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J. Math. Anal. Appl. 348 (2008) 938–970

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On estimates of the density of Feynman–Kac semigroups of α-stable-like processes

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Received 19 June 2007

Available online 20 February 2008

Submitted by M. Ledoux

Abstract

Suppose that $\alpha \in (0,2)$ and that X is an α-stable-like process on $\mathbb{R}^d$. Let F be a function on $\mathbb{R}^d$ belonging to the class $\mathbf{J}_{d,\alpha}$ (see Introduction) and A_t^F be $\sum_{s\leqslant t} F(X_{s-}, X_s)$, $t > 0$, a discontinuous additive functional of X. With neither F nor X being symmetric, under certain conditions, we show that the Feynman–Kac semigroup $\{S_t^F\colon t \geqslant 0\}$ defined by

$$S_t^F f(x) = \mathbb{E}_x\left(e^{-A_t^F} f(X_t)\right)$$

has a density q and that there exist positive constants C_1, C_2, C_3 and C_4 such that

$$C_1 e^{-C_2 t} t^{-\frac{d}{\alpha}}\left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha} \leqslant q(t,x,y) \leqslant C_3 e^{C_4 t} t^{-\frac{d}{\alpha}}\left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Keywords: α-Stable-like processes; Nonsymmetric; Kato class; Feynman–Kac semigroups; Discontinuous additive functionals

1. Introduction

Suppose $X = (X_t, \mathbb{P}_t)$ is a Hunt process on $\mathbb{R}^d$ with a Lévy system (N, H) given by $H_t = t$ and

$$N(x,dy) = 2C(x,y)|x-y|^{-(d+\alpha)} m(dy),$$

where m is a measure on $\mathbb{R}^d$ given by $m(dx) = M(x)\,dx$ with $M(x)$ bounded between two positive numbers. That is for any nonnegative function f on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal

$$\mathbb{E}_x\left(\sum_{s\leqslant t} f(X_{s-}, X_s)\right) = \mathbb{E}_x \int_0^t \int_{R^d} \frac{2C(X_s,y) f(X_s,y)}{|X_s-y|^{d+\alpha}} m(dy)\,ds$$

for every $x \in \mathbb{R}^d$ and $t > 0$.

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0022-247X/$ – see front matter © 2008 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2008.02.012

We introduce α-stable-like processes as follows.

Definition 1.1. We say that X is an α-stable-like process if $C(x, y)$ is bounded between two positive numbers.

In this paper, we assume that X admits a transition density $p(t, x, y)$ with respect to m and $p(t, x, y)$ is jointly continuously on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and satisfies the condition

$$M_1 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha} \leqslant p(t, x, y) \leqslant M_2 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha}, \quad \forall (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.1}$$

where M_1 and M_2 are positive constants.

Here we do not assume that X is symmetric. When X is symmetric, it is called a symmetric α-stable-like process, which was introduced in [3], where a symmetric Hunt process is associated with a regular Dirichlet form and thus Dirichlet form method can be applied. It was also shown in [3] that the transition densities of symmetric α-stable-like processes satisfy (1.1).

We list some examples which are α-stable-like processes and satisfy (1.1). For one-dimensional strictly α-stable processes with Lévy measure ν concentrated neither on $(0, \infty)$ nor on $(-\infty, 0)$, the Lévy measure $\nu(dx) = c_1 x^{-1-\alpha}\,dx$ on $(0, \infty)$ and $\nu(dx) = c_2 x^{-1-\alpha}\,dx$ on $(-\infty, 0)$ with $c_1 > 0$ and $c_2 > 0$, which implies $C(x, y)$ in the Lévy system as above is bounded between two positive numbers. We set $c = c_1 + c_2$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$. Let $\rho = (1+\beta)/2$ or $= (1 - \beta\frac{2-\alpha}{2})/2$, according to $\alpha < 1$ or > 1. Without loss of generality, we can fix the parameter c and assume that c equals $\cos(\frac{\pi\beta\alpha}{2})$, $\frac{\pi}{2}$ or $\cos(\pi\beta\frac{2-\alpha}{2})$ for $\alpha < 1$, $= 1$, or > 1, respectively. The following estimates for the continuous transition density $p(t, 0, x)$, which equals $t^{-1/\alpha} p(1, 0, t^{-1/\alpha} x)$, is given in [4]:

1. As $x \to \infty$,
$$p(1, 0, x) \sim \frac{1}{\pi} \Gamma(\alpha+1)\big(\sin(\pi\rho\alpha)\big) x^{-\alpha-1}, \quad \text{if } \alpha \neq 1,$$
$$p(1, 0, x) \sim \frac{1+\beta}{2} x^{-2}, \quad \text{if } \alpha = 1.$$
2. As $x \to 0$,
$$p(1, 0, x) \to \frac{1}{\pi} \Gamma(1/\alpha + 1)(\sin\pi\alpha), \quad \text{if } \alpha \neq 1,$$
$$p(1, 0, x) \to \frac{1}{\pi} b_1, \quad \text{if } \alpha = 1,\ \beta > 0,$$
where b_1 is a positive constant.
See (14.37), (14.30), (14.33) and (14.32) in [4] for details. It is clear that the dual process of the one-dimensional strictly α-stable process has the transition density $p(t, 0, -x)$. Thus applying the above estimates to $p(t, 0, -x)$, we get:
3. As $x \to -\infty$,
$$p(1, 0, x) \sim \frac{1}{\pi} \Gamma(\alpha+1)\big(\sin(\pi\rho\alpha)\big) |x|^{-\alpha-1}, \quad \text{if } \alpha \neq 1,$$
$$p(1, 0, x) \sim \frac{1+\beta}{2} |x|^{-2}, \quad \text{if } \alpha = 1.$$
4. As $x \to 0$,
$$p(1, 0, x) \to \frac{1}{\pi} \tilde{b}_1, \quad \text{if } \alpha = 1,\ \beta < 0,$$
where $\tilde{b}_1$ is a positive constant.

One-dimensional strictly α-stable process with $\alpha = 1$ and $\beta = 0$ is a Cauchy process with drift 0. It is easy to see that when $x \to 0$, $p(1, 0, x) \to$ a positive constant.

It is also pointed out in [4] that $p(t,0,x)$ is positive when the Lévy measure ν is concentrated neither on $(0,\infty)$ nor on $(-\infty,0)$.

Therefore when the Lévy measure ν is concentrated neither on $(0,\infty)$ nor on $(-\infty,0)$, the transition density p satisfies (1.1).

For higher dimensions, a large class of nonsymmetric strictly α-stable processes with $0<\alpha<2$ is considered in [7]. It has Lévy measure ν satisfying

$$\nu(B)=\int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)r^{-(1+\alpha)}\,dr,$$

for every Borel set B in $\mathbb{R}^d$, where λ is a finite measure on the unit sphere $S=\{x\in\mathbb{R}^d\colon |x|=1\}$ and is called the spherical part of the Lévy measure ν. λ is assumed to have a density $\phi: S\to(0,\infty)$ such that

$$\phi=\frac{d\lambda}{d\sigma}\quad\text{and}\quad \kappa\leqslant\phi(\xi)\leqslant\kappa^{-1},\quad \forall\xi\in S,$$

where σ is the surface measure on the unit sphere and $\kappa>0$ is a positive constant. The assumption on ϕ implies the transition density $p(t,0,x)>0$ for all $t>0$ and all $x\in\mathbb{R}^d$. It is known that $p(1,0,x)$ is uniformly bounded in $x\in\mathbb{R}^d$. It is pointed out in [7] that the Lévy measure ν has a density $f(x)=\phi(x/|x|)|x|^{-(d+\alpha)}$ with respect to the d-dimensional Lebesgue measure, and

$$\kappa|x|^{-(d+\alpha)}\leqslant f(x)\leqslant\kappa^{-1}|x|^{-(d+\alpha)}$$

for every $x\in\mathbb{R}^d\setminus\{0\}$. Then the transition density $p(t,0,x)$ of the processes satisfy

$$p(t,0,x)\leqslant\tilde{C}t^{-\frac{d}{\alpha}},\quad x\in\mathbb{R}^d,\ t>0,$$

and

$$p(t,0,x)\leqslant\tilde{C}t|x|^{-(\alpha+d)},\quad x\in\mathbb{R}^d\setminus\{0\},\ t>0,$$

where $\tilde{C}$ is a positive constant. See (2.6) and (2.7) in [7] for these two inequalities. Thus we have

$$p(t,0,x)\leqslant\tilde{C}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha},\quad x\in\mathbb{R}^d,\ t>0. \tag{1.2}$$

(2.12) in [7] gave the following estimate,

$$\tilde{c}|x|^{-(\alpha+d)}\leqslant p(1,0,x)\leqslant\tilde{C}|x|^{-(\alpha+d)},\quad\text{for large } x,$$

where $\tilde{c}$ is a positive constant and $\tilde{C}$ is the same constant as above. This implies that

$$\tilde{c}\left|t^{-\frac{1}{\alpha}}x\right|^{-(\alpha+d)}\leqslant p\left(1,0,t^{-\frac{1}{\alpha}}x\right)\leqslant\tilde{C}\left|t^{-\frac{1}{\alpha}}x\right|^{-(\alpha+d)},\quad\text{for large } t^{-\frac{1}{\alpha}}x.$$

Thus

$$\tilde{c}t^{-\frac{d}{\alpha}}\left|t^{-\frac{1}{\alpha}}x\right|^{-(\alpha+d)}\leqslant t^{-\frac{d}{\alpha}}p\left(1,0,t^{-\frac{1}{\alpha}}x\right)=p(t,0,x),\quad\text{for large } t^{-\frac{1}{\alpha}}x. \tag{1.3}$$

For small $t^{-\frac{1}{\alpha}}x$, since $p(1,0,x)$ is positive and continuous in $x\in\mathbb{R}^d$, there exists a positive constant $\tilde{c}_0$ such that

$$\tilde{c}_0\leqslant p\left(1,0,t^{-\frac{1}{\alpha}}x\right),$$

which implies

$$\tilde{c}_0t^{-\frac{d}{\alpha}}\leqslant t^{-\frac{d}{\alpha}}p\left(1,0,t^{-\frac{1}{\alpha}}x\right)=p(t,0,x),\quad\text{for small } t^{-\frac{1}{\alpha}}x. \tag{1.4}$$

Combining (1.2)–(1.4), we can see that the transition density p satisfies (1.1). It is clear that $C(x,y)=\phi(\frac{x-y}{|x-y|})|x-y|^{-(d+\alpha)}$ is bounded between two positive numbers.

Next we introduce the Kato class discussed in [2] and [3].

We say that a function V on $\mathbb{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} p(s,x,y)\big|V(y)\big|\,dy\,ds=0,$$

and we say that a signed measure μ on $\mathbb{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} p(s,x,y)|\mu|(dy)\,ds=0.$$

Suppose that F is a function on $\mathbb{R}^d\times\mathbb{R}^d$.

Definition 1.2. We say F belongs to $\mathbf{J}_{d,\alpha}$ if F is bounded, vanishing on the diagonal, and the function

$$x\mapsto\int_{\mathbb{R}^d}\frac{|F(x,y)|}{|x-y|^{d+\alpha}}\,dy$$

belongs to $\mathbf{K}_{d,\alpha}$.

For any $F\in\mathbf{J}_{d,\alpha}$, we set

$$A_t^F=\sum_{s\leqslant t}F(X_{s-},X_s),\quad t>0.$$

We can define a non-local Feynman–Kac semigroup as follows

$$S_t^F f(x)=\mathbb{E}_x\big(e^{-A_t^F}f(X_t)\big),$$

where f is a measurable function on $\mathbb{R}^d$. This semigroup was studied in [5] and [2].

Let $\tilde{F}(x,y)=F(y,x)$, for any $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$. In this paper, we always assume both F and $\tilde{F}\in\mathbf{J}_{d,\alpha}$.

Recently, sharp two-sided estimates of the density of the semigroup $\{S_t^F, t\geqslant 0\}$ were established in [6]. Under the assumption that X is a symmetric α-stable-like process, using a martingale argument and results from [2], the following result was established in [6]: Suppose that $F\in\mathbf{J}_{d,\alpha}$ is a symmetric function, the semigroup $\{S_t^F, t\geqslant 0\}$ admits a density $q(t,x,y)$ with respect to m and that q is jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Furthermore, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$C_1e^{-C_2t}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha}\leqslant q(t,x,y)\leqslant C_3e^{C_4t}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

The question that we are going to address in this paper is the following: can we establish the same two-sided estimates for the density of the Feynman–Kac semigroup of nonsymmetric α-stable-like process X when $F\in\mathbf{J}_{d,\alpha}$ is nonsymmetric. The proof of the above result in [6] cannot be adapted to the case where neither F nor X is symmetric. It seems that, to answer the question, one has to use some new ideas. In this paper, we are going to tackle the question above by combining the generalization of an idea of [1], which was used to deal with the estimates of the density of continuous functionals of Brownian motion, with some results on discontinuous additive functionals.

The content of this paper is organized as follows. In Section 2, we present some preliminary results on discontinuous additive functionals. In Section 3, we establish the two-sided estimates on the density of Feynman–Kac semigroup under certain assumptions of $F(x,y)$.

2. Preliminary results on discontinuous additive functionals

For convenience, we use A_t to denote $\sum_{s\leqslant t}F(X_{s-},X_s)$ instead of A_t^F. We have the following formulae for A_t^2:

$$A_t^2=2\int_0^t A_s\,dA_s-\int_0^t F(X_{s-},X_s)\,dA_s,$$

and

$$A_t^2 = 2\int_0^t (A_t - A_s)\,dA_s + \int_0^t F(X_{s-}, X_s)\,dA_s.$$

The proof is straightforward.

