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Operational rules and a generalized Hermite polynomials

Hamza Chaggara

Institut Préparatoire aux Études d'Ingénieur de Monastir, Département de Préparation en Math-Physique, 5019 Monastir, Tunisia

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Abstract

In this paper, we use operational rules associated with three operators corresponding to a generalized Hermite polynomials introduced by Szegö to derive, as far as we know, new proofs of some known properties as well as new expansions formulae related to these polynomials.

Keywords: Lowering operators; Raising operators; Transfer operators; Quasi-monomiality; Brenke polynomials; Generalized Hermite polynomials; Dunkl operator; Connection coefficients; Linearization coefficients

1. Introduction

A sequence of polynomials $\{P_n\}_{n\geqslant 0}$, with coefficients in $\mathbb{C}$, is called polynomial set if $\deg P_n = n$ for all $n = 0, 1, \ldots$.

For a given polynomial set $\{P_n\}_{n\geqslant 0}$, we define three operators, not depending on n, by

$$\lambda P_n = nP_{n-1}, \quad \rho P_n = P_{n+1} \quad \text{and} \quad \tau B_n = P_n, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $P_{-1} = 0$ and $\{B_n\}_{n\geqslant 0}$ verifies

$$\lambda B_n = nB_{n-1}, \quad B_0(0) = 1 \quad \text{and} \quad B_n(0) = 0, \quad n = 1, 2, \ldots. \tag{1.2}$$

λ, ρ and τ are called respectively the *lowering*, the *raising* and the *transfer* operators associated to the polynomial set $\{P_n\}_{n\geqslant 0}$ while $\{B_n\}_{n\geqslant 0}$ is called *basic sequence associated to* λ.

E-mail address: hamza.chaggara@ipeim.rnu.tn.

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These operators satisfy the following commutation formulae

$$\lambda\rho - \rho\lambda = 1 \quad \text{and} \quad \tau\lambda = \lambda\tau.$$

The lowering and the raising operators play the role analogous to that of derivative and multiplicative operators on monomials. The operational rules associated to these two operators are known as *quasi-monomiality principle*. The polynomial set $\{P_n\}_{n\geqslant 0}$ is called quasi-monomial under the action of λ and ρ.

The monomiality principle was introduced by Dattoli et al. [18] in order to derive properties of special polynomials starting from the corresponding ones of monomials. The associated operational rules were used to explore new classes of isospectral problems leading to nontrivial generalizations of special functions [15,16].

Ben Cheikh [2] proved that every polynomial set can be viewed as quasi-monomial and [1] that any operator acting on analytic functions and reducing the degree of polynomials by exactly one has a sequence of basic polynomials. Furthermore, for a given polynomial set there exists a unique power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$, such that $\tau = A(\lambda)$, where τ is the transfer operator corresponding to the considered polynomial set. This last is then called a *λ-Appell polynomial set of transfer power series A*.

The operational rules associated to lowering, transfer and raising operators were used to study many problems arising in the theory of polynomials. Most of the properties of polynomial sets can be deduced by using operational rules with these operators. The main properties considered are related to generating functions, differential equations, Rodrigues formula and orthogonality. Sufficient condition, in terms of the lowering and raising operators, to ensure the orthogonality of any polynomial set, was given in [30]. The linear functional for which the orthogonality holds was also stated by means of the lowering operator and the transfer power series. These results were applied to recover the orthogonality of the ordinary Sheffer orthogonal polynomials.

In many works, new and known results related to Hermite polynomials, Laguerre polynomials [17], Laguerre–Konhauser polynomials [8], Legendre polynomials [19], Appell polynomials [22], Gould–Hopper polynomials [23], Boas–Buck polynomials [5], Sheffer polynomials [3] and d-orthogonal polynomials [7] were derived by using the monomiality principle.

