J. Math. Anal. Appl. 364 (2010) 384–394

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Log-convexity and log-concavity of hypergeometric-like functions

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ARTICLE INFO

Article history:
Received 2 March 2009
Available online 24 October 2009
Submitted by B.C. Berndt

Keywords:
Hypergeometric function
Kummer function
Gauss function
Log-convexity
Log-concavity
Turán inequality

ABSTRACT

We find sufficient conditions for log-convexity and log-concavity for the functions of the forms $a \mapsto \sum f_k(a)_k x^k$, $a \mapsto \sum f_k \Gamma(a+k)x^k$ and $a \mapsto \sum f_k x^k/(a)_k$. The most useful examples of such functions are generalized hypergeometric functions. In particular, we generalize the Turán inequality for the confluent hypergeometric function recently proved by Barnard, Gordy and Richards and log-convexity results for the same function recently proved by Baricz. Besides, we establish a reverse inequality which complements naturally the inequality of Barnard, Gordy and Richards. Similar results are established for the Gauss and the generalized hypergeometric functions. A conjecture about monotonicity of a quotient of products of confluent hypergeometric functions is made.

1. Introduction

This paper is motivated by some recent results dealing with log-convexity and log-concavity of hypergeometric functions as functions of parameters. More specifically, Baricz showed in [2] that the Kummer function (or the confluent hypergeometric function, see [10])

$$ {}_1F_1(a;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{x^k}{k!}, \tag{1}$$

where $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol, is log-convex in c on $(0,\infty)$ for $a, x > 0$ as well as the function $\mu \mapsto {}_1F_1(a+\mu; c+\mu; x)$ on $[0,\infty)$. This implies, in particular, the reverse Turán type inequality

$$ {}_1F_1(a;c+1;x)^2 \leqslant {}_1F_1(a;c;x)\,{}_1F_1(a;c+2;x). \tag{2}$$

(This sort of inequalities is called "Turán type" after Paul Turán in a 1946 letter to Szegő proved the inequality $[P_n(x)]^2 > P_{n-1}(x)P_{n+1}(x)$, $-1 < x < 1$, for Legendre polynomials P_n, which has a similar look as (2) but different nature (see [20]).) Baricz's other results [3,4] deal with log-convexity and some more general comparisons of means for the Bessel functions (expressible in terms of ${}_0F_1$) and the Gauss function ${}_2F_1$. Many of his proofs hinge on the additivity of concavity and logarithmic convexity. This method does not work, however, for proving logarithmic concavity since in general it is not additive.

Closely related results were given a bit earlier by Ismail and Laforgia in [13]. In particular, they showed that the determinant

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.057

$$D_n(x) = \begin{vmatrix} h(a,c,x) & h(a,c+1,x) & \cdots & h(a,c+n,x) \\ h(a,c+1,x) & h(a,c+2,x) & \cdots & h(a,c+n+1,x) \\ \vdots & \vdots & & \vdots \\ h(a,c+n,x) & h(a,c+n+1,x) & \cdots & h(a,c+n,x) \end{vmatrix},$$

where $h(a,c,x) = \Gamma(c-a){}_1F_1(a;c;x)/\Gamma(c)$, has positive power series coefficients. For $n=1$ this leads to an inequality which is weaker than (2).

Carey and Gordy conjectured in [8] that the Turán type inequality

$$\left[{}_1F_1(a;c;x)\right]^2 > {}_1F_1(a+1;c;x){}_1F_1(a-1;c;x),$$

holds for $a>0$, $c>a+2$, $x>0$. Using a clever combination of contiguous relations and telescoping sums Barnard, Gordy and Richards have recently shown in [5] that this is indeed true and even more general inequality

$$\left[{}_1F_1(a;c;x)\right]^2 \geqslant {}_1F_1(a+\nu;c;x){}_1F_1(a-\nu;c;x) \tag{3}$$

holds for $a>0$, $c>a\geqslant\nu-1$ and $x\in\mathbf{R}$ or $a\geqslant\nu-1$, $c>-1$ ($c\neq0$), $x>0$, and positive integer ν. In fact, the authors show that the difference of the left-hand and the right-hand sides of (3) has positive power series coefficients for $a>0$, $a\geqslant\nu-1$, $c>-1$. They also indicate that a similar result is true for the generalized hypergeometric function (see [10] for its various properties)

$${}_pF_q\big((a_p);(b_q);x\big) = {}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix}\,\middle|\,x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n n!}x^n \tag{4}$$

if $p\leqslant q+1$, $a_i>b_i$, $i=2,\ldots,p$ and $b_i>0$, $i=1,\ldots,q$.

