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J. Math. Anal. Appl. 329 (2007) 445–451
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On the boundedness of solutions of the Chen system

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Received 10 May 2006
Available online 28 July 2006
Submitted by G. Chen

Abstract

By constructing a suitable Lyapunov function, we show that for the system parameters in some specified regions, the solutions of the Chen system are globally bounded.

Keywords: Lyapunov function; Chen system; Boundedness of solution

1. Introduction

As a dual system to the classical Lorenz system, the Chen system has been seriously studied in recent years (see [1–3,5,8,10–12] and some references therein).

The Chen system is described by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{1.1}$$

where $a > 0$, $b > 0$, and $c > 0$ are constant parameters.

Despite the fact that many qualitative and quantitative results on the Chen system have been obtained, there is a fundamental question that has not been completely answered so far: are the

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1 Supported by the Special Funds for Major State Basic Research Projects, National Natural Science Foundation of China and Natural Science Foundation of Jiangsu Province.
2 Supported by the Hong Kong Research Grants Council under the CERG Grant CityU 1114/05E.

doi:10.1016/j.jmaa.2006.06.091

solutions of the Chen system globally bounded? In other words, is there a global trapping region where the system attractor exists?

It is well known that the orbits of the Lorenz system are ultimately trapped in a bounded region for all positive parameters [6]. One can easily show ([7, Appendix C] and [4]) that there is a bounded ellipsoid in R^3, which all orbits of the Lorenz system will eventually enter. For the case of $c < 0$ in the Chen system, one can similarly show [8] that there is an ellipsoid in R^3, which traps all system orbits. For the case of $c > 0$, however, the situation is totally different and so far the answer is unknown. Li et al. only gave in [4] the two dimensional bounds with respect to $x - z$ for the Chen system. Generally speaking, a chaotic attractor is ensured by two things: one is that the system must have a trapping region which guarantees the existence of a attractor, the other is that the system displays chaotic behavior on the attractor. For the Chen system, chaotic behavior has been confirmed in [11,12], while the problem of the existence of a trapping region remains open. In this paper, by constructing a suitable Lyapunov function, we show that for the case of $c > 0$ the solutions of the Chen system are globally bounded if the system parameters are restricted to certain regions.

In searching for a global bounded region, one generally would like to choose a Lyapunov function, as simple as possible, and apply the Lyapunov stability criteria. However, for the case of $c > 0$ in the Chen system, it seems that a quadratic Lyapunov function is not sufficient for this purpose, which is quite different from the Lorenz system. Note that the coefficient of variable y in the second equation is $c > 0$, which is different from the Lorenz system. Therefore, the approach applicable to the Lorenz system does not work for the Chen system. We overcome this difficulty by introducing a quartic term and a cross term.

2. Main result and its proof

The main result of this paper is summarized as follows.

Theorem 2.1. *All solutions of system* (1.1) *with* $a > c > 0$ *and* $b > 2c > 0$ *are globally bounded for* $t \in [0, +\infty)$. *In particular, if* $a > 2c > 0$ *and* $b > 2c > 0$, *then the system solutions* $(x(t), y(t), z(t)) \to (0, 0, 0)$ *as* $t \to +\infty$.

To prove the theorem, some preliminaries are first needed.

Throughout the paper, assume $a, b, c > 0$ in system (1.1). Let

$$A = \begin{pmatrix} -a & a \\ c-a & c \end{pmatrix},$$

and consider the following matrix equation:

$$A^T B + BA = C, \tag{2.1}$$

where, with new parameters $p, q > 0$ and r, α, β, γ,

$$C = \begin{pmatrix} -p & r \\ r & -q \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

It is easy to see that Eq. (2.1) is equivalent to

$$\begin{cases} -a\alpha + (c-a)\beta = -p/2, \\ a\beta + c\gamma = -q/2, \\ a\alpha + (c-a)\beta + (c-a)\gamma = r, \end{cases} \tag{2.2}$$

which has a unique solution (α, β, γ), if $a \neq c$ and $a \neq 2c$, given by

$$\alpha = \frac{(a-c)(p+q) - 2cr}{2a(a-2c)}, \tag{2.3}$$

$$\beta = -\frac{cp + (a-c)q - 2cr}{2(a-c)(a-2c)}, \tag{2.4}$$

$$\gamma = \frac{ap + 2(a-c)q - 2ar}{2(a-c)(a-2c)}. \tag{2.5}$$

There are three cases to discuss:

Case 1: $a > 2c$. In this case, matrix A has two negative eigenvalues, so that Eq. (2.1) has a positive definite matrix solution if the matrix C is negative definite [9]; that is, for $p > 0$, $q > 0$, and $r = 0$, Eq. (2.2) has a solution (α, β, γ) satisfying $\alpha > 0$, $\gamma > 0$, and $\alpha\gamma > \beta^2$.

Case 2: $a = 2c$. In this case, let $p > 0$, $q > 0$ with $p > q$, and $r = (p+q)/2$. Then, Eq. (2.2) has a solution,

$$\alpha = p/2c, \qquad \beta = -p/2c, \qquad \gamma = p/c - q/2c,$$

satisfying $\alpha > 0$, $\gamma > 0$, and $\alpha\gamma > \beta^2$.

