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Sandwich pairs for p-Laplacian systems

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ABSTRACT

We solve boundary value problems for p-Laplacian systems using sandwich pairs.

1. Introduction

The notion of sandwich pairs introduced by Schechter [6] is a useful tool for finding critical points of a functional. Let W be a Banach space and $\Phi \in C^1(W, \mathbb{R})$. Recall that a sequence $(u^j) \subset W$ such that

$$\Phi(u^j) \to c, \qquad \Phi'(u^j) \to 0 \tag{1.1}$$

is called a Palais–Smale sequence for Φ at the level c, or a $(\mathrm{PS})_c$ sequence for short, and that Φ satisfies the compactness condition $(\mathrm{PS})_c$ if every such sequence has a convergent subsequence.

Definition 1.1. We say that $A, B \subset W$ form a sandwich pair if for any $\Phi \in C^1(W, \mathbb{R})$,

$$-\infty < b := \inf_B \Phi \leqslant \sup_A \Phi =: a < +\infty \tag{1.2}$$

implies that Φ has a $(\mathrm{PS})_c$ sequence for some $c \in [b, a]$.

Thus, if A, B form a sandwich pair and Φ satisfies (1.2) as well as $(\mathrm{PS})_c$ for all $c \in [b, a]$, then Φ has a critical point. In [6] sandwich pairs constructed using the eigenspaces of a linear operator were used to solve semilinear elliptic boundary value problems, and in [4,5] the authors solved quasilinear problems using cones as sandwich pairs. In the present paper we use more general curved sandwich pairs made up of orbits of a certain group action on product spaces to solve systems of quasilinear equations.

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We consider the class of problems

$$\begin{cases} -\Delta_p u = \nabla_u F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, $p = (p_1, \ldots, p_m)$ with each $p_i \in (1, \infty)$, $u = (u_1, \ldots, u_m)$, $\Delta_p u = (\Delta_{p_1} u_1, \ldots, \Delta_{p_m} u_m)$ where $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2} \nabla u_i)$ is the p_i-Laplacian of u_i, $F \in C^1(\Omega \times \mathbb{R}^m)$, and $\nabla_u F = (\partial F/\partial u_1, \ldots, \partial F/\partial u_m)$. We assume that

$$\left| \frac{\partial F}{\partial u_i} \right| \leqslant C \left(\sum_{j=1}^{m} |u_j|^{r_{ij}-1} + 1 \right) \quad \forall (x,u) \in \Omega \times \mathbb{R}^m \tag{1.4}$$

for some $C > 0$ and $r_{ij} \in (1, p_j^*(p_i^* - 1)/p_i^*)$, where

$$p_i^* = \begin{cases} np_i/(n-p_i), & p_i < n, \\ \infty, & p_i \geqslant n \end{cases} \tag{1.5}$$

is the critical exponent for the Sobolev space $W_0^{1,p_i}(\Omega)$ with the norm

$$\|u_i\|_i = \left(\int_\Omega |\nabla u_i|^{p_i} \right)^{\frac{1}{p_i}}. \tag{1.6}$$

Let

$$W = W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_m}(\Omega) = \left\{ u = (u_1, \ldots, u_m)\colon\ u_i \in W_0^{1,p_i}(\Omega) \right\} \tag{1.7}$$

with the norm

$$\|u\| = \left(\sum_{i=1}^{m} \|u_i\|_i^2 \right)^{\frac{1}{2}}. \tag{1.8}$$

Then solutions of (1.3) coincide with critical points of

$$\Phi(u) = I(u) - \int_\Omega F(x,u), \quad u \in W, \tag{1.9}$$

where

$$I(u) = \sum_{i=1}^{m} \frac{1}{p_i} \int_\Omega |\nabla u_i|^{p_i} = \sum_{i=1}^{m} \frac{1}{p_i} \|u_i\|_i^{p_i}. \tag{1.10}$$

Under additional assumptions on F, we will obtain critical points of Φ using suitable sandwich pairs.

2. Sandwich pairs

In this section we construct sandwich pairs applicable to our problem (1.3). Let W be a Banach space and let Σ be the class of maps $\sigma \in C(W \times [0,1], W)$ such that, writing $\sigma_t = \sigma(\cdot, t)$,

(i) $\sigma_0 = id$,
(ii) $\sup_{(u,t) \in W \times [0,1]} \|\sigma_t(u) - u\| < \infty$.