In this paper we shall demonstrate that log-convexity and log-concavity properties of hypergeometric functions in their upper and lower parameters serve as an illustration of a more general phenomenon. Namely, we give sufficient conditions for the function $x\mapsto f(a+\delta,x)f(b,x)-f(b+\delta,x)f(a,x)$ to have positive power series coefficients if $f(a,x)=\sum f_k(a)_k x^k$, $f(a,x)=\sum f_k\Gamma(a+k)x^k$ or $f(a,x)=\sum f_k x^k/(a)_k$, where $f_k>0$ for all k. Log-convexity or log-concavity then follow immediately. Section 2 of the paper contains three general theorems for these three types of functions and a corollary which includes direct and reverse Turán type inequalities. Section 3 collects some applications to hypergeometric functions. In particular, we extend the result Barnard, Gordy and Richards (3) to non-integer positive ν and complement it with a reverse inequality giving asymptotically precise lower bound for the quantity

$$\frac{{}_1F_1(a+\nu;c;x){}_1F_1(a-\nu;c;x)}{[{}_1F_1(a;c;x)]^2}$$

bounded by 1 from above according to (3). We also extend some results of Baricz and provide sufficient conditions for log-concavity and log-convexity of the generalized hypergeometric functions which are less restrictive then the conditions $a_i>b_i$ for the (less general) Turán type inequality given in [5]. We use the generalized Stieltjes transform representation for ${}_{q+1}F_q$ from [15] to extend this results to negative x. One curious corollary of these results is positivity of certain finite hypergeometric sums evaluated at -1.

We note in passing that the true Turán type inequalities for the classical orthogonal polynomials [11,20] have been also shown to exemplify a more general phenomenon. Namely, it has been demonstrated in [19] that they are dependent on certain monotonicity properties of the coefficients of three-term recurrence relations. See further development in [6].

2. General theorems

A function $f:[a,b]\mapsto\mathbf{R}$ is said to be Wright-convex (strictly Wright-convex) if $F_h(x)=f(x+h)-f(x)$ is non-decreasing (increasing) on $[a,b-h]$ for any fixed $h>0$. If $F_h(x)$ is non-increasing (decreasing) then f is Wright-concave (strictly Wright-concave). This notion was introduced by Wright in the 1950s and well studied (see [9, p. 246] and [16, p. 3]). Clearly, Wright-convexity implies mid-point convexity and it can be shown (see [16, p. 2]) that convexity implies Wright-convexity so that by the celebrated result of Jensen (see, for instance, [17, Theorem 1.1.4]) for continuous functions all three notions (convexity, Wright-convexity and mid-point convexity) are equivalent. In general the inclusions Convex $\subset$ Wright-convex $\subset$ Midpoint convex are proper. We only deal here with log-convexity (log-concavity) of continuous functions so for our purposes we record

Proposition 1. *Suppose $f:[a,b]\mapsto\mathbf{R}$ is continuous and has constant sign. Then f is log-convex (strictly log-convex) iff $x\mapsto f(x+h)/f(x)$ is non-decreasing (increasing) on $[a,b-h]$ for each fixed $h>0$ and f is log-concave (strictly log-concave) iff $x\mapsto f(x+h)/f(x)$ is non-increasing (decreasing) on $[a,b-h]$ for each fixed $h>0$.*

Using Proposition 1 we will formulate our results in terms of more common log-convexity (log-concavity) while if fact we prove Wright log-convexity (log-concavity).

In what follows in this section the power series expansions are understood as formal, so that no questions of convergence are discussed. It is usually clear in specific applications which variable range should be considered. We will use the standard notation $(a)_n = a(a+1)\cdots(a+n-1)$ for the shifted factorial or the Pochhammer symbol. The main idea in the proof of the next theorem belongs to Fedor Nazarov (University of Wisconsin).

Theorem 1. *Let*

$$f(a,x) = \sum_{n=0}^{\infty} f_n \frac{(a)_n}{n!} x^n, \tag{5}$$

where $f_n > 0$ (and is independent of a). Suppose $b > a > 0$, $\delta > 0$. Then the function

$$\varphi_{a,b,\delta}(x) = f(a+\delta,x)f(b,x) - f(b+\delta,x)f(a,x) = \sum_{m=2}^{\infty} \varphi_m x^m$$

has positive power series coefficients $\varphi_m > 0$ so that $a \mapsto f(a,x)$ is strictly log-concave for $x > 0$ if the sequence $\{f_n/f_{n-1}\}_{n=0}^{\infty}$ is decreasing and negative power series coefficients $\varphi_m < 0$ so that $a \mapsto f(a,x)$ is strictly log-convex for $x > 0$ if the sequence $\{f_n/f_{n-1}\}_{n=0}^{\infty}$ is increasing.

Remark 1. Since, clearly $\varphi_{a,b,\delta}(x) = -\varphi_{b,a,\delta}(x)$, the sign of φ_m is reversed for $a > b > 0$.

Remark 2. The log-concavity condition from Theorem 1 can be rewritten as the Turán inequality $f_n^2 > f_{n+1}f_{n-1}$. This implies that the log-concavity of $f(a,x)$ is assured if $f(x) = \sum f_n x^n/n!$ belongs to the Laguerre–Pólya class $\mathcal{L}-\mathcal{P}^{+}$. See details on Laguerre–Pólya classes in [18].