In this paper, we consider the *generalized Hermite polynomials* generated by [26]

$$e^{-t^2} e_\mu(2xt) = \sum_{n=0}^{\infty} \mathcal{H}_n^\mu(x)\frac{t^n}{n!}, \quad \mu \in \mathbb{C},\ \mu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots, \tag{1.3}$$

where

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)},$$

with

$$\gamma_\mu(2m+\epsilon) = 2^{2m+\epsilon} m! \left(\mu + \epsilon + \frac{1}{2}\right)_m, \quad \epsilon = 1, 2, \tag{1.4}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

This polynomial set was introduced by Szegö [32] as a set of real polynomials orthogonal with respect to the weight $|x|^{2\mu} e^{-x^2}$, $\mu > -\frac{1}{2}$, then investigated by Chihara in his PhD thesis [13] and further studied by Rosenblum in [26] in connection with a Bose-like oscillator calculus. This family reduces to the ordinary Hermite polynomial set when $\mu = 0$.

Many other authors investigated properties of these polynomials, using classical methods well known in the special functions theory. For instance, some characterization problems related to this polynomial set were given in [6,20].

The purpose of this work is, starting from the generating function (1.3), to give the lowering, the transfer and the raising operators associated with the generalized Hermite polynomials. Then we show how to derive some corresponding properties using operational rules related to these operators. The method we propose allows us to obtain other proofs of some well-known properties of these polynomials as well as some expansions formulae.

The paper is organized as follows: In Section 2, following the prescription of the monomiality point of view, we derive some properties of the generalized Hermite polynomials, a recurrence relation from which we deduce the orthogonality, the functional vector for which the orthogonality holds and a differential equation satisfied by these polynomials. In Section 3, using a technique based on lowering and transfer operators, we give some expansion formulae for *Brenke polynomial sets*. We use the resulting formulae to derive duplication, connection and linearization coefficients associated to generalized Hermite polynomials as well as a convolution type formula which generalizes an old formula for ordinary Hermite polynomials due to Runge [27]. The expansion formulae derived in this section appear to be new.

2. Properties of generalized Hermite polynomials

2.1. Operators associated to generalized Hermite polynomials

A useful tool, based on a suitable generating function of the considered polynomial set to derive the three aforementioned operators, is given by

Proposition 2.1. [1,2] *Let $\{P_n\}_{n\geqslant 0}$ be a polynomial set generated by*

$$G(x,t) = A(t)G_0(x,t) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!}t^n, \quad \text{with } G_0(0,t) = 1.$$

Then λ, ρ and τ are characterized by

$$\begin{cases} \lambda G_0(x,t) = t\, G_0(x,t), & \lambda := \lambda_x, \\ \rho G(x,t) = \dfrac{\partial G}{\partial t}(x,t), & \rho := \rho_x, \\ \tau = A(\lambda). \end{cases} \tag{2.1}$$

As application, we consider the so-called *Brenke polynomials* which will be largely exploited in this work.

Proposition 2.2. *The Brenke polynomial set $\{P_n\}_{n\geqslant 0}$ generated by* [14]

$$A(t)B(xt) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!}t^n, \tag{2.2}$$

where $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ such that $a_0 b_k \neq 0\ \forall k$, is quasi-monomial under the action of

$$\lambda = D_b \quad \text{and} \quad \rho = A'(D_b)\widehat{A}(D_b) + xDD_b^{-1}, \tag{2.3}$$

where $\widehat{A}(t) = \frac{1}{A(t)}$ *and* D_b *is given by*

$$D_b(1) = 0, \quad D_b\big(x^n\big) = \frac{b_{n-1}}{b_n} x^{n-1} \quad \textit{and} \quad D_b^{-1}\big(x^n\big) = \frac{b_n}{b_{n-1}} x^{n+1}, \quad n = 1, 2, \ldots. \tag{2.4}$$

The transfer power series associated to $\{P_n\}_{n\geqslant 0}$ *is* $b_0 A$.

According to (1.3), the generalized Hermite polynomial set can be viewed as a Brenke set with $A(t) = e^{-t^2}$ and $B(x) = e_\mu(2x)$. Hence, by virtue of (2.3), $\{\mathcal{H}_n^\mu\}_{n\geqslant 0}$ is quasi-monomial under the action of

$$\lambda = \frac{\mathcal{D}_\mu}{2} \quad \text{and} \quad \rho = -\mathcal{D}_\mu + 2xD\mathcal{D}_\mu^{-1}, \tag{2.5}$$

where

$$\mathcal{D}_\mu\big(x^n\big) = \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} x^{n-1} = (n + \mu\theta_n)x^{n-1}, \quad \theta_n = 1 + (-1)^{n+1}. \tag{2.6}$$

