



Recurrence relations for semilocal convergence of a Newton-like method in Banach spaces[☆]

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ABSTRACT

The aim of this paper is to establish the semilocal convergence of a multipoint third order Newton-like method for solving $F(x) = 0$ in Banach spaces by using recurrence relations. The convergence of this method is studied under the assumption that the second Fréchet derivative of F satisfies Hölder continuity condition. This continuity condition is milder than the usual Lipschitz continuity condition. A new family of recurrence relations are defined based on the two new constants which depend on the operator F . These recurrence relations give a priori error bounds for the method. Two numerical examples are worked out to demonstrate the applicability of the method in cases where the Lipschitz continuity condition over second derivative of F fails but Hölder continuity condition holds.

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1. Introduction

Let $F : \Omega \subseteq \mathbb{X} \rightarrow \mathbb{Y}$ be a nonlinear operator on an open convex subset Ω of a Banach space \mathbb{X} with values in a Banach space \mathbb{Y} . Many researchers [1,2,4,5] have developed and studied the convergence of one point third order iterative methods such as Chebyshev's method, Halley's method and super-Halley's method to solve nonlinear operator equations $F(x) = 0$. These methods are useful in solving stiff systems of equations [15], where a quick convergence is required. They have established the convergence of these methods by considering the second Fréchet derivative of F to be Lipschitz continuous. One drawback of these methods is that they involve the second derivative of the operator F which at times is very difficult to compute. This restricts the use of these methods as their informational efficiency is less than or equal to unity.

Multipoint third order iterative methods [3,6,7,10–12,16] are also discussed to find the solutions of $F(x) = 0$ in Banach spaces. These methods require information at more than one point and are easy to use as they need only the operator F and its first derivative F' . The convergence of one such iterative method studied by Wu and Zhao [11] is given as

$$\left. \begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= x_n - \left(\frac{F'(x_n) + F'(y_n)}{2} \right)^{-1} F(x_n), \quad n = 0, 1, \dots \end{aligned} \right\} \quad (1)$$

They constructed a majorizing function to study the convergence of this method under the assumption that F'' satisfies Lipschitz continuity condition. Parida and Gupta [10] developed a semilocal convergence analysis of the above iterative method by using recurrence relations under the same assumptions on the operator F .

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But there are certain problems for which the convergence analysis cannot be studied under the assumption that the second derivative of F be Lipschitz continuous.

Example. Let us consider the integral equation of Fredholm type [13]

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+p} dt, \quad (2)$$

with $s \in [0, 1]$, $x, f \in C[0, 1]$, $p \in (0, 1]$ and λ is a real number. Now for instance when we choose sup norm, the corresponding second derivative of the operator F satisfies

$$\|F''(x) - F''(y)\| \leq |\lambda|(1+p)(2+p) \log 2 \|x - y\|^p, \quad x, y \in \Omega. \quad (3)$$

Hence for $p \in (0, 1)$, F'' does not satisfy the Lipschitz continuity condition. For example, for $p = 1/2$, F'' satisfies the Hölder continuity condition given by

$$\|F''(x) - F''(y)\| \leq \frac{15}{4} |\lambda| \log 2 \|x - y\|^{1/2}, \quad x, y \in \Omega,$$

but not the Lipschitz continuity condition. For $p = 1$, the condition (3) reduces to Lipschitz continuity condition. Under the assumption that F'' satisfies Hölder continuity condition, Hernández and Salanova [9] and Hernández [7] studied the convergence of Chebyshev's method and its second derivative free version by using recurrence relations. Also under similar conditions, by using the technique of recurrence relations, Ye and Li [12] studied the convergence of Euler–Halley method.

In this paper, we establish the semilocal convergence of the method (1) by using recurrence relations under the assumption that F'' satisfies Hölder continuity condition. In order to increase its R -order of convergence from 2 to $(2+p)$, we have also given a reformulation of the method (1).

The paper is organized as follows: In Section 2, some preliminary results are given. The recurrence relations for the method (1) are established in Section 3. An existence-uniqueness theorem is given in Section 4 to establish the R -order of the method to be 2 and a priori error bounds. In Section 4.1, the convergence of a modified version of the method (1) is given to establish that the R -order of the method can be increased. In Section 5, two numerical examples are worked out to show the efficacy of our convergence analysis. Finally, conclusions form Section 6.

2. Preliminary results

In this section, we shall discuss the convergence of the iterative method (1) for solving the nonlinear operator equation

$$F(x) = 0. \quad (4)$$

Let F be twice Fréchet differentiable operator in Ω and $\mathcal{BL}(\mathbb{Y}, \mathbb{X})$ be the set of bounded linear operators from \mathbb{Y} into \mathbb{X} . Let $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{BL}(\mathbb{Y}, \mathbb{X})$ exists at some point $x_0 \in \Omega$ and assume that the following conditions hold on F :

$$\left. \begin{aligned} 1. \quad & \|\Gamma_0\| \leq \beta, \\ 2. \quad & \|\Gamma_0 F(x_0)\| \leq \eta, \\ 3. \quad & \|F''(x)\| \leq M, \quad \forall x \in \Omega, \\ 4. \quad & \|F''(x) - F''(y)\| \leq N \|x - y\|^p, \quad \forall x, y \in \Omega, \quad p \in (0, 1]. \end{aligned} \right\} \quad (5)$$

If $a_0 = M\beta\eta$, $b_0 = N\beta\eta^{1+p}$ be two parameters, then for $n \in \mathbb{Z}_+$, define two real sequences

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n) \quad \text{and} \quad b_{n+1} = b_n f(a_n)^{2+p} g(a_n, b_n)^{1+p}, \quad (6)$$

where

$$f(x) = (2-x)/(2-3x), \quad (7)$$

$$g(x, y) = \left[\frac{x(4-x)(3-x)}{(2-x)^2} + \frac{(p+4)y}{2(p+1)(p+2)} \right]. \quad (8)$$

Let r_0 be the smallest positive zero of the polynomial $h(x) = -x^3 + 16x^2 - 24x + 4$, then $r_0 = 0.1905960896 \dots$. The following lemmas will prove a number of properties of the sequences $\{a_n\}$ and $\{b_n\}$.

Lemma 2.1. Let f and g be the functions defined by (7) and (8) respectively, then for $x \in (0, r_0)$,

- (i) f is increasing and $f(x) > 1$,
- (ii) g is increasing in both arguments for $y > 0$,
- (iii) $f(\delta x) < f(x)$ and $g(\delta x, \delta^p y) < \delta^p g(x, y)$, for $\delta \in (0, 1)$ and $p \in (0, 1]$.

Proof. The proof is simple and can be omitted. \square

Lemma 2.2. For fixed $p \in (0, 1]$, let f and g be the functions defined by (7) and (8), respectively. Let

$$\Phi_p(x) = \frac{2(p+1)(p+2)(-x^3 + 16x^2 - 24x + 4)}{(p+4)(2-x)^2}. \quad (9)$$

If $0 < a_0 \leq r_0$ and $0 \leq b_0 \leq \Phi_p(a_0)$, then

- (i) $f(a_n)^2 g(a_n, b_n) \leq 1$,
- (ii) $\{a_n\}, \{b_n\}$ are decreasing and $a_n < 1, \forall n$.