Proof of Theorem 1. By direct multiplication we have

$$\varphi_m = \sum_{k=0}^{m} f_k f_{m-k} \left\{ \frac{(a+\delta)_k (b)_{m-k}}{k!(m-k)!} - \frac{(b+\delta)_k (a)_{m-k}}{k!(m-k)!} \right\}.$$

This shows, on inspection, that $\varphi_0 = \varphi_1 = 0$, which explains why summation starts from $m = 2$ in the expansion for $\varphi_{a,b,\delta}(x)$. Further, we can write φ_m in the form

$$\varphi_m = \sum_{k=0}^{[m/2]} f_k f_{m-k} M_k \tag{6}$$

with

$$M_k = \begin{cases} [(a+\delta)_k(b)_{m-k} + (a+\delta)_{m-k}(b)_k - (a)_k(b+\delta)_{m-k} - (a)_{m-k}(b+\delta)_k]/[k!(m-k)!], & k < m/2, \\ [(a+\delta)_k(b)_{m-k} - (a)_k(b+\delta)_{m-k}]/[k!(m-k)!], & k = m/2. \end{cases}$$

Next we see that

$$\sum_{k=0}^{[m/2]} M_k = 0 \tag{7}$$

since for $f_n = 1$, $n = 0, 1, \ldots$, we will have $f(a,x) = (1-x)^{-a}$ by binomial theorem and hence the left-hand side of (7) is the coefficient at x^m in the power series expansion of

$$(1-x)^{-a-\delta}(1-x)^{-b} - (1-x)^{-a}(1-x)^{-b-\delta} = 0.$$

We aim to show that the sequence $\{M_k\}_{k=0}^{[m/2]}$ has exactly one change of sign, namely some number of initial terms are negative while all further terms are positive. To establish the claim note that $(a+\delta)_l(b)_l > (a)_l(b+\delta)_l$ for all l (since $b > a$ and $x \mapsto (x+\gamma)/x$ is decreasing for positive x and γ) and hence $M_0 < 0$. Now assume that $M_k \leqslant 0$ for some $k \leqslant n/2$, i.e.

$$\underbrace{(a+\delta)_k(b)_{n-k}}_{=r} + \underbrace{(a+\delta)_{n-k}(b)_k}_{=s} \leqslant \underbrace{(a)_k(b+\delta)_{n-k}}_{=v} + \underbrace{(a)_{n-k}(b+\delta)_k}_{=u}.$$

We want to show that the same inequality is true for $k-1$. We have by inspection $rs \geqslant uv$ and $v \geqslant u$. A short reflection shows that together with the above inequality $r+s \leqslant u+v$ this yields either $v \geqslant r \geqslant s \geqslant u$ or $v \geqslant s \geqslant r \geqslant u$ (another apparent possibility $r > v \geqslant u > s$ is discarded by noting that it implies that $u' = u/r$, $v' = v/r$, $s' = s/r$ all belong to $(0, 1)$ and satisfy $u' + v' \geqslant 1 + s'$ and $s' \geqslant u'v'$ so that $u' + v' \geqslant 1 + u'v'$ which contradicts the elementary inequality $u' + v' < 1 + u'v'$, similarly for $s > v \geqslant u > r$). We need to prove that

$$M_{k-1}(\delta) = r\frac{b+m-k}{a+\delta+k-1} + s\frac{a+\delta+m-k}{b+k-1} - v\frac{b+\delta+m-k}{a+k-1} - u\frac{a+m-k}{b+\delta+k-1} \leqslant 0.$$

For $\delta = 0$, we clearly have $\frac{b+m-k}{a+k-1} \geqslant \frac{a+m-k}{b+k-1}$, so the desired inequality is just a combination of $v \geqslant r$ and $u + v \geqslant r + s$ with positive coefficients. Treating u, v, r, s as constants and differentiating with respect to δ, we get

$$M'_{k-1}(\delta) = u\frac{a+m-k}{(b+\delta+k-1)^2} - v\frac{1}{a+k-1} + s\frac{1}{b+k-1} - r\frac{b+m-k}{(a+\delta+k-1)^2}$$

which is obviously non-positive since $v \geqslant s$ and $r \geqslant u$, which proves that $M_{k-1} \leqslant 0$ and hence that $\{M_k\}_{k=0}^{[m/2]}$ changes sign exactly once. Now if $\{f_n/f_{n-1}\}_{n=0}^{\infty}$ is decreasing, then for $k < m - k + 1$

$$\frac{f_k}{f_{k-1}} > \frac{f_{m-k+1}}{f_{m-k}} \quad \Leftrightarrow \quad f_k f_{m-k} > f_{k-1} f_{m-k+1}$$

which combined with (6) and (7) shows that $\varphi_m > 0$. Similarly, if $\{f_n/f_{n-1}\}_{n=0}^{\infty}$ is increasing, then for $k < m - k + 1$

$$\frac{f_k}{f_{k-1}} < \frac{f_{m-k+1}}{f_{m-k}} \quad \Leftrightarrow \quad f_k f_{m-k} < f_{k-1} f_{m-k+1}$$

and $\varphi_m < 0$. □

Theorem 2. *Let*

$$g(a, x) = \sum_{n=0}^{\infty} g_n \Gamma(a+n) x^n,$$

where $g_n > 0$ (and is independent of a) and $\Gamma(\cdot)$ is Euler's gamma function. Suppose $b > a > 0$, $\delta > 0$. Then the function

$$\psi_{a,b,\delta}(x) = g(a+\delta, x) g(b, x) - g(b+\delta, x) g(a, x) = \sum_{m=0}^{\infty} \psi_m x^m$$

has negative power series coefficients $\psi_m < 0$ so that the function $a \mapsto g(a, x)$ is strictly log-convex for $x > 0$.