We use the customary notation

$$\Phi^a = \left\{ u \in W\colon\ \Phi(u) \leqslant a \right\}, \qquad \Phi_a = \left\{ u \in W\colon\ \Phi(u) \geqslant a \right\} \tag{2.1}$$

for the sublevel and superlevel sets of a functional.

Lemma 2.1. *$A, B \subset W$ form a sandwich pair if*

$$\sigma_1(A) \cap B \neq \emptyset \quad \forall \sigma \in \Sigma. \tag{2.2}$$

Proof. Let $\Phi \in C^1(W, \mathbb{R})$ satisfy (1.2) and set

$$c := \inf_{\sigma \in \Sigma} \sup_{u \in \sigma_1(A)} \Phi(u). \tag{2.3}$$

Then $c \geqslant b$ by (2.2) and $c \leqslant a$ since the identity $\sigma_t(u) \equiv u$ is in Σ.

We claim that Φ has a $(\mathrm{PS})_c$ sequence. If not, the $(\mathrm{PS})_c$ condition holds vacuously and c is not a critical value of Φ, so there are $\varepsilon > 0$ and $\eta \in \Sigma$ such that $\eta_1(\Phi^{c+\varepsilon}) \subset \Phi^{c-\varepsilon}$ (see, e.g., Brezis and Nirenberg [1]). Take a $\sigma \in \Sigma$ such that $\sigma_1(A) \subset \Phi^{c+\varepsilon}$ and define $\widetilde{\sigma} \in \Sigma$ by

$$\widetilde{\sigma}_t(u) = \begin{cases} \sigma_{2t}(u), & 0 \leqslant t \leqslant 1/2, \\ \eta_{2t-1}(\sigma_1(u)), & 1/2 < t \leqslant 1. \end{cases} \tag{2.4}$$

Then $\widetilde{\sigma}_1(A) \subset \Phi^{c-\varepsilon}$, contradicting the definition (2.3) of c. $\square$

Let

$$S = \{u \in W \colon \|u\| = 1\} \tag{2.5}$$

be the unit sphere in W and let

$$\pi_S : W \setminus \{0\} \to S, \qquad u \mapsto \frac{u}{\|u\|} \tag{2.6}$$

be the radial projection onto S. Now let $\mathcal{M}$ be a bounded symmetric subset of $W \setminus \{0\}$ radially homeomorphic to S, i.e., $g = \pi_S|_{\mathcal{M}} : \mathcal{M} \to S$ is a homeomorphism. Then the radial projection from $W \setminus \{0\}$ onto $\mathcal{M}$ is given by $\pi_{\mathcal{M}} = g^{-1} \circ \pi_S$. For $A \subset \mathcal{M}$ and $r \geqslant 0$, we set

$$rA = \{ru \colon u \in A\} \tag{2.7}$$

and

$$\widetilde{A} = \pi_{\mathcal{M}}^{-1}(A) \cup \{0\} = \bigcup_{r \geqslant 0} rA. \tag{2.8}$$

We denote by SA the suspension of $A \subset W$, obtained from $A \times [-1, 1]$ by collapsing $A \times \{1\}$ and $A \times \{-1\}$ to different points, which can be realized in $W \oplus \mathbb{R}$ as the union of all line segments joining the two points $(0, \pm 1) \in W \oplus \mathbb{R}$ to points of A. For a symmetric subset A of $W \setminus \{0\}$, we denote by $i(A)$ the cohomological index of A and recall that

$$i(SA) = i(A) + 1 \tag{2.9}$$

when A is closed (see Fadell and Rabinowitz [2]).

Theorem 2.2. *If A_0, B_0 is a pair of disjoint nonempty closed symmetric subsets of $\mathcal{M}$ such that*

$$i(A_0) = i(\mathcal{M} \setminus B_0) < \infty \tag{2.10}$$

and h is an odd homeomorphism of W such that

$$\operatorname{dist}\bigl(h(rA_0), h(\widetilde{B_0})\bigr) \to \infty \quad \text{as} \quad r \to \infty, \tag{2.11}$$

then $A = h(\widetilde{A_0})$, $B = h(\widetilde{B_0})$ form a sandwich pair.