Then the generalized Hermite polynomials is a $(\frac{1}{2}\mathcal{D}_\mu)$-Appell polynomial set of transfer power series $A(t) = e^{-t^2}$. The sequence of basic polynomials for $\mathcal{D}_\mu$ is $\{B_n(x) = \frac{n!}{\gamma_\mu(n)} x^n\}_{n\geqslant 0}$. This means that

$$e^{-\frac{\mathcal{D}_\mu^2}{4}}\big(x^n\big) = \frac{\gamma_\mu(n)}{2^n n!} \mathcal{H}_n^\mu(x). \tag{2.7}$$

$\mathcal{D}_\mu$ is the well-known *Dunkl operator* associated with the parameter μ on the real line [26]. In fact, for a given analytic function $f(x) = \sum_{n=0}^\infty \rho_n x^n$, we have

$$\begin{aligned} \mathcal{D}_\mu(f)(x) &= \sum_{n=1}^\infty (n + \mu\theta_n)\rho_n x^{n-1} = \sum_{n=1}^\infty n\rho_n x^{n-1} + 2\mu \sum_{n=0}^\infty \rho_{2n+1} x^{2n} \\ &= Df(x) + \frac{\mu}{x}\big(f(x) - f(-x)\big). \end{aligned} \tag{2.8}$$

The differential-difference operator $\mathcal{D}_\mu$, recovered in (2.8), is a special case of operators set down by Dunkl in his work on root systems associated with finite reflection groups [21].

2.2. Recurrence relation and orthogonality

In this section, we give a new proof of the three term recurrence relation satisfied by the generalized Hermite polynomials and then we recover the orthogonality and the linear functional for which the orthogonality holds. First, we recall the following.

We denote by $\mathscr{P}$ the vector space of polynomials with coefficients in $\mathbb{C}$ and by $\mathscr{P}'$ its algebraic dual. $\langle \mathscr{L}, f \rangle$ designates the effect of the linear functional $\mathscr{L} \in \mathscr{P}'$ on the polynomial $f \in \mathscr{P}$.

A polynomial sequence $\{P_n\}_{n\geqslant 0}$ is called *orthogonal* polynomial set with respect to the functional $\mathscr{L}$ if it satisfies the following condition

$$\langle \mathscr{L}, P_m P_n \rangle = c_n \delta_{n,m}, \quad c_n \neq 0, \ n, m \in \mathbb{N}. \tag{2.9}$$

The so-called Favard Theorem asserts that the orthogonality may be deduced from the fact that the sequence $\{P_n\}_{n\geqslant 0}$ satisfies a three term recurrence relation with regularity conditions [14].

The three term recurrence relation satisfied by the generalized Hermite polynomials can be derived directly from the expression of the associated raising operator given by (2.5).

$$\begin{aligned}
\mathcal{H}_{n+1}^{\mu}(x) &= \rho\big(\mathcal{H}_n^{\mu}\big)(x) = -\mathcal{D}_\mu \mathcal{H}_n^{\mu}(x) + 2xD\mathcal{D}_\mu^{-1}\mathcal{H}_n^{\mu}(x) \\
&= -2n\mathcal{H}_{n-1}^{\mu}(x) + \frac{1}{n+1}xD\mathcal{H}_{n+1}^{\mu}(x) \\
&= -2n\mathcal{H}_{n-1}^{\mu}(x) + \frac{1}{n+1}\big(x\mathcal{D}_\mu\mathcal{H}_{n+1}^{\mu}(x) - \mu\mathcal{H}_{n+1}^{\mu}(x) + \mu\mathcal{H}_{n+1}^{\mu}(-x)\big) \\
&= -2n\mathcal{H}_{n-1}^{\mu}(x) + 2x\mathcal{H}_n^{\mu}(x) - \frac{\mu\theta_{n+1}}{n+1}\mathcal{H}_{n+1}^{\mu}(x).
\end{aligned}$$

Then, we obtain

$$2x\mathcal{H}_n^{\mu} = \left(1 + \frac{\mu\theta_{n+1}}{n+1}\right)\mathcal{H}_{n+1}^{\mu} + 2n\mathcal{H}_{n-1}^{\mu}. \tag{2.10}$$

It follows, from the Favard Theorem, that $\{\mathcal{H}_n^{\mu}\}_{n\geqslant 0}$ is an orthogonal polynomial set.

