \mathbb{T}$ be arbitrarily given and we choose any discrete measure chain $\tilde{\mathbb{T}} = \{t_k\}_{k\in\mathbb{Z}} \in \mathbb{S}^h_{h_0}(\mathbb{T})$ like in assumption (ii) (such a measure chain exists because of [13, p. 2, Lemma 1.1.7]). Then $\tilde{c}, \tilde{d} \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$, and one can easily show $\tilde{c} \lhd \tilde{d}$. In addition, we have

$$\begin{aligned}\frac{\ln(1 + \mu(t_{k+1}, t_k)\tilde{c}(t_k))}{\mu(t_{k+1}, t_k)} &= \frac{\ln e_c(t_{k+1}, t_k)}{\mu(t_{k+1}, t_k)} \overset{(1.2)}{=} \frac{1}{\mu(t_{k+1}, t_k)} \int_{t_k}^{t_{k+1}} \xi_{\mu^*(s)}\big(c(s)\big)\,\Delta s \\ &\leqslant \frac{1}{\mu(t_{k+1}, t_k)} \int_{t_k}^{t_{k+1}} \sup_{t\in\mathbb{T}} \xi_{\mu^*(t)}\big(c(t)\big)\,\Delta s \\ &= \sup_{t\in\mathbb{T}} \xi_{\mu^*(t)}\big(c(t)\big) \quad \text{for } k \in \mathbb{Z},\end{aligned}$$

and accordingly

$$\sup_{k\in\mathbb{Z}} \frac{\ln(1 + \mu(t_{k+1}, t_k)\tilde{c}(t_k))}{\mu(t_{k+1}, t_k)} \leqslant \sup_{t\in\mathbb{T}} \xi_{\mu^*(t)}\big(c(t)\big). \tag{3.20}$$

Now define the mapping $P_{t_0} : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, $P_{t_0}(t) := \Phi_B(t, t_0)\tilde{Q}_{t_0}(t_0)\Phi_B(t_0, t)$, which satisfies $P_{t_0}(t) \equiv P_{t_0}(t)^2$, $P_{t_0}(t)\Phi_B(t, t_0) \equiv \Phi_B(t, t_0)P_{t_0}(t_0)$ on $\mathbb{T}$ (cf. (2.3)); for this reason, P_{t_0} is also an invariant projector of the linear system (3.18). As a result of the identity $I_{\mathcal{X}} + \mu(t_{k+1}, t_k)\tilde{B}(t_k) \equiv \Phi_B(t_{k+1}, t_k)$ on $\mathbb{Z}$, the mapping $\tilde{B} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$ is regressive and one inductively obtains $\Phi_{\tilde{B}}(t_k, t_l) = \Phi_B(t_k, t_l)$ for $k, l \in \mathbb{Z}$. With a given $t \in \mathbb{T}$, $t_0 \preceq t$, we choose $k \in \mathbb{N}_0$ maximally such that $t_0 \preceq t_k \preceq t$ holds, and the assumptions (i) and (ii) imply

$$\begin{aligned}\big\|\Phi_B(t, t_0)P_{t_0}(t_0)\big\| &\overset{(2.3)}{\leqslant} \big\|\Phi_B(t, t_k)\big\|\big\|\Phi_B(t_k, t_0)\tilde{Q}_{t_0}(t_0)\big\| \\ &= \big\|\Phi_B(t, t_k)\big\|\big\|\Phi_{\tilde{B}}(t_k, t_0)\tilde{Q}_{t_0}(t_0)\big\| \leqslant C_2 e_{c_2}(t, t_k)L_1\tilde{e}_{\tilde{c}}(t_k, t_0).\end{aligned}$$

On the basis of Lemma 1.2, the monotonicity properties of $\vartheta_{\mu^*(t)} : \mathbb{R} \to \mathbb{R}_{\mu^*(t)}$, $t \in \mathbb{T}$, as well as (3.20), this leads to

$$\left\| \Phi_B(t, t_0) P_{t_0}(t_0) \right\| \overset{(1.3)}{\leqslant} L_1 C_2 e_{c_2}(t, t_k) e_{\bar{c}}(t_k, t_0) \leqslant L_1 C_2 e_{c_2 \ominus \bar{c}}(t, t_k) e_{\bar{c}}(t, t_0)$$
$$\leqslant L_1 C_2 E^+_{c_2 \ominus \bar{c}}(h_0, h) e_{\bar{c}}(t, t_0) \quad \text{for } t_0 \preceq t.$$