In general, the formulae for A_t^n are given by the following theorem.

Theorem 2.1.

$$\begin{aligned}A_t^n = {}& C_n^1 \int_0^t A_s^{n-1}\,dA_s - C_n^2 \int_0^t A_s^{n-2} F(X_{s-}, X_s)\,dA_s + C_n^3 \int_0^t A_s^{n-3} F^2(X_{s-}, X_s)\,dA_s + \cdots \\ & + (-1)^{i-1} C_n^i \int_0^t A_s^{n-i} F^{i-1}(X_{s-}, X_s)\,dA_s + \cdots + (-1)^{n-1} C_n^n \int_0^t F^{n-1}(X_{s-}, X_s)\,dA_s,\end{aligned}$$

$$\begin{aligned}A_t^n = {}& C_n^1 \int_0^t (A_t - A_s)^{n-1}\,dA_s + C_n^2 \int_0^t (A_t - A_s)^{n-2} F(X_{s-}, X_s)\,dA_s \\ & + C_n^3 \int_0^t (A_t - A_s)^{n-3} F^2(X_{s-}, X_s)\,dA_s + \cdots + C_n^i \int_0^t (A_t - A_s)^{n-i} F^{i-1}(X_{s-}, X_s)\,dA_s + \cdots \\ & + C_n^n \int_0^t F^{n-1}(X_{s-}, X_s)\,dA_s,\end{aligned}$$

where $C_n^i = \frac{n!}{i!(n-i)!}$.

Proof. We use induction to show these two formulae for A_t^n hold for all $n > 1$. It is clear that they are true for $n = 2$. Suppose they hold for $n \leqslant m - 1$, we show they hold for $n = m$.

It follows from the integration by parts formula,

$$A_t^m = \int_0^t A_{s-}\,dA_s^{m-1} + \int_0^t A_s^{m-1}\,dA_s$$

where

$$\begin{aligned}\int_0^t A_{s-}\,dA_s^{m-1} &= \int_0^t \bigl(A_s - F(X_{s-}, X_s)\bigr)\,dA_s^{m-1} \\ &= \int_0^t A_s\,dA_s^{m-1} - \int_0^t F(X_{s-}, X_s)\,dA_s^{m-1} \\ &= \int_0^t A_s \left(\sum_{i=1}^{m-1} (-1)^{i-1} C_{m-1}^i A_s^{m-1-i} F^{i-1}(X_{s-}, X_s)\right) dA_s \\ &\quad - \int_0^t F(X_{s-}, X_s) \left(\sum_{j=1}^{m-1} (-1)^{j-1} C_{m-1}^j A_s^{m-1-j} F^{j-1}(X_{s-}, X_s)\right) dA_s \\ &\quad \bigl(\text{by the first formula for } A_t^n \text{ when } n = m-1\bigr)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m-1} (-1)^{i-1} C_{m-1}^{i} \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s)\, dA_s \\
&\quad - \sum_{j=1}^{m-1} (-1)^{j-1} C_{m-1}^{j} \int_0^t A_s^{m-1-j} F^{j}(X_{s-}, X_s)\, dA_s \\
&= \sum_{i=1}^{m-1} (-1)^{i-1} C_{m-1}^{i} \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s)\, dA_s \\
&\quad - \sum_{i=2}^{m} (-1)^{i-2} C_{m-1}^{i-1} \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s)\, dA_s \\
&\quad (\text{let } j = i-1) \\
&= \sum_{i=2}^{m-1} (-1)^{i-1} \big(C_{m-1}^{i} + C_{m-1}^{i-1}\big) \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s)\, dA_s \\
&\quad + C_{m-1}^{1} \int_0^t A_s^{m-1}\, dA_s - (-1)^{m-2} \int_0^t F^{m-1}(X_{s-}, X_s)\, dA_s \\
&= \sum_{i=2}^{m-1} (-1)^{i-1} C_{m}^{i} \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s)\, dA_s \\
&\quad + C_{m-1}^{1} \int_0^t A_s^{m-1}\, dA_s - (-1)^{m} \int_0^t F^{m-1}(X_{s-}, X_s)\, dA_s \quad \big(\text{by } C_{m-1}^{i} + C_{m-1}^{i-1} = C_m^i\big),
\end{aligned}
$$

thus

$$
A_t^m = \int_0^t A_{s-}\, dA_s^{m-1} + \int_0^t A_s^{m-1}\, dA_s = \sum_{i=1}^{m} (-1)^{i-1} C_m^i \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s)\, dA_s,
$$

i.e. the first formula for A_t^n holds for $n = m$.

Now we go to the second formula for A_t^n, for $n = m$,

$$
\begin{aligned}
&C_m^1 \int_0^t (A_t - A_s)^{m-1}\, dA_s \\
&= C_m^1 \int_0^t \sum_{i=0}^{m-1} C_{m-1}^{i} A_t^i (-1)^{m-1-i} A_s^{m-1-i}\, dA_s = \sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^1 C_{m-1}^i A_t^i \int_0^t A_s^{m-1-i}\, dA_s \\
&= \sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^i (m-i) A_t^i \int_0^t A_s^{m-1-i}\, dA_s \quad \big(\text{by } C_m^1 C_{m-1}^i = C_m^i (m-i)\big) \\
&= \sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^i A_t^i \left((m-i) \int_0^t A_s^{m-1-i}\, dA_s \right)
\end{aligned}
$$

$$= \sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^i A_t^i \left(A_t^{m-i} + \int_0^t \sum_{k=2}^{m-i} (-1)^k C_{m-i}^k A_s^{m-i-k} F^{k-1}(X_{s-}, X_s)\, dA_s \right)$$

$\big($by the first formula for A_t^n when $n = m - i\big)$

$$= \sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^i A_t^i A_t^{m-i} + \int_0^t \sum_{i=0}^{m-1} (-1)^{m-1-i} \sum_{k=2}^{m-i} (-1)^k C_m^i C_{m-i}^k A_t^i A_s^{m-i-k} F^{k-1}(X_{s-}, X_s)\, dA_s,$$

where

$$\sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^i A_t^i A_t^{m-i} = \left(\sum_{i=0}^{m-1} (-1)^{m-1-i} C_m^i \right) A_t^m = (1) A_t^m,$$

and

$$\int_0^t \sum_{i=0}^{m-1} (-1)^{m-1-i} \sum_{k=2}^{m-i} (-1)^k C_m^i C_{m-i}^k A_t^i A_s^{m-i-k} F^{k-1}(X_{s-}, X_s)\, dA_s$$

$$= \int_0^t \sum_{k=2}^{m} \sum_{i=0}^{m-k} (-1)^{m-k-i-1} C_m^k C_{m-k}^i A_t^i A_s^{m-k-i} F^{k-1}(X_{s-}, X_s)\, dA_s$$

$\big($by $C_m^i C_{m-i}^k = C_m^k C_{m-k}^i$ and $(-1)^{m-1-i+k} = (-1)^{m-k-i-1}\big)$

$$= \sum_{k=2}^{m} C_m^k (-1)^{-1} \int_0^t (A_t - A_s)^{m-k} F^{k-1}(X_{s-}, X_s)\, dA_s$$

$$= -\sum_{k=2}^{m} C_m^k \int_0^t (A_t - A_s)^{m-k} F^{k-1}(X_{s-}, X_s)\, dA_s,$$

therefore

$$C_m^1 \int_0^t (A_t - A_s)^{m-1}\, dA_s = A_t^m - \sum_{k=2}^{m} C_m^k \int_0^t (A_t - A_s)^{m-k} F^{k-1}(X_{s-}, X_s)\, dA_s,$$

i.e. the second formula of A_t^n holds for $n = m$. □

3. Density of Feynman–Kac semigroups given by discontinuous additive functionals

From now on we define $q_0(t, x, y) = p(t, x, y)$ where $p(t, x, y)$ is the transition density of α-stable-like process X and satisfies (1.1). By the second formula for A_t^n, we have for any bounded measurable function g

$$\begin{aligned}
\mathbb{E}_x\big[A_t^n g(X_t)\big] &= \sum_{i=1}^{n} C_n^i \mathbb{E}_x\left[\int_0^t (A_t - A_s)^{n-i} F^{i-1}(X_{s-}, X_s) g(X_t)\, dA_s \right] \\
&= \sum_{i=1}^{n} C_n^i \mathbb{E}_x\left[\int_0^t \mathbb{E}_{X_s}\big(A_{t-s}^{n-i} g(X_{t-s})\big)\, d\left(\sum_{r \leqslant s} F^i(X_{r-}, X_r) \right) \right] \\
&= \sum_{i=1}^{n} C_n^i \mathbb{E}_x\left[\int_0^t \int_{\mathbb{R}^d} \frac{2C(X_s, y) F^i(X_s, y)}{|X_s - y|^{d+\alpha}} \mathbb{E}_y\big(A_{t-s}^{n-i} g(X_{t-s})\big) m(dy)\, ds \right].
\end{aligned}$$

We define $q_n(t,x,z)$ as follows,

$$q_n(t,x,z)=\sum_{i=1}^{n}C_n^i\int_0^t\int_{\mathbb{R}^d}p(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)m(dy)\,ds.$$

Then by induction, we can show that for any $n>0$,

$$\int_{\mathbb{R}^d}q_n(t,x,z)g(z)m(dz)=\mathbb{E}_x\left[A_t^n g(X_t)\right]$$

and

$$\begin{aligned}\mathbb{E}_x\left[A_t^n g(X_t)\right]&=\sum_{i=1}^{n}C_n^i\mathbb{E}_x\left[\int_0^t\int_{\mathbb{R}^d}\frac{2C(X_s,y)F^i(X_s,y)}{|X_s-y|^{d+\alpha}}\int_{\mathbb{R}^d}q_{n-i}(t-s,y,z)g(z)m(dz)m(dy)\,ds\right]\\&=\sum_{i=1}^{n}C_n^i\int_0^t\int_{\mathbb{R}^d}p(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\int_{\mathbb{R}^d}q_{n-i}(t-s,y,z)g(z)m(dz)m(dy)\,ds.\end{aligned}$$

We assume that there exist positive constants $\overline{C}$, L, M_0 and $\overline{M}$ such that $2C(x,y)\leqslant\overline{C}$, $|F(x,y)|\leqslant\frac{L}{2}$ and $0<M_0\leqslant M(y)\leqslant\overline{M}$ where $m(dy)=M(y)\,dy$. Define $\overline{F}(w,y)=|F(w,y)|+|F(y,w)|$, which is symmetric and satisfies $|\overline{F}(w,y)|\leqslant L$. Define $\overline{p}(t,x,y)=p(t,x,y)+p(t,y,x)$. Then $\overline{p}(t,x,y)$ is symmetric and satisfies

$$2M_1t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha}\leqslant\overline{p}(t,x,y)\leqslant 2M_2t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{d+\alpha},\quad\forall(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d.$$

Denote $(\int_{\mathbb{R}^d}\frac{|\overline{F}(w,y)|}{|w-y|^{d+\alpha}}\,dy)\,dw$ by $\mu(dw)$ and let $C_t=\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)\mu(dw)\,ds$. Then $C_t\downarrow 0$ as $t\downarrow 0$. It is clear that there exist two positive constants D_1 and D_2 such that $D_1\leqslant\int_{\mathbb{R}^d}\overline{p}(t,x,y)m(dy)\leqslant D_2$, as $\overline{p}(t,x,y)$ is comparable to $p(t,x,y)$. Let $\bar{q}_0(t,x,z)=\overline{p}(s,x,z)$ and define $\bar{q}_n(t,x,z)$ by

$$\bar{q}_n(t,x,z)=\sum_{i=1}^{n}C_n^i\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\bar{q}_{n-i}(t-s,y,z)m(dy)\,ds.$$

We can see that $|q_n(t,x,z)|\leqslant\bar{q}_n(t,x,z)$.

Before we move on to the main results, two lemmas are needed.

