Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 342 (2008) 386–393
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Norm inequalities for sums of two basic elementary operators

Mohamed Boumazgour

Faculté des Sciences Juridiques, Economiques et Sociales, B.P. 8658 Cité Dakhla, Agadir, Morocco

Received 21 August 2007
Available online 5 December 2007
Submitted by R. Timoney

Abstract

We use certain norm inequalities for 2×2 operator matrices to establish norm inequalities for sums of two basic elementary operators on a Hilbert space. Further, we give necessary and sufficient conditions under which the norm of the above sum of elementary operators attains its optimal value. Applications of the inequalities obtained are also considered.

Keywords: Norm; Elementary operators; Numerical ranges; $\mathcal{C}^*$-algebras

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra. An elementary operator on $\mathcal{A}$ is a linear mapping of the form

$$x \mapsto \sum_{i=1}^{n} a_i x b_i,$$

where $a_i, b_i \in \mathcal{A}$ $(1 \leqslant i \leqslant n)$. Interesting examples of elementary operators on $\mathcal{A}$ are left multiplication L_a by a, right multiplication R_b by b, two-sided multiplication $M_{a,b} := L_a R_b$ ($M_{a,b}$ is also called the basic elementary operator induced by a and b), generalized derivation $\delta_{a,b} := L_a - R_b$ and the Jordan elementary operator $\mathcal{U}_{a,b} := M_{a,b} + M_{b,a}$, here a and b are elements in $\mathcal{A}$.

The norm problem of an elementary operator consists in finding a formula which describes the norm of an elementary operator in terms of its coefficients. It has been considered by many authors, see [2–4,9–11,13,14,16–18] and references therein. We refer to the recent paper [19] for a formula for the norm of an elementary operator on a $\mathcal{C}^*$-algebra, involving matrix-valued numerical ranges and a kind of tracial geometric mean.

In the second section of the present paper, we shall be concerned with some useful estimates for the norm of the elementary operator $M_{A,B} + M_{C,D}$, when A, B, C and D are bounded linear operators on a complex Hilbert space H. As applications, we derive some operator norm inequalities. The third section is devoted to the triangle "equality," that is, we establish necessary and sufficient conditions on the operators A, B, C and D, under which $\|M_{A,B} + M_{C,D}\|$ equals its optimal value $\|A\|\|B\| + \|C\|\|D\|$. For the special case of a Jordan elementary operator,

E-mail address: boumazgour@hotmail.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.12.001

we provide an answer to [14, Problem 9] (see Corollary 3.6 below). In the last section, we give an estimate for the norm of a Jordan elementary operator on arbitrary $\mathcal{C}^*$-algebra.

We conclude this section with some notations and terminology.

Given a complex Hilbert space H, we denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators acting on H. If $A \in \mathcal{B}(H)$, we denote by $W(A)$ its numerical range defined by

$$W(A) = \{\langle Ax, x\rangle\colon\ x \in H,\ \|x\| = 1\}.$$

It is known that the closure of the set $W(A)$ is compact, convex and $\|A\| = \sup\{|\lambda|\colon \in W(A)\}$ whenever A is a self-adjoint operator on H (see [5]).

Let $A, B \in \mathcal{B}(H)$. Following [8], the numerical range of A^*B relative to B is defined to be the set

$$W_B(A^*B) = \Bigl\{\lambda \in \mathbb{C}\colon \text{ there exists } \{x_n\} \subseteq H,\ \|x_n\| = 1 \text{ such that } \lim_n \langle A^*Bx_n, x_n\rangle = \lambda \text{ and } \lim_n \|Bx_n\| = \|B\|\Bigr\}.$$

In the case $A = I$ this reduces to the Stampfli maximal numerical range $W_0(B)$ of B, see [16]. The set $W_B(A^*B)$ enjoys a number of exceptional properties listed below (see [8]):

(i) $W_B(A^*B)$ is a closed convex subset of the complex plane $\mathbb{C}$ for each $A, B \in \mathcal{B}(H)$,
(ii) The relation $\|B\| = \inf_{\lambda\in\mathbb{C}} \|B - \lambda A\|$ holds if and only if $0 \in W_B(A^*B)$.