Proof. Again, by direct multiplication we have

$$\psi_m = \sum_{k=0}^{m} g_k g_{m-k} \big\{\Gamma(a+\delta+k)\Gamma(b+m-k) - \Gamma(b+\delta+k)\Gamma(a+m-k)\big\}.$$

Just like in the proof of Theorem 1, we can write ψ_m in the form

$$\psi_m = \sum_{k=0}^{[m/2]} g_k g_{m-k} M_k \tag{8}$$

with

$$M_k = \begin{cases} \underbrace{\Gamma(a+\delta+k)\Gamma(b+m-k)}_{=r} + \underbrace{\Gamma(a+\delta+m-k)\Gamma(b+k)}_{=s} & \\ \quad - \underbrace{\Gamma(a+k)\Gamma(b+\delta+m-k)}_{=v} - \underbrace{\Gamma(a+m-k)\Gamma(b+\delta+k)}_{=u}, & k < m/2, \\ \Gamma(a+\delta+k)\Gamma(b+m-k) - \Gamma(a+k)\Gamma(b+\delta+m-k), & k = m/2. \end{cases}$$

We aim to show that $M_k < 0$ for $k = 0, 1, \ldots, [m/2]$. The basic fact that we need is that $x \mapsto \Gamma(x+\alpha)/\Gamma(x+\beta)$ is strictly increasing for $x > 0$ when $\alpha > \beta \geqslant 0$. This immediately implies that $M_k < 0$ for $k = m/2$ on taking $\alpha = \delta + k = \delta + m - k$, $\beta = k = m - k$. Further for $k < m - k$ we have

$$r<v \quad\Leftrightarrow\quad \frac{\Gamma(a+\delta+k)}{\Gamma(a+k)}<\frac{\Gamma(b+\delta+m-k)}{\Gamma(b+m-k)},$$
$$s<v \quad\Leftrightarrow\quad \frac{\Gamma(b+k)}{\Gamma(a+k)}<\frac{\Gamma(b+\delta+m-k)}{\Gamma(a+\delta+m-k)},$$
$$u<v \quad\Leftrightarrow\quad \frac{\Gamma(b+\delta+k)}{\Gamma(a+k)}<\frac{\Gamma(b+\delta+m-k)}{\Gamma(a+m-k)},$$
$$rs<uv \quad\Leftrightarrow\quad \frac{\Gamma(b+k)\Gamma(a+\delta+m-k)}{\Gamma(a+k)\Gamma(a+m-k)}<\frac{\Gamma(b+\delta+k)\Gamma(b+\delta+m-k)}{\Gamma(a+\delta+k)\Gamma(b+m-k)}.$$

Altogether these inequalities imply $r+s<u+v \Leftrightarrow M_k<0$. Indeed, dividing by v we can rewrite $r+s<u+v$ as $r'+s'<1+u'$, where $r'=r/v\in(0,1)$, $s'=s/v\in(0,1)$, $u'=u/v\in(0,1)$. Since $r's'<u'$ from $rs<uv$, the required inequality follows from the elementary inequality $r'+s'<1+r's'$. $\square$

Corollary 1. *Let $f(a,x)$ be given by* (5) *with decreasing sequence $\{f_n/f_{n-1}\}$, then for $b>a>0$ and $x>0$*

$$\frac{\Gamma(a+\delta)\Gamma(b)}{\Gamma(b+\delta)\Gamma(a)}<\frac{f(b+\delta,x)f(a,x)}{f(a+\delta,x)f(b,x)}<1.$$

Proof. Indeed, since $(a)_k=\Gamma(a+k)/\Gamma(a)$ we can choose g in Theorem 2 in the form $g(a,x)=\Gamma(a)f(a,x)$, where f is given by (5). Now the estimate from above is just Theorem 1 while the estimate from below is Theorem 2. $\square$

Remark 3. Choosing $b=a+\delta$ and $\delta=1$ we get the direct and reverse Turán type inequalities for $f(a,x)$ given by (5) with decreasing sequence $\{f_n/f_{n-1}\}$:

$$\frac{a}{a+1}<\frac{f(a+2,x)f(a,x)}{f(a+1,x)^2}<1.$$

Theorem 3. *Let*

$$h(a,x)=\sum_{n=0}^{\infty}\frac{h_n}{(a)_n}x^n,$$

where $h_n>0$ (and is independent of a). Suppose $b>a>0$. Then the function

$$\lambda_{a,b,\delta}(x)=h(a+\delta,x)h(b,x)-h(b+\delta,x)h(a,x)=\sum_{m=1}^{\infty}\lambda_m x^m$$

has negative power series coefficients $\lambda_m<0$ so that the function $a\mapsto h(a,x)$ is strictly log-convex for $x>0$.