When $\{P_n\}_{n\geqslant 0}$ is a λ-Appell polynomial set of transfer power series A, an explicit expression of its dual sequence was given by means of its corresponding lowering and transfer operators.

Recall first, that the dual sequence $\{\mathbb{P}_n\}_{n\geqslant 0}$ associated to a given polynomial set $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle \mathbb{P}_n, P_m\rangle = \delta_{nm}, \quad n, m \geqslant 0. \tag{2.11}$$

The action of this dual sequence on any polynomial is given by [1]

$$\langle \mathbb{P}_n, f\rangle = \frac{1}{n!}\big[\lambda^n \widehat{A}(\lambda)(f)(x)\big]_{x=0}, \quad n = 0, 1, \ldots, \ f \in \mathscr{P}. \tag{2.12}$$

Moreover, if $\{P_n\}_{n\geqslant 0}$ is a λ-Appell orthogonal polynomial set of transfer power series A, then the linear functional $\mathscr{L}$, for which we have the orthogonality, is given by $\mathscr{L} = \mathbb{P}_0$: The first vector of the dual sequence defined by (2.11). It follows from (2.12) that

$$\langle \mathscr{L}, f\rangle = \widehat{A}(\lambda)(f)(x)|_{x=0} \quad (f \in \mathscr{P}). \tag{2.13}$$

In the following proposition we give, by means of the transfer power series, an explicit expression of the moments associated to any orthogonal Brenke polynomial set.

Proposition 2.3. *Let $\{P_n\}_{n\geqslant 0}$ be an orthogonal polynomial set of Brenke type generated by* (2.2)*. Then the moments associated to the functional $\mathscr{L}$, for which the orthogonality holds, are given by*

$$\big\langle \mathscr{L}, x^n\big\rangle = \frac{\hat{a}_n}{b_n}, \quad \textit{where } \widehat{A}(t) = \frac{1}{A(t)} = \sum_{k=0}^{\infty}\hat{a}_k t^k \textit{ and } B(t) = \sum_{k=0}^{\infty} b_k t^k. \tag{2.14}$$

Proof. $\{P_n\}_{n\geqslant 0}$ is D_b-Appell of transfer power series $b_0 A(t)$. It follows from (2.13) that

$$\begin{aligned}
\big\langle \mathscr{L}, x^n\big\rangle &= \frac{1}{b_0}\widehat{A}(D_b)\big(x^n\big)\big|_{x=0} = \frac{1}{b_0}\sum_{k=0}^{\infty}\hat{a}_k D_b^k x^n\big|_{x=0} \\
&= \frac{1}{b_0}\sum_{k\geqslant n}\hat{a}_k \frac{b_{n-k}}{b_n} x^{n-k}\big|_{x=0} = \frac{\hat{a}_n}{b_n}. \qquad \square
\end{aligned}$$

Applying Proposition 2.3 to generalized Hermite polynomials with $A(t) = e^{-t^2}$ and $B(t) = e_\mu(2t)$ and according to (1.4), we obtain

$$\langle \mathscr{L}, x^n \rangle = \begin{cases} \frac{\gamma_\mu(2p)}{p!\gamma_\mu(0)2^{2p}} = (\mu + \frac{1}{2})_p & \text{if } n = 2p, \\ 0 & \text{if } n = 2p+1, \end{cases} \tag{2.15}$$

which can be written as [26]

$$\langle \mathscr{L}, x^n \rangle = \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_{-\infty}^{+\infty} x^n e^{-x^2} |x|^{2\mu} \, dx.$$

2.3. Differential equation

In this section, we derive, by means of the lowering and the raising operators, a new proof of the second order differential equation satisfied by the generalized Hermite polynomials. We have

Proposition 2.4. *The following relation holds*

$$2xn\mathcal{H}_n^\mu = 2(x^2 - \mu) D\mathcal{H}_n^\mu - xD^2\mathcal{H}_n^\mu + \frac{\mu\theta_n}{x}\mathcal{H}_n^\mu. \tag{2.16}$$

Proof. As any polynomial set $\{P_n\}_{n\geqslant 0}$ is quasi-monomial, it follows that P_n is an eigenfunction of the operator $\rho\lambda$ associated to the eigenvalues n. That is to say:

$$\rho\lambda P_n = nP_n, \tag{2.17}$$

which can be viewed, if λ and ρ have a differential realization, as a differential equation satisfied by the considered polynomial set.

