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On the characteristics of meromorphic functions with three weighted sharing values ☆

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Abstract

In this paper, we deal with the relation between the characteristic function of two nonconstant meromorphic functions with three weighted sharing values, which improves a result given by H.X. Yi and Y.H. Li. From this we establish a theorem which improves a result given by P. Li and C.C. Yang.

Keywords: Meromorphic functions; Shared values; Characteristics of meromorphic functions; Uniqueness

1. Introduction and main results

Let f and g be two nonconstant meromorphic functions in the complex plane. It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ and so on, which can be found in [4]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. The notation $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ $(r \to \infty, r \notin E)$.

Let a be a complex number, we say that f and g share the value a CM provided $f - a$ and $g - a$ have the same zeros counting multiplicities (see [13]). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM. Similarly, we say that f and g share the value a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say

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that f and g share ∞ CM, if $1/f$ and $1/g$ share the value 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share the value 0 IM. In this paper, we also need the following one definition.

Definition 1.1. (See [1, Definition 1].) Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_{p)}(r, \frac{1}{f-a})$ we denote the counting function of those zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r, \frac{1}{f-a})$ we denote the counting function of those zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities).

In 1975, C.F. Osgood and C.C. Yang [11] proved the following theorems.

Theorem A. *Let f and g be two nonconstant entire functions of finite order. If f and g share* 0, 1 *CM, then*

$$T(r, f) \sim T(r, g) \quad (r \to \infty). \tag{1.1}$$

In the paper of C.F. Osgood and C.C. Yang [11], they proposed the following conjecture.

Osgood–Yang's conjecture. [11, p. 409] *Let f and g be two nonconstant entire functions sharing* 0, 1 *CM. Then*

$$T(r, f) \sim T(r, g) \quad (r \to \infty,\ r \notin E). \tag{1.2}$$

In 1989, G. Brosch [3] proved the following two theorems.

Theorem B. *Let f and g be two nonconstant meromorphic functions sharing three values CM, then*

$$\left(\frac{3}{8} + o(1)\right) \leqslant \frac{T(r, f)}{T(r, g)} \leqslant \left(\frac{3}{8} + o(1)\right) \quad (r \to \infty,\ r \notin E).$$

In 1990, W. Bergweiler [2] proved the following theorem.

Theorem C. (See [2, Theorem 1].) *There exists a set $I \subset (0, \infty)$ of infinite Lebesgue measure and there exist meromorphic functions f and g sharing* 0, 1 *and* ∞ *CM such that*

$$\frac{T(r, f)}{T(r, g)} \geqslant 2.$$

Regarding Theorem C, E. Mues [10] proposed the following one conjecture in 1995.

Mues' conjecture. [10, p. 28] *Let f and g be two nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. Then*

$$\frac{1}{2} \cdot \bigl(1 + o(1)\bigr) \leqslant \frac{T(r, f)}{T(r, g)} \leqslant 2 \cdot \bigl(1 + o(1)\bigr) \quad (r \to \infty,\ r \notin E). \tag{1.3}$$

In 1998, P. Li and C.C. Yang [8] proved the following theorem.

Theorem D. (See [8, Theorem 1].) *Let f and g be two nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM, then for any positive number* ε

$$T(r,g) \leqslant (2+\varepsilon)T(r,f) + S(r,f). \tag{1.4}$$

In 2003, H.X. Yi and Y.H. Li affirmatively settle the above two conjectures, and so remove the ε in (1.4). In fact, they proved the following two theorems.

Let f and g share 0, 1 and ∞ IM, next we denote by $N_0(r)$ the counting function of $f-g$ not containing the zeros of f, $1/f$ and $f-1$.

Theorem E. (See [12, Theorem 2.1].) *Let f and g be two nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If*

$$\limsup_{\substack{r\to\infty\\ r\notin E}} \frac{N_0(r)}{T(r,f)} > \frac{1}{2}, \tag{1.5}$$

then (1.1) *holds. If*

$$0 < \limsup_{\substack{r\to\infty\\ r\notin E}} \frac{N_0(r)}{T(r,f)} \leqslant \frac{1}{2}, \tag{1.6}$$

then (1.2) *holds. If*

$$\limsup_{\substack{r\to\infty\\ r\notin E}} \frac{N_0(r)}{T(r,f)} = 0, \tag{1.7}$$

then (1.3) *holds.*

Theorem F. (See [12, Corollary 2.2].) *Let f and g be two nonconstant entire functions sharing two finite values CM. Then* (1.2) *can occur.*

Regarding Theorems E and F, it is natural to ask the following two questions.