Lemma 3.1. *For any two positive constants $K<1$ and L, there exists a constant $C_0(K,L)$ which depends on K and L, such that*

$$K^{n-1}+K^{n-2}\frac{L}{2!}+K^{n-3}\frac{L^2}{3!}+\cdots+K^{n-i}\frac{L^{i-1}}{i!}+\cdots+\frac{L^{n-1}}{n!}\leqslant C_0(K,L)K^n,\quad\textit{for all } n>0. \tag{3.1}$$

Proof. Use the fact that there exists $i_0\geqslant 0$, such that

$$\frac{L^{l-1}}{l!}\leqslant\left(\frac{K}{2}\right)^l,\quad\text{for } l\geqslant i_0.\qquad\square$$

Remark 3.2. For any given K and L, we can choose a small t_0 so that for a given constant M_1, $C_tM_1C_0(K,L)\leqslant 1$ for $0\leqslant t\leqslant t_0$, where C_t is defined as $\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)\mu(dw)\,ds$ in above. Thus

$$C_tM_1\left(K^{n-1}+K^{n-2}\frac{L}{2!}+K^{n-3}\frac{L^2}{3!}+\cdots+\frac{L^{n-1}}{n!}\right)\leqslant C_tM_1C_0(K,L)K^n\leqslant K^n,\quad\text{for } 0\leqslant t\leqslant t_0.$$

Lemma 3.3. *$\bar{q}_n(t,x,y)$ is symmetric in x and y.*

Proof. We know that

$$
\begin{aligned}
\bar q_n(t,z,x) &= \sum_{i_1=1}^{n} C_n^{i_1} \int_0^t \int_{\mathbb{R}^d} \bar p(s_1,z,w_1)m(dw_1)\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^{i_1}(w_1,y_1)}{|w_1-y_1|^{d+\alpha}}\bar q_{n-i}(t-s_1,y_1,x)m(dy_1)\,ds_1\\
&= \sum_{i_1=1}^{n} C_n^{i_1} \sum_{i_2=1}^{n-i_1} C_{n-i_1}^{i_2}\Bigg[\int_0^t \int_{\mathbb{R}^d} \bar p(s_1,z,w_1)m(dw_1)\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^{i_1}(w_1,y_1)}{|w_1-y_1|^{d+\alpha}}m(dy_1)\,ds_1 \int_0^{t-s_1}\\
&\quad\cdot \int_{\mathbb{R}^d} \bar p(s_2,y_1,w_2)m(dw_2)\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^{i_2}(w_2,y_2)}{|w_2-y_2|^{d+\alpha}}\bar q_{n-i_1-i_2}(t-s_1-s_2,y_2,x)m(dy_2)\,ds_2\Bigg]\\
&\ \ \vdots\\
&= \sum_{i_1+i_2+\cdots+i_k=n} C_n^{i_1}C_{n-i_1}^{i_2}\cdot\cdots\cdot C_{n-i_1-i_2-\cdots-i_{k-1}}^{i_k}\Bigg[\int_0^t\int_0^{t-s_1}\cdots\int_0^{t-s_1-\cdots-s_{k-1}}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\\
&\quad\cdot \bar p(s_1,z,w_1)\frac{\overline{C}\,\overline{F}^{i_1}(w_1,y_1)}{|w_1-y_1|^{d+\alpha}}\bar p(s_2,y_1,w_2)\frac{\overline{C}\,\overline{F}^{i_2}(w_2,y_2)}{|w_2-y_2|^{d+\alpha}}\cdot\cdots\cdot \bar p(s_k,y_{k-1},w_k)\\
&\quad\cdot \frac{\overline{C}\,\overline{F}^{i_k}(w_k,y_k)}{|w_k-y_k|^{d+\alpha}}\bar p(t-s_1-\cdots-s_k,y_k,x)\,ds_1\cdots ds_k\,m(dw_1)\cdots m(dw_k)\cdots m(dy_1)\cdots m(dy_k)\Bigg]\\
&= \sum_{i_1+i_2+\cdots+i_k=n} C_n^{i_1,\dots,i_k}\Bigg[\int_0^t\int_0^{t-s_1}\cdots\int_0^{t-s_1-\cdots-s_{k-1}}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\bar p(s_1,z,w_1)\frac{\overline{C}\,\overline{F}^{i_1}(w_1,y_1)}{|w_1-y_1|^{d+\alpha}}\\
&\quad\cdot \bar p(s_2,y_1,w_2)\frac{\overline{C}\,\overline{F}^{i_2}(w_2,y_2)}{|w_2-y_2|^{d+\alpha}}\cdot\cdots\cdot \bar p(s_k,y_{k-1},w_k)\frac{\overline{C}\,\overline{F}^{i_k}(w_k,y_k)}{|w_k-y_k|^{d+\alpha}}\\
&\quad\cdot \bar p(t-s_1-\cdots-s_k,y_k,x)\,ds_1\cdots ds_k\,m(dw_1)\cdots m(dw_k)m(dy_1)\cdots m(dy_k)\Bigg].
\end{aligned}
$$

Put $t-s_1-\cdots-s_k=\tilde s_1, s_k=\tilde s_2,\dots,s_l=\tilde s_{k+2-l},\dots,s_2=\tilde s_k$. It is easy to see the absolute value of the Jacobian of this transformation is 1. Let $y_k=\tilde w_1,\dots,y_l=\tilde w_{k-l+1},\dots,y_1=\tilde w_k$, $w_k=\tilde y_1,\dots,w_l=\tilde y_{k-l+1},\dots,w_1=\tilde y_k$ and $j_k=i_1,\dots,j_l=i_{k-l+1},\dots,j_1=i_k$.

Thus the above equality becomes

$$
\begin{aligned}
\bar q_n(t,z,x) = \sum_{j_1+j_2+\cdots+j_k=n} C_n^{j_1,\dots,j_k}\Bigg[&\int_0^{t-\tilde s_1-\cdots-\tilde s_{k-1}}\int_0^{t-\tilde s_1-\cdots-\tilde s_{k-2}}\cdots\int_0^t\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\\
&\cdot \bar p(t-\tilde s_1-\cdots-\tilde s_k,z,\tilde y_k)\frac{\overline{C}\,\overline{F}^{j_k}(\tilde y_k,\tilde w_k)}{|\tilde y_k-\tilde w_k|^{d+\alpha}}\bar p(\tilde s_k,\tilde w_k,\tilde y_{k-1})\\
&\cdot \frac{\overline{C}\,\overline{F}^{j_{k-1}}(\tilde y_{k-1},\tilde w_{k-1})}{|\tilde y_{k-1}-\tilde w_{k-1}|^{d+\alpha}}\cdot\cdots\cdot \bar p(\tilde s_2,\tilde w_2,\tilde y_1)\frac{\overline{C}\,\overline{F}^{j_1}(\tilde y_1,\tilde w_1)}{|\tilde y_1-\tilde w_1|^{d+\alpha}}\\
&\cdot \bar p(\tilde s_1,\tilde w_1,x)\,d\tilde s_k\cdots d\tilde s_1\,m(d\tilde y_k)\cdots m(d\tilde y_1)m(d\tilde w_k)\cdots m(d\tilde w_1)\Bigg].
\end{aligned}
$$

Rearranging the components of the integrand and using the fact that $\overline{F}(x,y)$ and $\bar p(t,x,y)$ are symmetric in x and y, it is easy to see that the above expression for $\bar q_n(t,z,x)$ is equal to $\bar q_n(t,x,z)$. $\Box$

In the proof of the following theorem, we use an idea similar to that used in [1] for Brownian motions.

Theorem 3.4. *There exist two positive constants $K < 1$ and M, and there exists $t_1 > 0$ such that for $0 < t \leqslant t_1$,*

$$\overline{q}_n(t, x, z) \leqslant n! M K^n t^{-\frac{d}{\alpha}}, \quad \textit{for all } n, \tag{3.2}$$

and

$$\int_0^t \int_{\mathbb{R}^d} \overline{q}_n(s, x, z)\mu(dz)\, ds \leqslant C_t n! K^n, \quad \textit{for all } n. \tag{3.3}$$

Proof. It is clear that when $n = 0$, (3.2) and (3.3) hold. We assume they hold for $n \leqslant m - 1$, and consider the case $n = m$. Writing $\overline{q}_m(t, x, y)$ into two terms in the following way:

$$\begin{aligned}
\overline{q}_m(t, x, z) = {} & \sum_{i=1}^m C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy)\, ds \\
& + \sum_{i=1}^m C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy)\, ds.
\end{aligned}$$

Since (3.2) and (3.3) hold for $n \leqslant m - 1$, we have

$$\begin{aligned}
& \sum_{i=1}^m C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy)\, ds \\
& \quad \leqslant \sum_{i=1}^m C_m^i M \overline{M}^2 \overline{C} L^{i-1} (m - i)! K^{m-i} \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s, x, w)\, \mu(dw)\, ds \\
& \quad \leqslant \sum_{i=1}^m C_t C_m^i M \overline{M}^2 \overline{C} L^{i-1} (m - i)! K^{m-i} \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \sum_{i=1}^m C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy)\, ds \\
& \quad \leqslant \sum_{i=1}^m C_m^i M \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} m(dw) \overline{q}_{m-i}(t - s, y, z) m(dy)\, ds \\
& \quad \leqslant \sum_{i=1}^m C_m^i M \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \overline{C} L^{i-1} \overline{M}^2 \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{q}_{m-i}(t - s, y, z) \mu(dy)\, ds \\
& \quad \leqslant \sum_{i=1}^m C_m^i M \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \overline{C} L^{i-1} \overline{M}^2 C_t (m - i)! K^{m-i} \\
& \qquad \big(\text{by symmetry, } \overline{q}_{m-i}(t - s, y, z) = \overline{q}_{m-i}(t - s, z, y) \text{ and (3.3)}\big) \\
& \quad = \sum_{i=1}^m C_t C_m^i M \overline{M}^2 \overline{C} L^{i-1} (m - i)! K^{m-i} \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}.
\end{aligned}$$

Therefore,

$$\bar{q}_m(t,x,z) \leqslant \sum_{i=1}^{m} C_t C_m^i 2^{1+\frac{d}{\alpha}} M\overline{M}^2\overline{C}L^{i-1}(m-i)!K^{m-i}t^{-\frac{d}{\alpha}} = m!2^{1+\frac{d}{\alpha}}M\overline{M}^2\overline{C}t^{-\frac{d}{\alpha}}C_t\left(\sum_{i=1}^{m} K^{m-i}\frac{L^{i-1}}{i!}\right).$$

Let $M_1 = 2^{1+\frac{d}{\alpha}}\overline{M}^2\overline{C}$. Then by Remark 3.2, we can choose a small t_1 such that for $0 < t \leqslant t_1$,

$$C_t M_1\left(\sum_{i=1}^{m} K^{m-i}\frac{L^{i-1}}{i!}\right) \leqslant K^m, \quad \text{for any } m.$$

Thus

$$\bar{q}_m(t,x,y) \leqslant m!MK^m t^{-\frac{d}{\alpha}},$$

i.e. (3.2) holds for $n = m$.

Now we show (3.3) holds for $n = m$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \bar{q}_m(s,x,z)\mu(dz)\,ds \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \left(\sum_{i=1}^{m} C_m^i \int_0^s \int_{\mathbb{R}^d} \bar{p}(u,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\bar{q}_{m-i}(s-u,y,z)m(dy)\,du\right)\mu(dz)\,ds.
\end{aligned}$$

Let $s - u = v$, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \bar{q}_m(s,x,z)\mu(dz)\,ds \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \left(\sum_{i=1}^{m} C_m^i \int_0^{t-u} \int_{\mathbb{R}^d} \bar{p}(u,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\bar{q}_{m-i}(v,y,z)\mu(dz)\,dv\right)m(dy)\,du \\
&\quad = \sum_{i=1}^{m} C_m^i \int_0^t \int_{\mathbb{R}^d} \bar{p}(u,x,w)\left(\int_0^{t-u} \int_{\mathbb{R}^d} \bar{q}_{m-i}(v,y,z)\mu(dz)\,dv\right) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dy)m(dw)\,du \\
&\quad \leqslant \sum_{i=1}^{m} C_m^i \int_0^t \int_{\mathbb{R}^d} \bar{p}(u,x,w)C_t(m-i)!K^{m-i} \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dy)m(dw)\,du \\
&\quad = \sum_{i=1}^{m} C_m^i C_t(m-i)!K^{m-i} \int_0^t \int_{\mathbb{R}^d} \bar{p}(u,x,w) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dy)m(dw)\,du \\
&\quad \leqslant \sum_{i=1}^{m} C_m^i C_t(m-i)!K^{m-i}\overline{C}L^{i-1}\overline{M}^2 \int_0^t \int_{\mathbb{R}^d} \bar{p}(u,x,w)\mu(dw)\,du \\
&\quad \leqslant C_t C_t \overline{M}^2\overline{C}m!\left(\sum_{i=1}^{m} K^{m-i}\frac{L^{i-1}}{i!}\right).
\end{aligned}$$

It is clear that for the previous t_1, when $0 < t \leqslant t_1$,

$$C_t\overline{M}^2\overline{C}\left(\sum_{i=1}^{m} K^{m-i}\frac{L^{i-1}}{i!}\right) \leqslant K^m.$$

Thus

$$C_tC_t\overline{M}^2\overline{C}m!\left(\sum_{i=1}^{m}K^{m-i}\frac{L^{i-1}}{i!}\right)\leqslant C_tm!K^m,$$

i.e.

$$\int_0^t\int_{\mathbb{R}^d}\overline{q}_m(s,x,z)\mu(dz)\,ds\leqslant C_tm!K^m.$$

Therefore (3.3) holds for $n=m$. $\square$

Next we obtain a better upper bound of $\overline{q}_n(t,x,z)$.

It is clear that for any $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, there exists a positive constant D_2 such that $\int_{R^d}\overline{p}(t,x,y)m(dy)\leqslant D_2$.

We can also see that for the positive constants L and $K<1$ given in Remark 3.2 and Theorem 3.4, and $\overline{M}$, $\overline{C}$, which are the upper bounds for $M(y)$ and $2C(x,y)$ respectively, there exists a constant $\tilde{C}\geqslant 1$ such that

$$L^{n-1}\overline{M}^2\overline{C}\leqslant\tilde{C}\frac{1}{2}n!K^n,\quad\forall n\geqslant 1.$$

Suppose that $g\geqslant 0$ is a bounded measurable function and $g\leqslant C_g\min(\frac{1}{D_2},1)$, where $C_g\geqslant 1$ is a constant, then we have the following:

Proposition 3.5. *There exists $t_2\geqslant 0$ such that when $0<t\leqslant t_2$,*

$$\int_{\mathbb{R}^d}\overline{q}_n(t,x,z)g(z)m(dz)\leqslant\tilde{C}C_gC_tn!K^n,\quad\forall n\geqslant 1.\tag{3.4}$$

Proof. When $n=1$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\overline{q}_1(t,x,z)g(z)m(dz)\\
&\quad=\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}}\overline{p}(t-s,y,z)m(dy)\,ds\,g(z)m(dz)\\
&\quad\leqslant\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}}C_g\overline{M}\,dy\,ds\quad\left(\text{by}\int_{\mathbb{R}^d}\overline{p}(t-s,y,z)g(z)m(dz)\leqslant C_g\right)\\
&\quad\leqslant\overline{M}^2\overline{C}C_gC_t\leqslant\tilde{C}\frac{1}{2}KC_gC_t\leqslant\tilde{C}C_gC_tK.
\end{aligned}$$

Thus (3.4) holds for $n=1$.

