



Euler–Bernoulli beams from a symmetry standpoint-characterization of equivalent equations

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ABSTRACT

We completely solve the equivalence problem for Euler–Bernoulli equation using Lie symmetry analysis. We show that the quotient of the symmetry Lie algebra of the Bernoulli equation by the infinite-dimensional Lie algebra spanned by solution symmetries is a representation of one of the following Lie algebras: $2A_1$, $A_1 \oplus A_2$, $3A_1$, or $A_{3,3} \oplus A_1$. Each quotient symmetry Lie algebra determines an equivalence class of Euler–Bernoulli equations. Save for the generic case corresponding to arbitrary lineal mass density and flexural rigidity, we characterize the elements of each class by giving a determined set of differential equations satisfied by physical parameters (lineal mass density and flexural rigidity). For each class, we provide a simple representative and we explicitly construct transformations that maps a class member to its representative. The maximally symmetric class described by the four-dimensional quotient symmetry Lie algebra $A_{3,3} \oplus A_1$ corresponds to Euler–Bernoulli equations homeomorphic to the uniform one (constant lineal mass density and flexural rigidity). We rigorously derive some non-trivial and non-uniform Euler–Bernoulli equations reducible to the uniform unit beam. Our models extend and emphasize the symmetry flavor of Gottlieb's iso-spectral beams [H.P.W. Gottlieb, Isospectral Euler–Bernoulli beam with continuous density and rigidity functions, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 413 (1987) 235–250].

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1. Introduction

Da Vinci and Galileo foresaw the need for a theory of vibrating thin beams. However they suggested theories that were either incomplete or erroneous. Da Vinci's theory was more descriptive and based on detailed sketches rather than physical laws and equations: he lacked tools such as Hooke's law, Newton's laws, and calculus which postdate him. In Galileo's approach, the nemesis was an incorrect calculation of the load carrying capacity of transversely loaded beams. We owe the first consistent thin beams theory to the Bernoullis. Jacob Bernoulli developed an elasticity theory in which the curvature of an elastic beam is proportional to its bending moment. Relying on his uncle elasticity theory, Daniel Bernoulli derived a partial differential equation governing the motion of a thin vibrating beam. Leonard Euler extended and applied the Bernoullis theory to loaded beams.

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¹ In loving memory of my brother Léopold Fotso Simo.

In Euler–Bernoulli beam theory, the transversal motion of an unloaded thin elastic beam is governed by the partial differential equation

$$\frac{\partial^2}{\partial x^2} \left(f(x) \frac{\partial^2 u}{\partial x^2} \right) + m(x) \frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0, \quad 0 < x < L, \tag{1}$$

where $f(x) > 0$ is the flexural rigidity, $m(x) > 0$ is the lineal mass density, and $u(t, x)$ is the transversal displacement at time t and position x from one end of the beam taken as origin. Eq. (1) must be solved subject to initial and boundary conditions such as clamped ends, hinged ends, and free ends boundary conditions.

In this paper, our focus is on the equivalence problem for Eq. (1): we seek necessary and sufficient conditions under which two equations of the form (1) can be mapped to each other using an invertible change of the dependent and independent variables. A particular case of this equivalence problem was tackled by Gottlieb [1] who was interested in equations of the form (1) that are equivalent to the uniform (constant f and m) beam equation. In the same vein, Bluman and Kumei [2, Chapter 6, Section 6.5] studied the problem of reducing a linear partial differential equation to a constant coefficient one. In [3], the authors obtained some general results about the symmetries of linear partial differential equations of any order.

The layout of this paper is the following. There are four sections including this introduction. Section 2 deals with the complete Lie symmetry classification of Eq. (1). Section 3 is dedicated to the construction of equivalence transformations. We recapitulate our findings in Section 4.

2. Symmetry analysis of Euler–Bernoulli equation

Our goal in this section is to study the symmetry breaking of Eq. (1). We assume that the reader is familiar with the rudiments of Lie’s symmetry theory [2,4,5]. Detailed implementations of Lie’s symmetry algorithm on models arising in applications may be found in the papers [6–10] and references therein.