Proof. By Lemma 2.1, it suffices to verify (2.2), so suppose there is a $\sigma \in \Sigma$ with

$$\sigma_1(A) \cap B = \emptyset. \tag{2.12}$$

By (2.11), there is an $R > 1$ such that

$$\operatorname{dist}\bigl(h(RA_0), h(\widetilde{B_0})\bigr) > \sup_{(u,t) \in W \times [0,1]} \bigl\|\sigma_t(u) - u\bigr\| \tag{2.13}$$

and hence

$$\sigma_t\bigl(h(RA_0)\bigr) \cap B = \emptyset \quad \forall t \in [0, 1]. \tag{2.14}$$

By (2.12) and (2.14), we can define a map $\eta \in C(A_0 \times [0, 1], W \setminus B)$ by

$$\eta(u,t)=\begin{cases} h((1-3t+3Rt)u), & u\in A_0,\ 0\leqslant t\leqslant 1/3,\\ \sigma_{3t-1}(h(Ru)), & u\in A_0,\ 1/3<t\leqslant 2/3,\\ \sigma_1(h(3(1-t)Ru)), & u\in A_0,\ 2/3<t\leqslant 1. \end{cases} \tag{2.15}$$

Since $\eta|_{A_0\times\{0\}}=h|_{A_0}$ is odd and $\eta(A_0\times\{1\})$ is the single point $\sigma_1(h(0))$, η can be extended to an odd map $\widetilde{\eta}\in C(SA_0,W\setminus B)$. Then $\pi_{\mathcal{M}}\circ h^{-1}\circ\widetilde{\eta}$ is an odd continuous map from SA_0 into $\mathcal{M}\setminus B_0$ and hence

$$i(\mathcal{M}\setminus B_0)\geqslant i(SA_0)=i(A_0)+1 \tag{2.16}$$

by the monotonicity of the index, contradicting (2.10). □

3. Eigenvalue problems for *p*-Laplacian systems

In this section we recall some results on eigenvalue problems for p-Laplacian systems proved in Perera et al. [3]. Define a continuous flow on W, as well as on $\mathbb{R}^m$, by

$$(\alpha,u)\mapsto u_\alpha:=\big(|\alpha|^{1/p_1-1}\alpha u_1,\dots,|\alpha|^{1/p_m-1}\alpha u_m\big) \tag{3.1}$$

for $\alpha\in\mathbb{R}$. Noting that the functional in (1.10) satisfies

$$I(u_\alpha)=|\alpha|I(u)\quad\forall\alpha\in\mathbb{R},\ u\in W, \tag{3.2}$$

we consider the eigenvalue problem

$$\begin{cases} -\Delta_p u=\lambda\nabla_u J(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

associated with our problem (1.3), where $J\in C^1(\Omega\times\mathbb{R}^m)$ is positive somewhere and satisfies

$$J(x,u_\alpha)=|\alpha|J(x,u)\quad\forall\alpha\in\mathbb{R},\ (x,u)\in\Omega\times\mathbb{R}^m \tag{3.4}$$

and the growth condition (1.4) with J in place of F.

For example, taking

$$J(x,u)=|u_1|^{r_1}\cdots|u_m|^{r_m} \tag{3.5}$$

with $r_i\in(1,p_i)$ and

$$\sum_{i=1}^m\frac{r_i}{p_i}=1 \tag{3.6}$$

gives

$$\begin{cases} -\Delta_{p_i}u_i=\lambda r_i|u_1|^{r_1}\cdots|u_i|^{r_i-2}u_i\cdots|u_m|^{r_m} & \text{in } \Omega,\ i=1,\dots,m,\\ u_1=\cdots=u_m=0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Let

$$J(u)=\int_\Omega J(x,u),\quad u\in W \tag{3.8}$$

and

$$\mathcal{M}=\big\{u\in W:\ I(u)=1\big\},\qquad \mathcal{M}^+=\big\{u\in\mathcal{M}:\ J(u)>0\big\}. \tag{3.9}$$