The normalized maximal numerical range of a bounded operator $T \in \mathcal{B}(H)$ is given by

$$W_N(T) = \begin{cases} W_0(\frac{T}{\|T\|}) & \text{if } T \neq 0, \\ 0 & \text{if } T = 0. \end{cases}$$

The set $W_N(T)$ is nonempty, closed, convex and contained in the closure of the numerical range of T, see [16].

Let E be a normed space. If $T : E \to E$ is a bounded linear operator, we denote by $\|T : E \to E\|$ the norm of T, or simply by $\|T\|$ when there is no confusion.

Throughout, I stands for the identity operator of $\mathcal{B}(H)$. If $A \in \mathcal{B}(H)$, we denote its adjoint by A^*. The real part and complex conjugate of a complex number z are denoted by $\Re(z)$ and $\bar{z}$, respectively.

2. A general norm inequality

Before we state and prove our first result we need a technical lemma.

Lemma 2.1. *Let A and B be positive operators on H. If T is the operator matrix on $H \oplus H$ defined by $T = \left[\begin{smallmatrix} A & C^* \\ C & B\end{smallmatrix}\right]$, then*

$$\|T\| \leqslant \max\{\|A\|, \|B\|\} + \|C\|.$$

Proof. Write $T = \left[\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & C^* \\ C & 0\end{smallmatrix}\right]$. It is easy to check that $\left\|\left[\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right]\right\| \leqslant \max\{\|A\|, \|B\|\}$ and $\left\|\left[\begin{smallmatrix} 0 & C^* \\ C & 0\end{smallmatrix}\right]\right\| \leqslant \|C\|$. Hence we get

$$\|T\| \leqslant \max\{\|A\|, \|B\|\} + \|C\|. \qquad \square$$

Besides some special cases, the norm of the operator $M_{A,B} + M_{C,D}$ is usually difficult to compute (see [10,17] for examples). The following estimation could be more useful.

Theorem 2.2. *If A, B, C and D are operators in $\mathcal{B}(H)$, then*

$$\|M_{A,B} + M_{C,D}\| \leqslant \bigl[\bigl(\max\{\|B\|^2, \|D\|^2\} + \|BD^*\|\bigr)\bigl(\max\{\|A\|^2, \|C\|^2\} + \|C^*A\|\bigr)\bigr]^{\frac{1}{2}}.$$

Proof. Let $X \in \mathcal{B}(H)$ and $x \in H$ be given such that $\|X\| = \|x\| = 1$. We have

$$\|AXBx + CXDx\|^2 = \|AXBx\|^2 + \|CXDx\|^2 + 2\Re(\langle AXBx, CXDx\rangle)$$

and

$$\begin{aligned}\|AXBx\|^2+\|CXDx\|^2 &\leqslant \|A\|^2\|Bx\|^2+\|C\|^2\|Dx\|^2\\ &\leqslant \max\{\|A\|^2,\|C\|^2\}\big(\|Bx\|^2+\|Dx\|^2\big)\\ &\leqslant \max\{\|A\|^2,\|C\|^2\}\big\|\left[\begin{smallmatrix}B&0\\D&0\end{smallmatrix}\right]\left(\begin{smallmatrix}x\\0\end{smallmatrix}\right)\big\|^2\\ &\leqslant \max\{\|A\|^2,\|C\|^2\}\big\|\left[\begin{smallmatrix}B&0\\D&0\end{smallmatrix}\right]\big\|^2.\end{aligned}$$

Since $\big\|\left[\begin{smallmatrix}B&0\\D&0\end{smallmatrix}\right]\big\|^2=\big\|\left[\begin{smallmatrix}B&0\\D&0\end{smallmatrix}\right]\left[\begin{smallmatrix}B^*&D^*\\0&0\end{smallmatrix}\right]\big\|=\big\|\left[\begin{smallmatrix}BB^*&BD^*\\DB^*&DD^*\end{smallmatrix}\right]\big\|$, we obtain by virtue of Lemma 2.1,

$$\big\|\left[\begin{smallmatrix}B&0\\D&0\end{smallmatrix}\right]\big\|^2\leqslant \max\{\|B\|^2,\|D\|^2\}+\|BD^*\|.$$