Proof. From definitions of f and g , one can easily conclude that

$$f(a_n)^2 g(a_n, b_n) \leq 1 \quad \text{iff} \quad b_n \leq \frac{2(p+1)(p+2)(-a_n^3 + 16a_n^2 - 24a_n + 4)}{(p+4)(2-a_n)^2} = \Phi_p(a_n).$$

Induction will be used to prove this lemma. For $0 < a_0 \leq r_0$ and $0 \leq b_0 \leq \Phi_p(a_0)$, one can easily conclude from above that $f(a_0)^2 g(a_0, b_0) \leq 1$. Thus, from Eq. (6), we obtain

$$a_1 = a_0 f(a_0)^2 g(a_0, b_0) \leq a_0 < 1,$$

and as $f(x) > 1$ in $(0, r_0]$, we get

$$b_1 = b_0 f(a_0)^{2+p} g(a_0, b_0)^{1+p} \leq b_0 f(a_0)^2 g(a_0, b_0) [f(a_0)^2 g(a_0, b_0)]^p \leq b_0.$$

Let the statements hold for $n = k$. Then proceeding similarly, one can easily prove that $a_{k+1} \leq a_k < 1$, and $b_{k+1} \leq b_k$. Since f and g are increasing functions, we get

$$f(a_{k+1})^2 g(a_{k+1}, b_{k+1}) \leq f(a_k)^2 g(a_k, b_k) \leq 1.$$

Thus, the above statements also hold for $n = k + 1$. Hence, it is true for all n . This proves Lemma 2.2. \square

Lemma 2.3. Let us suppose $0 < a_0 < r_0$ and $0 < b_0 < \Phi_p(a_0)$. Define $\gamma = a_1/a_0$, then for $n \geq 1$, we have

- (i) $a_n \leq \gamma^{2^{n-1}} a_{n-1} \leq \gamma^{2^n - 1} a_0$, strictly holds for $n \geq 2$,
- (ii) $b_n < \gamma^{2^{n-1}} b_{n-1} < \gamma^{2^n - 1} b_0$,
- (iii) $f(a_n)g(a_n, b_n) < \gamma^{2^n} / f(a_0)$.

Proof. We will prove (i) and (ii) by induction. Since $a_1 = \gamma a_0$ and $a_1 < a_0$ from Lemma 2.2(i), we get $\gamma < 1$. By Lemmas 2.1(i) and 2.2(i), we get

$$\begin{aligned} b_1 &= b_0 f(a_0)^{2+p} g(a_0, b_0)^{1+p} = b_0 (f(a_0)^2 g(a_0, b_0)) (f(a_0)g(a_0, b_0))^p \\ &< (f(a_0)^2 g(a_0, b_0)) (f(a_0)^2 g(a_0, b_0))^p b_0 < (f(a_0)^2 g(a_0, b_0)) b_0 = \gamma b_0. \end{aligned}$$

Also,

$$a_2 = a_1 f(a_1)^2 g(a_1, b_1) < \gamma a_0 f(\gamma a_0)^2 g(\gamma a_0, \gamma b_0) < \gamma a_0 f(a_0)^2 \gamma g(a_0, b_0) = \gamma^2 a_1.$$

Suppose (i) and (ii) hold for $n = k$, then

$$\begin{aligned} a_{k+1} &= a_k f(a_k)^2 g(a_k, b_k) < \gamma^{2^{k-1}} a_{k-1} f(\gamma^{2^{k-1}} a_{k-1})^2 g(\gamma^{2^{k-1}} a_{k-1}, \gamma^{2^{k-1}} b_{k-1}) \\ &< \gamma^{2^{k-1}} a_{k-1} f(a_{k-1})^2 \gamma^{2^{k-1}} g(a_{k-1}, b_{k-1}) = \gamma^{2^k} a_k. \end{aligned}$$

Also as $f(x) > 1$ in $(0, r_0]$, we get

$$\begin{aligned} b_{k+1} &= b_k f(a_k)^{2+p} g(a_k, b_k)^{1+p} = b_k (f(a_k)^2 g(a_k, b_k)) (f(a_k)g(a_k, b_k))^p \\ &< b_k (f(a_k)^2 g(a_k, b_k)) (f(a_k)^2 g(a_k, b_k))^p < b_k \left(\frac{a_{k+1}}{a_k} \right) < \gamma^{2^k} b_k. \end{aligned}$$

Hence,

$$a_{k+1} < \gamma^{2^k} a_k < \gamma^{2^k} \gamma^{2^{k-1}} \dots \gamma^{2^0} a_0 = \gamma^{2^{k+1} - 1} a_0$$

and

$$b_{k+1} < \gamma^{2^k} b_k < \gamma^{2^k} \gamma^{2^{k-1}} \dots \gamma^{2^0} b_0 = \gamma^{2^{k+1} - 1} b_0.$$

Thus (i) and (ii) hold for all n by induction. Condition (iii) follows from

$$f(a_n)g(a_n, b_n) < f(\gamma^{2^n-1}a_0)g(\gamma^{2^n-1}a_0, \gamma^{2^n-1}b_0) < \gamma^{2^n-1}f(a_0)g(a_0, b_0) = \gamma^{2^n}/f(a_0)$$

as $\gamma = a_1/a_0 = f(a_0)^2 g(a_0, b_0)$. \square

3. Recurrence relations for the method

In this section, the recurrence relations are derived for the method given by (1) under the assumptions mentioned in the previous section.

Since existence of I_0 gives existence of y_0 , we get $M\|I_0\|\|y_0 - x_0\| \leq M\beta\eta = a_0$. Also,

$$\left\| I - \Gamma_0 \frac{F'(y_0) + F'(x_0)}{2} \right\| = \left\| \Gamma_0 \frac{F'(x_0) - F'(y_0)}{2} \right\| \leq \frac{1}{2} M\|I_0\|\|x_0 - y_0\| \leq \frac{a_0}{2} < 1.$$

Hence, $(\frac{F'(y_0) + F'(x_0)}{2})^{-1} F'(x_0)$ exists by Banach theorem [14, p. 155], and

$$\left\| \left(\frac{F'(y_0) + F'(x_0)}{2} \right)^{-1} F'(x_0) \right\| \leq \frac{1}{1 - \frac{1}{2} M\|I_0\|\|x_0 - y_0\|} \leq \frac{2}{2 - a_0}.$$

Thus, we have

$$\|x_1 - x_0\| \leq \left\| \left(\frac{F'(y_0) + F'(x_0)}{2} \right)^{-1} F'(x_0) \right\| \|F'(x_0)^{-1} F(x_0)\| \|y_0 - x_0\| \leq \frac{2}{2 - a_0} \|y_0 - x_0\| \quad (10)$$

and

$$\|x_1 - y_0\| \leq \|x_1 - x_0\| + \|x_0 - y_0\| \leq \frac{4 - a_0}{2 - a_0} \|y_0 - x_0\|. \quad (11)$$