Case 3: $c < a < 2c$. In this case, it follows from (2.3)–(2.5) that

$$\alpha\gamma - \beta^2 = \frac{f(p,q,r)}{4(a-2c)^2(a-c)},$$

where

$$f(p,q,r) = \frac{1}{a}\big((a-c)(p+q) - 2cr\big)\big(ap + 2(a-c)q - 2ar\big) - \frac{1}{a-c}\big(cp + (a-c)q - 2cr\big)^2.$$

Assuming $q = 0$, one has

$$f(p,0,r) = \frac{1}{a}\big((a-c)p - 2cr\big)(ap - 2ar) - \frac{1}{a-c}(cp - 2cr)^2.$$

Choosing $r = p/2 + p/2m$ with $m > 0$ satisfying

$$m(2c-a) + c > c^2/(a-c),$$

one has $\alpha > 0$, $\beta > 0$, and

$$f(p,0,r) = \frac{p^2}{m^2}\left[m(2c-a) + c - \frac{c^2}{a-c}\right] > 0,$$

which implies that $\alpha\gamma > \beta^2$.

Since f and the solutions α, β, and γ are all continuous in q, one may choose $q_0 > 0$ such that for $0 < q < q_0$, $p > 0$, and $r = p/2 + p/2m$, Eq. (2.2) has a solution (α, β, γ) satisfying $\alpha > 0$, $\gamma > 0$, and $\alpha\gamma > \beta^2$.

In summary, for $p > 0$, $0 < q < q^*$, where q^* is a positive constant depending on the chosen p, and for some real constant r, Eq. (2.1) has a positive definite matrix solution provided that $a > c > 0$.

It should be noted that in all the above cases, $\beta < 0$.

From the above discussion, for $a > c > 0$, one also has

$$\lim_{q\to 0^+} \frac{-\beta}{\gamma} = \frac{c}{a}. \tag{2.6}$$

Therefore, if there is a constant $K > c/a$, then one may choose $q > 0$ sufficiently small such that $K > -\beta/\gamma$.

Next, let

$$\mu = \gamma - \frac{2a\delta\beta}{2a+b}, \qquad \theta = \frac{\delta\beta}{2a+b}, \qquad \tau = \frac{-\delta\beta}{2a(2a+b)}, \tag{2.7}$$

where constant $\delta > 0$ is to be determined later. Also, let

$$\omega = 2b\mu - \frac{(\delta+1)^2}{4a\tau}\beta^2. \tag{2.8}$$

For notational simplicity, denote

$$k = \frac{4b}{2a+b}, \qquad h = \frac{8ab}{(2a+b)^2}.$$

Lemma 2.2. *If $a > c > 0$ and $b > 2c > 0$, then $\omega > 0$ for a suitably chosen $\delta > 0$.*

Proof. If $h < 1$, then taking $\delta = 1/\sqrt{1-h} > 0$ leads to

$$\begin{aligned}
\omega > 0 \quad &\Leftrightarrow \quad 8ab\mu\tau > (\delta+1)^2\beta^2 \\
&\Leftrightarrow \quad k\delta\gamma(-\beta) > \big[(1-h)\delta^2 + 2\delta + 1\big]\beta^2 \\
&\Leftrightarrow \quad \frac{k\delta}{(1-h)\delta^2 + 2\delta + 1} > \frac{-\beta}{\gamma} \\
&\Leftrightarrow \quad \frac{2b}{2a+b+|2a-b|} > \frac{-\beta}{\gamma}.
\end{aligned}$$

Note that

$$\frac{2b}{2a+b+|2a-b|} = \begin{cases} b/2a & \text{if } a \geqslant b/2, \\ 1 & \text{if } a < b/2. \end{cases}$$

If $b > 2c$, then $b/2a > c/a$. Consequently, if $b > 2c$, then

$$\frac{2b}{2a+b+|2a-b|} > \frac{c}{a}$$

since $a > c$. Thus, one may choose q sufficiently small so that

$$\frac{2b}{2a+b+|2a-b|} > \frac{-\beta}{\gamma},$$

and, hence, $\omega > 0$.

If $h = 1$, then $k = 2$ and

$$\begin{aligned}
\omega > 0 \quad &\Leftrightarrow \quad \frac{k\delta}{2\delta+1} > \frac{-\beta}{\gamma} \\
&\Leftrightarrow \quad \frac{2\delta}{2\delta+1} > \frac{-\beta}{\gamma}.
\end{aligned}$$