Consequently, the first dichotomy estimate for (3.18) is shown. To prove the corresponding estimate in negative time, we fix $t \preceq t_0$ and choose $l \leqslant 0$, $l \in \mathbb{Z}$, minimally with $t \preceq t_l \preceq t_0$. Analogously we get from Lemma 1.2 that $\|\Phi_B(t, t_0)[I_{\mathcal{X}} - P_{t_0}(t_0)]\| \leqslant L_2 C_2 E^+_{\bar{d} \ominus d_2}(h_0, h) e_{\bar{d}}(t, t_0)$ for $t \preceq t_0$. Hence the proof is finished, if one defines the invariant projector for (3.18) by $Q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, $Q(t) := P_t(t)$. □

Now we arrive at the main result of this paper. In case of infinite dimensional differential equations it goes back to [6, pp. 240–241, Theorem 7.6.12]. However, [16, Theorem 1] contains a more accessible approach for ODEs in $\mathbb{R}^N$.

Theorem 3.4. *Let $\mathcal{Q}$ denote a nonempty set and consider the mappings $A(\cdot, q) \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, $q \in \mathcal{Q}$, $B \in \mathcal{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, reals $C_1, C_2, K_1, K_2 \geqslant 1$ and functions a, b, $c_1, c_2, d_2 \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \lhd b$, b bounded above, c_1, c_2 discretely bounded below, such that for any $q \in \mathcal{Q}$ the following conditions hold*:

(i) *The linear system*
$$x^\Delta = A(t, q)x \tag{3.21}$$
has c_1^+-bounded growth with constant C_1,
(ii) *the linear system* (3.21) *possesses an ED with a, b, K_1, K_2 and the invariant projector $P_q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$,*
(iii) *the linear system*
$$x^\Delta = B(t)x \tag{3.22}$$
has (c_2, d_2)-bounded growth with constant C_2.

Moreover, for arbitrarily fixed functions $c, d \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ with
$$a \lhd c \lhd d \lhd b, \qquad \sup_{s \in \mathbb{T}} \xi_{\mu^*(s)}\big(c(s)\big) < \inf_{s \in \mathbb{T}} \xi_{\mu^*(s)}\big(d(s)\big), \tag{3.23}$$
we choose reals $0 < h_0 \leqslant h$, $\lceil \mu^* \rceil \leqslant h$ so large that

(iv) *$K_1 K_2 < E^-_{b \ominus a}(h_0, h)$, $K_1 < E^-_{c \ominus a}(h_0, h)$ and $K_2 < E^-_{b \ominus d}(h_0, h)$,*
(v) *$(\mathbb{T}, \preceq, \mu)$ is a (h_0, h)-measure chain.*

Then there exist reals $\epsilon_0, \epsilon_1 > 0$, depending on $h_0, h, a, b, c, c_1, c_2, d, d_2, C_1, C_2, K_1, K_2$, such that under the additional assumption

(vi) *there exists a mapping $q_* : \mathbb{T} \to \mathcal{Q}$ with*

$$\big\| A\big(t, q_*(\tau)\big) - B(t) \big\| \leqslant \epsilon_0 \quad \textit{for } t, \tau \in \mathbb{T},\ 0 \leqslant \mu(t,\tau) \leqslant h, \tag{3.24}$$

$$\big\| P_{q_*(t)}(t) - P_{q_*(\tau)}(t) \big\| \leqslant \epsilon_1 \quad \textit{for } t, \tau \in \mathbb{T},\ h_0 \leqslant \mu(t,\tau) \leqslant h, \tag{3.25}$$

also the linear dynamic equation (3.22) *possesses an ED with* $\bar{c}, \bar{d} : \mathbb{T} \to \mathbb{R}$ *given in* (3.19), *constants* $\bar{L}_1, \bar{L}_2 \geqslant 1$ *and an invariant projector* $Q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, *satisfying*

$$\big\| Q(t) - P_{q_*(t)}(t) \big\| \leqslant \epsilon_1 + \big\| Q(t) - P_{q_*(\tau)}(t) \big\| \tag{3.26}$$

for $t, \tau \in \mathbb{T}$, $h_0 \leqslant \mu(t,\tau) \leqslant h$.