Also,

$$N\|I_0\|\|y_0 - x_0\|^{1+p} \leq N\beta\eta^{1+p} = b_0.$$

Based on these results, the following inequalities can be proved for $n \geq 1$,

$$\left. \begin{aligned} \text{(I)} \quad & \|I_n\| = \|F'(x_n)^{-1}\| \leq f(a_{n-1})\|I_{n-1}\|, \\ \text{(II)} \quad & \|y_n - x_n\| = \|\Gamma_n F(x_n)\| \leq f(a_{n-1})g(a_{n-1}, b_{n-1})\|y_{n-1} - x_{n-1}\|, \\ \text{(III)} \quad & M\|I_n\|\|y_n - x_n\| \leq a_n, \\ \text{(IV)} \quad & \|x_{n+1} - x_n\| \leq \frac{2}{2 - a_n} \|y_n - x_n\|, \\ \text{(V)} \quad & \|x_{n+1} - y_n\| \leq \frac{4 - a_n}{2 - a_n} \|y_n - x_n\|, \\ \text{(VI)} \quad & N\|I_n\|\|y_n - x_n\|^{1+p} \leq b_n. \end{aligned} \right\} \quad (12)$$

Now induction can be used to prove the conditions (I)–(VI). Assume that $x_1 \in \Omega$. We now have

$$\|I - \Gamma_0 F'(x_1)\| \leq M\|I_0\|\|x_0 - x_1\| \leq M\beta \frac{2}{2 - a_0} \|y_0 - x_0\| \leq \frac{2a_0}{2 - a_0} < 1.$$

Hence by Banach theorem, $I_1 = F'(x_1)^{-1}$ exists and

$$\|I_1\| \leq \frac{\|I_0\|}{1 - M\|I_0\|\|x_0 - x_1\|} \leq \frac{2 - a_0}{2 - 3a_0} \|I_0\| = f(a_0)\|I_0\|. \quad (13)$$

Thus y_1 exists. Now by applying Taylor's method, one can easily obtain

$$\begin{aligned} F(x_{n+1}) &= \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t) dt (x_{n+1} - y_n)^2 + \frac{1}{2} \int_0^1 F''(x_n + t(y_n - x_n)) dt (y_n - x_n)(x_{n+1} - y_n) \\ &\quad - \frac{1}{2} \int_0^1 F''(x_n + t(y_n - x_n)) dt (y_n - x_n)^2 + \int_0^1 F''(x_n + t(y_n - x_n))(1 - t) dt (y_n - x_n)^2. \end{aligned} \quad (14)$$

It is to be noted that

$$\begin{aligned} & \left\| \int_0^1 F''(x_0 + t(y_0 - x_0))(1-t) dt (y_0 - x_0)^2 - \frac{1}{2} \int_0^1 F''(x_0 + t(y_0 - x_0)) dt (y_0 - x_0)^2 \right\| \\ & \leq \left\| \int_0^1 F''(x_0 + t(y_0 - x_0))(1-t) dt (y_0 - x_0)^2 - \frac{1}{2} F''(x_0)(y_0 - x_0)^2 \right\| \\ & \quad + \left\| \frac{1}{2} \int_0^1 F''(x_0 + t(y_0 - x_0)) dt (y_0 - x_0)^2 - \frac{1}{2} F''(x_0)(y_0 - x_0)^2 \right\| \\ & \leq \frac{(p+4)N}{2(p+1)(p+2)} \|y_0 - x_0\|^{2+p}. \end{aligned}$$

Using this result in (14), we get

$$\begin{aligned} \|\Gamma_1 F(x_1)\| & \leq \|\Gamma_1\| \|F(x_1)\| \\ & \leq f(a_0) \|\Gamma_0\| \left[M \frac{(4-a_0)(3-a_0)}{(2-a_0)^2} \|y_0 - x_0\|^2 + \frac{(p+4)N}{2(p+1)(p+2)} \|y_0 - x_0\|^{2+p} \right] \\ & \leq f(a_0) \left[\frac{a_0(4-a_0)(3-a_0)}{(2-a_0)^2} + \frac{(p+4)b_0}{2(p+1)(p+2)} \right] \|y_0 - x_0\| \\ & = f(a_0)g(a_0, b_0) \|y_0 - x_0\|. \end{aligned} \quad (15)$$

Also,

$$M \|\Gamma_1\| \|y_1 - x_1\| \leq M \|\Gamma_0\| f(a_0)^2 g(a_0, b_0) \|y_0 - x_0\| \leq a_0 f(a_0)^2 g(a_0, b_0) = a_1. \quad (16)$$

Hence, for $n = 1$, the conditions (I)–(III) follow from Eqs. (13), (15) and (16), respectively. Proceeding as in (10) and (11), one can easily prove (IV) and (V) for $n = 1$.

We also have

$$N \|\Gamma_1\| \|y_1 - x_1\|^{1+p} \leq N \|\Gamma_0\| f(a_0) f(a_0)^{1+p} g(a_0, b_0)^{1+p} \|y_0 - x_0\|^{1+p} \leq b_1. \quad (17)$$

Thus, condition (VI) follows from the above equation. Now by considering that the statements (I)–(VI) hold for $n = k$ and $x_k \in \Omega$, one can easily prove that the conditions (I)–(VI) also hold for $n = k + 1$. Hence, by induction it holds for all n .

4. Convergence analysis

The following theorem will establish the convergence of the sequence $\{x_n\}$ and give a priori error bounds for it. Let us denote $\gamma = a_1/a_0$ and $\Delta = 1/f(a_0)$, $R = 2/((2-a_0)(1-\gamma\Delta))$, $\mathcal{B}(x_0, R\eta) = \{x \in \mathbb{X}: \|x - x_0\| < R\eta\}$ and $\bar{\mathcal{B}}(x_0, R\eta) = \{x \in \mathbb{X}: \|x - x_0\| \leq R\eta\}$.

Theorem 4.1. Let r_0 be the smallest positive zero of the polynomial $h(x) = -x^3 + 16x^2 - 24x + 4$. If $\Phi_p(x)$ be the function defined by Eq. (9), then let us assume that $0 < a_0 \leq r_0$ and $0 \leq b_0 \leq \Phi_p(a_0)$ hold. Under the assumptions given in (5) on F and $\bar{\mathcal{B}}(x_0, R\eta) \subseteq \Omega$, the method (1) starting from x_0 generates a sequence $\{x_n\}$ converging to the root x^* of (4) with R -order at least 2. In this case x_n, y_n and x^* lie in $\bar{\mathcal{B}}(x_0, R\eta)$ and the solution x^* is unique in $\mathcal{B}(x_0, 2/(M\beta) - R\eta) \cap \Omega$. Furthermore, the error bounds on x^* is given by

$$\|x^* - x_n\| \leq \frac{2\gamma^{2^n-1}}{(2-\gamma^{2^n-1}a_0)} \frac{\Delta^n}{(1-\gamma^{2^n}\Delta)} \eta. \quad (18)$$

Proof. It is sufficient to show that $\{x_n\}$ is a Cauchy sequence in order to establish the convergence of $\{x_n\}$.