Proof. By direct multiplication we have

$$\lambda_m=\sum_{k=0}^{m}h_kh_{m-k}\left\{\frac{1}{(a+\delta)_k(b)_{m-k}}-\frac{1}{(b+\delta)_k(a)_{m-k}}\right\}.$$

This shows, on inspection, that $h_0=0$, which explains why summation starts from $m=1$ in the expansion for $\lambda_{a,b,\delta}(x)$. Just like in the proof of Theorem 1, we can write λ_m in the form

$$\lambda_m=\sum_{k=0}^{[m/2]}h_kh_{m-k}M_k \tag{9}$$

with

$$M_k=\begin{cases}\underbrace{[(a+\delta)_k(b)_{m-k}]^{-1}}_{=r}+\underbrace{[(a+\delta)_{m-k}(b)_k]^{-1}}_{=s}-\underbrace{[(a)_{m-k}(b+\delta)_k]^{-1}}_{=v}-\underbrace{[(a)_k(b+\delta)_{m-k}]^{-1}}_{=u}, & k<m/2,\\ [(a+\delta)_k(b)_{m-k}]^{-1}-[(a)_k(b+\delta)_{m-k}]^{-1}, & k=m/2.\end{cases}$$

We aim to show that $M_k<0$ for $k=0,1,\dots,[m/2]$. First $M_k<0$ for $k=m-k=m/2$, since

$$\frac{(b+\delta)_k}{(b)_k}<\frac{(a+\delta)_k}{(a)_k}$$

because $x \mapsto (x+\alpha)/(x+\beta)$, $\alpha > \beta \geqslant 0$, is decreasing for $x > 0$. Further for $k < m-k$ we have the inequalities

$$\begin{aligned} r < v \quad &\Leftrightarrow \quad \frac{(b+\delta)_k}{(a+\delta)_k} < \frac{(b)_{m-k}}{(a)_{m-k}}, \\ s < v \quad &\Leftrightarrow \quad \frac{(b+\delta)_k}{(b)_k} < \frac{(a+\delta)_{m-k}}{(a)_{m-k}}, \\ u < v \quad &\Leftrightarrow \quad \frac{(b+\delta)_k}{(a)_k} < \frac{(b+\delta)_{m-k}}{(a)_{m-k}}, \\ rs < uv \quad &\Leftrightarrow \quad \frac{(b+\delta)_k(b+\delta)_{m-k}}{(a+\delta)_k(a+\delta)_{m-k}} < \frac{(b)_k(b)_{m-k}}{(a)_k(a)_{m-k}}. \end{aligned}$$

Altogether these inequalities imply $r+s < u+v \Leftrightarrow M_k < 0$ as shown in the proof of Theorem 2. □

3. Applications to hypergeometric functions

First consider the Kummer function (1). For $a, c > 0$ it satisfies the conditions of Theorem 1 with $f_n = 1/(c)_n$ and conditions of Theorem 3 with $h_n = (a)_n/n!$. Besides, $\Gamma(a)_1F_1(a;c;x)$ satisfies the conditions of Theorem 2 with $g_n = 1/[(c)_n n!]$. Clearly, $f_n/f_{n-1} = 1/(c+n-1)$ is decreasing and we are also in the position to apply Corollary 1. Using the Kummer transformation ${}_1F_1(a;c;x) = e^{-x}{}_1F_1(c-a;c;-x)$ we can extend some of the results to negative x. We collect the consequences of the general theorems for the Kummer function in the following two statements.

Theorem 4. *Suppose $\delta > 0$. Then*

a) *for $b > a \geqslant 0$, $c > 0$ the function*

$$x \mapsto {}_1F_1(a+\delta;c;x)\,{}_1F_1(b;c;x) - {}_1F_1(b+\delta;c;x)\,{}_1F_1(a;c;x)$$

has positive power series coefficients (starting with the coefficient at x^2);

b) *the function $a \mapsto {}_1F_1(a+\delta;c;x)/{}_1F_1(a;c;x)$ is monotone decreasing on $[0,\infty)$ for fixed $c, x > 0$ and on $(-\infty, c-\delta]$ for fixed $c > 0 > x$, so that $a \mapsto {}_1F_1(a;c;x)$ is log-concave,*

$${}_1F_1(a+\delta;c;x)^2 \geqslant {}_1F_1(a;c;x)\,{}_1F_1(a+2\delta;c;x),$$

on $[0,\infty)$ for fixed $c, x > 0$ and on $(-\infty, c]$ for fixed $c > 0 > x$;

c) *for $b > a > 0$ and $c, x > 0$*

$$\frac{\Gamma(a+\delta)\Gamma(b)}{\Gamma(b+\delta)\Gamma(a)} < \frac{{}_1F_1(b+\delta;c;x)\,{}_1F_1(a;c;x)}{{}_1F_1(a+\delta;c;x)\,{}_1F_1(b;c;x)} < 1; \tag{10}$$

for $a < b < c-\delta$ and $c > 0 > x$

$$\frac{\Gamma(c-a-\delta)\Gamma(c-b)}{\Gamma(c-b-\delta)\Gamma(c-a)} < \frac{{}_1F_1(b+\delta;c;x)\,{}_1F_1(a;c;x)}{{}_1F_1(a+\delta;c;x)\,{}_1F_1(b;c;x)} < 1; \tag{11}$$

both sides of both inequalities are sharp in the sense that the upper bound is attained at $x = 0$ and the lower bounds in (10), (11) are attained at $x = +\infty$ and $x = -\infty$, respectively;

d) *for $a > b > 0$, $c > 0$ and integer $m \geqslant 2$*

$${}_4F_3\left(\left.\begin{matrix} -m, a, 1-c-m, 1-am/(a+b) \\ c, 1-b-m, -am/(a+b) \end{matrix}\right| -1\right) > 0.$$

For $b > a > 0$ the sign of the above inequality is reversed.