Remark 3.1. (1) In general we have the inequalities $c \trianglelefteq \bar{c}$, $\bar{d} \trianglelefteq d$ and thus the exponential dichotomy with growth functions $\bar{c}, \bar{d}$ guaranteed from Theorem 3.4 is weaker than a dichotomy with c, d. Nevertheless, one has $c = \bar{c}, d = \bar{d}$ for the special case of the time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \bar{h}\mathbb{Z}$, $\bar{h} > 0$, and constant functions c, d, like usually assumed for ODEs and OΔEs. In particular, under these assumptions the right inequality in (3.23) becomes redundant. Moreover, for $\mathbb{T} = \bar{h}\mathbb{Z}$, $\bar{h} > 0$, we can replace hypothesis (v) by the inequality $\bar{h} \leqslant h_0$, while (v) can be dropped in case of $\mathbb{T} = \mathbb{R}$. A similar remark also holds for the subsequent Corollary 3.6.

(2) Even in the special case of ODEs, our Theorem 3.4 generalizes [16, Theorem 1] with regard to the following aspects: On the one hand, Theorem 3.4 holds true in infinite dimensional Banach spaces, we only need that (3.21) has bounded growth in forward time, and finally, beyond the inequalities (3.23) we do not assume any hyperbolicity conditions on the growth functions c, d.

(3) For a set $\mathcal{Q}$ with exactly one element, the inequality (3.25) is redundant and one can consider Theorem 3.4 as a roughness theorem for exponentially dichotomous systems with bounded growth. However, on discrete measure chains, Theorem 2.4 is more general then Theorem 3.4.

(4) In case of homogeneous time scales it is possible to derive a relatively handy explicit estimate for the maximal size of ϵ_0, ϵ_1 in terms of the growth constants for (3.21), the dichotomy data for (3.22), as well as $h_0, h > 0$. This can be found in [13, pp. 125–126, Korollar 2.3.10] or in [14] for OΔEs.

Proof of Theorem 3.4. Let $\Phi_A(\cdot\,; q)$, $q \in \mathcal{Q}$, denote the parameter-dependent transition operator of (3.21). We subdivide the present proof into four steps:

(I) Since b and, by virtue of (3.23) also the growth function $d \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$, is bounded above, we obtain that a, d are discretely bounded above and the inequalities $0 \lhd b \ominus a$, $0 \lhd c \ominus a$, $0 \lhd b \ominus d$. Due to Lemma 1.1(a) one can choose $h_0 > 0$ so large that the assumption (iv) is satisfied. Eventually, we pick reals $0 < \theta_1 < 1 < \theta_2$, such that $(\theta_2/\theta_1) K_1 K_2 < E^-_{b \ominus a}(h_0, h)$ holds.

(II) Let $s \in \mathbb{T}$ be arbitrary, but fixed. Then, due to assumption (ii), the linear dynamic equation

$$x^\Delta = A\big(t, q_*(s)\big)x \tag{3.27}$$

has an exponential dichotomy with an invariant projector $P_{q_*(s)} : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, which in particular satisfies the regularity condition (2.5) on $\mathbb{T}$. Hence [12, Proposition 2.3] guarantees that $\Phi_A(t, s; q_*(s))|_{\mathcal{N}(P_{q_*(s)}(s))} : \mathcal{N}(P_{q_*(s)}(s)) \to \mathcal{N}(P_{q_*(s)}(t))$, $s \preceq t$, is bijective.