Suppose it holds for $n\leqslant m-1$, we show that it holds for $n=m$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\overline{q}_m(t,x,z)g(z)m(dz)\\
&\quad=\int_{\mathbb{R}^d}\sum_{i=1}^{m}C_m^i\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\overline{q}_{m-i}(t-s,y,z)m(dy)\,ds\,g(z)m(dz)\\
&\quad=\sum_{i=1}^{m}C_m^i\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\int_{\mathbb{R}^d}\overline{q}_{m-i}(t-s,y,z)g(z)m(dz)m(dy)\,ds
\end{aligned}$$

$$\leqslant \sum_{i=1}^{m-1} C_m^i \int_0^t \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} m(dy)\,ds\, \tilde{C} C_g C_t (m-i)! K^{m-i}$$

$$+ \int_0^t \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^m(w,y)}{|w-y|^{d+\alpha}} m(dy)\,ds\, C_g$$

$$= \sum_{i=1}^{m-1} C_m^i C_t L^{i-1} \overline{M}^2 \overline{C} \tilde{C} C_g C_t (m-i)! K^{m-i} + L^{m-1} \overline{M}^2 \overline{C} C_g C_t$$

$$= m! \sum_{i=1}^{m-1} \left(\frac{L^{i-1} K^{m-i}}{i!} \right) C_t \overline{M}^2 \overline{C} \tilde{C} C_g C_t + L^{m-1} \overline{M}^2 \overline{C} C_g C_t.$$

Since $C_t \downarrow 0$ as $t \downarrow 0$, $\exists t_2 \geqslant 0$ such that when $0 < t \leqslant t_2$

$$\sum_{i=1}^{n-1} \left(\frac{L^{i-1} K^{n-i}}{i!} \right) C_t \overline{M}^2 \overline{C} \leqslant \frac{1}{2} K^n, \quad \forall n \geqslant 2,$$

by the choice of $\tilde{C}$,

$$L^{n-1} \overline{M}^2 \overline{C} \leqslant \tilde{C} \frac{1}{2} n! K^n, \quad \forall n \geqslant 1.$$

Thus

$$\int_{\mathbb{R}^d} \overline{q}_m(t,x,z) g(z)\,dz \leqslant \frac{1}{2} m! K^m \tilde{C} C_g C_t + \frac{1}{2} m! K^m \tilde{C} C_g C_t = \tilde{C} C_g C_t m! K^m,$$

i.e. the statement holds for $n = m$. $\square$

For the L, K, $\overline{C}$ and D_2 given above, it is clear that there exists $\tilde{C}_2 \geqslant 1$ such that

$$\frac{L^n}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha}} \frac{\overline{C} D_2^2}{2} \leqslant \frac{1}{8} \tilde{C}_2 n! K^n, \quad \forall n \geqslant 0.$$

We claim that

Theorem 3.6. *There exist $t_3 > 0$ and $\tilde{C}_1 \geqslant 1$ such that when $0 < t \leqslant t_3$,*

$$\overline{q}_n(t,x,z) \leqslant \tilde{C}_1 n! K^n t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|} \right)^{d+\alpha}, \quad \forall n \geqslant 0. \tag{3.5}$$

Proof. Since $\overline{q}_0(t,x,z) = \overline{p}(t,x,z)$, there exist $t_{13} > 0$ and $\tilde{C}_1 \geqslant \tilde{C}_2$ such that when $0 < t \leqslant t_{13}$,

$$\overline{q}_0(t,x,z) \leqslant \tilde{C}_1 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|} \right)^{d+\alpha},$$

i.e. the statement holds for $n = 0$. Suppose it is true for $n \leqslant m - 1$. We show that it holds for $n = m$. We write $\overline{q}_m(t,x,z)$ into two terms

$$\overline{q}_m(t,x,z) = \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z) m(dy)\,ds$$

$$+ \sum_{i=1}^{m} C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z) m(dy)\,ds.$$

First we look at the first term. There are two cases:

Case 1. When $|x-z| \leqslant t^{\frac{1}{\alpha}}$,

$$\begin{aligned}
&\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \\
&\quad \leqslant \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}}\,dy\,ds\, L^{i-1}\overline{M}\tilde{C}_1(m-i)!K^{m-i}\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \\
&\quad \leqslant \sum_{i=1}^{m} C_m^i C_t \overline{M}^2 \overline{C} L^{i-1}\tilde{C}_1(m-i)!K^{m-i}\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} = m! \sum_{i=1}^{m}\left(\frac{L^{i-1}K^{m-i}}{i!}\right)\overline{M}^2\overline{C}\left(\frac{1}{2}\right)^{-\frac{d}{\alpha}} C_t\tilde{C}_1 t^{-\frac{d}{\alpha}}.
\end{aligned}$$

Since there exists $t_{23} > 0$ and $t_{23} \leqslant t_{13}$ such that when $0 < t \leqslant t_{23}$,

$$\sum_{i=1}^{n}\left(\frac{L^{i-1}K^{n-i}}{i!}\right)\overline{M}^2\overline{C}\left(\frac{1}{2}\right)^{-\frac{d}{\alpha}} C_t \leqslant \frac{1}{2}K^n, \quad \forall n \geqslant 1,$$

thus in Case 1, when $0 < t \leqslant t_{23}$,

$$\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \leqslant \frac{1}{2}\tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}}.$$

Case 2. When $|x-z| \geqslant t^{\frac{1}{\alpha}}$. Let $B_1 = \{y \in \mathbb{R}^d \mid |y-z| \geqslant \frac{1}{10}|x-z|\}$, $B_2 = \{w \in \mathbb{R}^d \mid |w-x| \geqslant 2^{-\frac{1}{2}}|x-z|\}$ and $B_3 = \{(w,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |y-z| < \frac{1}{10}|x-z|,\ |w-x| < 2^{-\frac{1}{2}}|x-z|\}$. On B_3, we have $|w-y| \geqslant (1-\frac{1}{10}-2^{-\frac{1}{2}})|x-z|$,

$$\begin{aligned}
&\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \\
&\quad \leqslant \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)1_{B_1}(y)m(dy)\,ds \\
&\qquad + \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)1_{B_2}(w)m(dy)\,ds \\
&\qquad + \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)1_{B_3}(w,y)m(dy)\,ds \\
&\quad \leqslant \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}} \overline{M}L^{i-1}\tilde{C}_1(m-i)!K^{m-i}10^{d+\alpha}\frac{(t-s)}{|x-z|^{d+\alpha}}\,dy\,ds \\
&\qquad + \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} m(dw)\tilde{C}_1\overline{M}^2 2^{\frac{1}{2}(d+\alpha)}\frac{s}{|x-z|^{d+\alpha}} \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)\,dy\,ds\, L^{i-1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1} C_m^i \frac{L^i}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{1}{|x-z|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z) \\
&\cdot m(dy)\,ds + \frac{L^m}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} D_2^2 \frac{\frac{t}{2}}{|x-z|^{d+\alpha}} \\
\leqslant{}& \sum_{i=1}^{m} C_m^i C_t \overline{M}^2 \overline{C} L^{i-1} \tilde{C}_1 (m-i)! K^{m-i} 10^{d+\alpha} \frac{t}{|x-z|^{d+\alpha}} \\
&+\sum_{i=1}^{m} C_m^i \tilde{C}_1 \overline{M}^2 \overline{C} 2^{\frac{1}{2}(d+\alpha)} \frac{t}{|x-z|^{d+\alpha}} C_t (m-i)! K^{m-i} L^{i-1} \\
&\bigl(\text{by symmetry of } \overline{q}_{m-i}(t-s,y,z) \text{ and Theorem 3.4}\bigr) \\
&+\sum_{i=1}^{m-1} C_m^i \frac{L^i}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} D_2 \frac{t}{|x-z|^{d+\alpha}} \tilde{C} C_g C_t (m-i)! K^{m-i} \\
&\left(\text{by symmetry of } \overline{q}_{m-i}(t-s,y,z),\ \text{Proposition 3.5 and} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \leqslant D_2\right) \\
&+\frac{L^m}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} D_2^2 \frac{\frac{t}{2}}{|x-z|^{d+\alpha}}.
\end{aligned}
$$

It is easy to see that there exists $t_{33} > 0$ and $t_{33} \leqslant \min(t_1, t_2)$ such that when $0 < t \leqslant t_{33}$, the first three terms in above inequality $\leqslant \frac{1}{4}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}$, for all $m > 0$. We can also have the fourth term in the above inequality $\leqslant \frac{1}{8}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}$, for all $m > 0$, by the choice of $\tilde{C}_2$ and $\tilde{C}_1 \geqslant \tilde{C}_2$. Thus in Case 2 when $0 < t \leqslant t_{33}$,

$$
\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \leqslant \frac{1}{2}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}.
$$

Combining Cases 1 and 2, when $0 < t \leqslant \min(t_{23}, t_{33})$,

$$
\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \leqslant \frac{1}{2}\tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha}.
$$

For the second term in the expression of $\overline{q}_m(t,x,z)$:

$$
\sum_{i=1}^{m} C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds.
$$

Letting $t - s = \tilde{s}$, the second term becomes

$$
\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)m(dy)\,d\tilde{s}.
$$

There are two cases.

Case (a). When $|x-z| \leqslant t^{\frac{1}{\alpha}}$,

$$\begin{aligned}
&\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)m(dy)\,d\tilde{s} \\
&\quad\leqslant \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)m(dy)\,d\tilde{s} \\
&\quad\leqslant \sum_{i=1}^{m} C_m^i \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} L^{i-1}\overline{M}^2\overline{C} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{q}_{m-i}(\tilde{s},y,z)\mu(dy)\,d\tilde{s} \\
&\quad\leqslant \sum_{i=1}^{m} C_m^i \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} L^{i-1}\overline{M}^2\overline{C} C_t (m-i)! K^{m-i} \quad \bigl(\text{by symmetry of } \overline{q}_{m-i}(\tilde{s},y,z) \text{ and Theorem 3.4}\bigr) \\
&\quad= m! \sum_{i=1}^{m} \left(\frac{L^{i-1}K^{m-i}}{i!}\right) \overline{M}^2\overline{C} \left(\frac{1}{2}\right)^{-\frac{d}{\alpha}} C_t \tilde{C}_1 t^{-\frac{d}{\alpha}}.
\end{aligned}$$

Since there exists $\tilde{t}_{23} > 0$ and $\tilde{t}_{23} \leqslant \min(t_1, t_{13})$ such that when $0 < t \leqslant \tilde{t}_{23}$,

$$\sum_{i=1}^{n} \left(\frac{L^{i-1}K^{n-i}}{i!}\right) \overline{M}^2\overline{C} \left(\frac{1}{2}\right)^{-\frac{d}{\alpha}} C_t \leqslant \frac{1}{2} K^n, \quad \forall n \geqslant 1,$$

we have in Case (a) when $0 < t \leqslant \tilde{t}_{23}$,

$$\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)m(dy)\,d\tilde{s} \leqslant \frac{1}{2}\tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}},$$

i.e.

$$\sum_{i=1}^{m} C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \leqslant \frac{1}{2}\tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}}.$$

Case (b). When $|x-z| \geqslant t^{\frac{1}{\alpha}}$. Let $\tilde{B}_1 = \{y \in \mathbb{R}^d \mid |y-z| \geqslant \frac{1}{10}|x-z|\}$, $\tilde{B}_2 = \{w \in \mathbb{R}^d \mid |w-x| \geqslant 2^{-\frac{1}{2}}|x-z|\}$ and $\tilde{B}_3 = \{(w,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid |y-z| < \frac{1}{10}|x-z|,\ |w-x| < 2^{-\frac{1}{2}}|x-z|\}$. On $\tilde{B}_3$, we have $|w-y| \geqslant (1-\frac{1}{10}-2^{-\frac{1}{2}})|x-z|$,

$$\begin{aligned}
&\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)m(dy)\,d\tilde{s} \\
&\quad\leqslant \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)1_{\tilde{B}_1}(y)m(dy)\,d\tilde{s} \\
&\qquad+ \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)1_{\tilde{B}_2}(w)m(dy)\,d\tilde{s} \\
&\qquad+ \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)1_{\tilde{B}_3}(w,y)m(dy)\,d\tilde{s}
\end{aligned}$$

$$\leqslant \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}} \overline{M} L^{i-1} \tilde{C}_1 (m-i)! K^{m-i} \frac{10^{d+\alpha}\tilde{s}}{|x-z|^{d+\alpha}}\,dy\,d\tilde{s}$$

$$+ \sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} m(dw) \tilde{C}_1 \overline{M} 2^{\frac{1}{2}(d+\alpha)} \frac{(t-\tilde{s})}{|x-z|^{d+\alpha}} \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)\,dy\,ds\,L^{i-1}$$

$$+ \sum_{i=1}^{m-1} C_m^i \frac{L^i}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{1}{|x-z|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)$$

$$\cdot\, m(dy)\,d\tilde{s} + \frac{L^m}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} D_2^2 \frac{\frac{t}{2}}{|x-z|^{d+\alpha}}$$

$$\leqslant \sum_{i=1}^{m} C_m^i C_t \overline{M}^2 \overline{C} L^{i-1} \tilde{C}_1 (m-i)! K^{m-i} 10^{d+\alpha} \frac{t}{|x-z|^{d+\alpha}}$$

$$+ \sum_{i=1}^{m} C_m^i \tilde{C}_1 \overline{M}^2 \overline{C} 2^{\frac{1}{2}(d+\alpha)} \frac{t}{|x-z|^{d+\alpha}} C_t (m-i)! K^{m-i} L^{i-1}$$