Proof. Most statements follow immediately from Theorem 1. We only need to prove d) and the parts of b) and c) pertaining to negative x. To prove b) we need to show that

$${}_1F_1\big(a'+\delta;c;x\big)\,{}_1F_1(a;c;x) - {}_1F_1\big(a';c;x\big)\,{}_1F_1(a+\delta;c;x) < 0$$

for $a' < a < c-\delta$, $c > 0 > x$. Following the idea from [5] we apply the Kummer transformation ${}_1F_1(a;c;x) = e^x{}_1F_1(c-a;c;-x)$ so that the left-hand side of the above inequality equals

$$e^{2x}\big[{}_1F_1\big(c-a'-\delta;c;-x\big)\,{}_1F_1(c-a;c;-x) - {}_1F_1\big(c-a';c;-x\big)\,{}_1F_1(c-a-\delta;c;-x)\big].$$

Since $-x > 0$ and $a' < a < c-\delta$ implies positivity of all upper parameters in this formula the claim follows from Theorem 1.

This yields for positive a, c, δ and x:

$$\begin{aligned}&{}_1F_1(c-a-\delta;c;-x)^2-{}_1F_1(c-a;c;-x){}_1F_1(c-a-2\delta;c;-x)\\&\quad=e^{-2x}\big({}_1F_1(a+\delta;c;x)^2-{}_1F_1(a;c;x){}_1F_1(a+2\delta;c;x)\big)\geqslant 0.\end{aligned}$$

Now changing notation $c-a-\delta\to a$ and $-x\to x$ we arrive at the claim b).

In a similar fashion (11) follows from (10). The sharpness of (10) and (11) at infinity is seen from [1, Corollary 4.2.3].

Finally, d) is a restatement of a) for $\delta=1$, since the ${}_4F_3$ from d) is the coefficient at x^m in the Taylor series expansion of the function

$${}_1F_1(a+1;c;x){}_1F_1(b;c;x)-{}_1F_1(b+1;c;x){}_1F_1(a;c;x).$$

To see this apply the following easily verifiable identities

$$\begin{aligned}&(m-k)!=(-1)^k\frac{m!}{(-m)_k},\qquad (c)_{m-k}=\frac{(-1)^k(c)_m}{(1-c-m)_k},\\&(a+1)_k(b)_{m-k}-(a)_k(b+1)_{m-k}=-m(b+1)_{m-1}\frac{(-1)^k(a)_k(1-am/(a+b))_k}{(1-b-m)_k(-am/(a+b))_k}.\qquad\square\end{aligned}$$

Numerical tests suggests that the following enhancement of Theorem 4c) is true:

Conjecture. *The ratio in the middle of* (10) *is monotone decreasing from* $(0,\infty)$ *onto* $(1,A)$, *where A is the left-hand side of* (10); *the ratio in the middle of* (11) *is monotone increasing from* $(-\infty,0)$ *onto* $(B,1)$, *where B is the left-hand side of* (11).

Remark 4. Theorem 4, a) and b), for integer δ and $b=a+\delta$ recovers [5, Theorem 1, Corollary 2]. The lower bounds in Theorem 4c) are presumably new. Note also that although we allow any positive δ in b), for integer δ our parameter ranges are slightly more restrictive than those from [5].

Remark 5. For the ratio of two Kummer functions with different denominator parameters Bordelon found in [7, formula (5)] the inequality

$$1>\frac{{}_1F_1(a;c;x)}{{}_1F_1(a;d;x)}>\frac{\Gamma(c)\Gamma(d-a)}{\Gamma(d)\Gamma(c-a)}$$

valid for $d>c>a>0$, $x<1$.

Theorem 5. *Suppose* $\delta>0$. *Then*

a) *for* $d>c>0$ *and* $a>0$ *the function*

$$x\mapsto {}_1F_1(a;c+\delta;x){}_1F_1(a;d;x)-{}_1F_1(a;d+\delta;x){}_1F_1(a;c;x)$$

has negative power series coefficients (*starting with the coefficient at* x);

b) *the function* $c\mapsto {}_1F_1(a;c+\delta;x)/{}_1F_1(a;c;x)$ *is monotone increasing* (= *the function* $c\mapsto {}_1F_1(a;c;x)$ *is log-convex*) *on* $(0,\infty)$ *for fixed* $a,x>0$ *or fixed* $a,x<0$;

c) *the inequality* ${}_1F_1(a+\delta;c+\delta;x)^2\leqslant {}_1F_1(a+2\delta;c+2\delta;x){}_1F_1(a;c;x)$ *holds true for* $a\geqslant c>0$, $x>0$ *and* $a\leqslant c$, $c>0$, $x\leqslant 0$, *so that* $\mu\mapsto {}_1F_1(a+\mu;c+\mu;x)$ *is log-convex on* $[0,\infty)$ *under these restrictions on parameters;*

d) *for* $c>d>0$, $a>0$ *and integer* $m\geqslant 1$

$${}_4F_3\left(\begin{matrix}-m,-d-m,a,1-cm/(c+d)\\1-a-m,c+1,-cm/(c+d)\end{matrix}\,\middle|\,-1\right)>0.$$