A vector field

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \tag{2}$$

is a Lie symmetry of Eq. (1) if

$$X^{[4]} \left(\frac{\partial^2}{\partial x^2} \left(f(x) \frac{\partial^2 u}{\partial x^2} \right) + m(x) \frac{\partial^2 u}{\partial t^2} \right) \Big|_{\text{Eq. (1)}} = 0, \tag{3}$$

where $X^{[4]}$ is the fourth prolongation of X which is calculated using the formulas

$$X^{[k]} = X + \sum_{1 \leq |J| \leq k} \eta_J \frac{\partial}{\partial u_J}, \quad J = (j_1, j_2), \quad |J| = j_1 + j_2, \tag{4}$$

$$u_J = \partial^{|J|} u / \partial t^{j_1} \partial x^{j_2}, \tag{5}$$

$$\eta_J = D_J(\eta - \tau u_t - \xi u_x) + \tau u_{J,t} + \xi u_{J,x}, \quad u_{J,r} = \partial u_J / \partial r, \tag{6}$$

$$D_J = D_{j_1} D_{j_2}, \quad D_{j_1} = (D_t)^{j_1}, \quad D_{j_2} = (D_x)^{j_2}, \tag{7}$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \dots, \tag{8}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \dots. \tag{9}$$

Since the symmetry coefficients τ , ξ and η do not depend explicitly on the derivatives of u , the left-hand side of Eq. (2) is a polynomial in the derivatives of u . Thus we may set its coefficients to zero to obtain an over-determined system of linear partial differential equations. In order to avoid the appearance of the integral $\int (m/f)^{1/4} dx$ in our calculations, we express the lineal mass density as follows.

$$m(x) = (g'(x))^4 f(x). \tag{10}$$

After some calculations, the determining equations for the symmetries simplify to

$$\tau = 4c_1 t + c_2, \tag{11}$$

$$\xi = \frac{2c_1 g}{g'} + \frac{2c_3}{g'}, \tag{12}$$

$$\eta = - \left(\frac{c_1 f' g}{f g'} + \frac{c_3 f'}{f g'} + \frac{3c_1 g g''}{g'^2} + \frac{3c_3 g''}{g'^2} + c_4 \right) u + a(t, x), \tag{13}$$

$$(f a_{xx})_{xx} + g'^4 f a_{tt} = 0, \tag{14}$$

$$c_1 H_{11} + c_3 H_{12} = 0, \tag{15}$$

$$c_1 H_{21} + c_3 H_{22} = 0, \tag{16}$$

$$(c_1 H_{11} + c_3 H_{12})_{xx} = 0, \tag{17}$$

where c_1 to c_4 are integration constants, and the differential functions H_{ij} are relegated to Appendix A due to their size. It can be readily seen that Eq. (17) is a mere differential consequence of Eq. (15). Thus the equations we have to solve are Eqs. (15) and (16).

It is well-known (see for example [2]) that the symmetry Lie algebra of a scalar linear partial differential equation is of the form $L^r \oplus L^\infty$, where L^r is a finite-dimensional Lie algebra and L^∞ is an infinite-dimensional ideal of the symmetry Lie algebra spanned by the so-called solution symmetries. In our case

$$L^\infty = \langle a(t, x)\partial_u \rangle,$$

where $a(t, x)$ solves Eq. (14), i.e. Euler–Bernoulli equation. Thus we are left with characterizing L^r . The determining equations contain four constants viz. c_1 to c_4 with two constants that are unconstrained. Therefore the dimension of L^r , r , lies between two and four. Below we elucidate all the possibilities.

Case I: c_1 and c_3 are arbitrary constants

In this case, Eqs. (15)–(16) split into the following system.

$$H_{11} = 0, \tag{18}$$

$$H_{12} = 0, \tag{19}$$

$$H_{21} = 0, \tag{20}$$

$$H_{22} = 0. \tag{21}$$

The system (18)–(21) is an over-determined system of nonlinear ordinary differential equations for f and g . Thus, *a priori*, we are not guaranteed of a solution. Replace Eq. (18) by the combination Eq. (18) – $g \times$ Eq. (19). Solve the resulting equation for $g^{(3)}$ to obtain

$$g^{(3)} = \frac{3}{10} \frac{g' f'^2}{f^2} - \frac{2}{5} \frac{g' f''}{f} + \frac{3}{2} \frac{g''^2}{g'}. \tag{22}$$

Use Eq. (22) to eliminate the derivative $g^{(3)}$ and $g^{(4)}$ from Eq. (19). It results the equation

$$f^{(4)} = \frac{f' f^{(3)}}{f} + \frac{11}{10} \frac{f''^2}{f} - \frac{12}{5} \frac{f'^2 f''}{f^2} + \frac{9}{10} \frac{f'^4}{f^3}. \tag{23}$$