Now for $a_0 = r_0$, $b_0 = \Phi_p(a_0) = 0$ and $f(a_0)^2 g(a_0, b_0) = 1$. Hence from (6), $a_n = a_{n-1} = \dots = a_0$ and $b_n = b_{n-1} = \dots = b_0 = 0$.

Also from (12), we have

$$\begin{aligned} \|y_n - x_n\| & \leq f(a_{n-1})g(a_{n-1}, b_{n-1}) \|y_{n-1} - x_{n-1}\| = f(a_0)g(a_0, b_0) \|y_{n-1} - x_{n-1}\| \\ & \leq \dots \leq (f(a_0)g(a_0, b_0))^n \|y_0 - x_0\| = \Delta^n \eta \end{aligned}$$

and

$$\|x_{n+1} - x_n\| \leq \frac{2}{2-a_n} \|y_n - x_n\| \leq \frac{2}{2-a_0} \Delta^n \eta.$$

Thus,

$$\|x_{m+n} - x_m\| \leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{m+1} - x_m\| \leq \frac{2}{2-a_0} [\Delta^{m+n-1} + \cdots + \Delta^m] \eta = \frac{2\Delta^m}{2-a_0} \left(\frac{1-\Delta^n}{1-\Delta} \right) \eta. \quad (19)$$

Hence, if we take $m=0$, $x_n \in \bar{\mathcal{B}}(x_0, R\eta)$. Similarly one can prove that $y_n \in \bar{\mathcal{B}}(x_0, R\eta)$. Also from the above equation, we can conclude that $\{x_n\}$ is a Cauchy sequence.

Let $0 < a_0 < r_0$ and $0 < b_0 < \Phi_p(a_0)$. Now from (12) and Lemma 2.3(iii), for $n \geq 1$, we have

$$\|y_n - x_n\| \leq f(a_{n-1})g(a_{n-1}, b_{n-1})\|y_{n-1} - x_{n-1}\| \leq \cdots \leq \|y_0 - x_0\| \prod_{j=0}^{n-1} f(a_j)g(a_j, b_j) < \prod_{j=0}^{n-1} (\gamma^{2^j} \Delta) \eta = \gamma^{2^n-1} \Delta^n \eta,$$

where $\gamma = a_1/a_0 < 1$ and $\Delta = 1/f(a_0) < 1$. Hence,

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq \frac{2}{2-a_{m+n-1}} \|y_{m+n-1} - x_{m+n-1}\| + \cdots + \frac{2}{2-a_m} \|y_m - x_m\| \\ &< \frac{2}{2-a_{m+n-1}} \gamma^{2^{m+n-1}-1} \Delta^{m+n-1} \eta + \cdots + \frac{2}{2-a_m} \gamma^{2^m-1} \Delta^m \eta \\ &< \frac{2\Delta^m}{2-a_m} [\gamma^{2^{m+n-1}-1} \Delta^{n-1} + \cdots + \gamma^{2^m-1}] \eta \\ &< \frac{2\gamma^{2^m-1} \Delta^m}{2-\gamma^{2^m-1} a_0} [\gamma^{2^m[2^n-1]-1} \Delta^{n-1} + \cdots + \gamma^{2^m[2-1]} \Delta + 1] \eta. \end{aligned}$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \geq 0$, we have $(1+x)^k - 1 \geq kx$. Thus,

$$\|x_{m+n} - x_m\| < \frac{2\gamma^{2^m-1} \Delta^m}{(2-\gamma^{2^m-1} a_0)} \frac{1-\gamma^{2^m n} \Delta^n}{(1-\gamma^{2^m} \Delta)} \eta. \quad (20)$$

Thus, for $m=0$, we get

$$\|x_n - x_0\| < \frac{2}{2-a_0} \frac{1-\gamma^n \Delta^n}{1-\gamma \Delta} \eta < R\eta. \quad (21)$$

Hence, $x_n \in \mathcal{B}(x_0, R\eta)$. Also, $y_n \in \mathcal{B}(x_0, R\eta)$ results from

$$\begin{aligned} \|y_{n+1} - x_0\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \cdots + \|x_1 - x_0\| \\ &\leq \|y_{n+1} - x_{n+1}\| + \frac{2}{2-a_n} \|y_n - x_n\| + \cdots + \frac{2}{2-a_0} \|y_0 - x_0\| \\ &< \frac{2}{2-a_{n+1}} \|y_{n+1} - x_{n+1}\| + \frac{2}{2-a_n} \|y_n - x_n\| + \cdots + \frac{2}{2-a_0} \|y_0 - x_0\| \\ &< \cdots < \frac{2}{2-a_0} \frac{1-\gamma^{n+2} \Delta^{n+2}}{1-\gamma \Delta} \eta < R\eta. \end{aligned}$$

On taking limit as $n \rightarrow \infty$ in (19) or (21), we get $x^* \in \bar{\mathcal{B}}(x_0, R\eta)$. To show that x^* is a solution of $F(x) = 0$, we see that $\|F(x_n)\| \leq \|F'(x_n)\| \|x_n - x_0\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded as

$$\|F'(x_n)\| \leq \|F'(x_0)\| + M\|x_n - x_0\| < \|F'(x_0)\| + MR\eta.$$

Since F is continuous, so by taking limit as $n \rightarrow \infty$ we get $F(x^*) = 0$. To show the uniqueness of x^* , let y^* be the another root of (4) in $\mathcal{B}(x_0, 2/(M\beta) - R\eta) \cap \mathcal{Q}$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*)) dt (y^* - x^*).$$

If we can show that $\int_0^1 F'(x^* + t(y^* - x^*)) dt$ is invertible, then $y^* = x^*$. From

$$\begin{aligned} \|\Gamma_0\| \left\| \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)] dt \right\| &\leq M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\| dt \leq M\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|) dt \\ &< \frac{M\beta}{2} \left(R\eta + \frac{2}{M\beta} - R\eta \right) = 1, \end{aligned}$$

and the Banach theorem [14, p. 155], the operator $\int_0^1 F'(x^* + t(y^* - x^*)) dt$ is invertible. Hence, $y^* = x^*$. \square

4.1. On the R-order of convergence $(2 + p)$

The R-order convergence of the method in Section 4 is found to be 2, where as the method is of order 3. In [8], it is established that the R-order of the Halley's method can be increased from 2 to $(2 + p)$. In order to increase the R-order of our method from 2 to $(2 + p)$, we rewrite the method (1) in the following way:

$$\left. \begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ H(x_n, y_n) &= F'(x_n)^{-1} [F'(y_n) - F'(x_n)], \\ Q(x_n, y_n) &= -\frac{1}{2} H(x_n, y_n) \left[I + \frac{1}{2} H(x_n, y_n) \right]^{-1}, \\ x_{n+1} &= y_n + Q(x_n, y_n)(y_n - x_n), \quad n = 0, 1, \dots \end{aligned} \right\} \quad (22)$$

Now consider the sequences

$$\tilde{a}_{n+1} = \tilde{a}_n \tilde{f}(\tilde{a}_n)^2 \tilde{g}(\tilde{a}_n, \tilde{b}_n) \quad \text{and} \quad \tilde{b}_{n+1} = \tilde{b}_n \tilde{f}(\tilde{a}_n)^{2+p} \tilde{g}(\tilde{a}_n, \tilde{b}_n)^{1+p}, \quad (23)$$

where

$$\tilde{f}(x) = (2 - x)/(2 - 3x), \quad (24)$$

$$\tilde{g}(x, y) = \left[\frac{x^2}{(2 - x)^2} + \frac{(p + 4)y}{2(p + 1)(p + 2)} \right], \quad (25)$$

$\tilde{a}_0 = M\beta\eta$ and $\tilde{b}_0 = N\beta\eta^{1+p}$ are two parameters.