$\big($by symmetry of $\overline{q}_{m-i}(\tilde{s},y,z)$ and Theorem 3.4$\big)$

$$+ \sum_{i=1}^{m-1} C_m^i \frac{L^i}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} D \frac{t}{|x-z|^{d+\alpha}} \tilde{C} C_g C_t (m-i)! K^{m-i}$$

$\left(\text{by symmetry of } \overline{q}_{m-i}(\tilde{s},y,z),\ \text{Proposition 3.5 and } \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \leqslant D_2\right)$

$$+ \frac{L^m}{(1-\frac{1}{10}-2^{-\frac{1}{2}})^{d+\alpha}} \overline{C} D_2^2 \frac{\frac{t}{2}}{|x-z|^{d+\alpha}}.$$

It is easy to see that there exists $\tilde{t}_{33} > 0$ and $\tilde{t}_{33} \leqslant \min(t_1, t_2)$ such that when $0 < t \leqslant \tilde{t}_{33}$, the first three terms in above inequality $\leqslant \frac{1}{4}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}$, for any $m > 0$. We can also have the fourth term in the above inequality $\leqslant \frac{1}{8}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}$, for any $m > 0$, by the choice of $\tilde{C}_2$ and $\tilde{C}_1 \geqslant \tilde{C}_2$. Thus in Case (b) when $0 < t \leqslant \tilde{t}_{33}$,

$$\sum_{i=1}^{m} C_m^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(t-\tilde{s},x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(\tilde{s},y,z)m(dy)\,d\tilde{s} \leqslant \frac{1}{2}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}},$$

i.e.

$$\sum_{i=1}^{m} C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \leqslant \frac{1}{2}\tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}.$$

Combining Cases (a) and (b), when $0 < t \leqslant \min(\tilde{t}_{23}, \tilde{t}_{33})$,

$$\sum_{i=1}^{m} C_m^i \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^d} \overline{p}(s,x,w)m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{m-i}(t-s,y,z)m(dy)\,ds \leqslant \frac{1}{2}\tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha}.$$

Therefore when $0 < t < t_3 = \min(t_{23}, t_{33}, \tilde{t}_{23}, \tilde{t}_{33})$,

$$\bar{q}_m(t,x,z) \leqslant \tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha},$$

i.e. the statement holds for $n = m$. □

By the above theorem, we have for $0 < t \leqslant t_3$,

$$\sum_{n=0}^{\infty} \frac{\bar{q}_n(t,x,z)}{n!} \leqslant \sum_{n=0}^{\infty} \tilde{C}_1 K^n t^{-\frac{d}{\alpha}} = \tilde{C}_1 \frac{1}{1-K} t^{-\frac{d}{\alpha}}.$$

Since $|q_n(t,x,z)| \leqslant \bar{q}_n(t,x,z)$, $\sum_{n=0}^{\infty}(-1)^n \frac{q_n(t,x,z)}{n!}$ is uniformly convergent on $[\epsilon, t_3] \times \mathbb{R}^d \times \mathbb{R}^d$, for any $\epsilon > 0$. Let $q(t,x,z) = \sum_{n=0}^{\infty}(-1)^n \frac{q_n(t,x,z)}{n!}$. Then $q(t,x,z)$ is well defined on $(0, t_3] \times \mathbb{R}^d \times \mathbb{R}^d$.

Next we show that $q(t,x,z)$ is joint continuous on $(0,t_3) \times \mathbb{R}^d \times \mathbb{R}^d$.

Since $q(t,x,z) = \sum_{n=0}^{\infty}(-1)^n \frac{q_n(t,x,z)}{n!}$ is uniformly convergent on $[\epsilon, t_3] \times \mathbb{R}^d \times \mathbb{R}^d$ for any $\epsilon > 0$, to show $q(t',x',z') \to q(t,x,z)$ as $(t',x',z') \to (t,x,z) \in (0,t_3) \times \mathbb{R}^d \times \mathbb{R}^d$, it is sufficient to show $q_n(t',x',z') \to q_n(t,x,z)$, i.e. $q_n(t,x,z)$ is joint continuous at (t,x,z).

To obtain joint continuity, the following lemma is needed.

Lemma 3.7. *For any (s,x,w) and $(s,x',w) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,*

1. *if $|x'-w| \geqslant \frac{1}{2}|x-w|$ or $|x'-w| < \frac{1}{2}|x-w|$ with $\frac{1}{2}|x-w| \leqslant s^{\frac{1}{\alpha}}$, then there exists some constant K_1 such that $p(s,x',w) \leqslant K_1 p(s,x,w)$,*
2. *if $|x'-w| < \frac{1}{2}|x-w|$ with $\frac{1}{2}|x-w| > s^{\frac{1}{\alpha}}$, then $|x-x'|^\alpha > s$,*

where $p(t,x,w)$ is the transition density of the α-stable-like process X.

Proof. There are three cases.

Case 1. When $|x'-w| \geqslant \frac{1}{2}|x-w|$,

$$1 \wedge \frac{s^{\frac{1}{\alpha}}}{|x'-w|} \leqslant 1 \wedge \frac{2s^{\frac{1}{\alpha}}}{|x-w|} \leqslant 2\left(1 \wedge \frac{s^{\frac{1}{\alpha}}}{|x-w|}\right).$$

Combining this with (1.1), there exists a positive constant K_{11} such that $p(s,x',w) \leqslant K_{11} p(s,x,w)$ in this case.

Case 2. When $|x'-w| < \frac{1}{2}|x-w|$ with $\frac{1}{2}|x-w| \leqslant s^{\frac{1}{\alpha}}$,

$$\frac{s^{\frac{1}{\alpha}}}{|x-w|} \geqslant \frac{1}{2} \quad \text{and} \quad \frac{s^{\frac{1}{\alpha}}}{|x'-w|} \geqslant 1.$$

This implies that

$$\frac{1}{2} \leqslant 1 \wedge \frac{s^{\frac{1}{\alpha}}}{|x-w|} \leqslant 1 \quad \text{and} \quad 1 \wedge \frac{s^{\frac{1}{\alpha}}}{|x'-w|} = 1.$$

Combining this with (1.1), there exists a positive constant K_{12} such that $p(s,x',w) \leqslant K_{12} p(s,x,w)$ in this case.

Case 3. When $|x'-w| < \frac{1}{2}|x-w|$ with $\frac{1}{2}|x-w| > s^{\frac{1}{\alpha}}$,

$$|x-x'| \geqslant \frac{1}{2}|x-w| \geqslant s^{\frac{1}{\alpha}},$$

because if $|x-x'| < \frac{1}{2}|x-w|$, then $|x-x'| + |x'-w| < |x-w|$, which contradicts the triangle inequality $|x-x'| + |x'-w| \geqslant |x-w|$. Thus $|x-x'| \geqslant s^{\frac{1}{\alpha}}$, i.e. $|x-x'|^\alpha > s$ in this case.

Define $K_1 = \max(K_{11}, K_{12})$. Then the statement holds. □

Next we show the following property holds.

Proposition 3.8. *$q_n(t,x,z)$ is joint continuous on $(0,t_3)\times\mathbb{R}^d\times\mathbb{R}^d$, for all $n\geqslant 0$.*

Proof. We use induction.

Since $p(t,x,z)$ is joint continuous and $q_0(t,x,z)=p(t,x,z)$, the statement holds for $n=0$.

We assume $q_i(t,x,z)$ is joint continuous on $(0,t_3)\times\mathbb{R}^d\times\mathbb{R}^d$, for all $i\leqslant n-1$. We want to show $q_n(t,x,z)$ is joint continuous on $(0,t_3)\times\mathbb{R}^d\times\mathbb{R}^d$.

Recall the definition of $q_n(t,x,z)$:

$$q_n(t,x,z)=\sum_{i=1}^n C_n^i\int_0^t\int_{\mathbb{R}^d} p(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)m(dy)\,ds.$$

Fix (t,x,z) in $(0,t_3)\times\mathbb{R}^d\times\mathbb{R}^d$. Let $(t',x',z')\to(t,x,z)$. Since $t'\to t$ and $t<t_3$, we can assume $\frac{2}{3}t<t'<t_3$. The difference between $q_n(t,x,z)$ and $q_n(t',x',z')$,

$$\begin{aligned}
&q_n(t,x,z)-q_n(t',x',z')\\
&\quad=\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d} p(s,x,w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)1_{\{s\leqslant t\}}m(dy)\,ds\\
&\qquad-\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t'-s,y,z')1_{\{s\leqslant t'\}}m(dy)\,ds\\
&\quad=\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d}\big(p(s,x,w)-p(s,x',w)\big)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)1_{\{s\leqslant t\}}m(dy)\,ds\\
&\qquad+\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)(1_{\{s\leqslant t\}}-1_{\{s\leqslant t\wedge t'\}})m(dy)\,ds\\
&\qquad+\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\big(q_{n-i}(t-s,y,z)-q_{n-i}(t'-s,y,z)\big)\\
&\qquad\cdot 1_{\{s\leqslant t\wedge t'\}}m(dy)\,ds\\
&\qquad+\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\big(q_{n-i}(t'-s,y,z)-q_{n-i}(t'-s,y,z')\big)\\
&\qquad\cdot 1_{\{s\leqslant t\wedge t'\}}m(dy)\,ds\\
&\qquad+\sum_{i=1}^n C_n^i\int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t'-s,y,z')(1_{\{s\leqslant t\wedge t'\}}-1_{\{s\leqslant t'\}})m(dy)\,ds.
\end{aligned}$$

In the following, we check the convergence of all the five terms in the above expression.

For convenience, we need to define some sets.

For given x and x' in $\mathbb{R}^d$, define A as the set $\{(s,w)\in(0,t_3)\times\mathbb{R}^d\mid |x'-w|\geqslant\frac{1}{2}|x-w|$, or $|x'-w|<\frac{1}{2}|x-w|$ with $\frac{1}{2}|x-w|\leqslant s^{\frac{1}{\alpha}}\}$.

For given x and x' in $\mathbb{R}^d$, define B as the set $\{(s,w)\in(0,t_3)\times\mathbb{R}^d\mid |x'-w|<\frac{1}{2}|x-w|$ with $\frac{1}{2}|x-w|>s^{\frac{1}{\alpha}}\}$.

For given z and z' in $\mathbb{R}^d$, define $\tilde{A}$ as the set $\{(\tilde{s}, y) \in (0, t_3) \times \mathbb{R}^d \mid |z' - y| \geqslant \frac{1}{2}|z - y|,\ \text{or } |z' - y| < \frac{1}{2}|z - y| \text{ with } \frac{1}{2}|z - y| \leqslant s^{\frac{1}{\alpha}}\}$.

For given z and z' in $\mathbb{R}^d$, define $\tilde{B}$ as the set $\{(\tilde{s}, y) \in (0, t_3) \times \mathbb{R}^d \mid |z' - y| < \frac{1}{2}|z - y| \text{ with } \frac{1}{2}|z - y| > s^{\frac{1}{\alpha}}\}$.

The first term in the expression of $q_n(t, x, z) - q_n(t', x', z')$ can be written into two parts

$$\sum_{i=1}^n C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} \big(p(s, x, w) - p(s, x', w)\big) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w - y|^{d+\alpha}} q_{n-i}(t - s, y, z) 1_{\{s \leqslant \frac{t}{2}\}} m(dy)\, ds$$

$$+ \sum_{i=1}^n C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} \big(p(s, x, w) - p(s, x', w)\big) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w - y|^{d+\alpha}} q_{n-i}(t - s, y, z) 1_{\{\frac{t}{2} < s \leqslant t\}} m(dy)\, ds.$$

The absolute value of the first part

$$\left| \sum_{i=1}^n C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \big(p(s, x, w) - p(s, x', w)\big) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w - y|^{d+\alpha}} q_{n-i}(t - s, y, z) m(dy)\, ds \right|$$

$$\leqslant \sum_{i=1}^n C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \big|p(s, x, w) - p(s, x', w)\big| m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \bar{q}_{n-i}(t - s, y, z) m(dy)\, ds$$

$\big($where $\overline{C}$, $\overline{F}$ and $\bar{q}_{n-i}(t, y, z)$ are the same ones in the proof of Theorem 3.4$\big)$

$$\leqslant \sum_{i=1}^n C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \big|p(s, x, w) - p(s, x', w)\big| 1_A(s, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \bar{q}_{n-i}(t - s, y, z) m(dy)\, ds$$

$$+ \sum_{i=1}^n C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \big|p(s, x, w) - p(s, x', w)\big| 1_B(s, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \bar{q}_{n-i}(t - s, y, z) m(dy)\, ds$$

$$\leqslant \sum_{i=1}^n C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} (K_1 + 1) p(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \bar{q}_{n-i}(t - s, y, z) m(dy)\, ds$$

$$+ \sum_{i=1}^n C_n^i \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \big|p(s, x, w) - p(s, x', w)\big| m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w, y)}{|w - y|^{d+\alpha}} \bar{q}_{n-i}(t - s, y, z) m(dy)\, ds$$

(Lemma 3.7 is used here)

$$\leqslant \sum_{i=1}^n C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n - i)! K^{n-i} (K_1 + 1) \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(s, x, w)\, \mu(dw)\, ds$$

$$+ \sum_{i=1}^n C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n - i)! K^{n-i} \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \big(\bar{p}(s, x, w) + \bar{p}(s, x', w)\big) \mu(dw)\, ds$$

$\big($(3.2) is used here. M, $\overline{M}$, $\overline{C}$, L, K and $\bar{p}$ are the same ones in the proof of Theorem 3.4$\big)$

$$\leqslant \sum_{i=1}^n C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n - i)! K^{n-i} (K_1 + 1) \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} C_t$$

$$+\sum_{i=1}^{n} C_n^i M\overline{M}^2\overline{C}L^{i-1}(n-i)!K^{n-i}\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}\int_0^{|x-x'|^\alpha}\int_{\mathbb{R}^d}\big(\overline{p}(s,x,w)+\overline{p}(s,x',w)\big)\mu(dw)\,ds$$

$$\left(\text{where } C_t=\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\overline{p}(s,x,w)\mu(dw)\,ds\right).$$

Notice that $C_t<\infty$ and as $x'\to x$,

$$\int_0^{|x-x'|^\alpha}\int_{\mathbb{R}^d}\big(\overline{p}(s,x,w)+\overline{p}(s,x'w)\big)\mu(dw)\,ds$$
$$=\int_0^{|x-x'|^\alpha}\int_{\mathbb{R}^d}\overline{p}(s,x,w)\,\mu(dw)\,ds+\int_0^{|x-x'|^\alpha}\int_{\mathbb{R}^d}\overline{p}(s,x',w)\,\mu(dw)\,ds\to 0.$$

Thus when $x'\to x$, by the dominated convergence theorem,

$$\sum_{i=1}^{n}C_n^i\int_0^{\frac{t}{2}}\int_{\mathbb{R}^d}\big|p(s,x,w)-p(s,x',w)\big|1_A(s,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\overline{q}_{n-i}(t-s,y,z)m(dy)\,ds\to 0,$$

and because of the convergent upper bound,

$$\sum_{i=1}^{n}C_n^i\int_0^{\frac{t}{2}}\int_{\mathbb{R}^d}\big|p(s,x,w)-p(s,x',w)\big|1_B(s,w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\overline{q}_{n-i}(t-s,y,z)m(dy)\,ds\to 0.$$

Therefore the first part of the first term goes to 0 as $x'\to x$.