Then $\mathcal{M}\subset W\setminus\{0\}$ is a bounded symmetric C^1-Finsler manifold radially homeomorphic to S, $\mathcal{M}^+$ is an open submanifold of $\mathcal{M}$, and positive eigenvalues of (3.3) coincide with critical values of

$$\Psi(u)=\frac{1}{J(u)},\quad u\in\mathcal{M}^+ \tag{3.10}$$

(see Lemmas 10.1.4 and 10.1.5 of Perera et al. [3]). Taking $\alpha=-1$ in (3.4) shows that $J(x,u)$ is even in u, so Ψ is even. Letting $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}^+$, we can define a positive, nondecreasing, and unbounded sequence of eigenvalues of (3.3) by

$$\lambda_k:=\inf_{\substack{M\in\mathcal{F}\\ i(M)\geqslant k}}\ \sup_{u\in M}\Psi(u), \tag{3.11}$$

and for this particular sequence of eigenvalues

$$i\big(\Psi^{\lambda_k}\big) = i\big(\mathcal{M}^+ \setminus \Psi_{\lambda_{k+1}}\big) = k \tag{3.12}$$

when $\lambda_k < \lambda_{k+1}$ (see Theorem 10.1.8 of Perera et al. [3]).

4. Main result

In this section we give sufficient conditions on F for the existence of a solution to our problem (1.3). Let $\mathcal{M}$ be as in (3.9). Identifying W with $\{\alpha u\colon u \in \mathcal{M},\ \alpha \geqslant 0\}$,

$$h(\alpha u) = u_\alpha \tag{4.1}$$

defines an odd homeomorphism of W. For $A \subset \mathcal{M}$ and $\widetilde{A}$ defined by (2.8),

$$h(\widetilde{A}) = \{u_\alpha\colon u \in A,\ \alpha \geqslant 0\}. \tag{4.2}$$

We also note that

$$I(u_\alpha) = \alpha, \qquad J(u_\alpha) = \alpha J(u) \quad \forall u \in \mathcal{M},\ \alpha \geqslant 0 \tag{4.3}$$

by (3.2) and (3.4), respectively.

Lemma 4.1. *If $\lambda_k < \lambda_{k+1}$ and*

$$\lambda_k J(x,u) - W(x) \leqslant F(x,u) \leqslant \lambda_{k+1} J(x,u) + W(x) \quad \forall (x,u) \in \Omega \times \mathbb{R}^m \tag{4.4}$$

for some $W \in L^1(\Omega)$, then Φ has a $(\mathrm{PS})_c$ sequence for some $c \in [-K, K]$ where $K = \int_\Omega W(x)$.

Proof. For $u \in \mathcal{M}$ and $\alpha \geqslant 0$, integrating (4.4) with u_α in place of u gives

$$\lambda_k J(u_\alpha) - K \leqslant \int_\Omega F(x, u_\alpha) \leqslant \lambda_{k+1} J(u_\alpha) + K, \tag{4.5}$$

and hence

$$\alpha\big(1 - \lambda_{k+1} J(u)\big) - K \leqslant \Phi(u_\alpha) \leqslant \alpha\big(1 - \lambda_k J(u)\big) + K \tag{4.6}$$

by (4.3).

Let $A_0 = \Psi^{\lambda_k}$ and $B_0 = \Psi_{\lambda_{k+1}} \cup (\mathcal{M} \setminus \mathcal{M}^+)$ where $\mathcal{M}^+$ and Ψ are as in (3.9) and (3.10). Then (3.12) implies (2.10), so $A = h(\widetilde{A_0})$, $B = h(\widetilde{B_0})$ form a sandwich pair by Theorem 2.2.

By (4.2),

$$A = \{u_\alpha\colon u \in A_0,\ \alpha \geqslant 0\}, \qquad B = \{u_\alpha\colon u \in B_0,\ \alpha \geqslant 0\}. \tag{4.7}$$

For $u \in A_0$ and $\alpha \geqslant 0$, $J(u) \geqslant 1/\lambda_k$ and hence $\Phi(u_\alpha) \leqslant K$ by (4.6), so $\Phi \leqslant K$ on A by (4.7). Similarly, $J(u) \leqslant 1/\lambda_{k+1}$ and hence $\Phi(u_\alpha) \geqslant -K$ for $u \in B_0$ and $\alpha \geqslant 0$, so $\Phi \geqslant -K$ on B. $\square$

Let

$$H(x,u) = F(x,u) - \sum_{i=1}^m \frac{u_i}{p_i} \frac{\partial F}{\partial u_i} \tag{4.8}$$

and

$$\tau(u) = \sum_{i=1}^m \frac{1}{p_i} |u_i|^{p_i}. \tag{4.9}$$

Note that

$$\tau(u_\alpha) = |\alpha| \tau(u) \quad \forall \alpha \in \mathbb{R},\ u \in \mathbb{R}^m. \tag{4.10}$$