Using this fact for the generalized Hermite polynomials $\{\mathcal{H}_n^\mu\}_{n\geqslant 0}$, we obtain

$$n\mathcal{H}_n^\mu = \rho\lambda\mathcal{H}_n^\mu = \frac{1}{2}\left(-\mathcal{D}_\mu + 2xD\mathcal{D}_\mu^{-1}\right)\mathcal{D}_\mu\mathcal{H}_n^\mu = xD\mathcal{H}_n^\mu - \frac{1}{2}\mathcal{D}_\mu^2\mathcal{H}_n^\mu.$$

Multiplying by $2x$ and using the trivial following relation, which can be easily derived from (2.8)

$$x\mathcal{D}_\mu^2 = xD^2 + (2\mu + 1)D - \mathcal{D}_\mu,$$

we obtain

$$2xn\mathcal{H}_n^\mu = 2x^2 D\mathcal{H}_n^\mu - x\mathcal{D}_\mu^2\mathcal{H}_n^\mu = 2x^2 D\mathcal{H}_n^\mu - xD^2\mathcal{H}_n^\mu - 2\mu D\mathcal{H}_n^\mu - (D - \mathcal{D}_\mu)\mathcal{H}_n^\mu,$$

which, by virtue of the relation $(D - \mathcal{D}_\mu)\mathcal{H}_n = -\frac{\mu}{x}\theta_n\mathcal{H}_n$, gives (2.16). □

Notice that Eqs. (2.10) and (2.16) were already given, respectively, in [26] and [32].

For $\mu = 0$, these relations reduce, respectively, to the well-known pure recurrence relation and second-order differential equation satisfied by the ordinary Hermite polynomials.

3. Connection and linearization coefficients

In this section, we deal with some expansion formulae associated to generalized Hermite polynomials. In particular, we solve the corresponding connection and linearization problems which are defined as follows:

Given two polynomial sets $\{S_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$. The so-called *connection problem* between them asks to find the coefficients $C_m(n)$ in the expression:

$$S_n(x)=\sum_{m=0}^{n} C_m(n)P_m(x), \tag{3.1}$$

which, for $S_n(x)=x^n$ is known as the *inversion problem* for the polynomial set $\{P_n\}_{n\geqslant 0}$, and for $S_n(x)=P_n(a\cdot x)$, a being a nonzero complex number, is reduced to *duplication or multiplication problem* associated with the polynomial set $\{P_n\}_{n\geqslant 0}$.

When $S_{i+j}(x)=Q_i(x)R_j(x)$ in (3.1), $\{Q_n\}_n$ and $\{R_n\}_n$ being two polynomial sets, we are faced to the *general linearization problem*

$$Q_i(x)R_j(x)=\sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{3.2}$$

Particular case of this problem is the *standard linearization problem* or *Clebsch–Gordan-type problem* if $Q_n=R_n=P_n$.

The computation of the connection and linearization coefficients plays an important role in many situations of pure and applied mathematics and also in physical and quantum chemical applications [28,29]. The literature on this topic is extremely vast and a wide variety of methods, based on specific properties of the involved polynomials, have been devised for computing the linearization coefficients $L_{ij}(k)$ either in closed form or by means of recursive relations.

A general method, based on lowering and transfer operators, generating functions and a simple manipulation of formal power series, was developed to solve connection [4,5], duplication [12] and linearization [3,11] problems. This approach does not need particular properties of the polynomials involved in the problems. In fact, according to (2.12), we have for every polynomial $f\in\mathscr{P}$ of degree n the generalized expansion

$$f(x)=\sum_{k=0}^{n}\big[\lambda^k\widehat{A}(\lambda)(f)(0)\big]\frac{P_k(x)}{k!}. \tag{3.3}$$

Then we derive a simple and general formula to compute $C_m(n)$ and $L_{ij}(k)$ which consists in putting, respectively, in (3.3) $f=S_n$, and $f=Q_iR_j$.