Question 1.1. *Is it possible to remove the condition "$r \notin E$" of* (1.2) *in Theorems* E *and* F?

Question 1.2. (See [5].) *Is it really impossible to relax in any way the nature of sharing any one of* 0, 1 *and* ∞ *in Theorems* E *and* F?

In this paper, we shall study the two problems. Next we shall explain the notion of weighted sharing by the following definition.

Definition 1.2. (See [6].) Let k be a nonnegative integer or infinity. For any $a \in C \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$, and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the value a with weight k.

Remark 1.1. Definition 1.2 implies that if f, g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity m ($\leqslant k$) if and only if it is a zero of $g-a$ with multiplicity m ($\leqslant k$), and z_0 is a zero of $f - a$ with multiplicity m $(> k)$, if and only if it is a zero of $g - a$ with multiplicity n $(> k)$, where m is not necessarily equal to n. Throughout this paper, we write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for all integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Using the idea of weighted sharing, we shall establish the following one theorem, which improves Theorem E.

Theorem 1.1. *Let f and g be distinct nonconstant meromorphic functions sharing $(a_1, 1)$, (a_2, ∞) and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$. If* (1.5) *holds, then* (1.1) *and*

$$T(r, f) = N_0(r) + S(r, f) \tag{1.8}$$

still hold, and f is a fractional linear transformation (*Möbius transformation*) *of g. Moreover, f and g assume one of the following three relations*:

(i) $f = e^{\gamma}$ *and* $g = e^{-\gamma}$;
(ii) $f = e^{\gamma} + 1$ *and* $g = e^{-\gamma} + 1$; *and*
(iii) $f = \frac{1}{e^{\gamma}+1}$ *and* $g = \frac{1}{e^{-\gamma}+1}$,

where γ is a nonconstant entire function. If (1.6) *holds, then there exists a nonconstant entire function γ, and two positive integers s and k $(\geqslant 2)$ satisfying $1 \leqslant s \leqslant k$ such that s and $k + 1$ are mutually prime, such that* (1.1) *and*

$$N_0(r) = \frac{1}{k} T(r, f) + S(r, f) \tag{1.9}$$

still hold, and f is not any fractional linear transformation (*Möbius transformation*) *of g, moreover, f and g assume one of the relations* (i)–(iii) *of Lemma* 2.5 *in Section* 2 *of this paper. If* (1.7) *holds and if f is a fractional linear transformation* (*Möbius transformation*) *of g, then* (1.1) *and*

$$N_0(r) = 0 \tag{1.10}$$

still hold. Moreover, f and g assume one of the following three relations:

(iv) $f = \frac{c(e^{\gamma}-1)}{e^{\gamma}-c}$ *and* $g = \frac{e^{\gamma}-1}{e^{\gamma}-c}$;
(v) $f = \frac{c-1}{e^{\gamma}-1}$ *and* $g = \frac{(c-1)e^{\gamma}}{c(e^{\gamma}-1)}$;
(vi) $f = \frac{e^{\gamma}-1}{c-1}$ *and* $g = \frac{c(e^{\gamma}-1)}{(c-1)e^{\gamma}}$;

where $c \in C \setminus \{0, 1\}$ is some finite complex constant. If (1.7) *holds and if f is not any fractional linear transformation* (*Möbius transformation*) *of g, then*

$$N_0(r) = S(r, f) \tag{1.11}$$

and (1.3) *are still valid.*

Using proceeding as in the proof of Theorem 1.1 in Section 3 of this paper we easily deduce the following one theorem, which improves Theorem F.

Theorem 1.2. *Let f and g be two distinct nonconstant entire functions sharing* $(a_1, 1)$ *and* (a_2, ∞)*, where* $\{a_1, a_2\} = \{0, 1\}$*. If* (1.5) *holds, then* (1.1) *and* (1.8) *are still valid, and f is a fractional linear transformation* (*Möbius transformation*) *of g. Moreover, f and g assume one of the relations* (i) *and* (ii) *in Theorem* 1.1*. If* (1.6) *holds, then* (1.1) *is still valid and that f is not any fractional linear transformation* (*Möbius transformation*) *of g. Moreover, there exists a nonconstant entire function γ, and two positive integers s and k* ($\geqslant 2$) *satisfying* $1 \leqslant s \leqslant k$ *such that s and $k+1$ are mutually prime, and* (1.9) *still holds. Moreover, f and g assume one of the following two relations*:

(iii) $f = e^{k\gamma} + e^{(k-1)\gamma} + \cdots + 1$, $g = e^{-k\gamma} + e^{-(k-1)\gamma} + \cdots + 1$;
(iv) $f = -e^{k\gamma} - e^{(k-1)\gamma} - \cdots - e^{\gamma}$, $g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \cdots - e^{-\gamma}$.