Employing Eqs. (22)–(23) to get rid of the derivatives $g^{(3)}$ to $g^{(6)}$, $f^{(4)}$, and $f^{(5)}$ from Eqs. (20)–(21), we discover that Eqs. (20)–(21) are identically satisfied. To sum up, we have established that the over-determined system of Eqs. (18)–(21) is equivalent to the determined system formed by Eqs. (22)–(23). Thus provided Eqs. (22)–(23) are satisfied, the finite part of the symmetry Lie algebra, L^4 , is spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = u \frac{\partial}{\partial u}, \tag{24}$$

$$X_3 = 4t \frac{\partial}{\partial t} + \frac{2g}{g'} \frac{\partial}{\partial x} - \left(\frac{f'g}{fg'} + \frac{3gg''}{g'^2} \right) u \frac{\partial}{\partial u}, \tag{25}$$

$$X_4 = \frac{2}{g'} \frac{\partial}{\partial x} - \left(\frac{f'}{fg'} + \frac{3g''}{g'^2} \right) u \frac{\partial}{\partial u}. \tag{26}$$

Simple computations show that the nonzero commutators of the symmetry generators are

$$[X_1, X_3] = 4X_1, \quad [X_3, X_4] = -2X_4.$$

By making the change of basis

$$e_1 = X_1, \quad e_2 = \frac{1}{\sqrt{2}} X_4, \quad e_3 = \frac{1}{4} X_3, \quad e_4 = X_2,$$

it can be seen that L^4 is equivalent to $A_{3,3} \oplus A_1$ in Patera and Winternitz [11] classification scheme.

Case II: c_1 is arbitrary and $c_3 = 0$

The determining equations (15)–(16) become

$$H_{11} = 0, \tag{27}$$

$$H_{21} = 0. \tag{28}$$

We aim at rewriting the constraints (27)–(28) in terms of lowest possible derivatives of f and g . In order to achieve this goal, we first solve Eq. (27) for $g^{(4)}$ to obtain

$$g^{(4)} = 6 \frac{g''g^{(3)}}{g'} - 2 \frac{g'g^{(3)}}{g} - \frac{2g'f^{(3)}}{f} + 6 \frac{g''^3}{g'^2} + 3 \frac{g''^2}{g} + \frac{4g''f''}{f} - \frac{3f'^2g''}{5g} + \frac{g'ff''}{f} - \frac{4g'^2f''}{5gf} - \frac{3g'f'^3}{5f^3} + \frac{3g'^2f'^2}{5f^2g}. \tag{29}$$

Employ Eq. (29) to get rid of the derivatives $g^{(4)}$ to $g^{(6)}$ from Eq. (28) and solve the resulting equation for $f^{(5)}$ to obtain

$$f^{(5)} = \frac{-18f'^5}{5f^4} + \frac{18f'^4g'}{5f^3g} + \frac{54f'^3f''}{5f^3} - \frac{48f'^2g'f''}{5f^2g} - \frac{7f'f''^2}{f^2} + \frac{22g'f''^2}{5fg} - \frac{18f'^4g''}{5f^3g'} + \frac{48f'^2f''g''}{5f^2g'} - \frac{22f''^2g''}{5fg'} - \frac{22f'^2f^{(3)}}{5f^2} + \frac{4f'g'f^{(3)}}{fg} + \frac{16f''f^{(3)}}{5f} - \frac{4f'g''f^{(3)}}{fg'} + \frac{2f'f^{(4)}}{f} - \frac{4g'f^{(4)}}{g} + \frac{4g''f^{(4)}}{g'}. \tag{30}$$

Thus the constraints Eqs. (27)–(28) are equivalent to the simplified constraints (29)–(30). Provided f and g fulfilled Eqs. (29)–(30), the finite part of the symmetry Lie algebra, $L^{3,1}$, is spanned by X_1, X_2 , and X_3 . This Lie algebra corresponds to $A_1 \oplus A_2$ in Patera and Winternitz [11] classification of lower-dimensional Lie algebras.