Now we look at the second part of the first term.

The absolute value of the second part

$$\left|\sum_{i=1}^{n}C_n^i\int_0^{t_3}\int_{\mathbb{R}^d}\big(p(s,x,w)-p(s,x',w)\big)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)1_{\{\frac{t}{2}<s\leqslant t\}}m(dy)\,ds\right|$$
$$\leqslant\sum_{i=1}^{n}C_n^i\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^d}\big|p(s,x,w)-p(s,x',w)\big|m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\overline{q}_{n-i}(t-s,y,z)m(dy)\,ds$$
$$\leqslant\sum_{i=1}^{n}C_n^i2M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dw)\overline{q}_{n-i}(t-s,y,z)m(dy)\,ds\quad\big((3.2)\text{ is used here}\big)$$
$$\leqslant\sum_{i=1}^{n}C_n^i2M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}\overline{C}L^{i-1}\overline{M}^2C_t(n-i)!K^{n-i}$$
$$\big(\text{by symmetry, } \overline{q}_{n-i}(t-s,y,z)=\overline{q}_{n-i}(t-s,z,y)\text{ and }(3.3)\big)$$
$$<\infty.$$

Thus by the dominated convergence theorem, the second part of the first term goes to 0 as $x'\to x$.

For the second term, its absolute value

$$\begin{aligned}
&\left|\sum_{i=1}^{n} C_n^i \int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t-s,y,z)(1_{\{s\leqslant t\}}-1_{\{s\leqslant t\wedge t'\}})m(dy)\,ds\right|\\
&\quad\leqslant \sum_{i=1}^{n} C_n^i \int_{t\wedge t'}^{t}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\bar{q}_{n-i}(t-s,y,z)m(dy)\,ds\\
&\quad\leqslant \sum_{i=1}^{n} C_n^i \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}\bar{q}_{n-i}(t-s,y,z)m(dy)\,ds\\
&\quad\leqslant \sum_{i=1}^{n} C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dw)\bar{q}_{n-i}(t-s,y,z)m(dy)\,ds \qquad \bigl((3.2)\text{ is used here}\bigr)\\
&\quad\leqslant \sum_{i=1}^{n} C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \overline{C}L^{i-1}\overline{M}^2 C_t(n-i)!K^{n-i}\\
&\qquad\bigl(\text{by symmetry, } \bar{q}_{n-i}(t-s,y,z)=\bar{q}_{n-i}(t-s,z,y) \text{ and } (3.3)\bigr)\\
&\quad<\infty.
\end{aligned}$$

Thus by the dominated convergence theorem, the second term goes to 0, as $t'\to t$ and $x'\to x$.

For the third term, we can write it into two parts:

$$\begin{aligned}
&\sum_{i=1}^{n} C_n^i \int_0^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\bigl(q_{n-i}(t-s,y,z)-q_{n-i}(t'-s,y,z)\bigr)1_{\{s\leqslant t\wedge t'\}}m(dy)\,ds\\
&\quad=\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\bigl(q_{n-i}(t-s,y,z)-q_{n-i}(t'-s,y,z)\bigr)m(dy)\,ds\\
&\qquad+\sum_{i=1}^{n} C_n^i \int_{\frac{t}{2}}^{t\wedge t'}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\bigl(q_{n-i}(t-s,y,z)-q_{n-i}(t'-s,y,z)\bigr)m(dy)\,ds.
\end{aligned}$$

The absolute value of the first part of the third term

$$\begin{aligned}
&\left|\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\bigl(q_{n-i}(t-s,y,z)-q_{n-i}(t'-s,y,z)\bigr)m(dy)\,ds\right|\\
&\quad\leqslant\left|\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}}\int_{\mathbb{R}^d} p(s,x',w)1_A(s,w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\bigl(q_{n-i}(t-s,y,z)\right.\\
&\qquad\left.-\,q_{n-i}(t'-s,y,z)\bigr)m(dy)\,ds\right|\\
&\qquad+\left|\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}}\int_{\mathbb{R}^d} p(s,x',w)1_B(s,w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}\bigl(q_{n-i}(t-s,y,z)\right.\\
&\qquad\left.-\,q_{n-i}(t'-s,y,z)\bigr)m(dy)\,ds\right|
\end{aligned}$$

$$\leqslant \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} K_1 p(s,x,w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \big(\overline{q}_{n-i}(t-s,y,z) + \overline{q}_{n-i}(t'-s,y,z)\big) m(dy)\,ds$$

$$+ \sum_{i=1}^{n} C_n^i \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} p(s,x',w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \big(\overline{q}_{n-i}(t-s,y,z) + \overline{q}_{n-i}(t'-s,y,z)\big) m(dy)\,ds$$

(Lemma 3.7 is used here)

$$\leqslant \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} K_1 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)\,\mu(dw)\,ds$$

$$+ \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w) \mu(dw)\,ds \quad \big((3.2) \text{ is used here}\big)$$

$$\leqslant \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} K_1 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} C_t$$

$$+ \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w) \mu(dw)\,ds.$$

It is clear that $C_t < \infty$ and as $x' \to x$, $\int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w)\mu(dw)\,ds \to 0$.

Thus when $t' \to t$ and $x' \to x$, by the dominated convergence theorem,

$$\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_A(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t-s,y,z) - q_{n-i}(t'-s,y,z)\big) m(dy)\,ds \to 0,$$

and because of the convergent upper bound,

$$\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_B(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t-s,y,z) - q_{n-i}(t'-s,y,z)\big) m(dy)\,ds \to 0.$$

Therefore the first part of the third term goes to 0 as $t' \to t$ and $x' \to x$.

The second part of the third term

$$\sum_{i=1}^{n} C_n^i \int_{\frac{t}{2}}^{t\wedge t'} \int_{\mathbb{R}^d} p(s,x',w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t-s,y,z) - q_{n-i}(t'-s,y,z)\big) m(dy)\,ds$$

$$= \sum_{i=1}^{n} C_n^i \int_{t-t\wedge t'}^{t-\frac{t}{2}} \int_{\mathbb{R}^d} p(t-\tilde{s},x',w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} q_{n-i}(\tilde{s},y,z) m(dy)\,d\tilde{s}$$

$$- \sum_{i=1}^{n} C_n^i \int_{t'-t\wedge t'}^{t'-\frac{t}{2}} \int_{\mathbb{R}^d} p(t'-\tilde{s},x',w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} q_{n-i}(\tilde{s},y,z) m(dy)\,d\tilde{s}$$

(by transformations $\tilde{s} = t - s$ and $\tilde{s} = t' - s$ respectively)

$$= \sum_{i=1}^{n} C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} \left(p(t-\tilde{s}, x', w) 1_{\{t-t\wedge t' \leqslant \tilde{s} \leqslant t-\frac{t}{2}\}} - p(t'-\tilde{s}, x', w) 1_{\{t'-t\wedge t' \leqslant \tilde{s} \leqslant t'-\frac{t}{2}\}}\right) m(dw)$$
$$\cdot \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} q_{n-i}(\tilde{s}, y, z) m(dy)\, d\tilde{s}.$$

Notice that

$$p(t-\tilde{s}, x', w) 1_{\{t-t\wedge t' \leqslant \tilde{s} \leqslant t-\frac{t}{2}\}} \leqslant M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \quad \text{and} \quad p(t'-\tilde{s}, x', w) 1_{\{t'-t\wedge t' \leqslant \tilde{s} \leqslant t'-\frac{t}{2}\}} \leqslant M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}.$$

Thus the absolute value of the second part of the third term

$$\leqslant \sum_{i=1}^{n} C_n^i 2M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_0^{t_3} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} m(dw) \overline{q}_{n-i}(\tilde{s}, y, z) m(dy)\, d\tilde{s}$$
$$\leqslant \sum_{i=1}^{n} C_n^i 2M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \overline{C} L^{i-1} \overline{M}^2 \int_0^{t_3} \int_{\mathbb{R}^d} \overline{q}_{n-i}(\tilde{s}, y, z) \mu(dy)\, ds$$
$$\leqslant \sum_{i=1}^{n} C_n^i 2M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \overline{C} L^{i-1} \overline{M}^2 C_{t_3} (n-i)! K^{n-i}$$

(by symmetry, $\overline{q}_{n-i}(t-s, y, z) = \overline{q}_{n-i}(t-s, z, y)$ and (3.3))

$$< \infty.$$

Therefore by the dominated convergence theorem, the second part of the third term goes to 0 as $t' \to t$ and $x' \to x$. For the fourth term, we can write it into two parts

$$\sum_{i=1}^{n} C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \left(q_{n-i}(t'-s, y, z) - q_{n-i}(t'-s, y, z')\right) 1_{\{s \leqslant t\wedge t'\}} m(dy)\, ds$$
$$= \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \left(q_{n-i}(t'-s, y, z) - q_{n-i}(t'-s, y, z')\right) m(dy)\, ds$$
$$+ \sum_{i=1}^{n} C_n^i \int_{\frac{t}{2}}^{t\wedge t'} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \left(q_{n-i}(t'-s, y, z) - q_{n-i}(t'-s, y, z')\right) m(dy)\, ds.$$

The absolute value of the first part of the fourth term

$$\left| \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \left(q_{n-i}(t'-s, y, z) - q_{n-i}(t'-s, y, z')\right) m(dy)\, ds \right|$$
$$\leqslant \left| \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s, x', w) 1_A(s, w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \left(q_{n-i}(t'-s, y, z)\right.\right.$$
$$\left.\left. - q_{n-i}(t'-s, y, z')\right) m(dy)\, ds \right|$$

$$+\left|\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_B(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t'-s,y,z) - q_{n-i}(t'-s,y,z')\big) m(dy)\, ds\right|$$

$$\leqslant \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} K_1 p(s,x,w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \big(\overline{q}_{n-i}(t'-s,y,z) + \overline{q}_{n-i}(t'-s,y,z')\big) m(dy)\, ds$$

$$+ \sum_{i=1}^{n} C_n^i \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} p(s,x',w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \big(\overline{q}_{n-i}(t'-s,y,z) + \overline{q}_{n-i}(t'-s,y,z')\big) m(dy)\, ds$$

(Lemma 3.7 is used here)

$$\leqslant \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} K_1 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)\, \mu(dw)\, ds$$

$$+ \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w) \mu(dw)\, ds \quad \big((3.2) \text{ is used here}\big)$$

$$\leqslant \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} K_1 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} C_t$$

$$+ \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} 2 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w) \mu(dw)\, ds.$$

It is clear that $C_t < \infty$ and as $x' \to x$, $\int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w)\mu(dw)\, ds \to 0$.

Thus when $t' \to t$, $x' \to x$ and $z' \to z$, by the dominated convergence theorem,

$$\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_A(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t'-s,y,z) - q_{n-i}(t'-s,y,z')\big) m(dy)\, ds \to 0,$$

and because of the convergent upper bound,

$$\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_B(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t'-s,y,z) - q_{n-i}(t'-s,y,z')\big) m(dy)\, ds \to 0.$$

Therefore the first part of the fourth term goes to 0 as $t' \to t$ and $x' \to x$.