Thus, for any $\xi \in \mathcal{N}(P_{q_*(s)}(t))$, $s \preceq t$, there exists a pre-image $\xi_0 \in \mathcal{N}(P_{q_*(s)}(s))$ with $\xi = \Phi_A(t,s;q_*(s))\xi_0$ and consequently we have the inclusion

$$\mathcal{N}\big(P_{q_*(s)}(t)\big) \subseteq \mathcal{R}\big(\Phi_A\big(t,s;q_*(s)\big)\big) \quad \text{for } s \preceq t. \tag{3.28}$$

(III) By assumption (v) we know that $(\mathbb{T}, \preceq, \mu)$ is a (h_0, h)-measure chain, and therefore for any $t_0 \in \mathbb{T}$ we get a discrete measure chain $\tilde{\mathbb{T}} = \{t_k\}_{k\in\mathbb{Z}} \in \mathbb{S}_{h_0}^h(\mathbb{T})$. We are going to verify that the operator sequences $\hat{A}, \hat{B}, \hat{P}_1, \hat{P}_2 : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$, $\hat{A}(k) := \Phi_A(t_{k+1}, t_k; q_*(t_k))$, $\hat{B}(k) := \Phi_B(t_{k+1}, t_k)$, $\hat{P}_1(k) := P_{q_*(t_k)}(t_k)$, $\hat{P}_2(k) := P_{q_*(t_{k-1})}(t_k)$ satisfy the assumptions of Lemma 3.2. Obviously $\hat{P}_1(k), \hat{P}_2(k) \in \mathcal{L}(\mathcal{X})$ are projections for every $k \in \mathbb{Z}$. Furthermore, we have $\hat{P}_2(k+1)\hat{A}(k) = \hat{A}(k)\hat{P}_1(k)$ for $k \in \mathbb{Z}$ and due to the inclusion (3.28) also $\mathcal{N}(\hat{P}_2(k+1)) \subseteq \mathcal{R}(\hat{A}(k))$ for $k \in \mathbb{Z}$. Now we define the functions $\tilde{a}, \tilde{b} : \tilde{\mathbb{T}} \to \mathbb{R}$,

$$\tilde{a}(t_k) := \frac{K_1 e_a(t_{k+1}, t_k) - \theta_1}{\theta_1 \mu(t_{k+1}, t_k)}, \quad \tilde{b}(t_k) := \frac{e_b(t_{k+1}, t_k) - \theta_2 K_2}{\theta_2 K_2 \mu(t_{k+1}, t_k)} \quad \text{for } k \in \mathbb{Z},$$

which satisfy $\tilde{a}, \tilde{b} \in \mathcal{C}_{\mathrm{rd}}^+\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$, as well as $\tilde{a} \lhd \tilde{b}$. Since b is bounded above, Lemma 1.1(b) guarantees that $\tilde{b}$ is bounded above. From assumption (ii) and

$$\begin{aligned}
\big\|\hat{A}(k)\eta\big\| &= \big\|\hat{A}(k)\hat{P}_1(k)\eta\big\| = \big\|\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big)P_{q_*(t_k)}(t_k)\eta\big\| \\
&\overset{(2.6)}{\leqslant} K_1 e_a(t_{k+1}, t_k)\|\eta\| \quad \text{for } \eta \in \mathcal{R}\big(\hat{P}_1(k)\big), \\
\|\xi\| &= \big\|\bar{\Phi}_A\big(t_k, t_{k+1}; q_*(t_k)\big)\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big)\big[I_{\mathcal{X}} - P_{q_*(t_k)}(t_k)\big]\xi\big\| \\
&= \big\|\bar{\Phi}_A\big(t_k, t_{k+1}; q_*(t_k)\big)\big[I_{\mathcal{X}} - P_{q_*(t_k)}(t_{k+1})\big]\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big)\xi\big\| \\
&\overset{(2.7)}{\leqslant} K_2 e_b(t_k, t_{k+1})\big\|\hat{A}(k)\xi\big\| \quad \text{for } \xi \in \mathcal{N}\big(\hat{P}_1(k)\big),
\end{aligned}$$

the above construction of $\tilde{a}, \tilde{b}$ yields $\|\hat{A}(k)\eta\| \leqslant \theta_1(1 + \mu(t_{k+1}, t_k)\tilde{a}(t_k))\|\eta\|$ for $\eta \in \mathcal{R}(\hat{P}_1(k))$, $\|\hat{A}(k)\xi\| \geqslant \theta_2(1 + \mu(t_{k+1}, t_k)\tilde{b}(t_k))\|\xi\|$ for $\xi \in \mathcal{N}(\hat{P}_1(k))$. Since the assumption (ii) implies for any $q \in \mathcal{Q}$ that $\|P_q(s)\| \leqslant K_1$, $\|I_{\mathcal{X}} - P_q(s)\| \leqslant K_2$ for $s \in \mathbb{T}$, one directly has $\|\hat{P}_1(k)\| \leqslant K_1$, $\|I_{\mathcal{X}} - \hat{P}_2(k)\| \leqslant K_2$, $\|\hat{P}_2(k)\| \leqslant K_1$ for $k \in \mathbb{Z}$. Finally, from assumption (i) we get