As the expression for \tilde{f} is the same as the expression of f of the previous section, the properties of \tilde{f} are the same as defined previously. If \tilde{r}_0 be the smallest positive zero of the polynomial $\tilde{h}(x) = 2x^2 - 3x + 1$, it is very easy to prove that for $x \in (0, \tilde{r}_0]$ and $y > 0$, \tilde{g} is increasing in both arguments, and $\tilde{g}(\delta x, \delta^{p+1}y) < \delta^{p+1}\tilde{g}(x, y)$, for $\delta \in (0, 1)$ and $p \in (0, 1]$.

Also it is very easy to prove properties similar to properties of Lemma 2.2 with new functions \tilde{g} and $\tilde{\Phi}_p(x)$ given by

$$\tilde{\Phi}_p(x) = \frac{8(p + 1)(p + 2)(2x^2 - 3x + 1)}{(p + 4)(2 - x)^2}. \quad (26)$$

Proceeding similarly as in Lemma 2.3, one can easily prove the following lemma.

Lemma 4.1. Let us suppose $\tilde{a}_0 \in (0, \tilde{r}_0)$ and $0 < \tilde{b}_0 < \tilde{\Phi}_p(\tilde{a}_0)$. Define $\gamma = \tilde{a}_1/\tilde{a}_0$, then for $n \geq 1$, we have

- (i) $\tilde{a}_n \leq \gamma^{(2+p)^{n-1}} \tilde{a}_{n-1} \leq \gamma^{((2+p)^n - 1)/(1+p)} \tilde{a}_0$, strictly holds for $n \geq 2$,
- (ii) $\tilde{b}_n < (\gamma^{(2+p)^{n-1}})^{1+p} \tilde{b}_{n-1} < \gamma^{(2+p)^n - 1} \tilde{b}_0$,
- (iii) $\tilde{f}(\tilde{a}_n) \tilde{g}(\tilde{a}_n, \tilde{b}_n) < \gamma^{(2+p)^n} / \tilde{f}(\tilde{a}_0)$.

Depending upon the properties of the sequences $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ we can easily have

$$\left. \begin{aligned} \text{(I)} \quad & \| \Gamma_n \| = \| F'(x_n)^{-1} \| \leq \tilde{f}(\tilde{a}_{n-1}) \| \Gamma_{n-1} \|, \\ \text{(II)} \quad & \| y_n - x_n \| \leq \tilde{f}(\tilde{a}_{n-1}) \tilde{g}(\tilde{a}_{n-1}, \tilde{b}_{n-1}) \| y_{n-1} - x_{n-1} \|, \\ \text{(III)} \quad & M \| \Gamma_n \| \| y_n - x_n \| \leq \tilde{a}_n, \\ \text{(IV)} \quad & \| x_{n+1} - y_n \| \leq \tilde{a}_n / (2 - \tilde{a}_n) \| y_n - x_n \|, \\ \text{(V)} \quad & \| x_{n+1} - x_n \| \leq 2 / (2 - \tilde{a}_n) \| y_n - x_n \|, \\ \text{(VI)} \quad & N \| \Gamma_n \| \| y_n - x_n \|^{1+p} \leq \tilde{b}_n. \end{aligned} \right\} \quad (27)$$

Now we give below a theorem to establish the R-order convergence of the method to be $(2 + p)$ and a priori error bounds. For this, let $\tilde{\Delta} = 1/\tilde{f}(\tilde{a}_0)$ and $\tilde{R} = 2/((2 - \tilde{a}_0)(1 - \gamma\tilde{\Delta}))$.

Theorem 4.2. Let $0 < \tilde{a}_0 \leq \tilde{r}_0$ and $0 \leq \tilde{b}_0 \leq \tilde{\Phi}_p(\tilde{a}_0)$ hold, where \tilde{r}_0 is the smallest positive zero of the polynomial $\tilde{h}(x) = 2x^2 - 3x + 1$ and $\tilde{\Phi}_p(x)$ be the function defined by (26). Under the assumptions given in (5) on F and $\tilde{B}(x_0, \tilde{R}\eta) \subseteq \Omega$, the method (22) starting from x_0 generates a sequence $\{x_n\}$ converging to the root x^* of (4) with R-order at least $(p + 2)$. In this case x_n , y_n and x^* lie in $\tilde{B}(x_0, \tilde{R}\eta)$ and the solution x^* is unique in $\tilde{B}(x_0, 2/(M\beta) - \tilde{R}\eta) \cap \Omega$. Furthermore, the error bounds on x^* are given by

$$\| x^* - x_n \| \leq \frac{2\gamma^{((2+p)^n - 1)/(1+p)}}{(2 - \gamma^{((2+p)^n - 1)/(1+p)} \tilde{a}_0)} \frac{\tilde{\Delta}^n}{(1 - \gamma^{(2+p)^n} \tilde{\Delta})} \eta. \quad (28)$$

Proof. Here it is sufficient to prove the R -order convergence of the sequence $\{x_n\}$. Let $0 < \tilde{a}_0 < \tilde{r}_0$ and $0 < \tilde{b}_0 < \tilde{\Phi}_p(\tilde{a}_0)$. Now from (27) and Lemma 4.1(iii), for $n \geq 1$, we obtain

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq \frac{2}{2 - \tilde{a}_m} \left[\left(\prod_{j=0}^{m+n-2} \tilde{f}(\tilde{a}_j) \tilde{g}(\tilde{a}_j, \tilde{b}_j) \right) \eta + \cdots + \left(\prod_{j=0}^{m-1} \tilde{f}(\tilde{a}_j) \tilde{g}(\tilde{a}_j, \tilde{b}_j) \right) \eta \right] \\ &< \frac{2}{2 - \tilde{a}_m} [\gamma^{((2+p)^{m+n-1}-1)/(1+p)} \tilde{\Delta}^{m+n-1} + \cdots + \gamma^{((2+p)^m-1)/(1+p)} \tilde{\Delta}^m] \eta \\ &= \frac{2\tilde{\Delta}^m}{2 - \tilde{a}_m} [\gamma^{((2+p)^{m+n-1}-1)/(1+p)} \tilde{\Delta}^{n-1} + \cdots + \gamma^{((2+p)^m-1)/(1+p)}] \eta \\ &< \frac{2\gamma^{((2+p)^m-1)/(1+p)} \tilde{\Delta}^m}{2 - \gamma^{((2+p)^m-1)/(1+p)} \tilde{a}_0} [\gamma^{((2+p)^m[(2+p)^{n-1}-1]/(1+p)} \tilde{\Delta}^{n-1} \\ &\quad + \gamma^{((2+p)^m[(2+p)^{n-2}-1]/(1+p)} \tilde{\Delta}^{n-2} + \cdots + \gamma^{((2+p)^m[(2+p)-1]/(1+p)} \tilde{\Delta} + 1] \eta. \end{aligned}$$

Thus, by Bernoulli's inequality, we get

$$\|x_{m+n} - x_m\| < \frac{2\gamma^{((2+p)^m-1)/(1+p)} \tilde{\Delta}^m}{(2 - \gamma^{((2+p)^m-1)/(1+p)} \tilde{a}_0)} \frac{1 - \gamma^{(2+p)^m n} \tilde{\Delta}^n}{(1 - \gamma^{(2+p)^m} \tilde{\Delta})} \eta. \quad (29)$$

Rest of the things are the same as given in Theorem 4.1. \square

Remark 4.1. Note that, if $p = 1$, then Hölder continuity condition reduces to Lipschitz continuity condition. So the R -order of convergence is at least 3 for this case.

