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Construction of circle bifurcations of a two-dimensional spatially periodic flow

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Abstract

The study by Yudovich [V.I. Yudovich, Example of the generation of a secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid, J. Math. Mech. 29 (1965) 587–603] on spatially periodic flows forced by a single Fourier mode proved the existence of two-dimensional spectral spaces and each space gives rise to a bifurcating steady-state solution. The investigation discussed herein provides a structure of secondary steady-state flows. It is constructed explicitly by an expansion that when the Reynolds number increases across each of its critical values, a unique steady-state solution bifurcates from the basic flow along each normal vector of the two-dimensional spectral space. Thus, at a single Reynolds number supercritical value, the bifurcating steady-state solutions arising from the basic solution form a circle.

Keywords: Steady-state bifurcation; Pitchfork bifurcation; Navier–Stokes equation; Kolmogorov flow

1. Introduction

Navier–Stokes equations mathematically model viscous fluid flows and from this theoretical base it has been proved (see Krasnoselskii [7], Rabinowitz [15], Nirenberg [11]) that secondary steady-state bifurcation flows arise from the basic flow when the Reynolds number varies across critical values under the condition that the corresponding real critical eigenvalue is an odd al-

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gebraic multiplicity. In Taylor (see [4,13]), Bénard (see [5,14]) and Kolmogorov (see [2,10,17]) problems, there exist a flow invariant space, in which the corresponding real critical eigenvalue has a single algebraic multiplicity or the eigenvalue is simple, and thus a bifurcating steady-state solution arises.

In this study, we examine the bifurcation problem of the Kolmogorov model [2] in which a spatially periodic flow is forced by a single Fourier mode. The model is expressed in the dimensionless form of the Navier–Stokes equations

$$\left.\begin{aligned} &\partial_t u - \Delta u + R\, u\cdot\nabla u + \nabla p = (\sin y, 0),\\ &u|_{x=0} = u|_{x=2\pi/\alpha}, \quad u|_{y=0} = u|_{y=2\pi},\\ &\nabla\cdot u = 0,\\ &\int_0^{2\pi}\int_0^{2\pi/\alpha} u\,dx\,dy = 0 \end{aligned}\right\} \tag{1}$$

in the domain $\Omega = (0, 2\pi/\alpha)\times(0, 2\pi)$ for $0<\alpha<1$, where the Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$, the gradient operator $\nabla = (\partial_x, \partial_y)$, $u = (u_1, u_2)$ and p denote the unknown fluid velocity and pressure, and R represents the Reynolds number controlling the dynamical behaviour of the fluid motion.

Yudovich [17] demonstrated that the critical eigenvector spaces of the linearized equation of (1) are two-dimensional and established a flow invariant subspace of (1), in which the reduced spectral problem has a simple critical eigenvalue. He proved the existence of a bifurcating solution using the Krasnoselskii theorem [7]. However, the dynamical behaviour close to a bifurcation point still remains unclear as discussed by Kirchgässner [6] in a review of bifurcation analyses associated with Navier–Stokes flows. That is, by using some invariant properties inherent to the Navier–Stokes equations, one reduces the underlying space until the critical eigenvalue is simple. Although such a method can give a rich set of solutions, any knowledge of their inter-relationship is lost.

It is the purpose of this investigation to construct the secondary flows with respect to a two-dimensional critical eigenvector space, of which each unit vector gives rise to a unique bifurcating steady-state solution, and to understand the inter-relationship of the bifurcating secondary flows. It is demonstrated that the bifurcation steady-state solutions form a circle in the mathematical space at a supercritical value of the Reynolds number. It is verified that each critical eigenvector generates a flow invariant space, in which the critical eigenvalue is simple. We can therefore construct the bifurcating solutions by an amplitude expansion technique as discussed in [6].

Recently, an analysis on the existence of bifurcating attractors has been established by Ma and Wang [8] with respect to any dimension of eigenvector spaces. This theory might be applicable to the present fluid motion problem. However, the approach of our study is rather different to the analysis of Ma and Wang [8].

2. Statement of the main result

For convenience, we apply the curl operator to (1) to obtain the vorticity equation

$$\partial_t\Delta\psi - \Delta^2\psi + R\,J(\psi, \Delta\psi) = \cos y, \tag{2}$$

$$\int_0^{2\pi}\int_0^{2\pi/\alpha} \psi\,dx\,dy = 0 \tag{3}$$

together with the periodic boundary condition

$$\psi|_{x=0} = \psi|_{x=2\pi/\alpha}, \qquad \psi|_{y=0} = \psi|_{y=2\pi}. \tag{4}$$

Here $J(\varphi, \phi) = \partial_y\varphi\,\partial_x\phi - \partial_x\varphi\,\partial_y\phi$ is the advection operator, $\psi = \psi(x, y, t)$ is the stream function and the fluid flow velocity is defined by

$$u = (\partial_y\psi, -\partial_x\psi).$$

We see that $u_0 = (\sin y, 0)$ is a solution of (1) giving $\psi_0 = -\cos y$, the solution of (2)–(4).

It is convenient to use the real Hilbert space

$$H^4 = \{\phi \in L_2(\Omega);\ \Delta^2\phi \in L_2(\Omega),\ \phi \text{ satisfies (3), (4)}\} \quad \text{with } \|\phi\|_{H^4} = \|\Delta^2\phi\|_{L_2},$$

the L_2 inner product

$$\langle\psi, \phi\rangle = \int_0^{2\pi}\int_0^{2\pi/\alpha} \psi\phi\,dx\,dy,$$

and the L_q-norm

$$\|\phi\|_{L_q} = \left(\int_0^{2\pi}\int_0^{2\pi/\alpha} |\phi|^q\,dx\,dy\right)^{1/q}.$$

The substitution of $\psi = \psi_0 + \phi(x, y)e^{t\rho}$ into (2)–(4) and the omission of the nonlinear term produce the spectral problem described by

$$L_R\phi = \rho\Delta\phi, \qquad L_R\phi \equiv \Delta^2\phi - R\sin y(\Delta + 1)\partial_x\phi, \tag{5}$$

of which, the conjugate spectral problem is expressed as

$$L_R^*\phi^* = \bar{\rho}\Delta\phi^*, \qquad L_R^*\phi^* \equiv \Delta^2\phi^* + R(\Delta + 1)\partial_x(\sin y\phi^*).$$

The main result of this study reads as follows:

Theorem 2.1. *For real value* $0 < \alpha < 1$ *and integer*

$$m_\alpha = \max\{k \geqslant 1;\ k\alpha < 1\},$$

Eqs. (2)–(4) *admit exactly* m_α *circle bifurcation points*

$$(\psi_0, R_{k\alpha}) \quad \text{for } k = 1, \dots, m_\alpha \text{ on } \{(\psi_0, R);\ R > 0\}.$$