If (1.7) *holds and if f is a fractional linear transformation* (*Möbius transformation*) *of g, then* (1.1) *and* (1.10) *are still valid. Moreover, f and g assume one of the relation* (vi) *of Theorem* 1.1*. If* (1.7) *holds and f is not any fractional linear transformation* (*Möbius transformation*) *of g, then* (1.3) *and* (1.11) *are still valid. Moreover, f and g assume the following one relation*:

(v) $f = \frac{e^{2\pi i\mu}-1}{h_0-1}$, $g = \frac{e^{-2\pi i\mu}-1}{h_0^{-1}-1}$, *where μ is a nonconstant entire function, and h_0 is a nonconstant meromorphic function such that if z_n is a zero of $h_0 - 1$ with multiplicity $\nu(n)$* $(n = 1, 2, \ldots)$*, then z_n is also a zero of μ at least with multiplicity $\nu(n)$, and such that $T(r, h_0) = S(r, f)$.*

P. Li and C.C. Yang [8] proved the following result.

Theorem G. (See [8, Theorem 2].) *Let f and g be two nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If*

$$\delta_{1)}(0, f) + \delta_{1)}(1, f) + \delta_{1)}(\infty, f) > \frac{3}{2}, \tag{1.12}$$

where

$$\delta_{1)}(a, f) := 1 - \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_{1)}(r, \frac{1}{f-a})}{T(r, f)},$$

$a \in \{0, 1, \infty\}$ *and* $N_{1)}(r, \frac{1}{f-a})$ *denotes the counting function of the simple a-points of f, then f is a Möbius transformation of g.*

Using the idea of weighted sharing, we shall establish the following one theorem, which improves Theorem G.

Theorem 1.3. *Let f and g be two distinct nonconstant meromorphic functions sharing* $(0, 1)$, $(1, \infty)$ *and* (∞, ∞)*. If* (1.12) *holds, then f is a fractional transformation of g.*

2. Some lemmas

Lemma 2.1. (See [7, Lemma 6].) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share* 0, 1 *and* ∞ *IM, If f is a fractional linear transformation* (*Möbius transformation*) *of g, then f and g satisfy one of the following relations*:

(i) $f \cdot g \equiv 1$,
(ii) $(f-1)(g-1) \equiv 1$,
(iii) $f+g \equiv 1$,
(iv) $f \equiv cg$,
(v) $f-1 \equiv c(g-1)$,
(vi) $[(c-1)f+1] \cdot [(c-1)g-c] \equiv -c$,

where c $(\neq 0, 1)$ is a finite complex constant.

Lemma 2.2. (See [4, p. 8] or [13, Theorem 1.11].) *Let f and g be two nonconstant meromorphic functions. If f is a fractional linear transformation of g, then*

$$T(r, f) = T(r, g) + O(1).$$

Lemma 2.3. (See [6, Lemma 4].) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0, 1)$, $(1, \infty)$ and (∞, ∞). Then*

$$\frac{f-1}{g-1} = e^{\alpha}, \tag{2.1}$$

$$\frac{f}{g} = H, \tag{2.2}$$

where α is an entire function and H is a meromorphic function with

$$\overline{N}(r, H) + \overline{N}(r, 1/H) = S(r, f). \tag{2.3}$$

Lemma 2.4. (See [14, Lemma 6].) *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying*

$$\overline{N}(r, f_i) + \overline{N}\left(r, \frac{1}{f_i}\right) = S(r), \quad i = 1, 2.$$

Then either

$$N_0(r, 1; f_1, f_2) = S(r)$$

or there exist two integers s, t $(|s| + |t| > 0)$ such that

$$f_1^s f_1^t \equiv 1,$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-point.