For $d>c>0$ *the sign of the above inequality is reversed.*

Proof. Statements a) and the part of b) for $a,x>0$ follow from Theorem 3. The claim c) for positive x and $a\geqslant c>0$ was proved by Baricz in [2, Theorem 2]. By the Kummer transformation this yields b) for negative x and $a<0$. Similarly, an application of the Kummer transformation to the part of b) with $x>0$ gives the part of c) with $x<0$. Finally, d) is a reformulation of a) for $\delta=1$. $\square$

For the Gauss function

$${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_kk!}x^k$$

two distinct cases present themselves: if $b > c > 0$ then the sequence

$$\frac{(b)_{k+1}/(c)_{k+1}}{(b)_k/(c)_k} = \frac{b+k}{c+k}, \quad k = 0, 1, 2, \ldots,$$

is decreasing, while for $c > b > 0$ it is increasing. We can combine Theorem 1 with Euler's and Pfaff's transformations,

$$\begin{aligned} {}_2F_1(a, b; c; x) &= (1-x)^{c-a-b} {}_2F_1(c-a, c-b; c; x) \\ &= (1-x)^{-a} {}_2F_1\big(a, c-b; c; x/(x-1)\big) = (1-x)^{-b} {}_2F_1\big(c-a, b; c; x/(x-1)\big), \end{aligned}$$

to get the following assertions.

Theorem 6.

a) *Suppose $a' > a \geqslant 0$ and $b > c > 0$, $\delta > 0$. Then the function*

$$x \mapsto {}_2F_1(a+\delta, b; c; x) {}_2F_1\big(a', b; c; x\big) - {}_2F_1\big(a'+\delta, b; c; x\big) {}_2F_1(a, b; c; x)$$

has positive power series coefficients (starting with the coefficient at x^2);

b) *the function $a \mapsto {}_2F_1(a, b; c; x)$ is log-concave,*

$${}_2F_1(a+\delta, b; c; x)^2 \geqslant {}_2F_1(a, b; c; x) {}_2F_1(a+2\delta, b; c; x),$$

on $[0, \infty)$ for $0 < x < 1$, $b > c > 0$ and $x < 0$, $c > 0 > b$ and on $(-\infty, c]$ for $0 < x < 1$, $c > 0 > b$ and $x < 0$, $b > c > 0$;

c) *for $0 < x < 1$:*

$$\frac{\Gamma(a+\delta)\Gamma(a')}{\Gamma(a)\Gamma(a'+\delta)} < \frac{{}_2F_1(a'+\delta, b; c; x) {}_2F_1(a, b; c; x)}{{}_2F_1(a+\delta, b; c; x) {}_2F_1(a', b; c; x)} < 1 \tag{12}$$

if $b > c > 0$, $a' > a > 0$ and

$$\frac{\Gamma(c-a-\delta)\Gamma(c-a')}{\Gamma(c-a)\Gamma(c-a'-\delta)} < \frac{{}_2F_1(a'+\delta, b; c; x) {}_2F_1(a, b; c; x)}{{}_2F_1(a+\delta, b; c; x) {}_2F_1(a', b; c; x)} < 1 \tag{13}$$

if $c > 0 > b$, $c - \delta > a' > a$. Further for $x < 0$ (12) holds true if $c > 0 > b$, $a' > a > 0$ and (13) holds true if $b > c > 0$, $c - \delta > a' > a$.

Theorem 7.

a) *Suppose $a' > a > 0$ and $c > b > 0$, $\delta > 0$. Then the function*

$$x \mapsto {}_2F_1(a+\delta, b; c; x) {}_2F_1\big(a', b; c; x\big) - {}_2F_1\big(a'+\delta, b; c; x\big) {}_2F_1(a, b; c; x)$$

has negative power series coefficients (starting with the coefficient at x^2);

b) *the function $a \mapsto {}_2F_1(a, b; c; x)$ is log-convex,*

$${}_2F_1(a+\delta, b; c; x)^2 \leqslant {}_2F_1(a, b; c; x) {}_2F_1(a+2\delta, b; c; x),$$

on $(-\infty, \infty)$ for $-\infty < x < 1$, $c > b > 0$.