5. Numerical examples

In this section, two numerical examples are given for demonstrating the efficacy of our convergence analysis described in the previous section.

Example 5.1. Let $\mathbb{X} = C[a, b]$ be the space of continuous functions on $[a, b]$ and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = x(s) - f(s) - \lambda \int_a^b G(s, t)x(t)^{2+p} dt, \quad (30)$$

with $s \in [a, b]$, $x, f \in \mathbb{X}$, $p \in (0, 1]$ and λ be a real number. The kernel G is continuous and nonnegative in $[a, b] \times [a, b]$.

Sol. The given integral equations are Hammerstein integral equations of second kind [17]. Here, we have taken norm as sup-norm and $G(s, t)$ as the Green's function

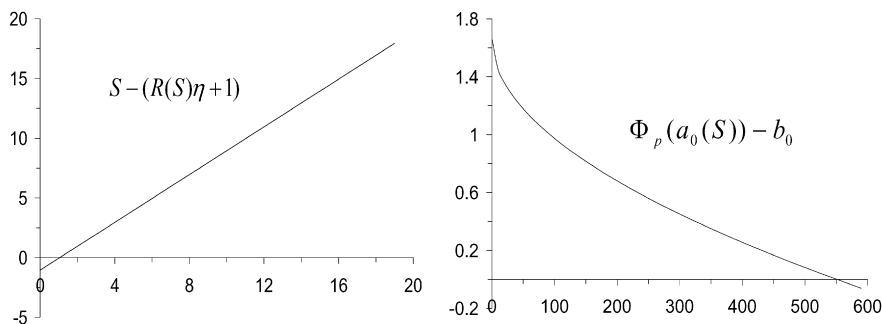
$$G(s, t) = \begin{cases} (b-s)(t-a)/(b-a), & t \leq s, \\ (s-a)(b-t)/(b-a), & s \leq t. \end{cases}$$

Now it is easy to find the first and the second derivatives of F as

$$\begin{aligned} F'(x)u(s) &= u(s) - \lambda(2+p) \int_a^b G(s, t)x(t)^{1+p}u(t) dt, \quad u \in \Omega, \\ F''(x)(uv)(s) &= -\lambda(1+p)(2+p) \int_a^b G(s, t)x(t)^p(uv)(t) dt, \quad u, v \in \Omega. \end{aligned}$$

Here F'' does not satisfy the Lipschitz continuity condition as, for $p \in (0, 1)$ and $\forall x, y \in \Omega$, we have

$$\begin{aligned} \|F''(x) - F''(y)\| &= \left\| \lambda(1+p)(2+p) \int_a^b G(s, t)[x(t)^p - y(t)^p] dt \right\| \leq |\lambda|(1+p)(2+p) \max_{s \in [a, b]} \left| \int_a^b G(s, t) dt \right| \|x(t)^p - y(t)^p\| \\ &\leq |\lambda|(1+p)(2+p) \|I\| \|x - y\|^p, \end{aligned}$$

Fig. 1. Conditions on the parameter S .

where

$$\|I\| = \max_{s \in [a, b]} \left| \int_a^b G(s, t) dt \right|.$$

But it satisfies the Hölder continuity condition as $p \in (0, 1)$. Hence, $N = |\lambda|(1+p)(2+p)\|I\|$. Note that it is easy to compute

$$\|F(x_0)\| \leq \|x_0 - f\| + |\lambda|\|I\|\|x_0\|^{2+p}$$

and

$$\|F''(x)\| \leq |\lambda|(1+p)(2+p)\|I\|\|x\|^p.$$

Hence, $M = |\lambda|(1+p)(2+p)\|I\|\|x\|^p = N\|x\|^p$. Also note that

$$\|I - F'(x_0)\| \leq |\lambda|(2+p)\|I\|\|x_0\|^{1+p}$$

and if $|\lambda|(2+p)\|I\|\|x_0\|^{1+p} < 1$, then by Banach theorem [14, p. 155], we obtain

$$\|I_0\| = \|F'(x_0)^{-1}\| \leq \frac{1}{1 - |\lambda|(2+p)\|I\|\|x_0\|^{1+p}} = \beta.$$

Hence,

$$\|I_0 F(x_0)\| \leq \frac{\|x_0 - f\| + |\lambda|\|I\|\|x_0\|^{2+p}}{1 - |\lambda|(2+p)\|I\|\|x_0\|^{1+p}} = \eta.$$

Now in the interval $[a, b] = [0, 1]$, we have

$$\|I\| = \max_{s \in [0, 1]} \left| \int_0^1 G(s, t) dt \right| = 1/8.$$

Here for $\lambda = 1/3$, $p = 1/2$, $f(s) = 1$, and initial point $x_0 = x_0(s) = 1$ in the interval $[a, b] = [0, 1]$, we have $\|I_0\| \leq \beta = 1.11628$, $\|I_0 F(x_0)\| \leq \eta = 0.0465116$, $N = 0.15625$. Now we look for a domain in the form of $\Omega = \mathcal{B}(x_0, S)$ such that

$$\Omega = \mathcal{B}(x_0, S) \subseteq C[0, 1] = \mathbb{X}.$$

So $M = M(S) = 0.15625S^p$. Hence, $a_0 = a_0(S) = M(S)\beta\eta = 0.00811249S^p$ and $b_0 = N\beta\eta^{1+p} = 0.00174959$. Now in order to calculate S from the conditions of Theorem 4.1, it is necessary that $\tilde{\mathcal{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Phi_p(a_0(S)) - b_0 > 0$. Hence, it is necessary that $S \in (1.04797, 550.87)$ as is evident from Fig. 1.