The critical values of the Reynolds number satisfy the property

$$\sqrt{\frac{2}{1-\alpha^2}}(1 + \alpha^2) < R_\alpha < \cdots < R_{m_\alpha\alpha}, \qquad \lim_{\alpha\to 0} R_\alpha = \sqrt{2}, \qquad \lim_{\alpha\to 0} R_{m_\alpha\alpha} = \infty,$$

and the circles of bifurcating steady-state solutions

$$\{\psi = \psi_{k\alpha,\theta,R};\ 0 \leqslant \theta < 2\pi\}$$

at a supercritical Reynolds number value R *are locally represented by the expressions*

$$\psi_{k\alpha,\theta,R} = \psi_0 + \epsilon\phi_{k\alpha,\theta} + \sum_{n=2}^{\infty}\epsilon^n\phi_{n,k\alpha,\theta},$$
$$\frac{1}{R} = \frac{1}{R_{k\alpha}} + \sum_{n=1}^{\infty}\epsilon^{2n}\lambda_{2n,k\alpha} \tag{6}$$

for small $\epsilon > 0$, such that

$$\lambda_{2,k\alpha} < 0, \qquad \langle\psi_{k\alpha,\theta,R}, \Delta^2\phi^*_{k\alpha,\theta}\rangle = \epsilon\langle\phi_{k\alpha,\theta}, \Delta^2\phi^*_{k\alpha,\theta}\rangle \neq 0,$$

where numbers $\lambda_{2n,k\alpha}$ are independent of θ, functions $\phi_{n,k\alpha,\theta} \in H^4$,

$$|\lambda_{2n,k\alpha}| \leqslant c^{2n}, \quad \|\phi_{n,k\alpha,\theta}\|_{H^4} \leqslant c^n \quad \text{for some constant } c,$$

and $\phi_{k\alpha,\theta}$ and $\phi^*_{k\alpha,\theta}$ are the critical eigenvectors defined by the spectral problems

$$L_{R_{k\alpha}}\phi_{k\alpha,\theta} = 0,$$
$$\phi_{k\alpha,\theta} = \cos(k\alpha x + \theta) + \sum_{n=-\infty,\ n\neq 0}^{\infty}\eta_{n,k\alpha}\cos(k\alpha x + ny + \theta) \in H^4 \tag{7}$$

and

$$L^*_{R_{k\alpha}}\phi^*_{k\alpha,\theta} = 0,$$
$$\phi^*_{k\alpha,\theta} = \cos(k\alpha x + \theta) + \sum_{n=-\infty,\ n\neq 0}^{\infty}\eta^*_{n,k\alpha}\cos(k\alpha x + ny + \theta) \in H^4 \tag{8}$$

with $\eta_{n,k\alpha}$ and $\eta^*_{n,k\alpha}$ independent of θ.

The expression in (6) with $\lambda_{2,k\alpha} < 0$ implies the circle bifurcation is supercritical. That is, the circle of steady-state solutions branches off the basic solution ψ_0 when R increases across a critical value $R_{k,\alpha}$. The constants $\lambda_{2n,k\alpha}$ and the functions $\psi_{k\alpha,\theta,R}$ are defined inductively in Section 4.

The existence of the m_α critical values $\{R_{k\alpha}\}_{k=1}^{m_\alpha}$ was derived by Yudovich [17]. He showed that the critical eigenvector space with respect to each $R_{k\alpha}$ is two-dimensional which he further reduced to a simple flow invariance space of even functions $\psi(x, y) = \psi(-x, -y)$. Therefore Yudovich [17] proved the existence of a bifurcating steady-state solution by using the Krasnoselskii theorem [7]. By examining the steady-state solutions in this even function space, Okamoto and Shoji [12] displayed computational results illustrating the occurrence of two steady-state solutions bifurcating from $(\psi_0, R_{k\alpha})$ supercritically, and Matsuda and Miyatake [9] proved the existence of two steady-state solutions branching off $(\psi_0, R_{k\alpha})$ supercritically. The existence of four steady-state solutions branching off $(\psi_0, R_{k\alpha})$ is also implied from the study of Chen and Wang [3].

In fact, if we define the projection operator

$$Q_{k\alpha}\psi = \frac{\langle\psi, \Delta^2\phi^*_{k\alpha,0}\rangle}{\langle\phi_{k\alpha,0}, \Delta^2\phi^*_{k\alpha,0}\rangle}\phi_{k\alpha,0} + \frac{\langle\psi, \Delta^2\phi^*_{k\alpha,\pi/2}\rangle}{\langle\phi_{k\alpha,\pi/2}, \Delta^2\phi^*_{k\alpha,\pi/2}\rangle}\phi_{k\alpha,\pi/2}$$

mapping H^4 onto the two-dimensional critical eigenvector space

$$\operatorname{span}\{\phi_{k\alpha,0}, \phi_{k\alpha,\pi/2}\},$$

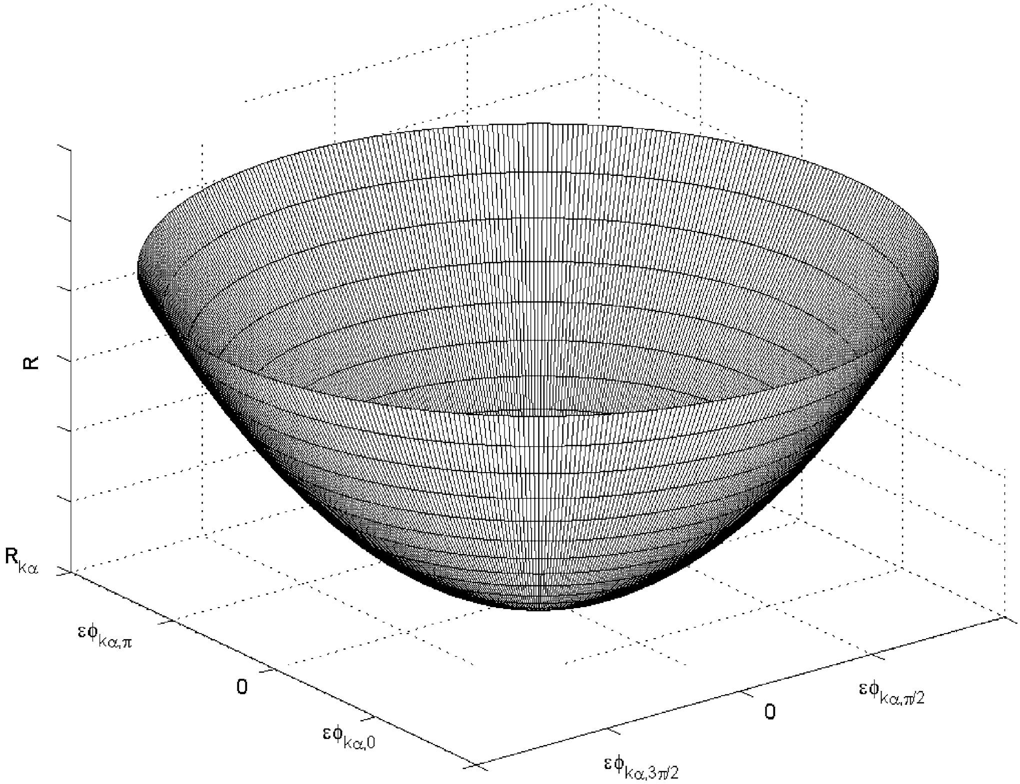

Fig. 1. Schematic circle bifurcation diagram over the critical two-dimensional eigenvector space span$\{\phi_{k\alpha,0}, \phi_{k\alpha,\pi/2}\}$ for $k = 1, \ldots, m_\alpha$.