Therefore

$$\|AXBx\|^2+\|CXDx\|^2\leqslant \max\{\|A\|^2,\|C\|^2\}\big[\max\{\|B\|^2,\|D\|^2\}+\|BD^*\|\big].$$

On the other hand, we have

$$2\Re\big(\langle AXBx, CXDx\rangle\big)\leqslant 2\|Bx\|\|Dx\|\|C^*A\|\leqslant \|C^*A\|\big(\|Bx\|^2+\|Dx\|^2\big).$$

As in the above, we deduce that

$$2\Re\big(\langle AXBx, CXDx\rangle\big)\leqslant \|C^*A\|\big[\max\{\|B\|^2,\|D\|^2\}+\|BD^*\|\big].$$

Hence

$$\|M_{A,B}+M_{C,D}\|^2\leqslant \big[\max\{\|B\|^2,\|D\|^2\}+\|BD^*\|\big]\big[\max\{\|A\|^2,\|C\|^2\}+\|C^*A\|\big].$$

This ends the proof. □

The majorisation in Theorem 2.2 can be used to derive several operator norm inequalities:

Corollary 2.3. *For $A, B\in\mathcal{B}(H)$, the following properties hold*:

(1) $\|A+B\|^2\leqslant 2(\max\{\|A\|^2,\|B\|^2\}+\|B^*A\|)$,
(2) $\|AA^*+BB^*\|\leqslant \max\{\|A\|^2,\|B\|^2\}+\|B^*A\|$.

Proof. The inequality in (1) follows from Theorem 2.2 by letting $B=D=I$. The second inequality follows by letting $B=A^*$ and $D=C^*$ in the same theorem. □

Remark 2.4. If A and B are operators in $\mathcal{B}(H)$ with orthogonal ranges, then

$$\|A+B\|^2\leqslant \max\{\|A\|^2,\|B\|^2\}+\|BA^*\|. \tag{2.1}$$

To see this, observe first that, since $A^*B=0$, we have $(A+B)^*(A+B)=A^*A+B^*B$. Thus, the inequality (2.1) follows from Corollary 2.3(2). We refer to [7] for other norm inequalities for sums of operators having orthogonal ranges.

Remark 2.5. Let $A, B\in\mathcal{B}(H)$ and let $A\otimes B$ denote the tensor product of A by B. It is well known (and easy to prove) that $\|A\otimes B\|=\|A\|\|B\|$ (here $\|A\otimes B\|$ denotes the norm of $A\otimes B$ induced by the usual inner product on the Hilbert space $H\otimes H$), hence it follows from Corollary 2.3(1) that

$$\|A\otimes B+B\otimes A\|\leqslant\sqrt{2\|A\|^2\|B\|^2+2\|B^*A\|^2}.$$

For the Jordan elementary operator $\mathcal{U}_{A,B}$ we get the following upper estimate which is sharper than the triangle inequality.

Corollary 2.6. *If $A, B \in \mathcal{B}(H)$, then*

$$\|\mathcal{U}_{A,B}\| \leqslant \big[\big(\|A\|\|B\| + \|AB^*\|\big)\big(\|A\|\|B\| + \|B^*A\|\big)\big]^{\frac{1}{2}}. \tag{2.2}$$

In particular, if $AB^* = B^*A = 0$, then

$$\|\mathcal{U}_{A,B}\| = \|A\|\|B\|.$$

Proof. By letting $D = A$ and $C = B$ in Theorem 2.2, we get

$$\|\mathcal{U}_{A,B}\|^2 \leqslant \big[\max\big\{\|A\|^2, \|B\|^2\big\} + \|AB^*\|\big]\big[\max\big\{\|A\|^2, \|B\|^2\big\} + \|B^*A\|\big]. \tag{2.3}$$

Since $\|\mathcal{U}_{A,B}\| = \|A\|\|B\|\|U_{\frac{A}{\|A\|}, \frac{B}{\|B\|}}\|$ (we may assume that A and B are nonzero), we derive from (2.3) that

$$\|\mathcal{U}_{A,B}\| \leqslant \big[\big(\|A\|\|B\| + \|AB^*\|\big)\big(\|A\|\|B\| + \|B^*A\|\big)\big]^{\frac{1}{2}}.$$

This proves the first part.