Therefore, one may choose $\delta > 0$ sufficiently large such that

$$\frac{2\delta}{2\delta+1} > \frac{c}{a}$$

since $a > c$. To this end, taking a small $q > 0$ yields

$$\frac{2\delta}{2\delta+1} > \frac{-\beta}{\gamma}. \qquad \square$$

Since it is required that $\alpha > 0$, $\gamma > 0$, and $\alpha\gamma > \beta^2$, one may choose positive constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$(\alpha - \varepsilon_1)(\gamma - \varepsilon_2) = \beta^2,$$

so that

$$\alpha x^2 + 2\beta xy + \gamma y^2 = \varepsilon_1 x^2 + (\sigma x + \rho y)^2 + \varepsilon_2 y^2,$$

where

$$\sigma = \sqrt{\alpha - \varepsilon_1}$$

and

$$\rho = \sqrt{\gamma - \varepsilon_2}.$$

Now, let

$$\begin{aligned} V(x,y,z) &= \alpha x^2 + 2\beta xy + \gamma y^2 + \mu z^2 + 2\theta x^2 z + \tau x^4 - 2rz + r^2/\gamma \\ &= \alpha x^2 + 2\beta xy + \gamma y^2 + \frac{\delta(-\beta)}{2a+b}\left(\sqrt{2a}\,z - \frac{x^2}{\sqrt{2a}}\right)^2 + \gamma\left(z - \frac{r}{\gamma}\right)^2 \\ &= \varepsilon_1 x^2 + (\sigma x + \rho y)^2 + \varepsilon_2 y^2 + \frac{2a\delta(-\beta)}{2a+b}\left(z - \frac{x^2}{2a}\right)^2 + \gamma\left(z - \frac{r}{\gamma}\right)^2. \end{aligned}$$

Thus, along the trajectories of the Chen system (1.1), one has

$$\begin{aligned} \dot V(x,y,z) &= 2\alpha x\dot x + 2\beta \dot x y + 2\beta x\dot y + 2\gamma y\dot y + 2\mu z\dot z + 4\theta xz\dot x + 2\theta x^2\dot z + 4\tau x^3\dot x - 2r\dot z \\ &= \big[-2a\alpha + 2(c-a)\beta\big]x^2 + [2a\beta + 2c\gamma]y^2 - 4a\tau x^4 - 2b\mu z^2 \\ &\quad + \big[2a\alpha + 2(c-a)\beta + 2(c-a)\gamma - 2r\big]xy + (2\mu + 4a\theta - 2\gamma)xyz \\ &\quad + (4a\tau + 2\theta)x^3 y - (4a\theta + 2b\theta + 2\beta)x^2 z + 2brz \\ &= -px^2 - qy^2 - 4a\tau x^4 - 2(\delta+1)\beta x^2 z - 2b\mu z^2 + 2brz \\ &= -px^2 - qy^2 - \left(2\sqrt{a\tau}x^2 + \frac{(\delta+1)\beta}{2\sqrt{a\tau}}z\right)^2 - \omega z^2 + 2brz. \end{aligned}$$

In the above, the last two equalities follow from (2.2), (2.7), and (2.8).

Proof of Theorem 2.1. By Lemma 2.2, $\omega > 0$ and

$$\dot{V}(x, y, z) \leqslant -px^2 - qy^2 - \omega\left(z - \frac{br}{\omega}\right)^2 + \frac{b^2r^2}{\omega}.$$

Hence, one may take d_0 sufficiently large such that

$$px^2 + qy^2 + \omega\left(z - \frac{br}{\omega}\right)^2 > \frac{b^2r^2}{\omega}$$

provided that (x, y, z) satisfies

$$V(x, y, z) = d$$

with $d > d_0$. Consequently, on the surface

$$\{(x, y, z) \mid V(x, y, z) = d\},$$

where $d > d_0$, one has $\dot{V}(x, y, z) < 0$, which implies that the set

$$\{(x, y, z) \mid V(x, y, z) \leqslant d\}$$

is a trapping region, implying that the solutions of system (1.1) are globally bounded.

If, furthermore, $a > 2c$, then one can choose $r = 0$, so that

$$V(x, y, z) = \varepsilon_1 x^2 + \varepsilon_2 y^2 + \gamma z^2 + (\sigma x + \rho y)^2 + \frac{2a\delta(-\beta)}{2a + b}\left(z - \frac{x^2}{2a}\right)^2,$$

and

$$\begin{aligned}\dot{V}(x, y, z) &= -px^2 - qx^2 - \left(2\sqrt{a\tau}x^2 + \frac{(\delta+1)\beta}{2\sqrt{a\tau}}z\right)^2 - \omega z^2 \\ &\leqslant -px^2 - qy^2 - \omega z^2 < 0.\end{aligned}$$

Therefore, V decreases to 0 along any orbit $(x(t), y(t), z(t))$ of the system, as $t \to +\infty$. Consequently, $(x(t), y(t), z(t)) \to (0, 0, 0)$ as $t \to +\infty$, since V is positive definite. In conclusion, the origin is a globally asymptotically stable equilibrium. □

Finally, it should be remarked that the above Lyapunov function is constructed only for qualitative analysis, which by no means gives an optimal result [4,6].

3. Conclusion

In this paper, we have shown the global boundedness of the Chen system but only for some cases with $c > 0$, which did not include the most interesting situation with the chaotic attractor of the Chen system (with parameters $a = 35$, $b = 3$, $c = 28$ [1,8,11]), leaving an important and yet nontrivial open problem for future research.

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