Case III: $c_1 = 0$ and c_3 is arbitrary

Here, Eqs. (27)–(28) are equivalent to the system of equations

$$H_{21} = 0, \tag{31}$$

$$H_{22} = 0. \tag{32}$$

By following the same *modus operandi* as in Case II, we arrive at the following simplified constraints on f and g .

$$g^{(4)} = \frac{-3f'^3g'}{5f^3} + \frac{f'g'f''}{f^2} - \frac{3f'^2g''}{5f^2} + \frac{4f''g''}{5f} - \frac{6g''^3}{g'^2} - \frac{2g'f^{(3)}}{5f} + \frac{6g''g^{(3)}}{g'}, \tag{33}$$

$$f^{(5)} = \frac{-18f'^5}{5f^4} + \frac{54f'^3f''}{5f^3} - \frac{7f'f''^2}{f^2} - \frac{18f'^4g''}{5f^3g'} + \frac{48f'^2f''g''}{5f^2g'} - \frac{22f''^2g''}{5fg'} - \frac{22f'^2f^{(3)}}{5f^2} + \frac{16f''f^{(3)}}{5f} - \frac{4f'g''f^{(3)}}{fg'} + \frac{2f'f^{(4)}}{f} + \frac{4g''f^{(4)}}{g'}. \tag{34}$$

The finite portion of the symmetry Lie algebra, $L^{3,2}$, is spanned by X_1, X_2 and X_4 . The Lie algebra $L^{3,2}$ is nothing but the three-dimensional Abelian Lie algebra denoted by $3A_1$ in Patera and Winternitz [11] classification of lower-dimensional Lie algebras.

Case IV: $c_1 = 0 = c_3$

This is the generic case: there are no constraints on f and g . The finite part of the Lie algebra, L^2 , is generated by X_1 and X_2 . It is the two-dimensional Abelian Lie algebra $2A_1$.

Remark. Note that in the analysis above, we have excluded the possibility $c_3 = mc_1, m \neq 0$, since the equivalence transformation $g(x) \mapsto g(x) + m$ reduces this case to Case II.

To sum up, we have established the following result.

Theorem 1. Denote the symmetry Lie algebra of Euler–Bernoulli equation by S , and L^∞ the infinite-dimensional Lie algebra generated by the solution symmetries. Then, $S/L^\infty = A_{3,3} \oplus A_1, A_1 \oplus A_2, 3A_1, 2A_1$, depending on whether f and g respectively satisfy Eqs. (22)–(23), Eqs. (29)–(30), Eqs. (33)–(34), or are arbitrary.

3. Equivalence classes and mapping to canonical elements

From a symmetry standpoint there are essentially four classes of Euler–Bernoulli equations. Equations of the same class share the same symmetry structure, and are homeomorphic. Our aim in this section is to select a simple representative for the non-generic classes (i.e. all the cases save the case where f and g are arbitrary), and constructively show how class members are mapped to their representative.

In the sequel, we shall use capitalized variables to describe representative of each class. The rationale behind this choice shall be apparent as this section unfolds. We denote by $[\mathcal{L}]$ the set of all Euler–Bernoulli equations having \mathcal{L} as the finite part of their symmetry Lie algebra. Notations not introduced here are those of the previous sections.

3.1. The class $[A_{3,3} \oplus A_1]$

3.1.1. Construction of similarity transformations

It can be readily verified that $f(x) = 1$ and $g(x) = x$ satisfy Eqs. (22)–(23). Thus, we select as representative element of this class the equation

$$U_{XXXX} + U_{TT} = 0. \tag{35}$$

The finite part of the symmetry Lie algebra of Eq. (35) is generated by the vectors

$$Y_1 = \frac{\partial}{\partial T}, \quad Y_2 = U \frac{\partial}{\partial U}, \quad Y_{3,1} = 4T \frac{\partial}{\partial T} + 2X \frac{\partial}{\partial X}, \quad Y_4 = 2 \frac{\partial}{\partial X}. \tag{36}$$

The invertible transformation

$$T = T(t, x, u), \quad X = X(t, x, u), \quad U = U(t, x, u) \tag{37}$$

maps an element of $[A_{3,3} \oplus A_1]$ to Eq. (35) if and only if the same transformation maps $\langle X_1, X_2, X_3, X_4 \rangle$ to $\langle Y_1, Y_2, Y_{3,1}, Y_4 \rangle$. We look for a transformation (37) such that

$$X_1 \mapsto Y_1, \tag{38a}$$

$$X_2 \mapsto Y_2, \tag{38b}$$

$$Y_{3,1} \mapsto X_3 + \mu_1 X_1 + \mu_2 X_4, \tag{38c}$$

$$Y_4 \mapsto X_4. \tag{38d}$$

Recall that the transformation defined by Eq. (37) maps a vector field Γ depending on t, x and u to the vector field $\Gamma(T)\partial_T + \Gamma(X)\partial_X + \Gamma(U)\partial_U$. Thus, in order to realize Eqs. (38a)–(38b), and Eq. (38c), we have to impose