Next, assume that $AB^* = B^*A = 0$. By the inequality in (2.2), it turns out that $\|\mathcal{U}_{A,B}\| \leqslant \|A\|\|B\|$. Since we always have $\|\mathcal{U}_{A,B}\| \geqslant \|A\|\|B\|$, then $\|\mathcal{U}_{A,B}\| = \|A\|\|B\|$, as required. □

Remark 2.7. In many cases the inequality $\|\mathcal{U}_{A,B}\| \leqslant \|A\|\|B\|(1 + \|AB\|)$ $(A, B \in \mathcal{B}(H))$ is satisfied. In [15], Stacho and Zalar asked whether this holds in general. The next example shows that the inequality fails to be true. Let $A \in \mathcal{B}(H)$ be nilpotent of order 2. Then clearly we have $\|U_{A,A}\| = 2\|A\|$, but $A^2 = 0$.

Proposition 2.8. *Let $A, B \in \mathcal{B}(H)$. If $\|\mathcal{U}_{A,B}\| = \|A\|\|B\|$, then $0 \in W_B(A^*B) \cup W_A(B^*A)$, if one of the following conditions is satisfied*:

(1) *there exists $\lambda \in W_B(A^*B)$ and there exists $\mu \in W_A(B^*A)$ such that $\Re(\lambda\mu) \geqslant 0$,*
(2) *either A^*B or $AB^* \geqslant 0$ is positive,*
(3) *both A and B are positive.*

Proof. Assume that $\|\mathcal{U}_{A,B}\| = \|A\|\|B\|$. We may suppose that $\|A\| = \|B\| = 1$. If x and y are unit elements in H, then

$$\begin{aligned}\|\mathcal{U}_{A,B}\| &\geqslant \|\mathcal{U}_{A,B}(x \otimes By)(y)\| \\ &\geqslant \big|\|Ax\|^2\|By\|^2 + \langle A^*Bx, x\rangle\langle B^*Ay, y\rangle\big|.\end{aligned}$$

On the other hand, we have

$$\|\mathcal{U}_{A,B}\| \geqslant \big|\langle Ax, x\rangle\langle By, y\rangle + \langle Ay, y\rangle\langle Bx, x\rangle\big|.$$

From these inequalities we may deduce the following:

(1) If $\lambda \in W_B(A^*B)$ and $\mu \in W_A(B^*A)$ are given such that $\Re(\lambda\mu) \geqslant 0$, then $\lambda\mu = 0$, that is, $0 \in W_B(A^*B) \cup W_A(B^*A)$.
(2) If either $A^*B \geqslant 0$ or $B^*A \geqslant 0$, then $\Re(\lambda\mu) \geqslant 0$ for all $\lambda \in W_B(A^*B)$ and $\mu \in W_A(B^*A)$. So the result follows by (1).
(3) If the operators A and B are positive, then $w(A) = w(B) = 1$. Hence, considering a subsequence if necessary, we deduce from the second above inequality that $0 \in W_B(AB) \cup W_A(BA)$. □

Remark 2.9. If $\|U_{A,B}\| = \|A\|\|B\|$, then $w(A^*B) \leqslant \frac{1}{2}\|A\|\|B\|$. This follows easily from the fact that if $x \in H$ is a unit vector, with $Ax \neq 0$, then

$$\|\mathcal{U}_{A,B}\| \geqslant \frac{1}{\|Ax\|}\big\|\mathcal{U}_{A,B}(x \otimes Ax)\big\| \geqslant 2\big|\langle Bx, Ax\rangle\big|.$$

In particular, we conclude that if $\|\mathcal{U}_{A,B}\| = \|A\|\|B\|$, then the operator A^*B may not be normaloid.