Next, we apply such a technique to solve inversion, connection, duplication and linearization problems for generalized Hermite polynomials.

We begin by recalling a result giving the connection coefficients between two λ-Appell polynomials.

Lemma 3.1. [4, Corollary 3.4] *Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be two λ-Appell polynomial sets of transfer power series, respectively, A_1 and A_2. Then*

$$Q_n(x)=\sum_{m=0}^{n}\frac{n!}{m!}\alpha_{n-m}P_m(x),\quad \textit{where } \frac{A_2(t)}{A_1(t)}=\sum_{k=0}^{\infty}\alpha_k t^k. \tag{3.4}$$

Since $\{\mathcal{H}_n^{\mu}\}_{n\geqslant 0}$ and $\{B_n(x)=\frac{2^n n!}{\gamma_\mu(n)}x^n\}_{n\geqslant 0}$ are two $(\frac{1}{2}\mathcal{D}_\mu)$-Appell of transfer power series, respectively, e^{-t^2} and 1, we obtain the following explicit, inversion and connection formulae

$$\frac{\mathcal{H}_n^\mu(x)}{n!}=\sum_{m=0}^{[\frac{n}{2}]}\frac{(-1)^m(2x)^{n-2m}}{m!\gamma_\mu(n-2m)},\tag{3.5}$$

$$\frac{(2x)^n}{\gamma_\mu(n)}=\sum_{m=0}^{[\frac{n}{2}]}\frac{\mathcal{H}_{n-2m}^\mu(x)}{m!(n-2m)!},\tag{3.6}$$

and, by composition,

$$\frac{\mathcal{H}_n^{\mu_2}(x)}{n!}=\sum_{m=0}^{[\frac{n}{2}]}\left(\sum_{p=0}^{m}\frac{(-1)^{m-p}}{(m-p)!p!}\frac{\gamma_{\mu_1}(n-2m+2p)}{\gamma_{\mu_2}(n-2m+2p)}\right)\frac{\mathcal{H}_{n-2m}^{\mu_1}(x)}{(n-2m)!}.\tag{3.7}$$

On the other hand, $\{\mathcal{H}_n^\mu\}_{n\geqslant 0}$ possess the generalized addition formula

$$T_y^\mu\mathcal{H}_n^\mu(x)=\sum_{k=0}^{n}\frac{2^{n-k}n!}{\gamma_\mu(n-k)k!}y^{n-k}\mathcal{H}_k^\mu(x),\tag{3.8}$$

and the relation of convolution type

$$e^{-\frac{\mathcal{D}_\mu^2}{4}}T_y^\mu\mathcal{H}_n^\mu(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k}^\mu(y)\mathcal{H}_k^\mu(x),\tag{3.9}$$

where $T_y^\mu:=e_\mu(y\mathcal{D}_\mu)$, T_y^μ is, in fact, a generalized translation operator which is reduced to the ordinary one when $\mu=0$.

The obtained formulae can be used to recover some known expansion associated to Hermite and Laguerre polynomials. For instance, according to (1.4) and the explicit formula (3.5), we obtain

$$H_{2n+\epsilon}^\mu(x)=\frac{(-1)^n(2n+\epsilon)!}{(\mu+\frac{1}{2})_{n+\epsilon}}x^\epsilon L_n^{\mu-\frac{1}{2}+\epsilon}\bigl(x^2\bigr),\quad \epsilon=0,1,\tag{3.10}$$

where L_n^α designates the Laguerre polynomial defined by [24]

$$L_n^\alpha(x)=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix},x\right),\tag{3.11}$$

where ${}_pF_q$ denotes the hypergeometric function with p numerator and q denominator parameters.