Lemma 4.2. *If (4.4) holds, then Φ satisfies $(\mathrm{PS})_c$ for all $c \in \mathbb{R}$ in the following cases:*

(i) $\quad H(x,u) \leqslant C\big(\tau(u) + 1\big)$ *and* $\overline{H}(x) := \varlimsup_{\tau(u)\to\infty} H(x,u)/\tau(u) < 0,$

(ii) $$H(x,u) \geqslant -C\big(\tau(u)+1\big) \quad \text{and} \quad \underline{H}(x) := \varliminf_{\tau(u)\to\infty} H(x,u)/\tau(u) > 0$$

for some $C > 0$.

Proof. We give the proof under assumption (i). The proof under (ii) is similar. Let (u^j) be a $(\mathrm{PS})_c$ sequence. By a standard argument, it suffices to show that $\{u^j\}$ is bounded, so suppose $\rho_j := I(u^j) \to \infty$ and set $\widetilde{u}^j := u^j_{1/\rho_j}$. Then $I(\widetilde{u}^j) = 1$ by (3.2) and hence a subsequence of $(\widetilde{u}^j)$ converges to some $\widetilde{u}$ weakly in W, strongly in $L^{p_1}(\Omega) \times \cdots \times L^{p_m}(\Omega)$, and a.e. in $\Omega \times \cdots \times \Omega$. We have

$$\int_\Omega \frac{H(x,u^j)}{\rho_j} = \frac{\langle \Phi'(u^j), (u^j_1/p_1, \ldots, u^j_m/p_m)\rangle - \Phi(u^j)}{\rho_j} \to 0 \tag{4.11}$$

by (1.1). On the other hand, $\tau(u^j)/\rho_j = \tau(\widetilde{u}^j)$ by (4.10) and hence

$$\varlimsup \int_\Omega \frac{H(x,u^j)}{\rho_j} \leqslant \int_{\{\widetilde{u}\neq 0\}} \varlimsup \frac{H(x,u^j)}{\tau(u^j)} \tau\big(\widetilde{u}^j\big) + \int_{\{\widetilde{u}=0\}} \lim C\left(\tau\big(\widetilde{u}^j\big) + \frac{1}{\rho_j}\right) = \int_{\{\widetilde{u}\neq 0\}} \overline{H}(x)\tau(\widetilde{u}) \leqslant 0. \tag{4.12}$$

It follows that $\widetilde{u} = 0$. But, passing to the limit in

$$1 - \frac{\Phi(u^j)}{\rho_j} = \int_\Omega \frac{F(x,u^j)}{\rho_j} \leqslant \int_\Omega \lambda_{k+1} J\big(x, \widetilde{u}^j\big) + \frac{W(x)}{\rho_j} \tag{4.13}$$

gives $1 \leqslant \lambda_{k+1} J(\widetilde{u})$, and hence $\widetilde{u} \neq 0$ since taking $\alpha = 0$ in (3.4) shows that $J(0) = 0$, a contradiction. □

We now have

Theorem 4.3. *Under the hypotheses of Lemmas* 4.1 *and* 4.2, *problem* (1.3) *has a solution.*

References

[1] Haím Brezis, Louis Nirenberg, Remarks on finding critical points, Comm. Pure Appl. Math. 44 (8–9) (1991) 939–963.
[2] Edward R. Fadell, Paul H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems., Invent. Math. 45 (2) (1978) 139–174.
[3] Kanishka Perera, Ravi P. Agarwal, Donal O'Regan, Morse-Theoretic Aspects of p-Laplacian Type Operators, in press.
[4] Kanishka Perera, Martin Schechter, Sandwich pairs in p-Laplacian problems, Topol. Methods Nonlinear Anal. 29 (1) (2007) 29–34.
[5] Kanishka Perera, Martin Schechter, Flows critical points, NoDEA Nonlinear Differential Equations Appl. 15 (4–5) (2008) 495–509.
[6] Martin Schechter, Sandwich pairs, in: Differential & Difference Equations and Applications, Hindawi Publ. Corp., New York, 2006, pp. 999–1007.