Remark 2.10. If A is an isometry and $\|\mathcal{U}_{A,B}\| = \|A\|\|B\|$, then $W_B(A^*B) = \{0\}$. Indeed, by [3, Theorem 1], we have

$$\|\mathcal{U}_{A,B}\| \geqslant \sup_{\lambda \in W_B(A^*B)} \left\| \|B\|A + \frac{\bar{\lambda}}{\|B\|} B \right\|. \tag{2.4}$$

Since A is isometry, we derive from (2.4) that $\|\mathcal{U}_{A,B}\|^2 \geqslant \|B\|^2 + 3|\lambda|^2$ for all $\lambda \in W_B(A^*B)$. Hence $W_B(A^*B) = \{0\}$.

3. The triangle "equality"

Our next aim is to establish necessary and sufficient conditions on the set $\{A, B, C, D\}$ of operators so that the norm of $M_{A,B} + M_{C,D}$ equals its optimal value.

The following theorem is a generalization of [2, Theorem 1].

Theorem 3.1. *For nonzero operators $A, B, C, D \in \mathcal{B}(H)$ the following properties are equivalent*:

(1) $\|M_{A,B} + M_{C,D}\| = \|A\|\|B\| + \|C\|\|D\|$,
(2) $W_N(C^*A) \cap W_N(DB^*) \neq \emptyset$, $\|C^*A\| = \|A\|\|C\|$ *and* $\|DB^*\| = \|B\|\|D\|$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\|M_{A,B} + M_{C,D}\| = \|A\|\|B\| + \|C\|\|D\|$. Then we can find two unit sequences $\{X_n\}_n \subseteq \mathcal{B}(H)$ and $\{x_n\}_n \subseteq H$ such that

$$\lim_n \|AX_nBx_n + CX_nDx_n\| = \|A\|\|B\| + \|C\|\|D\|. \tag{3.1}$$

We have, for each $n \geqslant 1$,

$$\|AX_nBx_n + CX_nDx_n\|^2 = \|AX_nBx_n\|^2 + \|CX_nDx_n\|^2 + 2\Re\big(\langle AX_nBx_n, CX_nDx_n\rangle\big). \tag{3.2}$$

Since $\|AX_nBx_n\| \leqslant \|A\|\|Bx_n\| \leqslant \|A\|\|B\|$ and $\|CX_nDx_n\| \leqslant \|C\|\|Dx_n\| \leqslant \|C\|\|D\|$, it follows that

$$\lim_n \|Bx_n\| = \|B\| \quad \text{and} \quad \lim_n \|Dx_n\| = \|D\|.$$

Therefore

$$\lim_n \big(B^*Bx_n - \|B\|^2x_n\big) = \lim_n \big(D^*Dx_n - \|D\|^2x_n\big) = 0.$$

Set $y_n = \frac{Bx_n}{\|B\|}$. Then one can write $x_n = \frac{B^*y_n}{\|B\|} + z_n$, where z_n is a vector in H such that $\lim_n z_n = 0$. So

$$\langle AX_nBx_n, CX_nDx_n\rangle = \big\langle C^*AX_ny_n, X_nDB^*y_n\big\rangle + \epsilon_n, \tag{3.3}$$

where $\epsilon_n \in \mathbb{C}$, with $\lim_n \epsilon_n = 0$. Combining (3.1)–(3.3) we obtain

$$\lim_n \big\langle C^*AX_ny_n, X_nDB^*y_n\big\rangle = \|A\|\|B\|\|C\|\|D\|.$$

From this we derive that

$$\lim_n \big\|\big(C^*A\big)X_ny_n\big\| = \|A\|\|C\| = \big\|C^*A\big\|, \qquad \lim_n \big\|X_n\big(DB^*\big)y_n\big\| = \|B\|\|D\| = \big\|DB^*\big\|,$$

so it follows immediately that $\|\delta_{C^*A,-DB^*}\| = \|A\|\|C\| + \|B\|\|D\| = \|C^*A\| + \|DB^*\|$. So from [16, Theorem 7] it turns out that

$$W_N\big(C^*A\big) \cap W_N\big(DB^*\big) \neq \emptyset.$$

(2) $\Rightarrow$ (1). Suppose that $\|C^*A\| = \|A\|\|C\|$, $\|DB^*\| = \|B\|\|D\|$ and $W_N(C^*A) \cap W_N(DB^*) \neq \emptyset$. Then by [16, Theorem 7], we get

$$\|\delta_{C^*A,-DB^*}\| = \|A\|\|C\| + \|B\|\|D\|.$$

Thus, there exist two sequences $\{X_n\}_n \subseteq \mathcal{B}(H)$ and $\{x_n\}_n \subseteq H$, with $\|X_n\| = \|x_n\| = 1$ for each n such that

$$\lim_n \big\|C^*AX_nx_n + X_nDB^*x_n\big\| = \|A\|\|C\| + \|B\|\|D\|. \tag{3.4}$$