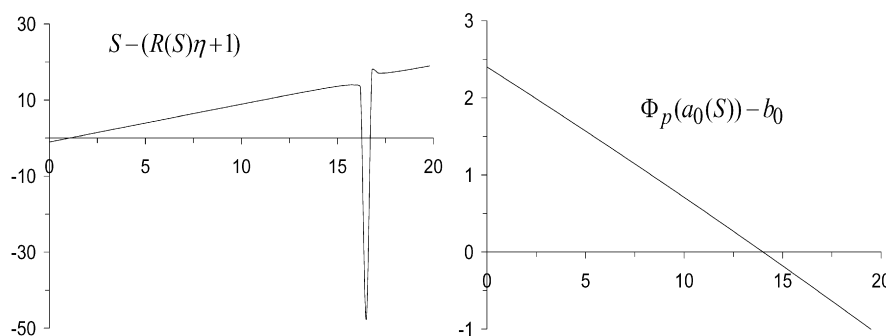
Also, $a_0(S) < r_0 = 0.1905960896$ if and only if $S < 551.9752734$. Hence, if we choose $S = 550$ then we have $\Omega = \mathcal{B}(1, 550)$, $M = 3.66439$, $a_0 = 0.190255$ and $b_0 = 0.00174959 < 0.00312851 = \Phi_p(0.190255)$. Thus, the conditions of Theorem 4.1 are satisfied. Hence, a solution of Eq. (30) exists in $\tilde{\mathcal{B}}(1, 0.243) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(1, 0.2459395) \cap \Omega$.

It is to be noted that if we consider $p = 1$ in Eq. (30), we get the following problem to solve

$$F(x)(s) = x(s) - f(s) - \lambda \int_a^b G(s, t)x(t)^3 dt, \quad (31)$$

then it is easy to conclude that F'' is Lipschitz continuous as evident from the inequality

$$\|F''(x) - F''(y)\| \leq 6|\lambda|\|I\|\|x - y\|.$$

Fig. 2. Conditions on the parameter S .

Thus, if we take $\lambda = 1/3$, $f(s) = 1$, and initial point $x_0 = x_0(s) = 1$ in the interval $[0, 1]$, we have $\|F_0\| \leq \beta = 1.14286$, $\|F_0 F(x_0)\| \leq \eta = 0.047619$ and $N = 0.25$. Now similarly as above we are looking for a domain in the form of $\Omega = \mathcal{B}(x_0, S)$ such that

$$\Omega = \mathcal{B}(x_0, S) \subseteq C[0, 1] = \mathbb{X}.$$

So $M = M(S) = 0.25S$. Hence, $a_0 = a_0(S) = M(S)\beta\eta = 0.0136054S$ and $b_0 = N\beta\eta^2 = 0.000647878$. Now in order to calculate S , from the conditions of Theorem 4.1, it is necessary that $\bar{\mathcal{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Phi_p(a_0(S)) - b_0 > 0$. Hence, it is necessary that $S \in (1.05017, 14.0052)$ as evident from Fig. 2.

Also $a_0(S) < r_0 = 0.1905960896$ if and only if $S < 14.008856$. Hence, if we choose $S = 9$, then we have $\Omega = \mathcal{B}(1, 9)$, $M = 2.25$, $a_0 = 0.122449$ and $b_0 = 0.000647878 < 0.884571878 = \Phi_p(0.122449)$. Thus the conditions of Theorem 4.1 are satisfied. Hence, a solution of Eq. (31) exists in $\bar{\mathcal{B}}(1, 0.09156) \subseteq \Omega$ which is unique in the ball $\mathcal{B}(1, 0.6862158) \cap \Omega$.

But by the work of Wu and Zhao [11], $K = M[1 + \frac{5N}{3M^2\beta}] = 2.4120366$. Hence, $h = K\beta\eta = 0.131267496$. So $t^* = \frac{1 - \sqrt{1 - 2h}}{h}\eta = 0.051237443$ and $t^{**} = \frac{1 + \sqrt{1 - 2h}}{h}\eta = 0.674288695$. Hence, by the convergence method given in [11], the solution of Eq. (31) exists in $\bar{\mathcal{B}}(1, 0.051237443) \subseteq \Omega$ which is unique in the ball $\mathcal{B}(1, 0.674288695) \cap \Omega$, both of which are inferior to our result.

Example 5.2. Let $\mathbb{X} = C[0, 1]$ be the space of continuous functions on $[0, 1]$ and let us consider the integral equation $F(x) = 0$ on \mathbb{X} , where

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+p} dt, \quad (32)$$

with $s \in [0, 1]$, $x, f \in \mathbb{X}$, $p \in (0, 1]$ and λ be a real number.

Sol. The given integral equations are called Fredholm-type integral equations studied by Davis [13]. Here the norm is taken as sup-norm.

Now, it is easy to find the first and the second derivatives of F as

$$F'(x)u(s) = u(s) - \lambda(2+p) \int_0^1 \frac{s}{s+t} x(t)^{1+p} u(t) dt, \quad u \in \Omega,$$

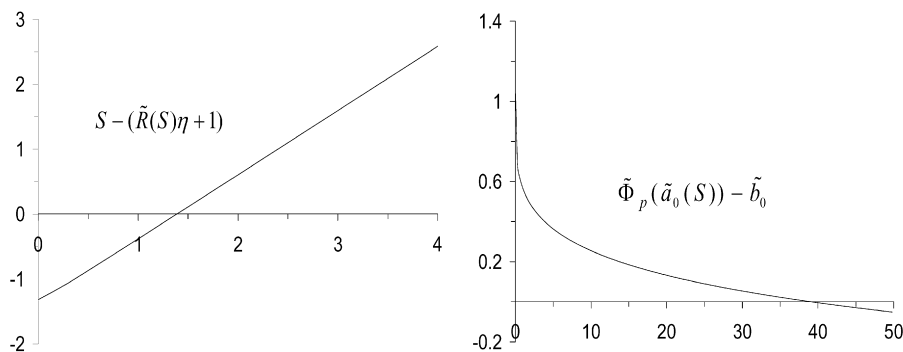
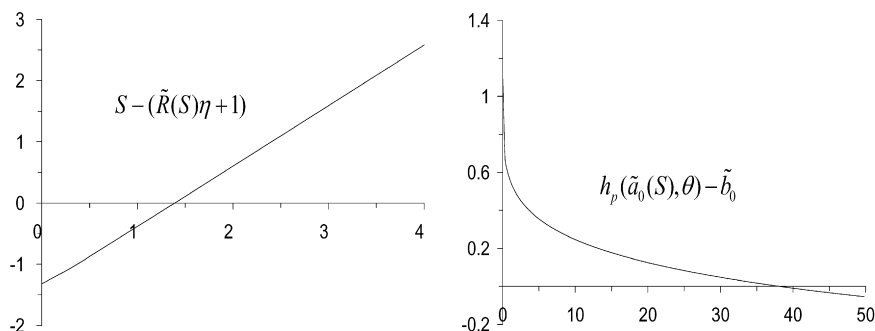
$$F''(x)(uv)(s) = -\lambda(1+p)(2+p) \int_0^1 \frac{s}{s+t} x(t)^p (uv)(t) dt, \quad u, v \in \Omega.$$

Here F'' does not satisfy the Lipschitz condition as, for $p \in (0, 1)$ and $\forall x, y \in \Omega$,

$$\begin{aligned} \|F''(x) - F''(y)\| &= \left\| \lambda(1+p)(2+p) \int_0^1 \frac{s}{s+t} [x(t)^p - y(t)^p] dt \right\| \leq |\lambda|(1+p)(2+p) \max_{s \in [0, 1]} \left| \int_0^1 \frac{s}{s+t} dt \right| \|x(t)^p - y(t)^p\| \\ &\leq |\lambda|(1+p)(2+p) \log 2 \|x - y\|^p. \end{aligned}$$