we produce the circles

$$Q_{k\alpha}\psi_{k\alpha,\theta,R} = \epsilon\phi_{k\alpha,\theta} = \epsilon\cos\theta\phi_{k\alpha,0} + \epsilon\sin\theta\phi_{k\alpha,\pi/2} \quad \text{for } 0 \leqslant \theta < 2\pi,$$

or

$$\langle Q_{k\alpha}\psi_{k\alpha,\theta,R}, Q_{k\alpha}\psi_{k\alpha,\theta,R}\rangle = \epsilon^2\|\phi_{k\alpha,0}\|^2 \quad \text{for } 0 \leqslant \theta < 2\pi.$$

Observing that $\lambda_{2,k\alpha} < 0$ and $1/R_{k\alpha} - 1/R = O(\epsilon^2)$, we may write

$$\epsilon = O\left(\sqrt{\frac{1}{R_{k\alpha}} - \frac{1}{R}}\right).$$

Thus the circle bifurcation diagram illustrated in Fig. 1 is produced for any integer $k = 1, \ldots, m_\alpha$.

In fact, the m_α circles of bifurcating steady-state solutions

$$\{\psi_{k\alpha,\theta,R};\ 0 \leqslant \theta < 2\pi\}, \quad k = 1, \ldots, m_\alpha,$$

exist globally for $R_{k\alpha} < R < \infty$ due to Rabinowitz global bifurcation theorem [15].

Based on spectral analysis and preliminary discussions in Section 3, we prove Theorem 2.1 and discuss the global branches of bifurcating circles of steady-state solutions in Section 4.

3. Preliminary discussions

3.1. Spectral problem

The spectral problem (5) was solved by Meshalkin and Sinai [10] and Yudovich [17]. By using a continued fraction approach [10], Yudovich [17] showed exactly m_α eigenvector spaces, which are two-dimensional. Meshalkin and Sinai [10] showed the nonexistence of nonreal critical eigenvalues. More exactly, their study of (5) can be expressed as follows.

Lemma 3.1 *(Meshalkin and Sinai [10]). For real value $\alpha \geqslant 0$, the critical spectral problem written in the form*

$$\left.\begin{aligned} &L_R\phi = \rho\Delta\phi, \quad \Re\rho = 0, \\ &\phi = \cos(\alpha x) + \sum_{n=-\infty,\, n\neq 0}^{\infty} \eta_{n,\alpha}\cos(\alpha x + ny), \quad \sum_{n\neq 0}|\eta_{n,\alpha}|^2 < \infty \end{aligned}\right\} \tag{9}$$

has no solution provided that $\Im\rho \neq 0$. Moreover, Eq. (9) *has no solution provided either $\alpha = 0$ or $\alpha \geqslant 1$.*

Based on a continued fraction approach [10], Yudovich [17] derived the result.

Lemma 3.2. *For real value $0 < \alpha < 1$, Eq.* (9) *with $\rho = 0$ has a unique solution $(\phi, R) = (\phi_\alpha, R_\alpha)$, which satisfies*

$$\begin{aligned} &\langle \phi_\alpha, \Delta^2\phi_\alpha^*\rangle \neq 0, \\ &\lim_{\alpha\to 0} R_\alpha = \sqrt{2}, \quad \lim_{\alpha\to 1} R_\alpha = \infty, \quad R_\alpha < R_{\alpha'} \quad \text{for } \alpha < \alpha' < 1, \\ &\eta_{\pm n,\alpha} = \frac{\alpha^2 - 1}{\alpha^2 + n^2 - 1}\gamma_{\pm 1}\cdots\gamma_{\pm n}, \quad n \geqslant 1, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\gamma_{\pm n} = \cfrac{\mp 1}{d_n + \cfrac{1}{d_{n+1} + \cfrac{1}{\ddots}}}, \\ &-\frac{d_0}{2} = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{\ddots}}}, \quad d_n = \frac{2(\alpha^2 + n^2)^2}{R_\alpha \alpha(\alpha^2 + n^2 - 1)} \end{aligned} \tag{11}$$

and $\phi_\alpha^* \in H^4$ is the solution of $L_{R_\alpha}^*\phi_\alpha^* = 0$, the conjugate equation of (9).

Yudovich [17] also proved that the critical eigenvector space of (5) with

$$(R, \rho) = (R_\alpha, 0)$$

is two-dimensional. By following his approach, we derive the result.

Lemma 3.3. *For real value* $0<\alpha<1$ *and* $\theta\geqslant 0$, *the critical spectral problem*

$$L_R\phi=0,\qquad \phi=\cos(\alpha x+\theta)+\sum_{n=-\infty,\ n\neq 0}^{\infty}\eta_{n,\alpha}\cos(\alpha x+ny+\theta)\in H^4 \tag{12}$$

has a unique solution $(\phi,R)=(\phi_{\alpha,\theta},R_\alpha)$ *such that*

$$\sqrt{\frac{2}{1-\alpha^2}}\big(1+\alpha^2\big)<R_\alpha,\qquad \lim_{\alpha\to 0}R_{m_\alpha\alpha}=\infty, \tag{13}$$

$$\big\langle\phi_{\alpha,\theta},\Delta^2\phi^*_{\alpha,\theta}\big\rangle\neq 0, \tag{14}$$

where the coefficients $\{\eta_{n,\alpha}\}$ *and the critical Reynolds number value* R_α *are uniquely defined in Lemma* 3.2, *and* $\phi^*_{\alpha,\theta}$, *defined in the form of* (8), *is the conjugate eigenfunction relating to* $\phi_{\alpha,\theta}$.

It is obvious that the spectral problem (12) is equivalent to the difference equation

$$2\big(\alpha^2+n^2\big)^2\eta_{n,\alpha}+R\alpha\big(\alpha^2+(n-1)^2-1\big)\eta_{n-1,\alpha}-R\alpha\big(\alpha^2+(n+1)^2-1\big)\eta_{n+1,\alpha}=0$$

with the initial condition $\eta_{0,\alpha}=1$, which is independent of the value $\theta\geqslant 0$. Thus the derivation of Lemma 3.2 with $\theta=0$ based on the continued fraction approach implies the validity of Lemma 3.3. The integration in (14) is independent of θ and hence Lemma 3.2 implies the validity of (14). It follows from Lemma 3.2 that (13) is an immediate consequence of the observation $\lim_{\alpha\to 0}m_\alpha\alpha=1$ and the inequality

$$-\frac{d_0}{2}<\frac{1}{d_1},$$

which is from (11).