$$\big\|\hat{A}(k)\big\| = \big\|\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big)\big\| \leqslant C_1 e_{c_1}(t_{k+1}, t_k) \leqslant C_1 E_{c_1}^+(h_0, h)$$

for $k \in \mathbb{Z}$, and assumption (iv) together with Lemma 2.1 leads to

$$\begin{aligned}
\big\|\hat{A}(k) - \hat{B}(k)\big\| &= \big\|\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big) - \Phi_B(t_{k+1}, t_k)\big\| \\
&\overset{(3.24)}{\leqslant} \frac{C_1^2 \epsilon_0}{\Gamma_-(c_1 + \epsilon_0 C_1)} h E_{c_1+\epsilon_0 C_1}^+(h_0, h) \quad \text{for } k \in \mathbb{Z},
\end{aligned} \tag{3.29}$$

as well as $\|\hat{P}_1(k) - \hat{P}_2(k)\| = \|P_{q_*(t_k)}(t_k) - P_{q_*(t_{k-1})}(t_k)\| \leqslant \epsilon_1$ for $k \in \mathbb{Z}$ (cf. (3.25)). Now $\tilde{c}, \tilde{d} : \tilde{\mathbb{T}} \to \mathbb{R}$,

$$\tilde{c}(t_k) := \frac{e_c(t_{k+1}, t_k) - 1}{\mu(t_{k+1}, t_k)}, \quad \tilde{d}(t_k) := \frac{e_d(t_{k+1}, t_k) - 1}{\mu(t_{k+1}, t_k)}, \quad k \in \mathbb{Z},$$

define functions in $\mathcal{C}_{\mathrm{rd}}^+\mathcal{R}(\tilde{\mathbb{T}}, \mathbb{R})$, which satisfy $\tilde{a} \lhd \tilde{c} \lhd \tilde{d} \lhd \tilde{b}$ by means of the assumption (iv).

(IV) As a result of step (III), for sufficiently small reals $\epsilon_0, \epsilon_1 > 0$, one can apply Lemma 3.2 and therefore the system

$$x^{\Delta} = \tilde{B}(t)x, \qquad \tilde{B}(t_k) := \frac{1}{\mu(t_{k+1}, t_k)}\big(\hat{B}(k) - I_{\mathcal{X}}\big), \quad k \in \mathbb{Z},$$

on $\tilde{\mathbb{T}}$ has an exponential dichotomy with $\tilde{c}, \tilde{d}, \bar{L}_1, \bar{L}_2 \geqslant 1$ and an invariant projector $\tilde{Q}_{t_0} : \tilde{\mathbb{T}} \to \mathcal{L}(\mathcal{X})$. The estimate (3.23) implies that d is bounded above and since $t_0 \in \mathbb{T}$, as well as the discrete measure chain $\tilde{\mathbb{T}} \in \mathbb{S}^h_{h_0}(\mathbb{T})$ had been arbitrary, Lemma 3.3 implies an exponential dichotomy of the linear system (3.22) on $\mathbb{T}$. Ultimately, the estimate (3.26) is a trivial consequence of (3.25). □

We have formulated hypothesis (vi) of Theorem 3.4 using the coefficient mappings of (3.21) and (3.22) to increase its applicability. In some situations it is desirable, though, to assume conditions on the transition operators or the $\mathcal{L}^1$-distance of the two linear systems.

Corollary 3.5. *The assumed inequality* (3.24) *can be replaced by*

$$\big\|\Phi_A\big(t, \tau; q_*(\tau)\big) - \Phi_B(t, \tau)\big\| \leqslant \epsilon_0 \quad \text{for } t, \tau \in \mathbb{T},\ 0 \leqslant \mu(t, \tau) \leqslant h, \tag{3.30}$$

or, in case $c_1 = c_2$, *by*

$$\int_{\tau}^{t} \frac{\|A(s; q_*(s)) - B(s)\|}{1 + \mu^*(s)c_1(s)}\,\Delta s \leqslant \epsilon_0 \quad \text{for } t, \tau \in \mathbb{T},\ 0 \leqslant \mu(t, \tau) \leqslant h, \tag{3.31}$$

without changing the conclusion of Theorem 3.4.