Combining (3.10) with (3.11) and using (1.4), we get

$$\frac{L_n^\beta(x)}{(\beta+1)_n}=\sum_{m=0}^{n}\left(\sum_{k=0}^{m}\frac{(-1)^k}{k!(m-k)!}\frac{(\alpha+1)_{n-m+k}}{(\beta+1)_{n-m+k}}\right)\frac{L_{n-m}^\alpha(x)}{(\alpha+1)_{n-m}},\tag{3.12}$$

which, in view of the useful identities

$$\frac{(-1)^m}{(n-m)!}=\frac{(-n)_m}{n!}\quad\text{and}\quad(\delta)_{n-m}=\frac{(-1)^m(\delta)_n}{(1-\delta-n)_m},\quad 0\leqslant m\leqslant n,\tag{3.13}$$

and the Chu–Vandermonde reduction formula [31],

$${}_2F_1\left(\begin{matrix}-k,b\\ c\end{matrix};1\right)=\frac{(c-b)_k}{(c)_k},\quad c\neq 0,-1,-2,\ldots,\tag{3.14}$$

gives the well-known connection relation [24]

$$L_n^{\beta}(x)=\sum_{k=0}^{n}\frac{(\beta-\alpha)_k}{k!}L_{n-k}^{\alpha}(x). \tag{3.15}$$

Corollary 3.2. *The Brenke polynomials* $\{P_n\}_{n\geqslant 0}$*, generated by* (2.2) *possess a multiplication formula of the form*

$$P_n(ax)=\sum_{m=0}^{n}\binom{n}{m}a^m\beta_{n-m}(a)P_m(x), \quad \textit{where } \frac{A(t)}{A(at)}=\sum_{k=0}^{\infty}\frac{\beta_k(a)}{k!}t^k. \tag{3.16}$$

As an immediate consequence of (3.16) we obtain the duplication formula

$$\frac{\mathcal{H}_n^{\mu}(ax)}{n!}=\sum_{m=0}^{[\frac{n}{2}]}\frac{a^n(1-a^{-2})^m}{m!}\frac{\mathcal{H}_{n-2m}^{\mu}(x)}{(n-2m)!}. \tag{3.17}$$

Using (2.7) and (3.17), we obtain the operational rule

$$e^{-\frac{\mathcal{D}_{\mu}^2}{4}}\mathcal{H}_n^{\mu}(x)=2^{\frac{n}{2}}\mathcal{H}_n^{\mu}\left(\frac{x}{\sqrt{2}}\right) \tag{3.18}$$

which, in view of (3.9), gives the convolution-type expression

$$2^{\frac{n}{2}}T_y^{\mu}\mathcal{H}_n^{\mu}\left(\frac{x}{\sqrt{2}}\right)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k}^{\mu}(y)\mathcal{H}_k^{\mu}(x). \tag{3.19}$$

That reduces, for $\mu=0$, to the known Runge formula [27]

$$2^{\frac{n}{2}}H_n\left(\frac{x+y}{\sqrt{2}}\right)=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(y)H_k(x).$$

The following result, which gives an explicit expression of the linearization coefficients associated to three Brenke polynomial sets, is a generalization of products formulae associated to Appell and q-Appell polynomials obtained by Carlitz in [10].

Corollary 3.3. *Let* $\{P_n\}_{n\geqslant 0}$*,* $\{Q_n\}_{n\geqslant 0}$ *and* $\{R_n\}_{n\geqslant 0}$ *be three Brenke polynomial sets generated, respectively, by*

$$A_1(t)B_1(xt), \quad A_2(t)B_2(xt) \quad \textit{and} \quad A_3(t)B_3(xt), \tag{3.20}$$

where

$$A_p(t)=\sum_{k=0}^{\infty}a_k^{(p)}t^k \quad \textit{and} \quad B_p(t)=\sum_{k=0}^{\infty}b_k^{(p)}t^k, \quad a_0^{(p)}b_k^{(p)}\neq 0,\ p=1,2,3. \tag{3.21}$$

Then the linearization coefficients in (3.2) *are given by*

$$L_{ij}(k)=\frac{i!j!}{k!}\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{b_r^{(2)}b_s^{(3)}}{b_{r+s}^{(1)}}a_{i-r}^{(2)}a_{j-s}^{(3)}\hat{a}_{r+s-k}^{(1)}, \quad k=0,1,\dots,i+j, \tag{3.22}$$

where $\widehat{A}_1(t)=\frac{1}{A_1(t)}=\sum_{k=0}^{\infty}\hat{a}_k^{(1)}t^k$.