3.2. Flow invariant spaces and a Fredholm decomposition

We now seek bifurcating solutions in the following complete subspaces of H^4:

$$H^4_{\alpha,\theta}=\Big\{\psi\in H^4;\ \psi=\sum_{m,n}\eta_{m,n}\cos(m\alpha x+ny+m\theta)\Big\},$$

where we define that

$$\sum_{m,n}\equiv\sum_{m=0}\sum_{n=1}^{\infty}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}.$$

Lemma 3.4. *For real value* $0<\alpha<1$, $\theta\geqslant 0$, *the following assertions hold true*:

(i) $\Delta^{-2}J(\psi_1,\Delta\psi_2)\in H^4_{\alpha,\theta}$, $\qquad \big\|J(\psi_1,\Delta\psi_2)\big\|_{L_2}\leqslant c\|\psi_1\|_{H^4}\|\psi_2\|_{H^4}$ *for* $\psi_i\in H^4_{\alpha,\theta}$.

(ii) *For* $f\in L_2(\Omega)$ *such that* $\Delta^{-2}f\in H^4_{\alpha,\theta}$ *and the projection operator*

$$P_{\alpha,\theta}\psi=\psi-\frac{\langle\psi,\Delta^2\phi^*_{\alpha,\theta}\rangle}{\langle\phi_{\alpha,\theta},\Delta^2\phi^*_{\alpha,\theta}\rangle}\phi_{\alpha,\theta},$$

the equation

$$\Delta^{-2}L_{R_\alpha}P_{\alpha,\theta}\phi=P_{\alpha,\theta}\Delta^{-2}f \tag{15}$$

has a unique solution $P_{\alpha,\theta}\phi = \phi \in H^4_{\alpha,\theta}$ *such that*

$$\|P_{\alpha,\theta}\phi\|_{H^4} = \left\| \left(\Delta^{-2} L_{R_\alpha}\right)^{-1} P_{\alpha,\theta} \Delta^{-2} f \right\|_{H^4} \leqslant c \|f\|_{L_2}.$$

Proof. (i) Let

$$\psi_i = \sum_{m,n} \eta^{(i)}_{m,n} \cos(m\alpha x + ny + m\theta) \in H^4_{\alpha,\theta} \quad \text{for } i = 1, 2.$$

Using the Sobolev imbedding principle (see, for example, [1]), we derive

$$\left\|\Delta^{-2} J(\psi_1, \Delta\psi_2)\right\|_{H^4} = \left\|J(\psi_1, \Delta\psi_2)\right\|_{L_2} \leqslant \|\nabla\psi_1\|_{L_\infty} \|\nabla\Delta\psi_2\|_{L_2} \leqslant c\|\psi_1\|_{H^4}\|\psi_2\|_{H^4}.$$

Letting $\varphi_{m,n} = \cos(m\alpha x + ny + m\theta)$ and $\hat{\varphi}_{m,n} = \sin(m\alpha x + ny + m\theta)$, we have

$$\begin{aligned} J(\psi_1, \Delta\psi_2) &= \sum_{m,n} \sum_{m',n'} \eta^{(1)}_{m,n} \eta^{(2)}_{m',n'} (\partial_y \varphi_{m,n} \partial_x \Delta\varphi_{m',n'} - \partial_x \varphi_{m,n} \partial_y \Delta\varphi_{m',n'}) \\ &= \sum_{m,n} \sum_{m',n'} (mn' - m'n)\alpha\left[(m'\alpha)^2 + n'^2\right] \eta^{(1)}_{m,n} \eta^{(2)}_{m',n'} \hat{\varphi}_{m,n} \hat{\varphi}_{m',n'} \\ &= \frac{\alpha}{2} \sum_{m,n} \sum_{m',n'} (mn' - m'n)\left[(m'\alpha)^2 + n'^2\right] \eta^{(1)}_{m,n} \eta^{(2)}_{m',n'} \varphi_{m-m',n-n'} \\ &\quad - \frac{\alpha}{2} \sum_{m,n} \sum_{m',n'} (mn' - m'n)\left[(m'\alpha)^2 + n'^2\right] \eta^{(1)}_{m,n} \eta^{(2)}_{m',n'} \varphi_{m+m',n+n'}. \end{aligned} \tag{16}$$

This becomes, after rearranging the coefficients,

$$J(\psi_1, \Delta\psi_2) = \sum_{m,n} \xi_{m,n} \cos(m\alpha x + ny + m\theta).$$

We thus have $\Delta^{-2} J(\psi_1, \Delta\psi_2) \in H^4_{\alpha,\theta}$.

(ii) By the definition of $H^4_{\alpha,\theta}$ and Lemmas 3.1, 3.2 and 3.3, the critical spectral problem

$$L_R\phi = 0, \quad \phi \in H^4,$$

has an eigenfunction solution if and only if when $R = R_{k\alpha}$ for $k = 1, \ldots, m_\alpha$. We note that $R_\alpha < R_{k\alpha}$ whenever $\alpha < k\alpha < 1$. Thus a function ϕ solves the critical spectral problem

$$L_{R_\alpha}\phi = 0, \quad \phi \in H^4_{\alpha,\theta},$$

if and only if $\phi = c\phi_{\alpha,\theta}$ for any constant c.

Since the operator

$$\Delta^{-2} L_{R_\alpha} - I = -R_\alpha \Delta^{-2} \sin y(\Delta + 1)\partial_x : H^4_{\alpha,\theta} \mapsto H^4_{\alpha,\theta}$$

is compact, it follows from Lemma 3.3 and the Fredholm alternative principle (see, for example, [16]) that (15) has a unique solution if and only if

$$\left\langle \Delta^2 P_{\alpha,\theta} \Delta^{-2} f, \Delta^2 \hat{\phi}_{\alpha,\theta} \right\rangle = 0,$$

where $\hat{\phi}_{\alpha,\theta} \neq 0$ is a conjugate eigenvector of the operator $\Delta^{-2} L_{R_\alpha}$ in $H^4_{\alpha,\theta}$. That is,

$$\hat{L}_{R_\alpha} \hat{\phi}_{\alpha,\theta} = 0, \quad \hat{\phi}_{\alpha,\theta} \in H^4_{\alpha,\theta},$$

where the conjugate operator $\hat{L}_{R_\alpha}$ is defined through the scalar product of $H^4_{\alpha,\theta}$ in the following sense:

$$\langle \Delta^2\Delta^{-2}L_{R_\alpha}\phi, \Delta^2\psi\rangle = \langle \Delta^2\phi, \Delta^2\hat{L}_{R_\alpha}\psi\rangle.$$

This implies $\hat{L}_{R_\alpha}\hat{\phi}_{\alpha,\theta} = \Delta^{-4}L^*_{R_\alpha}\Delta^2\hat{\phi}_{\alpha,\theta}$ and so $\Delta^2\hat{\phi}_{\alpha,\theta} = \phi^*_{\alpha,\theta}$. Hence the unique existence of (15) in the space $H^4_{\alpha,\theta}$ is proved whenever

$$\langle P_{\alpha,\theta}\Delta^{-2}f, \Delta^2\phi^*_{\alpha,\theta}\rangle = 0,$$

which is always valid due to the definition of the projection $P_{\alpha,\theta}$. We thus derive assertion (ii) and complete the proof.