Remark 3.2. The three papers [8, Theorem 3.1], [11, Theorem 2] and [18, Corollary 2] prove roughness theorems for an exponential dichotomy of finite dimensional differential equations under assumptions similar to (3.30). In this situation, Theorem 3.4 is sufficient for [8, Theorem 3.1] and equivalent to [11, Theorem 2], like shown in [15].

Proof of Corollary 3.5. Under each assumption, either (3.30) or (3.31), one is able to derive the estimate (3.29) in the proof of Theorem 3.4. Actually we have

$$\big\|\hat{A}(k) - \hat{B}(k)\big\| = \big\|\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big) - \Phi_B(t_{k+1}, t_k)\big\| \overset{(3.30)}{\leqslant} \epsilon_0 \quad \text{for } k \in \mathbb{Z},$$

or using Lemma 2.2, we obtain

$$\begin{aligned}
\big\|\hat{A}(k) - \hat{B}(k)\big\| &= \big\|\Phi_A\big(t_{k+1}, t_k; q_*(t_k)\big) - \Phi_B(t_{k+1}, t_k)\big\| \\
&\leqslant C_1 C_2 e_{c_1}(t_{k+1}, t_k) \int_{t_k}^{t_{k+1}} \frac{\|A(s; q_*(s)) - B(s)\|}{1 + \mu^*(s)c_1(s)}\,\Delta s \\
&\overset{(3.31)}{\leqslant} \epsilon_0 C_1 C_2 E^+_{c_1}(h_0, h)
\end{aligned}$$

for $k \in \mathbb{Z}$, and therefore only the condition determining the size of $\epsilon_0 > 0$ changes, but not the assertion of Theorem 3.4. □

At first glance the technical and abstract Theorem 3.4 might be a little hard to grasp. For that reason we apply it to derive a result showing that the notion of an exponential dichotomy is robust under slowly varying coefficients. More precisely, this result essentially states that, if an exponentially dichotomous system depends Hölder-continuously on a fixed parameter, then this parameter can be replaced by a time-dependent function possessing a sufficiently small global Hölder constant, without destroying the ED of the dynamic equation.

Corollary 3.6. *Consider some metric space* $(\mathcal{Q}, d)$*, an rd-continuous mapping* $A : \mathbb{T} \times \mathcal{Q} \to \mathcal{L}(\mathcal{X})$*, reals* $K_1, K_2 \geqslant 1$, $C_1, C_2, L \geqslant 0$, $\alpha, \beta \in (0, 1]$ *and functions* $a, b, c_1, c_2, d_2 \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$*,* $a \lhd b$*,* b *bounded above, such that for any* $q \in \mathcal{Q}$ *the following conditions hold*:

(i) *We have the Hölder estimate*

$$\left\|A(t, q) - A(t, \bar{q})\right\| \leqslant L d(q, \bar{q})^\alpha \quad \text{for } t \in \mathbb{T},\ \bar{q} \in \mathcal{Q}, \tag{3.32}$$

(ii) *the linear system* (3.21) *has* c_1^+*-bounded growth with constant* C_1*,*

(iii) *the linear system* (3.21) *possesses an ED with* a*,* b*,* K_1*,* K_2 *and the invariant projector* $P_q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$*.*

Moreover, for arbitrarily fixed functions $c, d \in \mathcal{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ *like in* (3.23)*, we choose reals* $0 < h_0 \leqslant h$*,* $\lceil \mu^* \rceil \leqslant h$ *so large that*

(iv) $K_1 K_2 < E^-_{b \ominus a}(h_0, h)$*,* $K_1 < E^-_{c \ominus a}(h_0, h)$ *and* $K_2 < E^-_{b \ominus d}(h_0, h)$*,*