$$X_1(T) = 1, \quad X_1(X) = 0, \quad X_1(U) = 0, \tag{39}$$

$$X_2(T) = 0, \quad X_2(X) = 0, \quad X_2(U) = U, \tag{40}$$

$$X_4(T) = 0, \quad X_4(X) = 2, \quad X_4(U) = 0. \tag{41}$$

Solving Eqs. (39)–(41), we obtain

$$T = t + k_1, \quad X = g + k_2, \quad U = k_3 u \sqrt{fg'^3}, \tag{42}$$

where k_1, k_2 , and $k_3 \neq 0$ are constants. It can be readily verified that Eq. (38b) is satisfied for $\mu_1 = k_1$ and $\mu_2 = k_2$. We summarize the findings of this subsection as follows.

Theorem 2. *Equations of $[A_{3,3} \oplus A_1]$ are homeomorphic to Eq. (35). A transformation that maps an arbitrary element of $[A_{3,3} \oplus A_1]$ to Eq. (35) is given by Eq. (42).*

3.1.2. Examples of non-uniform beams homeomorphic to uniform beams: generalization and symmetry justification of Gottlieb's iso-spectral models

Here we look for closed-form solutions of the uncoupled system of Eqs. (22)–(23).

We may rewrite Eq. (22) as

$$\{g, x\} = \frac{3}{10} \frac{f'^2}{f^2} - \frac{2}{5} \frac{f''}{f}, \tag{43}$$

where $\{y, x\} = y'''/y' - (3/2)(y''/y')^2$ is the so-called Schwartzian 'derivative' (it is not really a derivative but a differential invariant!) of y with respect to x . From the well-known result on inversion of the Schwartzian derivative [12], we infer that

$$g = \frac{y_2(x)}{y_1(x)}, \tag{44}$$

where y_1 and y_2 are two linearly independent solutions of the second-order linear ordinary differential equation

$$y'' + \frac{1}{20} \left(3 \frac{f'^2}{f^2} - 4 \frac{f''}{f} \right) y = 0. \quad (45)$$

We are now left with solving Eq. (23). Its symmetry Lie algebra is spanned by the operators

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = x \frac{\partial}{\partial x}, \quad \Gamma_3 = f \frac{\partial}{\partial f}. \quad (46)$$

Since the Lie algebra $\langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle$ is solvable (its second derived algebra is trivial), we may use successive reduction [13,14] to depress the order of Eq. (23) by three to obtain an Abelian equation of the second kind which we could not solve in closed-form. One has to be careful when doing successive reduction of order: a wrong reduction of order *viz.* a reduction of order without using an ideal of the symmetry Lie algebra, may result in the disappearance of point symmetries or/and the appearance of non-local symmetries [15]. For details about the successive reduction of Eq. (23), we refer the reader to Appendix B. Due to the lack of closed-form solution of the reduced equation, we look for an invariant solution of Eq. (23). The most general invariant solution under a linear combination of Γ_1 to Γ_3 are

$$f = K(Ax + B)^m, \quad (47a)$$

$$f = Ce^{Dx}, \quad (47b)$$

where K , A to D , and m are constants. Note that the second ansatz Eq. (47b) satisfies Eq. (23) if and only if $D = 0$, and this possibility is already included in the first ansatz. The substitution of Eq. (47a) into Eq. (23) yields the constraint

$$m(4m^3 - 32m^2 + 79m - 60) = 0. \quad (48)$$

Solving Eq. (48), we obtain

$$m \in \left\{ 0, \frac{3}{2}, \frac{5}{2}, 4 \right\}. \quad (49)$$

Using the ansatz Eq. (47a) into Eq. (46) yields

$$y'' + \frac{A^2 m(4-m)}{20(Ax+B)^2} y = 0. \quad (50)$$

The general solution of Eq. (50) is

$$y = \begin{cases} k_1 + k_2(Ax + B) & \text{if } m \in \{0, 4\}, \\ k_1(Ax + B)^{1/4} + k_2(Ax + B)^{3/4} & \text{if } m \in \{3/2, 5/2\}, \end{cases} \quad (51)$$

where k_1 and k_2 are arbitrary constants. We infer from Eq. (44) that

$$g = \begin{cases} \frac{L+M(Ax+B)}{P+Q(Ax+B)} & \text{if } m \in \{0, 4\}, \\ \frac{L(Ax+B)^{1/4} + M(Ax+B)^{3/4}}{P(Ax+B)^{1/4} + Q(Ax+B)^{3/4}} & \text{if } m \in \{3/2, 5/2\}, \end{cases} \quad (52)$$

where L , M , P and Q are constants satisfying $LQ - MP \neq 0$.