The projection $P_{\alpha,\theta}$ gives the following Fredholm decomposition of the Hilbert space

$$H^4_{\alpha,\theta} = P_{\alpha,\theta}H^4_{\alpha,\theta} \oplus \operatorname{span}\{\phi_{\alpha,\theta}\}$$

and assertion (ii) shows the invertibility of the linear operator $\Delta^{-2}L_{R_\alpha}$ over the subspace $P_{\alpha,\theta}H^4_{\alpha,\theta}$. □

4. Construction of the circle bifurcation

4.1. Proof of Theorem 2.1

We construct the solutions $(\psi, R) = (\psi_{\alpha,\theta,R}, R)$ bifurcating from (ψ_0, R_α) with supercritical Reynolds number value R independent of $\theta \geqslant 0$. By the absence of critical oscillatory eigenfunctions due to Lemma 3.1, it suffices to construct bifurcating steady-state solutions of (2) and (3).

Firstly, let us formally construct the bifurcating solutions in $H^4_{\alpha,\theta}$. We rewrite the stationary equation of (2) and (3) in the form

$$\left(\frac{1}{R} - \frac{1}{R_\alpha}\right)(\psi - \psi_0) + \frac{1}{R_\alpha}\Delta^{-2}L_{R_\alpha}(\psi - \psi_0) = \Delta^{-2}J\big(\psi - \psi_0, \Delta(\psi - \psi_0)\big),$$
$$\psi \in H^4_{\alpha,\theta},$$

and use the amplitude expansion technique (see, for example, [6]) for the unknown bifurcation steady-state solution

$$\frac{1}{R} - \frac{1}{R_\alpha} = \sum_{n=1}^{\infty}\epsilon^n\lambda_n, \qquad \psi - \psi_0 = \sum_{n=1}^{\infty}\epsilon^n\phi_n, \quad \phi_n \in H^4_{\alpha,\theta}, \tag{17}$$

for small $\epsilon > 0$ due to the bifurcation property $\psi \to \psi_0$ as $R \to R_\alpha$. Hence we derive

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\epsilon^{n+m}\lambda_n\phi_m + \sum_{n=1}^{\infty}\epsilon^n\frac{1}{R_\alpha}\Delta^{-2}L_{R_\alpha}\phi_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\epsilon^{n+m}\Delta^{-2}J(\phi_n, \Delta\phi_m).$$

This implies

$$L_{R_\alpha}\phi_1 = 0,$$
$$\lambda_n\phi_1 + \frac{1}{R_\alpha}\Delta^{-2}L_{R_\alpha}\phi_{n+1} = \sum_{m=1}^{n}\Delta^{-2}J(\phi_m, \Delta\phi_{n+1-m}) - \sum_{m=1}^{n-1}\lambda_m\phi_{n+1-m}, \quad n \geqslant 1.$$

That is, $\phi_1 = c\phi_{\alpha,\theta}$ for any real value $c \neq 0$ due to Lemma 3.3, and

$$\left.\begin{aligned} &\frac{\Delta^{-2}L_{R_\alpha}P_{\alpha,\theta}\phi_{n+1}}{R_\alpha} = \sum_{m=1}^{n} P_{\alpha,\theta}\Delta^{-2}J(\phi_m, \Delta\phi_{n+1-m}) - \sum_{m=1}^{n-1}\lambda_m P_{\alpha,\theta}\phi_{n+1-m},\\ &\lambda_n\langle\phi_1, \Delta^2\phi^*_{\alpha,\theta}\rangle = \left\langle \sum_{m=1}^{n}\Delta^{-2}J(\phi_m, \Delta\phi_{n+1-m}) - \sum_{m=1}^{n-1}\lambda_m\phi_{n+1-m}, \Delta^2\phi^*_{\alpha,\theta}\right\rangle, \end{aligned}\right\}$$

due to the Fredholm decomposition $H^4_{\alpha,\theta} = P_{\alpha,\theta} \oplus \operatorname{span}\{\phi_{\alpha,\theta}\}$ and the commutativity

$$P_{\alpha,\theta}\Delta^{-2}L_{R_\alpha} = \Delta^{-2}L_{R_\alpha}P_{\alpha,\theta}.$$

Thus, by Lemma 3.4,

$$\left.\begin{aligned} &\phi_1 = c\phi_{\alpha,\theta} \quad (c \neq 0),\\ &\phi_{n+1} = R_\alpha\big(\Delta^{-2}L_{R_\alpha}\big)^{-1}P_{\alpha,\theta}\left(\sum_{m=1}^{n}\Delta^{-2}J(\phi_m, \Delta\phi_{n+1-m}) - \sum_{m=1}^{n-1}\lambda_m\phi_{n+1-m}\right),\\ &\lambda_n = \left\langle \sum_{m=1}^{n}\Delta^{-2}J(\phi_m, \Delta\phi_{n+1-m}) - \sum_{m=1}^{n-1}\lambda_m\phi_{n+1-m}, \Delta^2\phi^*_{\alpha,\theta}\right\rangle c\langle\phi_{\alpha,\theta}, \Delta^2\phi^*_{\alpha,\theta}\rangle. \end{aligned}\right\} \tag{18}$$

Noting that $\phi_{\alpha,\theta} = -\phi_{\alpha,\theta+\pi}$, it suffices to examine the case $c > 0$ for the eigenfunction $\phi_1 = c\phi_{\alpha,\theta}$. Moreover, if we denote $(\lambda_{n,c}, \phi_{n,c})$ the solution of (18) depending on the constant $c > 0$, Eq. (18) shows that

$$\lambda_{n,c} = \lambda_{n,1}c^n \quad \text{and} \quad \phi_{n,c} = \phi_{n,1}c^n.$$

That is, we have the solution expression

$$\frac{1}{R} - \frac{1}{R_\alpha} = \sum_{n=1}^{\infty}\epsilon^n\lambda_{n,c} = \sum_{n=1}^{\infty}(\epsilon c)^n\lambda_{n,1}, \qquad \psi - \psi_0 = \sum_{n=1}^{\infty}\epsilon^n\phi_{n,c} = \sum_{n=1}^{\infty}(\epsilon c)^n\phi_{n,1}.$$

This gives the same branch of the bifurcating solution and therefore the constant c appearing in (18) is always supposed to be 1, since the bifurcating solution is uniquely determined by (17) and (18) with $c = 1$.