The application of Corollary 3.3 allows us to solve the linearization problem for the generalized Hermite polynomials.

Taking into account the orthogonality and the symmetry of this family ($\mathcal{H}_n^\mu(-x) = (-1)^n \mathcal{H}_n^\mu(x)$), the linearization formula (3.2) can be reduced to

$$\mathcal{H}_i^{\mu_3}(x)\mathcal{H}_j^{\mu_2}(x) = \sum_{k=0}^{\min(i,j)} L_{ij}(i+j-2k)\mathcal{H}_{i+j-2k}^{\mu_1}(x).$$

By virtue of (3.22) we obtain the coefficients explicitly

$$L_{ij}(i+j-2k) = \frac{i!j!}{(i+j-2k)!k!} \sum_{r=0}^{[\frac{i}{2}]} \sum_{s=0}^{[\frac{j}{2}]} \frac{\gamma_{\mu_1}(i+j-2(r+s))}{\gamma_{\mu_2}(i-2r)\gamma_{\mu_3}(j-2s)} \frac{(-k)_{r+s}}{r!s!}. \tag{3.23}$$

Note that, the linearization problem associated to generalized Hermite polynomials was already studied by recurrent approaches. Starting from the fact that this polynomial set is semi-classical, it was shown in [25] that the linearization coefficients $L_{ij}(k)$ satisfy a linear recurrence relation involving only the k index. The coefficients of this recurrence relation are very complicated and can only be obtained using a symbolic manipulation package like `Mathematica`, the obtained coefficients filled many pages [25].

Applying formula (3.23) to Laguerre polynomial sets, we obtain, in view of (1.4) and (3.10),

$$\begin{aligned} L_i^\beta(x) L_j^\gamma(x) = \binom{i+j}{j} \sum_{k=0}^{\min(i,j)} F_{2:0}^{1:2}\left(\begin{matrix} -k : -\beta - i, -i; -\gamma - j, -j; \\ -\alpha - i - j, -i - j : -; -; \end{matrix}\ 1, 1\right) \\ \times (-1)^k \frac{(-\alpha - i - j)_k}{k!} L_{i+j-k}^\alpha(x), \end{aligned} \tag{3.24}$$

where $F_{q:s}^{p:r}$ designates the Kampé de Fériet function defined as follows [31]:

$$F_{q:s}^{p:r}\left(\begin{matrix} (a_p) : (b_r); (c_r); \\ (\alpha_q) : (\beta_s); (\gamma_s); \end{matrix}\ x, y\right) = \sum_{n,m=0}^{\infty} \frac{[a_p]_{n+m}[b_r]_n[c_r]_m}{[\alpha_q]_{n+m}[\beta_s]_n[\gamma_s]_m} \frac{x^n}{n!} \frac{y^m}{m!}, \tag{3.25}$$

where $[a_p]_n = \prod_{j=1}^p (a_j)_n \ldots .$

We remark that there is no difficulty in proving the corresponding formula for the linearization of an arbitrary number of generalized Hermite polynomials.

$$\begin{aligned} &\mathcal{H}_{i_1}^{\mu_1}(x)\mathcal{H}_{i_2}^{\mu_2}(x)\cdots\mathcal{H}_{i_p}^{\mu_p}(x) \\ &\quad = \sum_{2k \leqslant i_1+\cdots+i_p} \frac{i_1!i_2!\cdots i_p!}{(i_1+i_2+\cdots+i_p-2k)!} \sum_{2r_1 \leqslant i_1, \ldots, 2r_p \leqslant i_p} \frac{(-1)^{r_1+\cdots+r_p}}{(k-r_1-\cdots-r_p)!} \\ &\qquad \times \frac{\gamma_\mu(i_1+i_2+\cdots+i_p-2(r_1+r_2+\cdots+r_p))}{r_1!\cdots r_p!\gamma_{\mu_1}(i_1-2r_1)\cdots\gamma_{\mu_p}(i_p-2r_p)} \mathcal{H}_{i_1+i_2+\cdots+i_p-2k}^\mu(x). \end{aligned}$$

The previous formula contains, as particular case, the product of several Hermite and Laguerre polynomials obtained by Carlitz in [9].

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