The absolute value of the second part of the fourth term

$$\left|\sum_{i=1}^n C_n^i \int_{\frac{t}{2}}^{t\wedge t'} \int_{\mathbb{R}^d} p(s,x',w)m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(t'-s,y,z) - q_{n-i}(t'-s,y,z')\big)m(dy)\,ds\right|$$

$$= \left|\sum_{i=1}^n C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{t'-t'\wedge t\leqslant \tilde{s}\leqslant t'-\frac{t}{2}\}} m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(\tilde{s},y,z) \right.$$

$$\left. - q_{n-i}(\tilde{s},y,z')\big)m(dy)\,d\tilde{s}\right| \quad \text{(by transformation } \tilde{s} = t'-s)$$

$$\leqslant \left|\sum_{i=1}^n C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{t'-t'\wedge t\leqslant \tilde{s}\leqslant t'-\frac{t}{2}\}} m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(\tilde{s},y,z) \right.$$

$$\left. - q_{n-i}(\tilde{s},y,z')\big)1_{\tilde{A}}(\tilde{s},y)m(dy)\,d\tilde{s}\right|$$

$$+ \left|\sum_{i=1}^n C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{t'-t'\wedge t\leqslant \tilde{s}\leqslant t'-\frac{t}{2}\}} m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(\tilde{s},y,z) \right.$$

$$\left. - q_{n-i}(\tilde{s},y,z')\big)1_{\tilde{B}}(\tilde{s},y)m(dy)\,d\tilde{s}\right|$$

$$\leqslant \sum_{i=1}^n C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_0^{t_3} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} m(dw)\big(\overline{q}_{n-i}(\tilde{s},y,z) + \overline{q}_{n-i}(\tilde{s},y,z')\big)1_{\tilde{A}}(\tilde{s},y)m(dy)\,d\tilde{s}$$

$$+ \sum_{i=1}^n C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \int_0^{t_3} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} m(dw)\big(\overline{q}_{n-i}(\tilde{s},y,z) + \overline{q}_{n-i}(\tilde{s},y,z')\big)1_{\tilde{B}}(\tilde{s},y)m(dy)\,d\tilde{s}$$

$$\leqslant \sum_{i=1}^n C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \tilde{C}_1 (n-i)! K^{n-i} \frac{1}{M_1} \int_0^{t_3} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} m(dw)\big(p(\tilde{s},y,z) + p(\tilde{s},y,z')\big)$$

$$\cdot 1_{\tilde{A}}(\tilde{s},y)m(dy)\,d\tilde{s}$$

$$+ \sum_{i=1}^n C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \tilde{C}_1 (n-i)! K^{n-i} \frac{1}{M_1} \int_0^{t_3} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} m(dw)\big(p(\tilde{s},z,y) + p(\tilde{s},y,z')\big)$$

$$\cdot 1_{\tilde{B}}(\tilde{s},y)m(dy)\,d\tilde{s} \quad \big(\text{by Theorem 3.6 and (1.1)}\big)$$

$$\leqslant \sum_{i=1}^n C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \tilde{C}_1 (n-i)! K^{n-i} \frac{1}{M_1} \overline{C} L^{i-1} \overline{M}^2 \int_0^{t_3} \int_{\mathbb{R}^d} (K_1+1)\overline{p}(\tilde{s},y,z)\mu(dy)\,d\tilde{s}$$

$$+ \sum_{i=1}^n C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \tilde{C}_1 (n-i)! K^{n-i} \frac{1}{M_1} \overline{C} L^{i-1} \overline{M}^2 \int_0^{|z-z'|^\alpha} \int_{\mathbb{R}^d} \big(\overline{p}(\tilde{s},y,z) + \overline{p}(\tilde{s},y,z')\big)\mu(dy)\,d\tilde{s}$$

(by Lemma 3.7)

$$\leqslant \sum_{i=1}^{n} C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \tilde{C}_1 (n-i)! K^{n-i} \frac{1}{M_1} \overline{C} L^{i-1} \overline{M}^2 (K_1+1) C_{t_3}$$
$$+ \sum_{i=1}^{n} C_n^i M\left(\frac{t}{2}\right)^{-\frac{d}{\alpha}} \tilde{C}_1 (n-i)! K^{n-i} \frac{1}{M_1} \overline{C} L^{i-1} \overline{M}^2 \int_0^{|z-z'|^\alpha} \int_{\mathbb{R}^d} \big(\overline{p}(\tilde{s}, y, z) + \overline{p}(\tilde{s}, y, z')\big) \mu(dy)\, d\tilde{s}.$$

Notice that $C_{t_3} < \infty$ and as $z' \to z$,

$$\int_0^{|z-z'|^\alpha} \int_{\mathbb{R}^d} \big(\overline{p}(\tilde{s}, y, z) + \overline{p}(\tilde{s}, y, z')\big)\, \mu(dy)\, d\tilde{s} = \int_0^{|z-z'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(\tilde{s}, y, z)\, \mu(dy)\, d\tilde{s} + \int_0^{|z-z'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(\tilde{s}, y, z')\, \mu(dy)\, d\tilde{s}$$
$$\to 0.$$

Thus, when $t' \to t$, $x' \to x$ and $z' \to z$, by the dominated convergence theorem,

$$\sum_{i=1}^{n} C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(t' - \tilde{s}, x', w) 1_{\{t'-t'\wedge t \leqslant \tilde{s} \leqslant t' - \frac{t}{2}\}} m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(\tilde{s}, y, z)$$
$$- q_{n-i}(\tilde{s}, y, z')\big) 1_{\tilde{A}}(\tilde{s}, y) m(dy)\, d\tilde{s}$$
$$\to 0,$$

and because of the convergent upper bound,

$$\sum_{i=1}^{n} C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(t' - \tilde{s}, x', w) 1_{\{t'-t'\wedge t \leqslant \tilde{s} \leqslant t' - \frac{t}{2}\}} m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} \big(q_{n-i}(\tilde{s}, y, z)$$
$$- q_{n-i}(\tilde{s}, y, z')\big) 1_{\tilde{B}}(\tilde{s}, y) m(dy)\, d\tilde{s}$$
$$\to 0.$$

Therefore the second part of the fourth term goes to 0 as $t' \to t$, $x' \to x$ and $z' \to z$.

Now we look at the fifth term. We can write it into two parts

$$\sum_{i=1}^{n} C_n^i \int_0^{t_3} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s, y, z') (1_{\{s \leqslant t \wedge t'\}} - 1_{\{s \leqslant t'\}}) m(dy)\, ds$$
$$= \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s, y, z') (1_{\{s \leqslant t \wedge t'\}} - 1_{\{s \leqslant t'\}}) m(dy)\, ds$$
$$+ \sum_{i=1}^{n} C_n^i \int_{\frac{t}{2}}^{t_3} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s, y, z') (1_{\{s \leqslant t \wedge t'\}} - 1_{\{s \leqslant t'\}}) m(dy)\, ds.$$

The absolute value of the first part

$$= \left| \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s, x', w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s, y, z') (1_{\{s \leqslant t \wedge t'\}} - 1_{\{s \leqslant t'\}}) m(dy)\, ds \right|$$
$$\leqslant \left| \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s, x', w) 1_A(s, w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s, y, z') \right.$$

$$\cdot (1_{\{s\leqslant t\wedge t'\}} - 1_{\{s\leqslant t'\}}) m(dy)\, ds \Bigg|$$

$$+ \Bigg| \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_B(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s,y,z')$$

$$\cdot (1_{\{s\leqslant t\wedge t'\}} - 1_{\{s\leqslant t'\}}) m(dy)\, ds \Bigg|$$

$$\leqslant \sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} K_1 p(s,x,w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{n-i}(t'-s,y,z') m(dy)\, ds$$

$$+ \sum_{i=1}^{n} C_n^i \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} p(s,x',w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{n-i}(t'-s,y,z') m(dy)\, ds \quad \text{(Lemma 3.7 is used here)}$$

$$\leqslant \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} K_1 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s,x,w)\, \mu(dw)\, ds$$

$$+ \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w) \mu(dw)\, ds \quad \big(\text{(3.2) is used here}\big)$$

$$\leqslant \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} K_1 \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} C_t$$

$$+ \sum_{i=1}^{n} C_n^i M \overline{M}^2 \overline{C} L^{i-1} (n-i)! K^{n-i} \left(\frac{t}{6}\right)^{-\frac{d}{\alpha}} \int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w) \mu(dw)\, ds.$$

It is clear that $C_t < \infty$ and as $x' \to x$, $\int_0^{|x-x'|^\alpha} \int_{\mathbb{R}^d} \overline{p}(s,x',w)\mu(dw)\, ds \to 0$.

Thus when $t' \to t$, $x' \to x$ and $z' \to z$, by the dominated convergence theorem,

$$\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_A(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s,y,z') (1_{\{s\leqslant t\wedge t'\}} - 1_{\{s\leqslant t'\}}) m(dy)\, ds$$
$$\to 0,$$

and because of the convergent upper bound,

$$\sum_{i=1}^{n} C_n^i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} p(s,x',w) 1_B(s,w) m(dw) \int_{\mathbb{R}^d} \frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}} q_{n-i}(t'-s,y,z') (1_{\{s\leqslant t\wedge t'\}} - 1_{\{s\leqslant t'\}}) m(dy)\, ds$$
$$\to 0.$$

Therefore the first part of the fifth term goes to 0 as $t' \to t$ and $x' \to x$.

The absolute value of the second part of the fifth term

$$
\begin{aligned}
&\left|\sum_{i=1}^{n} C_n^i \int_{\frac{t}{2}}^{t_3}\int_{\mathbb{R}^d} p(s,x',w)m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(t'-s,y,z')(1_{\{s\leqslant t\wedge t'\}}-1_{\{s\leqslant t'\}})m(dy)\,ds\right|\\
&\quad=\left|\sum_{i=1}^{n} C_n^i \int_{0}^{t_3}\int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{\tilde{s}\leqslant t'-t\wedge t'\}}m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(\tilde{s},y,z')m(dy)\,d\tilde{s}\right|\\
&\qquad\text{(by transformation } \tilde{s}=t'-s)\\
&\quad\leqslant\left|\sum_{i=1}^{n} C_n^i \int_{0}^{t_3}\int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{\tilde{s}\leqslant t'-t\wedge t'\}}m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(\tilde{s},y,z')1_{\tilde{A}}(\tilde{s},y)m(dy)\,d\tilde{s}\right|\\
&\qquad+\left|\sum_{i=1}^{n} C_n^i \int_{0}^{t_3}\int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{\tilde{s}\leqslant t'-t\wedge t'\}}m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(\tilde{s},y,z')1_{\tilde{B}}(\tilde{s},y)m(dy)\,d\tilde{s}\right|\\
&\quad\leqslant\sum_{i=1}^{n} C_n^i M\left(\frac{2}{3}t\right)^{-\frac{d}{\alpha}}\int_{0}^{t_3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dw)q_{n-i}(\tilde{s},y,z')1_{\tilde{A}}(\tilde{s},y)m(dy)\,d\tilde{s}\\
&\qquad+\sum_{i=1}^{n} C_n^i M\left(\frac{2}{3}t\right)^{-\frac{d}{\alpha}}\int_{0}^{t_3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\overline{C}\,\overline{F}^i(w,y)}{|w-y|^{d+\alpha}}m(dw)q_{n-i}(\tilde{s},y,z')1_{\tilde{B}}(\tilde{s},y)m(dy)\,d\tilde{s}\\
&\quad\leqslant\sum_{i=1}^{n} C_n^i M\left(\frac{2}{3}t\right)^{-\frac{d}{\alpha}}\tilde{C}_1(n-i)!K^{n-i}\frac{1}{M_1}\overline{C}L^{i-1}\overline{M}^2\int_{0}^{t_3}\int_{\mathbb{R}^d}K_1\overline{p}(\tilde{s},y,z)\mu(dz)\tilde{s}\\
&\qquad+\sum_{i=1}^{n} C_n^i M\left(\frac{2}{3}t\right)^{-\frac{d}{\alpha}}\tilde{C}_1(n-i)!K^{n-i}\frac{1}{M_1}\overline{C}L^{i-1}\overline{M}^2\int_{0}^{|z-z'|^\alpha}\int_{\mathbb{R}^d}\overline{p}(\tilde{s},y,z')m(dy)\,d\tilde{s}\\
&\qquad\big(\text{by Theorem 3.6 and (1.1), then by Lemma 3.7}\big)\\
&\quad\leqslant\sum_{i=1}^{n} C_n^i M\left(\frac{2}{3}t\right)^{-\frac{d}{\alpha}}\tilde{C}_1(n-i)!K^{n-i}\frac{1}{M_1}\overline{C}L^{i-1}\overline{M}^2C_{t_3}\\
&\qquad+\sum_{i=1}^{n} C_n^i M\left(\frac{2}{3}t\right)^{-\frac{d}{\alpha}}\tilde{C}_1(n-i)!K^{n-i}\frac{1}{M_1}\overline{C}L^{i-1}\overline{M}^2\int_{0}^{|z-z'|^\alpha}\int_{\mathbb{R}^d}\overline{p}(\tilde{s},y,z')\mu(dy)\,d\tilde{s}.
\end{aligned}
$$

Notice that $C_{t_3}<\infty$ and as $z'\to z$, $\int_0^{|z-z'|^\alpha}\int_{\mathbb{R}^d}\overline{p}(\tilde{s},y,z')\mu(dy)\,d\tilde{s}\to 0$.

Thus, when $t'\to t$, $x'\to x$ and $z'\to z$, by the dominated convergence theorem,

$$
\sum_{i=1}^{n} C_n^i \int_{0}^{t_3}\int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{\tilde{s}\leqslant t'-t\wedge t'\}}m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(\tilde{s},y,z')1_{\tilde{A}}(\tilde{s},y)m(dy)\,d\tilde{s}\to 0,
$$

and because of the convergent upper bound,

$$
\sum_{i=1}^{n} C_n^i \int_{0}^{t_3}\int_{\mathbb{R}^d} p(t'-\tilde{s},x',w)1_{\{\tilde{s}\leqslant t'-t\wedge t'\}}m(dw)\int_{\mathbb{R}^d}\frac{2C(w,y)F^i(w,y)}{|w-y|^{d+\alpha}}q_{n-i}(\tilde{s},y,z')1_{\tilde{B}}(\tilde{s},y)m(dy)\,d\tilde{s}\to 0.
$$

Therefore the second part of the fifth term goes to 0 as $t'\to t$, $x'\to x$ and $z'\to z$.