On the other hand, the scalar products appearing in (18) are independent of the choice of $\theta \geqslant 0$ due to the change of variable $\alpha x' = \alpha x + \theta$ in the respective integrations. This finding shows the independence of λ_n on θ.

Secondly, we prove the convergence of the expansion. It follows from (18) and Lemma 3.4 that

$$\begin{aligned} &\|\phi_{n+1}\|_{H^4} + |\lambda_n|\\ &\quad\leqslant c\left(\sum_{m=1}^{n}\big\|J(\phi_m, \Delta\phi_{n+1-m})\big\|_{L_2} + \sum_{m=1}^{n-1}|\lambda_m|\big\|\Delta^2\phi_{n+1-m}\big\|_{L_2}\right)\\ &\quad\leqslant c\left(\sum_{m=1}^{n}\|\phi_m\|_{H^4}\|\phi_{n+1-m}\|_{H^4} + \sum_{m=1}^{n-1}|\lambda_m|\|\phi_{n+1-m}\|_{H^4}\right). \end{aligned} \tag{19}$$

By denoting the constant c in (19) as c_1 and examination of the difference equation

$$\xi_{n+1} = c_1\left(\sum_{m=1}^{n}\xi_m\xi_{n+1-m} + \sum_{m=1}^{n-1}|\lambda_m|\xi_{n+1-m}\right), \qquad \xi_1 = \|\phi_{\alpha,\theta}\|_{H^4},$$

we find that

$$\xi_n \leqslant \xi_{n+1}, \qquad |\lambda_n| + \|\phi_{n+1}\|_{H^4} \leqslant \xi_{n+1}, \qquad \xi_2 = c_1 \|\phi_{\alpha,\theta}\|^2_{H^4}, \tag{20}$$

and for $n \geqslant 2$,

$$\begin{aligned}
\xi_{n+1} &\leqslant c_1 \left(\sum_{m=1}^{n} \xi_m \xi_{n+1-m} + \sum_{m=1}^{n-1} |\lambda_m| \xi_{n+1-m} \right) \\
&= c_1 \sum_{m=1}^{n-1} \big(\xi_m + |\lambda_m|\big) \xi_{n+1-m} + c_1 \xi_n \xi_1 \\
&\leqslant 2c_1 \sum_{m=1}^{n-1} \xi_{m+1} \xi_{n+1-m} + c_1 \xi_n \xi_2 \\
&\leqslant 3c_1 \sum_{m=1}^{n-1} \xi_{m+1} \xi_{n+1-m} \\
&= 3c_1 \sum_{m=2}^{n} \xi_m \xi_{n+2-m}.
\end{aligned}$$

Letting $\{a_n\}$ satisfy the difference equation

$$a_{n+1} = \sum_{m=2}^{n} a_m a_{n+2-m} \quad \text{with } a_2 = \sqrt{3c_1}\xi_2 = \sqrt{3c_1^3}\|\phi_{\alpha,\theta}\|^2_{H^4},$$

we find that

$$\xi_{n+1} \leqslant a_{n+1}. \tag{21}$$

It is proved in Appendix A that

$$a_{n+1} \leqslant 4^{n-1} a_2^n \leqslant \left(4\sqrt{3c_1^3}\|\phi_{\alpha,\theta}\|^2_{H^4}\right)^n. \tag{22}$$

This estimate together with (20), (21) show the existence of a constant $c > 0$ such that

$$|\lambda_n| + \|\phi_{n+1}\|_{H^4} \leqslant \xi_{n+1} \leqslant a_{n+1} \leqslant c^n,$$

which yields the convergence of the expansion for small $\epsilon > 0$.

Thirdly, we carry out the proof of $\lambda_{2n-1} = 0$. For $n \geqslant 1$, we adopt the two orthogonal subspaces of $H^4_{\alpha,\theta}$:

$$H^4_{\alpha,\theta,\mathrm{odd}} \equiv \left\{ \psi \in H^4;\ \psi = \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \xi_{2m-1,n} \cos\big[(2m-1)(\alpha x + \theta) + ny\big] \right\},$$

$$H^4_{\alpha,\theta,\mathrm{even}} \equiv \left\{ \psi \in H^4;\ \psi = \sum_{m,n} \xi_{2m,n} \cos\big[2m(\alpha x + \theta) + ny\big] \right\}.$$

We see that $\phi_{\alpha,\theta}, \phi^*_{\alpha,\theta} \in H^4_{\alpha,\theta,\mathrm{odd}}$,

$$\Delta^{-2} L_{R_\alpha} : H^4_{\alpha,\theta,\mathrm{odd}} \mapsto H^4_{\alpha,\theta,\mathrm{odd}}, \qquad \Delta^{-2} L_{R_\alpha} : H^4_{\alpha,\theta,\mathrm{even}} \mapsto H^4_{\alpha,\theta,\mathrm{even}}$$

and

$$P_{\alpha,\theta} : H^4_{\alpha,\theta,\mathrm{odd}} \mapsto H^4_{\alpha,\theta,\mathrm{odd}}, \qquad P_{\alpha,\theta} : H^4_{\alpha,\theta,\mathrm{even}} \mapsto H^4_{\alpha,\theta,\mathrm{even}}.$$

By the manipulation given in (16), we observe that

$$\Delta^{-2} J(\psi_1, \Delta\psi_2) \in H^4_{\alpha,\theta,\mathrm{odd}}, \quad \text{if either} \quad (\psi_1, \psi_2) \quad \text{or} \quad (\psi_2, \psi_1) \in H^4_{\alpha,\theta,\mathrm{odd}} \times H^4_{\alpha,\theta,\mathrm{even}}$$

and

$$\Delta^{-2} J(\psi_1, \Delta\psi_2) \in H^4_{\alpha,\theta,\mathrm{even}}, \quad \text{if either} \quad \psi_1, \psi_2 \in H^4_{\alpha,\theta,\mathrm{odd}} \quad \text{or} \quad \psi_1,\ \psi_2 \in H^4_{\alpha,\theta,\mathrm{even}}.$$

These findings together with (18) imply $\phi_1 = \phi_{\alpha,\theta} \in H^4_{\alpha,\theta,\mathrm{odd}}$,

$$\lambda_1 = \frac{1}{\langle \phi_{\alpha,\theta}, \Delta\phi^*_{\alpha,\theta} \rangle} \big\langle J(\phi_1, \Delta\phi_1), \phi^*_{\alpha,\theta} \big\rangle = 0,$$

and

$$\phi_2 = R_\alpha \big(\Delta^{-2} L_{R_\alpha}\big)^{-1} P_{\alpha,\theta} \Delta^{-2} J(\phi_1, \Delta\phi_1) \in H^4_{\alpha,\theta,\mathrm{even}}.$$