As in the above we deduce that $\lim_n \|B^*x_n\| = \|B\|$. Thus, one can write $BB^*x_n = \|B\|^2 x_n + z_n$, where $z_n \in H$ with $\lim_n z_n = 0$. From the equality (3.4) it follows that

$$\lim_n \langle C^*AX_n x_n, X_n DB^* x_n \rangle = \lim_n \left\langle (AX_n B)\frac{B^*x_n}{\|B\|}, (CX_n D)\frac{B^*x_n}{\|B\|} \right\rangle = \|A\|\|B\|\|C\|\|D\|.$$

From this we conclude that

$$\|M_{A,B} + M_{C,D}\| = \|A\|\|B\| + \|C\|\|D\|.$$

This completes the proof. □

Corollary 3.2. *For $A, B \in \mathcal{B}(H)$ the following properties are equivalent*:

(1) $\|A + B\| = \|A\| + \|B\|$,
(2) $\|A\|\|B\| \in W_0(A^*B)$.

Remark 3.3. Suppose that A and B are positive operators in $\mathcal{B}(H)$. By Corollary 3.2 and Propositions 3.3 and 3.4 in [7], we see that the following are also equivalent:

(1) $\|A + B\| = \|A\| + \|B\|$,
(2) $\|A^{\frac{1}{2}} B^{\frac{1}{2}}\| = \|A\|^{\frac{1}{2}} \|B\|^{\frac{1}{2}}$,
(3) $\|AB\| = \|A\|\|B\|$,
(4) $w(AB) = \|A\|\|B\|$.

Corollary 3.4. *For $A, B \in \mathcal{B}(H)$ the following properties are equivalent*:

(1) $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$,
(2) $W_N(A) \cap W_N(B^*) \neq \emptyset$,
(3) $\|A\| + \|B\| \leqslant \|A - \lambda\| + \|B - \lambda\|$ *for all* $\lambda \in \mathbb{C}$.

Proof. The equivalence (1) ⇔ (2) is an immediate consequence of Theorem 3.1.
The equivalence (2) ⇔ (3) follows from Theorems 7 and 8 of [16]. □

Corollary 3.5. *If $A, B \in \mathcal{B}(H)$ are given such that $\|A^*B\| = \|A\|\|B\|$, then*

$$\|M_{A,A^*} + M_{B,B^*}\| = \|A\|^2 + \|B\|^2.$$

The next result, obtained as a special case of Theorem 3.1, is a generalization of [12, Theorem 9]. It provides a solution to Problem 9 in [14].

Corollary 3.6. *If $A, B \in \mathcal{B}(H)$, then the following assertions are equivalent*:

(1) $\|\mathcal{U}_{A,B}\| = 2\|A\|\|B\|$,
(2) $\|AB^*\| = \|B^*A\| = \|A\|\|B\|$ *and* $W_N(AB^*) \cap W_N(B^*A) \neq \emptyset$.

Remark 3.7. If $AB = BA$ and B is normal, then we conclude from the above corollary and Putnam–Fuglede theorem that $\|\mathcal{U}_{A,B}\| = 2\|A\|\|B\|$ if and only if $\|A^*B\| = \|A\|\|B\|$. If we drop the commutativity condition this equivalence is no longer true. Indeed, let $\{e_n\}_{n\geqslant 0}$ be an orthonormal basis for the space $\ell^2(\mathbb{N})$. Let S be the right shift on $\ell^2(\mathbb{N})$ defined by $Se_n = e_{n+1}$ $(n = 0, 1, \ldots)$, and let P be the orthogonal projection on the space spanned by e_0. If we set $A = S^*$ and $B = P$, then $AB^* = 0$ and $\|A^*B\| = \|A\|\|B\| = 1$. Thus, it follows from Corollary 2.6 that $\|\mathcal{U}_{A,B}\| \leqslant \sqrt{2}\|A\|\|B\|$. This answers negatively Questions 1 and 2 in [12].