Lemma 2.5. (See [14, proof of Theorems 1 and 2].) *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM, and let $N_0(r) \neq S(r, f)$. If f is a fractional linear transformation of g, then*

$$N_0(r) = T(r, f) + S(r, f). \tag{2.4}$$

If f is not any fractional linear transformation of g, then

$$N_0(r) \leqslant \frac{1}{2} T(r, f) + S(r, f), \tag{2.5}$$

and f and g assume one of the following three relations:

(i) $f \equiv \frac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1}$, $g \equiv \frac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1}$;
(ii) $f \equiv \frac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}$, $g \equiv \frac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1}$;
(iii) $f \equiv \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}$, $g \equiv \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}$,

where γ is a nonconstant entire function, s and k $(\geqslant 2)$ are positive integers such that s and $k+1$ are mutually prime and $1 \leqslant s \leqslant k$.

Lemma 2.6. (See [9, Lemma 2.5].) *Let s (>0) and t are mutually prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.*

Lemma 2.7. (See [13, proof of Theorems 1.12 and 1.13].) *Let f be a nonconstant meromorphic function, and let*

$$F = \sum_{k=0}^{p} a_k f^k \Big/ \sum_{j=0}^{q} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then

$$T(r, F) = dT(r, f) + O(1),$$

where $d = \max\{p, q\}$.

Lemma 2.8. (See [1, Theorem 1].) *Let f and g be two distinct nonconstant meromorphic functions sharing $(a_1, 1)$, (a_2, ∞) and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$. If f is not any fractional linear transformation of g, then*

$$T(r, f) + T(r, g) = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}(r, f) + N_0(r) + S(r, f).$$

Lemma 2.9. (See [6, Lemma 2].) *Let f and g be two distinct nonconstant meromorphic functions sharing $(0, 1)$, $(1, \infty)$ and (∞, ∞). Then*

$$\overline{N}_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f-1}\right) + N_{(2}(r, f) = S(r, f).$$

3. Proof of theorems

Proof of Theorem 1.1.
We discuss the following two cases.

Case 1. Suppose that f is a fractional linear transformation of g. Noting that f and g share 0, 1 and ∞ IM, from Lemma 2.1 we easily deduce that f and g share 0, 1 and ∞ CM, so from Lemma 2.2 we get (1.1). We discuss the following two subcases.

Subcase 1.1. Suppose that

$$N_0(r) \neq S(r, f). \tag{3.1}$$

Then from (3.1) and Lemma 2.1 we easily deduce that f and g assume one of the relations (i)–(iii) in Lemma 2.1. From (i)–(iii) in Lemma 2.1 we easily deduce (i)–(iii) in Theorem 1.1, respectively, and from (i)–(iii) in Lemma 2.1 we also deduce (1.8).

Subcase 1.2. Suppose that (1.11) holds. Then from Lemma 2.1 we easily deduce that f and g assume one the relations (iv)–(vi) in Lemma 2.1. From (iv)–(vi) in Lemma 2.1 we easily deduce (iv)–(vi) in Theorem 1.1, respectively, and from (iv)–(vi) in Lemma 2.1 we also deduce (1.10).

Case 2. Suppose that f is not any fractional linear transformation of g. Noting that f and g share 0, 1 and ∞ IM, we easily deduce

$$S(r,g)=S(r,f). \tag{3.2}$$

Let (2.1), (2.2) and

$$h_0=\frac{e^{\alpha}}{H}. \tag{3.3}$$

Then from (2.1), (2.2), (3.2) and (3.3) we easily deduce

$$H\not\equiv 0,1,\infty,\qquad h_0\not\equiv 0,1,\infty, \tag{3.4}$$

and

$$\overline{N}\left(r,\frac{1}{h_0}\right)+\overline{N}(r,h_0)=S(r,f). \tag{3.5}$$

Again from (2.1), (2.2) and (3.3) and (3.4) we easily deduce

$$f=\frac{e^{\alpha}-1}{h_0-1} \tag{3.6}$$

and

$$g=\frac{e^{-\alpha}-1}{h_0^{-1}-1}. \tag{3.7}$$

Noting that f is not any fractional transformation of g, from (2.1), (2.2), (3.6) and (3.7) we easily see that none of e^{α}, H and h_0 is a constant, and that

$$f-g=\frac{(e^{\alpha}-1)(1-e^{-\alpha}h_0)}{h_0-1}. \tag{3.8}$$

From (3.2), (3.3), (3.5)–(3.8) we easily deduce

$$N_0(r)=N_0\big(r,1;e^{\alpha},h_0\big)+S(r,f)=N_0\big(r,1;e^{\alpha},H\big)+S(r,f). \tag{3.9}$$

We consider the following two subcases.