By induction, for a given odd integer $j \geqslant 3$, let us assume

$$\lambda_n = 0, \quad \phi_n \in H^4_{\alpha,\theta,\mathrm{odd}}, \quad \phi_{n+1} \in H^4_{\alpha,\theta,\mathrm{even}}, \quad \text{for } n \text{ odd and } n \leqslant j-2.$$

It remains to prove

$$\lambda_j = 0, \quad \phi_j \in H^4_{\alpha,\theta,\mathrm{odd}}, \quad \phi_{j+1} \in H^4_{\alpha,\theta,\mathrm{even}}.$$

Indeed, by (18), we have

$$\begin{aligned}
\phi_j &= R_\alpha \big(\Delta^{-2} L_{R_\alpha}\big)^{-1} P_{\alpha,\theta} \left(\sum_{m=1}^{j-1} \Delta^{-2} J(\phi_m, \Delta\phi_{j-m}) - \sum_{m=1}^{j-2} \lambda_m \phi_{j-m} \right) \\
&= R_\alpha \big(\Delta^{-2} L_{R_\alpha}\big)^{-1} P_{\alpha,\theta} \left(\sum_{m=1}^{j-1} \Delta^{-2} J(\phi_m, \Delta\phi_{j-m}) - \sum_{m=1}^{(j-3)/2} \lambda_{2m} \phi_{j-2m} \right).
\end{aligned}$$

This gives $\phi_j \in H^4_{\alpha,\theta,\mathrm{odd}}$ and moreover, by (18),

$$\begin{aligned}
\lambda_j &= \frac{1}{\langle \phi_{\alpha,\theta}, \Delta^2\phi^*_{\alpha,\theta} \rangle} \left\langle \sum_{m=1}^{j} J(\phi_m, \Delta\phi_{j+1-m}) - \sum_{m=1}^{j-1} \lambda_m \Delta^2 \phi_{j+1-m}, \phi^*_{\alpha,\theta} \right\rangle \\
&= \frac{1}{\langle \phi_{\alpha,\theta}, \Delta^2\phi^*_{\alpha,\theta} \rangle} \left\langle \sum_{m=1}^{j} J(\phi_m, \Delta\phi_{j+1-m}) - \sum_{m=1}^{(j-1)/2} \lambda_{2m} \Delta^2 \phi_{j+1-2m},\ \phi^*_{\alpha,\theta} \right\rangle \\
&= 0.
\end{aligned}$$

Similarly, Eq. (18) gives

$$\begin{aligned}
\phi_{j+1} &= R_\alpha \big(\Delta^{-2} L_{R_\alpha}\big)^{-1} P_{\alpha,\theta} \left(\sum_{m=1}^{j} \Delta^{-2} J(\phi_m, \Delta\phi_{j+1-m}) - \sum_{m=1}^{(j-1)/2} \lambda_{2m} \phi_{j+1-2m} \right) \\
&\in H^4_{\alpha,\theta,\mathrm{even}}.
\end{aligned}$$

We thus obtain the desired assertion.

It should be noted that for the function $R = R(\epsilon)$ close to R_α, Matsuda and Miyatake [9] proved $R^{('')}(0) > 0$. Thus

$$\lambda_{2,\alpha} = -\frac{R^{('')}(0)}{R_\alpha^2} < 0.$$

Finally, it follows from Lemmas 3.1 and 3.3 that (2)–(4) admit exactly m_α critical values $\{R_{k\alpha}\}_{k=1}^{m_\alpha}$. The previous steps provide the proof of Theorem 2.1 when $k = 1$ for any real value $0 < \alpha < 1$ and so substantiate the proof of Theorem 2.1 for any k with $0 < k\alpha < 1$ after taking (10) and Lemma 3.3 into account. The proof of Theorem 2.1 is complete.

4.2. Global branches of the bifurcating solutions

As a consequence of Rabinowitz global bifurcation theorem [15], we now extend the local branches of bifurcating solutions constructed in the previous subsection globally.

For given $\theta \geqslant 0$, we see that the spectral problem

$$L_R\phi = 0, \quad \phi \in H^4_{m_\alpha\alpha, m_\alpha\theta} = H^4_{m_\alpha\alpha, m_\alpha\theta+\pi} \subset H^4_{\alpha,\theta}$$

has a unique critical value $R_{m_\alpha\alpha}$. By Theorem 2.1 and the Rabinowitz global bifurcation theorem [15], bifurcating steady-state solutions exist in $H^4_{m_\alpha\alpha, m_\alpha\theta}$ for any $R > R_{m_\alpha\alpha}$, whenever R satisfies the condition

$$R + \|\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{H^4} + \|\psi_{m_\alpha\alpha, R, m_\alpha\theta+\pi}\|_{H^4} < \infty.$$

By (2), (3), we obtain

$$\begin{aligned}\left\|\Delta^2\psi_{m_\alpha\alpha, R, m_\alpha\theta}\right\|_{L_2} &\leqslant R\|\nabla\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{L_2}\|\nabla\Delta\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{L_2} + \|\cos y\|_{L_2} \\ &\leqslant \frac{R}{m_\alpha^2\alpha^2}\|\nabla\Delta\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{L_2}^2 + \|\cos y\|_{L_2}.\end{aligned}$$

Multiplying (2) by $\Delta\psi_{m_\alpha\alpha, R, m_\alpha\theta}$ and integrating by parts, we have

$$\begin{aligned}\|\nabla\Delta\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{L_2}^2 &= \langle\cos y, \Delta\psi_{m_\alpha\alpha, R, m_\alpha\theta}\rangle \\ &\leqslant \frac{1}{2m_\alpha^4\alpha^4}\|\cos y\|_{L_2}^2 + \frac{1}{2}\|\nabla\Delta\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{L_2}^2.\end{aligned}$$

We thus derive the result

$$\|\psi_{m_\alpha\alpha, R, m_\alpha\theta}\|_{H^4} \leqslant cR + c < \infty$$

for some constant c and therefore the steady-state solution

$$\psi_{m_\alpha\alpha, R, m_\alpha\theta} \in H^4_{m_\alpha\alpha, m_\alpha\theta}$$

exists for all $R > R_{m_\alpha\alpha}$. If $m_\alpha \geqslant 2$, then by the Rabinowitz global bifurcation theorem [15], we also obtain the global existence of the bifurcation solution

$$\psi_{(m_\alpha-1)\alpha, R, (m_\alpha-1)\theta} \in H^4_{(m_\alpha-1)\alpha, (m_\alpha-1)\theta}$$

for any $R > R_{(m_\alpha-1)\alpha}$. By induction, we obtain the existence of the bifurcating steady-state solution $\psi_{k\alpha, R, k\theta}$ in $H^4_{k\alpha, k\theta}$ of (2), (3) for all $R > R_{k\alpha}$. We thus obtain the secondary flows bifurcating from the m_α bifurcating points $\{(\psi_0, R_{k\alpha})\}_{k=1}^{m_\alpha}$.