In conclusion, all the five terms in the expression of $q_n(t,x,z)-q_n(t',x',z')$ go to 0 as $t'\to t$, $x'\to x$ and $z'\to z$. This means that $q_n(t,x,z)-q_n(t',x',z')$ goes to 0 as $t'\to t$, $x'\to x$ and $z'\to z$. Thus the statement holds for $i=n$.

By induction, the statement holds for any $n\geqslant 0$. $\square$

Since $q(t,x,z)=\sum_{n=0}^{\infty}(-1)^n\frac{q_n(t,x,z)}{n!}$ is uniformly convergent on $[\epsilon,t_3]\times\mathbb{R}^d\times\mathbb{R}^d$ for any $\epsilon>0$, Proposition 3.8 implies that $q(t,x,z)$ is joint continuous on $(0,t_3)\times\mathbb{R}^d\times\mathbb{R}^d$.

We have the following properties for $q(t,x,z)$:

Proposition 3.9.

(i) $\int_{\mathbb{R}^d} q(t,x,z)g(z)m(dz)=\mathbb{E}_x[e^{-A_t}g(X_t)]$, *for any g bounded measurable and any $t\in(0,t_3)$,*
(ii) $\int_{\mathbb{R}^d} q(t,x,y)q(s,y,z)m(dy)=q(t+s,x,z)$, *for any $t,s\in(0,t_3)$.*
(iii) $|q(t,x,z)|\leqslant\overline{L}_3 p(t,x,z)\leqslant\overline{L}_3\overline{p}(t,x,z)$, *for $t\in(0,t_3)$,*

where $\overline{L}_3$ is a positive constant depending on t_3, $p(t,x,z)$ is the transition density of the α-stable-like process X and $\overline{p}(t,x,z)=p(t,x,z)+p(t,z,x)$.

Proof. (i) is straightforward. (ii) comes from (i), continuity of $q(t,x,y)$ in x and y and the Markov property. (iii) comes from Theorem 3.6 and the fact that $|q_n(t,x,z)|\leqslant\bar{q}_n(t,x,z)$, and $\sum_{n=0}^{\infty}\frac{\bar{q}_n(t,x,z)}{n!}$ is uniformly convergent on $[\epsilon,t_3]\times\mathbb{R}^d\times\mathbb{R}^d$ for any $\epsilon>0$. $\square$

We can extend the joint continuity of $q(t,x,z)$ to $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. The argument is the following:

By (ii) of Proposition 3.9, for any given $t,s\in(0,t_3)$, $\int_{\mathbb{R}^d} q(t,x,y)q(s,y,z)m(dy)=q(t+s,x,z)$. Suppose $(t'+s,x',z')\to(t+s,x,z)$. Since $t'\to t$ and $z'\to z$, we can assume that $t'>\frac{t}{2}$ and $|z'-z|^\alpha<s$ for given $s>0$.

The difference between $q(t+s,x,z)$ and $q(t'+s,x',z')$ can be written as the sum of three terms:

$$\begin{aligned}q(t+s,x,z)-q(t'+s,x',z')&=\big(q(t+s,x,z)-q(t'+s,x,z)\big)+\big(q(t'+s,x,z)-q(t'+s,x',z)\big)\\&\quad+\big(q(t'+s,x',z)-q(t'+s,x',z')\big).\end{aligned}$$

The absolute value of the first term

$$\begin{aligned}\big|q(t+s,x,z)-q(t'+s,x,z)\big|&=\Big|\int_{\mathbb{R}^d}\big(q(t,x,y)q(s,y,z)-q(t',x,y)q(s,y,z)\big)m(dy)\Big|\\&=\Big|\int_{\mathbb{R}^d}\big(q(t,x,y)-q(t',x,y)\big)q(s,y,z)m(dy)\Big|\\&\leqslant\int_{\mathbb{R}^d}\overline{L}_3\big(p(t,x,y)+p(t',x,y)\big)\big|q(s,y,z)\big|m(dy)\\&\quad\big(\text{by (iii) of Proposition 3.9}\big)\\&\leqslant\int_{\mathbb{R}^d}\overline{L}_3 2\Big(\frac{t}{2}\Big)^{-\frac{d}{\alpha}}\big|q(s,y,z)\big|m(dy)=\overline{L}_3 2\Big(\frac{t}{2}\Big)^{-\frac{d}{\alpha}}\int_{\mathbb{R}^d}\big|q(s,y,z)\big|m(dy)\\&\leqslant\overline{L}_3 2\Big(\frac{t}{2}\Big)^{-\frac{d}{\alpha}}\overline{L}_3\int_{\mathbb{R}^d}\overline{p}(s,y,z)m(dy)<\infty.\end{aligned}$$

Thus, by the dominated convergence theorem, $q(t+s,x,z)-q(t'+s,x,z)$ goes to 0 as $t'\to t$.

The absolute value of the second term

$$\begin{aligned}
&\bigl|q(t'+s,x,z)-q(t'+s,x',z)\bigr|\\
&\quad=\biggl|\int\limits_{\mathbb{R}^d}\bigl(q(t',x,y)q(s,y,z)-q(t',x',y)q(s,y,z)\bigr)m(dy)\biggr|=\biggl|\int\limits_{\mathbb{R}^d}\bigl(q(t',x,y)-q(t',x',y)\bigr)q(s,y,z)m(dy)\biggr|\\
&\quad\leqslant\int\limits_{\mathbb{R}^d}\overline{L}_3\bigl(p(t,x',y)+p(t',x',y)\bigr)\bigl|q(s,y,z)\bigr|m(dy)\quad\bigl(\text{by (iii) of Proposition 3.9}\bigr)\\
&\quad\leqslant\int\limits_{\mathbb{R}^d}\overline{L}_3 2\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\bigl|q(s,y,z)\bigr|m(dy)=\overline{L}_3 2\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\int\limits_{\mathbb{R}^d}\bigl|q(s,y,z)\bigr|m(dy)\\
&\quad\leqslant\overline{L}_3 2\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\overline{L}_3\int\limits_{\mathbb{R}^d}\overline{p}(s,y,z)m(dy)<\infty.
\end{aligned}$$

Thus, by the dominated convergence theorem, $q(t'+s,x,z)-q(t'+s,x',z)$ goes to 0 as $t'\to t$ and $x'\to x$.

The absolute value of the third term

$$\begin{aligned}
&\bigl|q(t'+s,x',z)-q(t'+s,x',z')\bigr|\\
&\quad=\biggl|\int\limits_{\mathbb{R}^d}\bigl(q(t',x',y)q(s,y,z)-q(t',x',y)q(s,y,z')\bigr)m(dy)\biggr|=\biggl|\int\limits_{\mathbb{R}^d}q(t',x',y)\bigl(q(s,y,z)-q(s,y,z')\bigr)m(dy)\biggr|\\
&\quad\leqslant\int\limits_{\mathbb{R}^d}\overline{L}_3\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\bigl|q(s,y,z)-q(s,y,z')\bigr|m(dy)\leqslant\int\limits_{\mathbb{R}^d}\overline{L}_3\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\overline{L}_3\bigl(p(s,y,z)+p(s,y,z')\bigr)m(dy)\\
&\quad\leqslant\int\limits_{\mathbb{R}^d}\overline{L}_3\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\overline{L}_3\bigl\{(K_1+1)p(s,y,z)+\bigl(p(s,y,z)+p(s,y,z')\bigr)1_{\{|z-z'|>s\}}\bigr\}m(dy)\quad(\text{by Lemma 3.7})\\
&\quad=\int\limits_{\mathbb{R}^d}\overline{L}_3\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\overline{L}_3(K_1+1)p(s,y,z)m(dy)\quad\bigl(|z-z'|^\alpha<s\text{ implies }1_{\{|z-z'|^\alpha>s\}}=0\bigr)\\
&\quad\leqslant\overline{L}_3\biggl(\frac{t}{2}\biggr)^{-\frac{d}{\alpha}}\overline{L}_3(K_1+1)\int\limits_{\mathbb{R}^d}\overline{p}(s,y,z)m(dy)<\infty.
\end{aligned}$$

Thus, by the dominated convergence theorem, $q(t'+s,x',z)-q(t'+s,x',z')$ goes to 0 as $t'\to t$, $x'\to x$ and $z'\to z$.

In conclusion, $q(t+s,x,z)-q(t'+s,x',z')$ goes to 0 as $t'\to t$, $x'\to x$ and $z'\to z$.

This shows that we can extend the joint continuity of $q(t,x,z)$ to $(0,2t_3)\times\mathbb{R}^d\times\mathbb{R}^d$. It is easy to see that by (ii) and (iii) of Proposition 3.9, $|q(t,x,z)|\leqslant\overline{L}_3^2 p(t,x,z)$ for $t\in(0,2t_3)$. Repeating the above procedure, we can extend the joint continuity of $q(t,x,z)$ to $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and have the following property:

Proposition 3.10.

(i) $\int_{\mathbb{R}^d}q(t,x,z)g(z)m(dz)=\mathbb{E}_x[e^{-A_t}g(X_t)]$, *for any g bounded measurable and any $t>0$,*
(ii) $\int_{\mathbb{R}^d}q(t,x,y)q(s,y,z)m(dy)=q(t+s,x,z)$, *for any $t,s>0$,*
(iii) *there exist positive constants C_3 and C_4 such that*

$$q(t,x,z)\leqslant C_3e^{C_4t}t^{-\frac{d}{\alpha}}\biggl(1\wedge\frac{t^{\frac{1}{\alpha}}}{|x-z|}\biggr)^{d+\alpha},\quad\forall(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d.$$

Thus the upper bound of $q(t,x,z)$ is obtained.

Next we look at the lower bound of $q(t,x,z)$.

We consider $\frac{\bar{q}_1(t,x,z)}{k}$. By Theorem 3.6 and (1.1), there exists a large number k such that when $0 < t \leqslant t_3$,

$$\frac{\bar{q}_1(t,x,z)}{k} \leqslant \frac{1}{2} p(t,x,z), \quad \forall (x,z) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{3.6}$$

It is clear that $\frac{|q_1(t,x,z)|}{k} \leqslant \frac{\bar{q}_1(t,x,z)}{k}$. (3.6) implies that when $0 < t \leqslant t_3$,

$$p(t,x,z) - \frac{q_1(t,x,z)}{k} \geqslant p(t,x,z) - \frac{\bar{q}_1(t,x,z)}{k} \geqslant \frac{1}{2} p(t,x,z).$$

We know $\int_{\mathbb{R}^d} \frac{q_1(t,x,z)}{k} g(z) m(dz) = \mathbb{E}_x[\frac{A_t}{k} g(X_t)]$, for any g measurable. Since $1 - \frac{A_t}{k} \leqslant e^{-\frac{A_t}{k}}$, we have

$$\frac{1}{|B_r|} \mathbb{E}_x\left[\left(1 - \frac{A_t}{k}\right) 1_{B_r}(X_t)\right] \leqslant \frac{1}{|B_r|} \mathbb{E}_x\left[e^{-\frac{A_t}{k}} 1_{B_r}(X_t)\right].$$

Thus

$$\begin{aligned} \frac{1}{2} \frac{1}{|B_r|} \mathbb{E}_x\left[1_{B_r}(X_t)\right] &\leqslant \frac{1}{|B_r|} \mathbb{E}_x\left[e^{-\frac{A_t}{k}} 1_{B_r}(X_t)\right] \\ &\leqslant \left(\frac{1}{|B_r|} \mathbb{E}_x\left[e^{-A_t} 1_{B_r}(X_t)\right]\right)^{\frac{1}{k}} \left(\frac{1}{|B_r|} \mathbb{E}_x\left[1_{B_r}(X_t)\right]\right)^{1-\frac{1}{k}} \quad \text{(by Hölder inequality).} \end{aligned}$$

Therefore

$$\frac{\frac{1}{2} \frac{1}{|B_r|} \mathbb{E}_x[1_{B_r}(X_t)]}{(\frac{1}{|B_r|} \mathbb{E}_x[1_{B_r}(X_t)])^{1-\frac{1}{k}}} \leqslant \left(\frac{1}{|B_r|} \mathbb{E}_x\left[e^{-A_t} 1_{B_r}(X_t)\right]\right)^{\frac{1}{k}},$$

i.e.

$$\frac{1}{2} \left(\frac{1}{|B_r|} \mathbb{E}_x\left[1_{B_r}(X_t)\right]\right)^{\frac{1}{k}} \leqslant \left(\frac{1}{|B_r|} \mathbb{E}_x\left[e^{-A_t} 1_{B_r}(X_t)\right]\right)^{\frac{1}{k}},$$

i.e.

$$\frac{1}{2^k} \frac{1}{|B_r|} \mathbb{E}_x\left[1_{B_r}(X_t)\right] \leqslant \frac{1}{|B_r|} \mathbb{E}_x\left[e^{-A_t} 1_{B_r}(X_t)\right].$$

Let $r \downarrow 0$, we have

$$\frac{1}{2^k} p(t,x,z) \leqslant q(t,x,z).$$

Therefore when $0 < t \leqslant t_3$,

$$\frac{1}{2^k} M_1 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha} \leqslant q(t,x,z).$$

Applying (ii) of Proposition 3.10, we have

$$q(t,x,z) \geqslant C_5 e^{-C_6 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha}, \quad \forall (t,x,z) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

where C_5 and C_6 are positive constants.

Combining this with (iii) of Proposition 3.10, we establish the lower and upper estimates of $q(t,x,z)$ as follows.

Theorem 3.11. *There exist positive constants C_3, C_4, C_5 and C_6 such that*

$$C_5 e^{-C_6 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha} \leqslant q(t,x,z) \leqslant C_3 e^{C_4 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha} \tag{3.7}$$

for all $(t,x,z) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Acknowledgments

I am very grateful to my advisor Professor Renming Song for his encouragement and many suggestions. I really appreciate the comments and suggestions from the anonymous referees.

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