Subcase 2.1. Suppose that (3.1) holds. Then from (3.1) and (3.9) we deduce

$$\overline{N}_0\big(r,1;e^{\alpha},H\big)\neq S(r,f). \tag{3.10}$$

Thus from (2.3), (3.10) and Lemma 2.4 we easily see that there exist two integers s and t ($|s|+|t|>0$) such that

$$e^{s\alpha}H^{t}\equiv 1. \tag{3.11}$$

Substituting (2.1) and (2.2) into (3.11) we get

$$f^s(f-1)^t \equiv g^s(g-1)^t. \tag{3.12}$$

Noting that f is not any fractional transformation of g, from (3.12) we deduce that $s \neq 0$, and $t \neq 0$ and $|s| \neq |t|$, so from (3.12) we easily deduce that f and g share 0, 1 and ∞ CM. Thus from Lemma 2.5 we get (2.5), from (2.5) and (3.1) we easily deduce (1.6). Moreover, f and g assume one of the three relations (i)–(iii) in Lemma 2.5, and by simply calculating we easily deduce (1.9).

Suppose that f and g assume the relation (i) in Lemma 2.5, then

$$f \equiv \frac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1}, \qquad g \equiv \frac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1}, \tag{3.13}$$

where γ is a nonconstant entire function, s and k ($\geqslant 2$) are positive integers such that s and $k+1$ are mutually prime and $1 \leqslant s \leqslant k$. It follows by Lemma 2.6 that $\omega = 1$ is the only one common zero of $P_1(\omega) = \omega^{k+1} - 1$ and $P_2(\omega) = \omega^s - 1$, so from (3.13) and Lemma 2.7 we easily deduce (1.1).

Suppose that f and g assume one of the relations (ii) and (iii) in Lemma 2.5. In the same manner as above we easily deduce (1.1), (1.6) and (1.9).

Subcase 2.2. Suppose that

$$N_0(r) = S(r,f), \tag{3.14}$$

which implies (1.7). On the other hand, from (3.14) and Lemma 2.8 we easily deduce

$$\begin{aligned} T(r,f)+T(r,g) &= \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+N_0(r)+S(r,f) \\ &= \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}(r,g)+S(r,f) \\ &\leqslant 3T(r,g)+S(r,f), \end{aligned}$$

which implies that

$$T(r,f) \leqslant 2T(r,g)+S(r,f). \tag{3.15}$$

Similarly

$$T(r,g) \leqslant 2T(r,f)+S(r,g). \tag{3.16}$$

From (3.2), (3.15) and (3.16) we deduce (1.3).

Theorem 1.1 is thus completely proved. □

Proof of Theorem 1.3. Suppose that f is not any fractional transformation of g. Then from Theorem 1.1 we get (1.3). On the other hand, from Lemmas 2.8 and 2.9 we easily deduce

$$\begin{aligned} T(r,f)+T(r,g) &= \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+N_0(r)+S(r,f) \\ &= N_{1)}\left(r,\frac{1}{f}\right)+N_{1)}\left(r,\frac{1}{f-1}\right)+N_{1)}(r,f)+N_0(r)+S(r,f), \end{aligned}$$

namely

$$T(r,f)+T(r,g)=N_{1)}\left(r,\frac{1}{f}\right)+N_{1)}\left(r,\frac{1}{f-1}\right)+N_{1)}(r,f)+N_0(r)+S(r,f). \quad (3.17)$$

We discuss the following two cases.

Case 1. Suppose that (3.14) holds. Then from (1.3), (3.14), (3.17) and (3.2) we easily deduce

$$\frac{3}{2}T(r,f)\leqslant N_{1)}\left(r,\frac{1}{f}\right)+N_{1)}\left(r,\frac{1}{f-1}\right)+N_{1)}(r,f)+S(r,f), \quad (3.18)$$

which implies that

$$\delta_{1)}(0,f)+\delta_{1)}(1,f)+\delta_{1)}(\infty,f)\leqslant\frac{3}{2}, \quad (3.19)$$

which contradicts (1.12).

Case 2. Suppose that (3.1) holds. Then in the same manner as in Subcase 2.1 of the proof of Theorem 1.1 we easily deduce (2.5) and (3.12), and deduce that f and g share the 0, 1 and ∞ CM. From (3.12) and Lemma 2.7 we easily deduce

$$T(r,f)=T(r,g)+O(1). \quad (3.20)$$

From (2.5), (3.17) and (3.20) we easily deduce (3.18). From (3.18) we easily deduce (3.19), which contradicts (1.12).

Theorem 1.3 is thus completely proved. □

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