(v) $(\mathbb{T}, \preceq, \mu)$ *is a* (h_0, h)*-measure chain.*

Then there exist reals $\epsilon_0, \epsilon_1 > 0$*, depending only on* $h_0, h, a, b, c, d, c_1, c_2, d_2, C_1, C_2, K_1, K_2$*, such that for any mapping* $q_* : \mathbb{T} \to \mathcal{Q}$ *satisfying*

(vi) *the Hölder condition*

$$d\big(q_*(t), q_*(\tau)\big) \leqslant \theta \big|\mu(t, \tau)\big|^\beta \quad \text{for } t, \tau \in \mathbb{T}, \tag{3.33}$$

where $\theta \geqslant 0$ *satisfies* $L\theta^\alpha h^{\alpha\beta} \leqslant \epsilon_0$*,* $L\theta^\alpha h^{\alpha\beta} \max\{K_1, K_2\} C_{a,b}(c, d) \leqslant \epsilon_1$*,*

(vii) *the linear system*

$$x^\Delta = A\big(t, q_*(t)\big)x \tag{3.34}$$

has (c_2, d_2)*-bounded growth with* C_2*,*

also the linear system (3.34) *has an ED with* $\bar{c}, \bar{d} : \mathbb{T} \to \mathbb{R}$ *given in* (3.19)*,* $\bar{L}_1, \bar{L}_2 \geqslant 1$ *and an invariant projector* $Q : \mathbb{T} \to \mathcal{L}(\mathcal{X})$*.*

Remark 3.3. (1) The property that $q_* : \mathbb{T} \to \mathcal{Q}$ changes slowly in time has been formulated using the Hölder condition (3.33). In case of a Banach space $\mathcal{Q}$ and a differentiable mapping q_*, one can use the mean value theorem on measure chains (cf. [7, pp. 16–17, Corollary 3.3(i)]) to show that (3.33) is satisfied with $\beta = 1$, if the derivative $q_*^\Delta : \mathbb{T} \to \mathcal{Q}$

has sufficiently small values. This is usually fulfilled in applications from singular perturbation theory (cf. [13, pp. 219–226] for dynamic equations on measure chains, or [14] for OΔEs).

(2) One can also use Corollary 3.6 as a criterion for an exponential dichotomy of the linear system (2.1). In fact, one assumes that

- $\bar{h} \leqslant \mu^*(t) \leqslant \bar{H}$ for all $t \in \mathbb{T}$ with certain reals $\bar{h}, \bar{H} > 0$,
- there exist reals $\bar{\alpha} < \bar{\beta}$, $\bar{\alpha} \in \mathbb{R}_{\bar{h}}$, such that the spectrum of $A(t_0) \in \mathcal{L}(\mathcal{X})$, $t_0 \in \mathbb{T}$, can be decomposed into closed disjoint sets $\sigma_1(t_0), \sigma_2(t_0)$ with

$$\sup_{\lambda \in \sigma_1(t_0)} \Re_{\bar{H}} \lambda < \bar{\alpha} < \bar{\beta} < \inf_{\lambda \in \sigma_2(t_0)} \Re_{\bar{h}} \lambda \quad \text{for } t_0 \in \mathbb{T},$$

and gets from [13, p. 97, Satz 2.1.22] that the time-invariant systems $x^\Delta = A(t_0)x$, $t_0 \in \mathbb{T}$ fixed, possess an exponential dichotomy. Here

$$\Re_h z := \lim_{t \searrow h} \frac{|1 + tz| - 1}{t}, \quad z \in \mathbb{C}, \text{ with } 1 + hz \neq 0,$$

is the *Hilger real part*. Now the above Corollary 3.6 with $\mathcal{Q} = \mathbb{T}$, the metric $d(t,\tau) := |\mu(t,\tau)|$, as well as $q_*(t) := t$, implies that (2.1) possesses an exponential dichotomy under the assumption $\|A(t) - A(\tau)\| \leqslant L|\mu(t,\tau)|^\alpha$ for $t, \tau \in \mathbb{T}$ and a sufficiently small $L \geqslant 0$.