The models defined by Eqs. (47a), (49) and (52) generalize Gottlieb's [1] iso-spectral models.

3.2. The class $[A_1 \oplus A_2]$

A particular solution of the system (29)–(30) that does not satisfy Eqs. (22)–(23) is $f(x) = x$, $g(x) = x$. Based on this particular solution, we choose as representative of $[A_1 \oplus A_2]$ the equation

$$(XU_{XX})_{XX} + XU_{TT} = 0. \quad (53)$$

The finite portion of the Lie symmetry algebra of Eq. (53) is generated by Y_1 , Y_2 , and $Y_{3,2} = 4T\partial_T + 2X\partial_X - U\partial_U$. An invertible transformation (37) maps an element of $[A_1 \oplus A_2]$ to Eq. (53) if and only if it maps $\langle X_1, X_2, X_3 \rangle$ to $\langle Y_1, Y_2, Y_{3,2} \rangle$. We search for such a transformation by mapping the basis elements as follows.

$$X_1 \mapsto Y_1, \quad X_2 \mapsto Y_2, \quad X_3 \mapsto Y_{3,2}. \quad (54)$$

Simple calculations show that Eq. (54) is realized if and only if

$$T = t + l_1 g^2, \quad X = 2l_2 g, \quad U = l_3 u \sqrt{\frac{fg'^3}{g}}, \quad (55)$$

where $l_1, l_2 \neq 0$, and $l_3 \neq 0$ are constants.

Thus we have proved the following theorem.

Theorem 3. Equations of $[A_2 \oplus A_1]$ are homeomorphic to Eq. (53). A transformation that maps an arbitrary element of $[A_2 \oplus A_1]$ to Eq. (53) is given by Eq. (55).

3.3. The class $[3A_1]$

A simple solution of Eqs. (33)–(34) which does not satisfy Eqs. (22)–(23) is $f(x) = 1, g(x) = \ln x$. To this solution corresponds the representative

$$U_{XXX} + X^{-4}U_{TT} = 0. \tag{56}$$

The quotient symmetry Lie algebra S/L^∞ of Eq. (56) is spanned by $Y_1, Y_2,$ and $Y_{3,2} = 2X\partial_X + 3U\partial_U$. An element of $[3A_1]$ will be homeomorphic to Eq. (56) provided the Lie algebras $\langle X_1, X_2, X_4 \rangle$ and $\langle Y_1, Y_2, Y_{3,2} \rangle$ are similar. We look for a similarity that transforms the basis vectors as follows.

$$X_1 \mapsto Y_1, \quad X_2 \mapsto Y_1, \quad X_4 \mapsto Y_{3,2}. \tag{57}$$

Elementary reckoning shows that the mapping (57) is realized by the transformation

$$T = t + m_1, \quad X = m_2 e^g, \quad U = m_3 u \sqrt{f g'^3 e^{3g}}, \tag{58}$$

where $m_1, m_2 \neq 0,$ and $m_3 \neq 0$ are constants.

In summary, we have proved the following statement.

Theorem 4. Equations of $[3A_1]$ are homeomorphic to Eq. (56). A transformation that maps an arbitrary element of $[3A_1]$ to Eq. (56) is given by Eq. (58).

4. Conclusion

We have studied in details symmetry breaking of Euler–Bernoulli equation. We have shown that the Lie symmetry algebra of the Euler–Bernoulli equation is one of the following: $2A_1 \oplus L^\infty, 3A_1 \oplus L^\infty, A_1 \oplus A_2 \oplus L^\infty,$ or $A_{3,3} \oplus A_1 \oplus L^\infty,$ where L^∞ is the infinite-dimensional Lie algebra spanned by solution symmetries. Equations admitting a given symmetry class are characterized completely in terms of a determined set of non-linear ordinary differential equations that physical parameters (flexural rigidity and lineal mass density) must fulfill. Equations of the same class can be mapped to each other via invertible transformations. For each class we provided a simple representative and we explicitly constructed similarity mappings. For the particular class of equations equivalent to the uniform Euler–Bernoulli equation, we rigorously constructed explicit non-trivial examples that extend and generalize Gottlieb’s [1] iso-spectral models.