4. An estimate for the norm of a Jordan elementary operator on a $\mathcal{C}^*$-algebra

In this section, we establish an upper bound for the norm of the Jordan elementary operator $\mathcal{U}_{A,B}$ on a $\mathcal{C}^*$-algebra.

Proposition 4.1. *Let $\mathcal{A}$ be an arbitrary $\mathcal{C}^*$-algebra, and let $A, B \in \mathcal{A}$. Then*

$$\|\mathcal{U}_{A,B}\| \leqslant \left[\left(\|L_A R_B\| + \|AB^*\|\right)\left(\|L_A R_B\| + \|B^*A\|\right)\right]^{\frac{1}{2}}.$$

Proof. Let $A, B \in \mathcal{A}$. By [1, Theorem 5.3.12], we have

$$\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| = \sup_{\pi \in \mathrm{Irr}(\mathcal{A})} \left\|\mathcal{U}_{\pi(A),\pi(B)} : \pi(\mathcal{A}) \to \pi(\mathcal{A})\right\|, \tag{4.1}$$

where $\mathrm{Irr}(\mathcal{A})$ denotes a faithful family of irreducible representations of $\mathcal{A}$.

Let $\pi : \mathcal{A} \to \mathcal{B}(H_\pi)$ be an irreducible representation of $\mathcal{A}$. Since $\mathcal{C}^*$-algebras have metric approximate identities, we have

$$\left\|\mathcal{U}_{\pi(A),\pi(B)} : \pi(\mathcal{A}) \to \pi(\mathcal{A})\right\| = \left\|\mathcal{U}_{\pi(A),\pi(B)} : \pi(\mathcal{A}) + \mathbb{C}I_\pi \to \pi(\mathcal{A}) + \mathbb{C}I_\pi\right\|, \tag{4.2}$$

where I_π denotes the identity of $\mathcal{B}(H_\pi)$. Moreover, since $\pi(\mathcal{A})$ is irreducible, the commutant of $\pi(\mathcal{A}) + \mathbb{C}I_\pi$ is precisely $\mathcal{B}(H_\pi)$, and so, by the double commutant theorem and the Kaplanski density theorem [6, Theorems 5.3.1 and 5.3.5], we have

$$\left\|\mathcal{U}_{\pi(A),\pi(B)} : \pi(\mathcal{A}) + \mathbb{C}I_\pi \to \pi(\mathcal{A}) + \mathbb{C}I_\pi\right\| = \left\|\mathcal{U}_{\pi(A),\pi(B)} : B(H_\pi) \to \mathcal{B}(H_\pi)\right\|. \tag{4.3}$$

Combining (4.1)–(4.3), and then using Corollary 2.6, we obtain

$$\begin{aligned}\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| &= \sup_{\pi \in \mathrm{Irr}(\mathcal{A})} \left\|\mathcal{U}_{\pi(A),\pi(B)} : B(H_\pi) \to \mathcal{B}(H_\pi)\right\| \\ &\leqslant \sup_{\pi \in \mathrm{Irr}(\mathcal{A})} \left[\left(\|\pi(A)\|\|\pi(B)\| + \left\|\pi\left(AB^*\right)\right\|\right)\left(\|\pi(A)\|\|\pi(B)\| + \left\|\pi\left(B^*A\right)\right\|\right)\right]^{\frac{1}{2}}.\end{aligned}$$