Proof of Corollary 3.6. We successively verify the hypotheses of Theorem 3.4 applied to the mapping $B(t) := A(t, q_*(t))$. Due to the assumption (vi) we know that $q_* : \mathbb{T} \to \mathcal{Q}$ is continuous and consequently $B : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is rd-continuous. The assumptions (ii) and (vii) imply that the two systems (3.21) and (3.34) have bounded growth, and (vii) includes that (3.34) is regressive. In order to derive the inequalities (3.24) and (3.25), we pick $t_1, t_2 \in \mathbb{T}$ arbitrarily, use (3.32), (3.33) and arrive at

$$\left\| A\bigl(t, q_*(t_1)\bigr) - A\bigl(t, q_*(t_2)\bigr) \right\| \leqslant L\theta^\alpha h^{\alpha\beta} \quad \text{for } t \in \mathbb{T},\ 0 \leqslant \mu(t_1, t_2) \leqslant h. \tag{3.35}$$

Setting $t_1 = t$, $t_2 = \tau$ yields (3.24). Using the hypothesis (iii) we know that the linear system $x^\Delta = A(t, q_*(t_1))x$ has an exponential dichotomy with a, b, K_1, K_2 and $P_{q_*(t_1)}$. Similarly, $x^\Delta = A(t, q_*(t_2))x$ has an exponential dichotomy with the invariant projector $P_{q_*(t_2)}$, and weaker growth functions c, d. The relation (3.35), as well as Lemma 2.3 imply for $t_1 = t$, $t_2 = \tau$ the estimate

$$\left\| P_{q_*(t)}(t) - P_{q_*(\tau)}(t) \right\| \leqslant L\theta^\alpha h^{\alpha\beta} \max\{K_1, K_2\} C_{a,b}(c,d)$$

for $t, \tau \in \mathbb{T}$, $h_0 \leqslant \mu(t,\tau) \leqslant h$, and using Theorem 3.4 we obtain the assertion. □

References

[1] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, in: G.A. Leonov, et al. (Eds.), Nonlinear Dynamics and Quantum Dynamical Systems, Akademie-Verlag, Berlin, 1990, pp. 9–20.

[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales—An Introduction with Applications, Birkhäuser, Boston, 2001.

[3] W.A. Coppel, Dichotomies in Stability Theory, in: Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, Berlin, 1978.
[4] C.V. Coffman, J.J. Schäffer, Dichotomies for linear difference equations, Math. Ann. 172 (1967) 139–166.
[5] J.L. Daleckiĭ, M.G. Kreĭn, Stability of Solutions of Differential Equations in Banach Space, in: Translations of Mathematical Monographs, Vol. 43, American Mathematical Society, Providence, RI, 1974.
[6] D. Henry, Geometric Theory of Semilinear Parabolic Equations, in: Lecture Notes in Mathematics, Vol. 840, Springer-Verlag, Berlin, 1980.
[7] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[8] R.A. Johnson, Remarks on linear differential systems with measurable coefficients, Proc. Amer. Math. Soc. 100 (1987) 491–504.
[9] J. Kalkbrenner, Exponential Dichotomy and Chaotic Dynamic of Noninvertible Difference Equations, Ph.D. thesis, University of Augsburg, 1994, in German.
[10] J.S. Muldowney, Dichotomies and asymptotic behaviour for linear differential systems, Trans. Amer. Math. Soc. 283 (1984) 465–484.
[11] K.J. Palmer, A perturbation theorem for exponential dichotomies, Proc. Roy. Soc. Edinburgh Sect. A 106 (1987) 25–37.
[12] C. Pötzsche, Exponential dichotomies for linear dynamic equations, Nonlinear Anal. 47 (2001) 873–884.
[13] C. Pötzsche, Slow Fiber Bundles of Dynamic Equations on Measure Chains, Ph.D. thesis, University of Augsburg, 2002, in German.
[14] C. Pötzsche, Slow and fast variables in nonautonomous difference equations, J. Difference Equations Appl. 9 (2003) 473–487.
[15] K. Sakamoto, A remark on perturbation theorems for exponential dichotomies, private correspondence, March 2000.
[16] K. Sakamoto, Estimates on the strength of exponential dichotomies and application to integral manifolds, J. Differential Equations 107 (1994) 259–279.
[17] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differential Equations 27 (1978) 320–358.
[18] N. Van Minh, Spectral theory for linear non-autonomous differential equations, J. Math. Anal. Appl. 187 (1994) 339–351.