But it satisfies the Hölder continuity condition as $p \in (0, 1)$. Hence, $N = |\lambda|(1+p)(2+p) \log 2$. Now it is easy to compute

$$\|F(x_0)\| \leq \|x_0 - f\| + |\lambda| \log 2 \|x_0\|^{2+p}$$

Fig. 3. Conditions on the parameter S .Fig. 4. Conditions on the parameter S .

and

$$\|F''(x)\| \leq |\lambda|(1+p)(2+p) \log 2 \|x\|^p.$$

Hence, $M = |\lambda|(1+p)(2+p) \log 2 \|x\|^p = N \|x\|^p$. Also note that

$$\|I - F'(x_0)\| \leq |\lambda|(2+p) \log 2 \|x_0\|^{1+p}$$

and if $|\lambda|(2+p) \log 2 \|x_0\|^{1+p} < 1$, then by Banach theorem, we obtain

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leq \frac{1}{1 - |\lambda|(2+p) \log 2 \|x_0\|^{1+p}} = \beta.$$

Hence,

$$\|\Gamma_0 F(x_0)\| \leq \frac{\|x_0 - f\| + |\lambda| \log 2 \|x_0\|^{2+p}}{1 - |\lambda|(2+p) \log 2 \|x_0\|^{1+p}} = \eta.$$

Now for $\lambda = 1/4$, $p = 1/5$, $f(s) = 1$, and initial point $x_0 = x_0(s) = 1$ in the interval $[0, 1]$, we have $\|\Gamma_0\| \leq \beta = 1.61611$, $\|\Gamma_0 F(x_0)\| \leq \eta = 0.280051$ and $N = 0.457477$. Now we look for a domain in the form of $\Omega = \mathcal{B}(x_0, S)$ such that

$$\Omega = \mathcal{B}(x_0, S) \subseteq C[0, 1] = \mathbb{X}.$$

So $M = M(S) = 0.457477S^p$, $\tilde{a}_0 = \tilde{a}_0(S) = M(S)\beta\eta = 0.207051S^p$ and $\tilde{b}_0 = N\beta\eta^{1+p} = 0.160518$. Now in order to calculate S from the conditions of Theorem 4.2, it is necessary that $\tilde{\mathcal{B}}(x_0, \tilde{R}\eta) \subseteq \Omega$. For this, it is sufficient to check $S - (\tilde{R}(S)\eta + 1) > 0$ and $\tilde{\Phi}_p(\tilde{a}_0(S)) - \tilde{b}_0 > 0$. Hence, it is necessary that $S \in (1.38895, 39.0605)$ as is evident from Fig. 3.

Also $\tilde{a}_0(S) \leq \tilde{r}_0 = 0.5$ if and only if $S \leq 82.122726$. Hence, if we choose $S = 38$ then we have $\Omega = \mathcal{B}(1, 38)$, $M = 0.946948$, $\tilde{a}_0 = 0.428582$ and $\tilde{b}_0 = 0.160518 < 0.16620666 = \tilde{\Phi}_p(0.428582)$. Thus, the conditions of Theorem 4.2 are satisfied. Hence, a solution of Eq. (32) exists in $\tilde{\mathcal{B}}(1, 0.641723) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(1, 0.665147) \cap \Omega$.

But, for the case of Hernández [7], it is necessary that $S \in (1.39047, 38.1489)$ as seen in Fig. 4. For numerical computation here we have taken $\theta = 1$.

Also $\tilde{a}_0(S) \leq 0.5$ if and only if $S \leq 82.122726$. Hence, if we choose $S = 38$, then we have $\Omega = \mathcal{B}(1, 38)$, $M = 0.946948$, $\tilde{a}_0 = 0.428582$ and $\tilde{b}_0 = 0.160518 < 0.161305132 = h_p(0.428582, 1)$. Thus the conditions of Theorem 4 of that paper are satisfied. Hence, a solution of Eq. (32) exists in $\tilde{\mathcal{B}}(1, 0.6518) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(1, 0.65507) \cap \Omega$.

From both the results, we can conclude that the uniqueness ball of solution based on our result is larger than that of Hernández [7].

6. Conclusions

In this paper, recurrence relations are developed for establishing the convergence of third order Newton-like methods for solving $F(x) = 0$ in Banach spaces under the Hölder continuity condition over F'' . An existence-uniqueness theorem is given to establish the R -order of the method to be 2 and a priori error bounds. Also we give convergence analysis of another formulation of our method in order to increase the R -order of the method from 2 to $(2 + p)$. One important contribution of our approach is that the convergence analysis of the method is established even when Lipschitz condition over F'' fails.

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References

- [1] V. Candela, A. Marquina, Recurrence relations for rational cubic methods I: The Halley method, *Computing* 44 (1990) 169–184.
- [2] V. Candela, A. Marquina, Recurrence relations for rational cubic methods II: The Chebyshev method, *Computing* 45 (1990) 355–367.
- [3] J.A. Ezquerro, M.A. Hernández, Avoiding the computation of the second Fréchet-derivative in the convex acceleration of Newton's method, *J. Comput. Appl. Math.* 96 (1998) 1–12.
- [4] J.M. Gutiérrez, M.A. Hernández, Recurrence relations for the super-Halley method, *Comput. Math. Appl.* 36 (1998) 1–8.
- [5] M.A. Hernández, Reduced recurrence relations for the Chebyshev method, *J. Optim. Theory Appl.* 98 (1998) 385–397.
- [6] M.A. Hernández, Second-derivative-free variant of the Chebyshev method for nonlinear equations, *J. Optim. Theory Appl.* 104 (2000) 501–515.
- [7] M.A. Hernández, Chebyshev's approximation algorithms and applications, *Comput. Math. Appl.* 41 (2001) 433–445.
- [8] J.A. Ezquerro, M.A. Hernández, On the R -order of the Halley method, *J. Math. Anal. Appl.* 303 (2005) 591–601.
- [9] M.A. Hernández, M.A. Salanova, Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method, *J. Comput. Appl. Math.* 126 (2000) 131–143.
- [10] P.K. Parida, D.K. Gupta, Recurrence relations for a Newton-like method in Banach spaces, *J. Comput. Appl. Math.* 206 (2007) 873–887.
- [11] Q. Wu, Y. Zhao, Third-order convergence theorem by using majorizing function for a modified Newton method in Banach space, *Appl. Math. Comput.* 175 (2006) 1515–1524.
- [12] X. Ye, C. Li, Convergence of the family of the deformed Euler–Halley iterations under the Hölder condition of the second derivative, *J. Comput. Appl. Math.* 194 (2006) 294–308.
- [13] H.T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, 1962.
- [14] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.
- [15] J.D. Lambert, *Computational Methods in Ordinary Differential Equations*, Wiley, New York, 1979.
- [16] J.M. Ortega, W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, 1970.
- [17] A.D. Polyanin, A.V. Manzhirov, *Handbook of Integral Equations*, CRC Press, Boca Ratón, FL, 1998.