It is known (see [11, p. 364]) that, for every $\mathcal{C}^*$-algebra $\mathcal{A}$, $\|L_A R_B\| = \sup_{\pi \in \mathrm{Irr}(\mathcal{A})} \|\pi(A)\|\|\pi(B)\|$. Hence,

$$\begin{aligned}\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| &\leqslant \sup_\pi \left[\left(\|L_A R_B\| + \left\|\pi\left(AB^*\right)\right\|\right)\left(\|L_A R_B\| + \left\|\pi\left(B^*A\right)\right\|\right)\right]^{\frac{1}{2}} \\ &\leqslant \left[\left(\|L_A R_B\| + \|AB^*\|\right)\left(\|L_A R_B\| + \|B^*A\|\right)\right]^{\frac{1}{2}},\end{aligned}$$

because every representation π is a contraction [1, Proposition 1.2.3]. $\square$

Remark 4.2. Let $\mathcal{A}$, A and B as in the hypotheses of the last proposition. If $AB^* = B^*A = 0$, then $\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| \leqslant \|L_A R_B\|$. It is known [4, Corollary 3.9] that, $\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| \geqslant \|L_A R_B\|$, so it follows that $\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| = \|L_A R_B\|$. In particular, if $\mathcal{A}$ is prime, and $AB^* = B^*A = 0$, then $\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| = \|A\|\|B\|$.

We conclude with the following questions posed also in [12].

Questions 4.3.

(1) If $\|\mathcal{U}_{A,B} : \mathcal{A} \to \mathcal{A}\| = \|L_A R_B\|$, does it follow that $AB^* = B^*A = 0$?
(2) Does Proposition 2.8 remain true if we drop all conditions on the operators A and B?

References

[1] P. Ara, M. Mathieu, Local Multipliers of $\mathcal{C}^*$-Algebras, Monogr. Math., Springer, London, 2003.
[2] M. Barraa, M. Boumazgour, Norm equality for a basic elementary operator, J. Math. Anal. Appl. 286 (2003) 359–362.
[3] M. Barraa, M. Boumazgour, A lower bound for the norm of the operator $X \to AXB + BXA$, Extracta Math. 16 (2001) 223–227.
[4] A. Blanco, M. Boumazgour, T.J. Ransford, On the norm of elementary operators, J. London Math. Soc. (2) 70 (2004) 479–498.

[5] F.F. Bonsall, J. Duncan, Numerical Ranges of Operators on Normed Spaces and Elements of Normed Algebras, Cambridge Univ. Press, 1971.
[6] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, vol. I, Academic Press, New York, 1983.
[7] F. Kittaneh, Norm inequalities for sums of positive operators, J. Operator Theory 48 (2002) 95–103.
[8] B. Magajna, On the distance to finite-dimensional subspaces in operator algebras, J. London Math. Soc. (2) 47 (1993) 516–532.
[9] B. Magajna, A. Turnsek, On the norm of symmetrised two-sided multiplications, Bull. Austral. Math. Soc. 67 (2003) 27–38.
[10] B. Magajna, The norm of a symmetric elementary operator, Proc. Amer. Math. Soc. 132 (2004) 1747–1754.
[11] B. Magajna, The norm problem for elementary operators, in: K.D. Bierstedt, et al. (Eds.), Recent Progress in Functional Analysis, Elsevier, Amsterdam, 2001, pp. 363–368.
[12] A. Seddik, On the numerical range and norm of elementary operators, Linear Multilinear Algebra 52 (2004) 293–302.
[13] A. Seddik, On the norm of elementary operators in standard operator algebras, Acta Sci. Math. (Szeged) 70 (2004) 229–236.
[14] L.L. Stacho, B. Zalar, On the norm of Jordan elementary operators in standard operator algebras, Publ. Math. Debrecen 49 (1996) 127–134.
[15] L.L. Stacho, B. Zalar, Uniform primeness of Jordan algebra of symmetric elementary operators, Proc. Amer. Math. Soc. 126 (1998) 2241–2247.
[16] J. Stampfli, The norm of a derivation, Pacific J. Math. 33 (1970) 737–747.
[17] R.M. Timoney, Norms and CB norms of Jordan elementary operators, Bull. Sci. Math. 127 (2003) 597–609.
[18] R.M. Timoney, Computing the norm of elementary operators, Illinois J. Math. 47 (2003) 1207–1226.
[19] R.M. Timoney, Some formulae for norms of elementary operators, J. Operator Theory 57 (2007) 121–145.