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Appendix A. Differential functions appearing in Eqs. (15)–(17)

$$\begin{aligned}
 H_{11} &= \frac{6f'^2}{f} - \frac{6fgf'^3}{f^2g'} - 8f'' + \frac{10gf'f''}{fg'} - \frac{6gf'^2g''}{fg'^2} + \frac{8gf''g''}{g'^2} + \frac{30fg''^2}{g'^2} - \frac{60fgg''^3}{g'^4} - \frac{4gf^{(3)}}{g'} \\
 &\quad - \frac{20fg^{(3)}}{g'} + \frac{60fgg''g^{(3)}}{g'^3} - \frac{10fgg^{(4)}}{g'^2}, \\
 H_{12} &= \frac{-6f'^3}{f^2g'} + \frac{10f'f''}{fg'} - \frac{6f'^2g''}{fg'^2} + \frac{8f''g''}{g'^2} - \frac{60fg''^3}{g'^4} - \frac{4f^{(3)}}{g'} + \frac{60fg''g^{(3)}}{g'^3} - \frac{10fg^{(4)}}{g'^2}, \\
 H_{21} &= \frac{12f'^4}{f^3} - \frac{12gf'^5}{f^4g'} - \frac{28f'^2f''}{f^2} + \frac{34gf'^3f''}{f^3g'} + \frac{10f''^2}{f} - \frac{21gf'f''^2}{f^2g'} - \frac{12gf'^4g''}{f^3g'^2} + \frac{6f'^3g''}{f^2g'} + \frac{28gf'^2f''g''}{f^2g'^2} \\
 &\quad - \frac{11f'f''g''}{fg'} - \frac{10gf''^2g''}{fg'^2} - \frac{12gf'^3g''^2}{f^2g'^3} + \frac{6f'^2g''^2}{fg'^2} + \frac{22gf'f''g''^2}{fg'^3} - \frac{3f''g''^2}{g'^2} - \frac{12gf'^2g''^3}{fg'^4} - \frac{60f'g''^3}{g'^3} \\
 &\quad + \frac{6gf''g''^3}{g'^4} + \frac{120gf'g''^4}{g'^5} + \frac{180fg''^4}{g'^4} - \frac{360fgg''^5}{g'^6} + \frac{10f'f^{(3)}}{f} - \frac{12gf'^2f^{(3)}}{f^2g'} + \frac{9gf''f^{(3)}}{fg'} - \frac{10gf'g''f^{(3)}}{fg'^2} \\
 &\quad + \frac{6g''f^{(3)}}{g'} - \frac{12gg''^2f^{(3)}}{g'^3} + \frac{6gf'^3g^{(3)}}{f^2g'^2} - \frac{4f'^2g^{(3)}}{fg'} - \frac{11gf'f''g^{(3)}}{fg'^2} + \frac{2f''g^{(3)}}{g'} + \frac{12gf'^2g''g^{(3)}}{fg'^3} + \frac{70f'g''g^{(3)}}{g'^2}
 \end{aligned}$$