Appendix A. Proof of (22)

To prove (22), it is sufficient to verify that

$$a_{n+1} \leqslant 4a_2a_n, \quad n \geqslant 2,$$

for $\{a_n\}$ defined by the difference equation

$$a_{n+1} = \sum_{m=2}^{n} a_m a_{n+2-m} \quad \text{with the initial value } a_2 > 0 \ (n \geqslant 2). \tag{A.1}$$

Indeed, let $[a]$ be the integer part of a such that $[a] \leqslant a < [a] + 1$ and the summation $\sum_{m=i}^{j} b_m = 0$ whenever $j < i$. It follows from (A.1) that

$$a_{n+1} = 2a_2a_n + \sum_{m=3}^{n-1} a_m a_{n+2-m} \leqslant 2a_2a_n + 2\sum_{m=3}^{[n/2]+1} a_m a_{n+2-m}.$$

Thus it remains to show

$$a_2a_n \geqslant \sum_{m=3}^{[n/2]+1} a_m a_{n+2-m} \quad (n \geqslant 2),$$

which is valid if the following inequality

$$\begin{aligned} a_2a_n &- \sum_{m=3}^{[n/2]+1} a_m a_{n+2-m} \\ &\geqslant \sum_{m=2}^{k} \sum_{i=m}^{k} a_i a_{k+m-i} a_{n+3-k-m} - \sum_{m=2k}^{[n/2]} \sum_{i=k+1}^{m-k+1} a_i a_{m+2-i} a_{n+1-m} \end{aligned} \tag{A.2}$$

holds true for any integer $4 \leqslant 2k \leqslant [n/2] + 2$, since the second term on the right-hand side of (A.2) vanishes when $2k > [n/2]$.

To prove the validity of (A.2), an induction approach is introduced commencing at $k = 2$. By using (A.1) repeatedly, we find that

$$\begin{aligned} a_2a_n &= 2a_2^2a_{n-1} + \sum_{m=3}^{n-2} a_2 a_m a_{n+1-m} \\ &\geqslant 2a_2^2a_{n-1} + 2\sum_{m=3}^{[n/2]} a_2 a_m a_{n+1-m} \\ &= 2a_2^2a_{n-1} + \sum_{m=3}^{[n/2]} \left(a_{m+1} - \sum_{i=3}^{m-1} a_i a_{m+2-i} \right) a_{n+1-m} \\ &= 2a_2^2a_{n-1} + \sum_{m=3}^{[n/2]} a_{m+1}a_{n+1-m} - \sum_{m=4}^{[n/2]} \sum_{i=3}^{m-1} a_i a_{m+2-i} a_{n+1-m}, \end{aligned}$$

which becomes

$$\begin{aligned}
a_2 a_n &= a_2^2 a_{n-1} + a_3 a_{n-1} + \sum_{m=4}^{[n/2]+1} a_m a_{n+2-m} - \sum_{m=4}^{[n/2]} \sum_{i=3}^{m-1} a_i a_{m+2-i} a_{n+1-m} \\
&= a_2^2 a_{n-1} + \sum_{m=3}^{[n/2]+1} a_m a_{n+2-m} - \sum_{m=4}^{[n/2]} \sum_{i=3}^{m-1} a_i a_{m+2-i} a_{n+1-m}.
\end{aligned}$$

This gives (A.2) for the case $k = 2$. Suppose that (A.2) is valid for an integer $k \geqslant 2$. We verify that (A.2) is also valid for k replaced by $k + 1$. Let $A_{k,n}$ denote the right-hand side of (A.2). It is now sufficient to show that $A_{k,n} \geqslant A_{k+1,n}$. Applying (A.1), we have

$$\begin{aligned}
A_{k,n} &= \sum_{m=2}^{k} \sum_{i=m}^{k} a_i a_{k+m-i} a_{n+3-k-m} - \sum_{m=2k}^{[n/2]} \sum_{i=k+1}^{m-k+1} a_i a_{m+2-i} a_{n+1-m} \\
&= \sum_{m=2}^{k} \sum_{i=m}^{k} a_i a_{k+m-i} a_{n+3-k-m} - 2 \sum_{m=2k}^{[n/2]} a_{k+1} a_{m+1-k} a_{n+1-m} \\
&\quad + a_{k+1}^2 a_{n+1-2k} - \sum_{m=2k+2}^{[n/2]} \sum_{i=k+2}^{m-k} a_i a_{m+2-i} a_{n+1-m}.
\end{aligned} \tag{A.3}$$

The first three terms on the right-hand side of this equation can be written as

$$\begin{aligned}
&\sum_{m=2}^{k} \sum_{i=m}^{k} a_i a_{k+m-i} a_{n+3-k-m} - a_{k+1} \sum_{m=k+1}^{[n/2]+1-k} 2 a_m a_{n+2-k-m} + a_{k+1}^2 a_{n+1-2k} \\
&= a_{k+1} \left(a_{n+1-k} - \sum_{m=k+1}^{[n/2]+1-k} 2 a_m a_{n+2-k-m} \right) \\
&\quad + \sum_{m=3}^{k} \sum_{i=m}^{k} a_i a_{k+m-i} a_{n+3-k-m} + a_{k+1}^2 a_{n+1-2k} \\
&\geqslant \sum_{m=2}^{k} 2 a_{k+1} a_m a_{n+2-k-m} + \sum_{m=3}^{k} \sum_{i=m}^{k} a_i a_{k+m-i} a_{n+3-k-m} + a_{k+1}^2 a_{n+1-2k} \\
&= \sum_{m=2}^{k} 2 a_{k+1} a_m a_{n+2-k-m} + \sum_{m=2}^{k} \sum_{i=m+1}^{k} a_i a_{k+1+m-i} a_{n+2-k-m} + a_{k+1}^2 a_{n+1-2k} \\
&= \sum_{m=2}^{k} \sum_{i=m}^{k+1} a_i a_{k+1+m-i} a_{n+2-k-m} + a_{k+1}^2 a_{n+1-2k} \\
&= \sum_{m=2}^{k+1} \sum_{i=m}^{k+1} a_i a_{k+1+m-i} a_{n+2-k-m} \\
&= A_{k+1,n} + \sum_{m=2k+2}^{[n/2]} \sum_{i=k+2}^{m-k} a_i a_{m+2-i} a_{n+1-m}.
\end{aligned}$$

This together with (A.3) gives the desired assertion. The proof is complete.

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