$$\begin{aligned}
& -\frac{6gf''g''g^{(3)}}{g'^3} - \frac{180gf'g''^2g^{(3)}}{g'^4} - \frac{300fgg''^2g^{(3)}}{g'^3} + \frac{720fggg''^3g^{(3)}}{g'^5} + \frac{6gf^{(3)}g^{(3)}}{g'^2} + \frac{30gf'g^{(3)^2}}{g'^3} + \frac{60fg^{(3)^2}}{g'^2} \\
& - \frac{270fgg''g^{(3)^2}}{g'^4} - 4f^{(4)} + \frac{3gf'f^{(4)}}{fg'} + \frac{4gg''f^{(4)}}{g'^2} - \frac{2gf'^2g^{(4)}}{fg'^2} - \frac{15f'g^{(4)}}{g'} + \frac{gf''g^{(4)}}{g'^2} + \frac{40gf'g''g^{(4)}}{g'^3} \\
& + \frac{75fg''g^{(4)}}{g'^2} - \frac{180fggg''^2g^{(4)}}{g'^4} + \frac{60fggg^{(3)}g^{(4)}}{g'^3} - \frac{gf^{(5)}}{g'} - \frac{5gf'g^{(5)}}{g'^2} - \frac{12fg^{(5)}}{g'} + \frac{30fgg''g^{(5)}}{g'^3} - \frac{3fgg^{(6)}}{g'^2}, \\
H_{22} = & \frac{-12f'^5}{f^4g'} + \frac{34f'^3f''}{f^3g'} - \frac{21f'f''^2}{f^2g'} - \frac{12f'^4g''}{f^3g'^2} + \frac{28f'^2f''g''}{f^2g'^2} - \frac{10f''^2g''}{fg'^2} - \frac{12f'^3g''^2}{f^2g'^3} + \frac{22f'f''g''^2}{fg'^3} \\
& - \frac{12f'^2g''^3}{fg'^4} + \frac{6f''g''^3}{g'^4} + \frac{120f'g''^4}{g'^5} - \frac{360fgg''^5}{g'^6} - \frac{12f'^2f^{(3)}}{f^2g'} + \frac{9f''f^{(3)}}{fg'} - \frac{10f'g''f^{(3)}}{fg'^2} - \frac{12g''^2f^{(3)}}{g'^3} \\
& + \frac{6f'^3g^{(3)}}{f^2g'^2} - \frac{11f'f''g^{(3)}}{fg'^2} + \frac{12f'^2g''g^{(3)}}{fg'^3} - \frac{6f''g''g^{(3)}}{g'^3} - \frac{180f'g''^2g^{(3)}}{g'^4} + \frac{720fgg''^3g^{(3)}}{g'^5} + \frac{6f^{(3)}g^{(3)}}{g'^2} \\
& + \frac{30f'g^{(3)^2}}{g'^3} - \frac{270fg''g^{(3)^2}}{g'^4} + \frac{3f'f^{(4)}}{fg'} + \frac{4g''f^{(4)}}{g'^2} - \frac{2f'^2g^{(4)}}{fg'^2} + \frac{f''g^{(4)}}{g'^2} + \frac{40f'g''g^{(4)}}{g'^3} - \frac{180fgg''^2g^{(4)}}{g'^4} \\
& + \frac{60fg^{(3)}g^{(4)}}{g'^3} - \frac{f^{(5)}}{g'} - \frac{5f'g^{(5)}}{g'^2} + \frac{30fgg''g^{(5)}}{g'^3} - \frac{3fg^{(6)}}{g'^2}.
\end{aligned}$$

Appendix B. Successive reduction of the order of Eq. (23)

The Lie brackets of the symmetry operators are $[\Gamma_1, \Gamma_2] = \Gamma_1$, $[\Gamma_1, \Gamma_3] = 0$, $[\Gamma_2, \Gamma_3] = 0$. Thus $\langle \Gamma_1 \rangle$ and $\langle \Gamma_3 \rangle$ are ideals of the symmetry Lie algebra. We may start reduction using Γ_1 or Γ_3 . It is crucial start reduction using an ideal of the symmetry Lie algebra in order to preserve the remaining symmetries.

A basis of first-order differential invariants of Γ_1 is formed by f and f' . We define new dependent and independent variables by

$$y = f', \quad t = f. \quad (\text{B.1})$$

In the new variables, Eq. (23) reads

$$y^{(3)} = \frac{\ddot{y}}{t} - 4\frac{\dot{y}\ddot{y}}{y} - \frac{12}{5}\frac{\dot{y}}{t^2} + \frac{21}{20}\frac{\dot{y}^2}{y} - 7\frac{\dot{y}^3}{y^2} + \frac{9}{10}\frac{y}{t^3}, \quad (\text{B.2})$$

where the over-dot stands for differentiation with respect to t . In the new variables, the symmetries Γ_2 and Γ_3 are

$$\Gamma_2 = -y\partial_y, \quad \Gamma_3 = t\partial_t - y\partial_y.$$

It can be verified that Eq. (B.2) inherits the symmetries Γ_2 and Γ_3 as expected.

A basis of first-order differential invariant of Γ_2 is formed by t and

$$z = \frac{\dot{y}}{y}. \quad (\text{B.3})$$

In the new variables t and z , Eq. (B.2) becomes

$$\ddot{z} = \frac{\dot{z}}{t} - 7z\dot{z} - \frac{12}{5}z + \frac{41}{20}\frac{z^2}{t} - 16z^3 + \frac{9}{10}\frac{1}{t^3}. \quad (\text{B.4})$$

In the variables t and z , the symmetry operator Γ_3 becomes

$$\Gamma_3 = t\partial_t - z\partial_z.$$

A basis of first-order differential invariant of Γ_3 is

$$u = tz, \quad v = tz + t^2z'. \quad (\text{B.5})$$

In the new variables u and v , Eq. (B.5) is

$$\frac{dv}{du} = 5 - 7u - \frac{320u^3 - 181u^2 + 108u - 18}{20v}. \quad (\text{B.6})$$

Eq. (B.6) is an Abelian equation of the second kind.

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