Theorem 8.

a) *Suppose $d > c > 0$ and $a, b, \delta > 0$. Then the function*

$$x \mapsto {}_2F_1(a, b; c+\delta; x) {}_2F_1(a, b; d; x) - {}_2F_1(a, b; d+\delta; x) {}_2F_1(a, b; c; x)$$

has negative power series coefficients (starting with the coefficient at x);

b) *the function $c \mapsto {}_2F_1(a, b; c; x)$ is log-convex on $(0, \infty)$ if $a, b > 0$, $0 < x < 1$ or $a, x < 0$, $b > 0$ or $b, x < 0$, $a > 0$;*

c) *the function $\mu \mapsto {}_2F_1(a, b+\mu; c+\mu; x)$ is log-convex on $[0, \infty)$ if $a > 0$, $b > c > 0$, $0 < x < 1$ or $a < 0$, $b < c$, $c > 0$, $0 < x < 1$ or $a > 0$, $b < c$, $c > 0$, $x < 0$;*

d) *the function $\mu \mapsto {}_2F_1(a+\mu, b+\mu; c+\mu; x)$ is log-convex on $[0, \infty)$ if $a < c$, $b \geqslant c > 0$, $x < 0$ or $a < c$, $b < c$, $c > 0$, $0 < x < 1$.*

Remark 6. Theorems 6–8 do not cover the case when ${}_2F_1$ is expressed in terms of the Jacobi polynomials so that the original Turán type inequalities due to Szegő [20] and Gasper [11] cannot be derived from it.

Applications to generalized hypergeometric function ${}_{q+1}F_q$ (for its definition and properties see [10]) hinge on the following observation which might be of independent interest:

Lemma 1. *Let $A(x) = a_0 + a_1x + \cdots + a_nx^n$ and $B(x) = b_0 + b_1x + \cdots + b_nx^n$ have positive coefficients. Then $A'(x)B(x) - B'(x)A(x)$ has non-negative coefficients if*

$$\frac{a_n}{b_n} \geqslant \frac{a_{n-1}}{b_{n-1}} \geqslant \cdots \geqslant \frac{a_1}{b_1} \geqslant \frac{a_0}{b_0} \tag{14}$$

and non-positive coefficients if

$$\frac{a_n}{b_n} \leqslant \frac{a_{n-1}}{b_{n-1}} \leqslant \cdots \leqslant \frac{a_1}{b_1} \leqslant \frac{a_0}{b_0}. \tag{15}$$

If A and B are not identical then some of the coefficients are positive under (14) *(so that $A'(x)B(x) - B'(x)A(x) > 0$ for $x > 0$) and negative under* (15) *(so that $A'(x)B(x) - B'(x)A(x) < 0$ for $x > 0$).*

Proof. We have

$$\begin{aligned} A'(x)B(x) - B'(x)A(x) &= \sum_{k=1}^{n}\sum_{i=0}^{n} ka_kb_ix^{i+k-1} - \sum_{k=1}^{n}\sum_{i=0}^{n} kb_ka_ix^{i+k-1} \\ &= \frac{1}{x}\sum_{k=1}^{n}\sum_{i=0}^{n} x^{i+k}k(a_kb_i - a_ib_k) = \sum_{m=1}^{2n} x^{m-1} \sum_{\substack{i+k=m\\ i\geqslant 0,\, k\geqslant 1}}^{k,i\leqslant n} k(a_kb_i - a_ib_k). \end{aligned}$$

Since each term in the inner sum with $i = k$ is clearly zero, we may write:

$$\sum_{\substack{i+k=m\\ i\geqslant 0,\, k\geqslant 1}}^{k,i\leqslant n} k(a_kb_i - a_ib_k) = \sum_{\substack{i+k=m,\, k<i\\ i\geqslant 0,\, k\geqslant 1}}^{k,i\leqslant n} k(a_kb_i - a_ib_k) + \sum_{\substack{i+k=m,\, k>i\\ i\geqslant 0,\, k\geqslant 1}}^{k,i\leqslant n} k(a_kb_i - a_ib_k).$$

Due to condition (14) every term in the second sum is non-negative. For each term in the first sum (say indexed $k = k^*$, $i = i^*$, $k^* < i^*$), there is a term in the second sum with $k = i^*$, $i = k^*$ and

$$k^*(a_{k^*}b_{i^*} - a_{i^*}b_{k^*}) + i^*(a_{i^*}b_{k^*} - a_{k^*}b_{i^*}) = \left(i^* - k^*\right)(a_{i^*}b_{k^*} - a_{k^*}b_{i^*}) \geqslant 0.$$

Since A and B are not identical at least one of the inequalities (14) is strict which implies that at least one the terms above is strictly positive. The second statement follows from $A'(x)B(x) - B'(x)A(x) = -(B'(x)A(x) - A'(x)B(x))$ by exchanging the roles of A and B. □

Remark 7. It is easy to verify directly that conditions (14) are also necessary for $A'(x)B(x) - B'(x)A(x) > 0$ if $x > 0$ when $n = 1$ and $n = 2$ (similarly for conditions (15)).

Remark 8. After this paper was already written Árpád Baricz brought the recently published article [12] to our attention. Lemma 1 repeats almost precisely the contents of Theorem 4.4 from [12]. Interestingly, the proof in [12] is by induction and hence differs substantially from ours.

Now consider

$$R(x) = \frac{\prod_{k=1}^{q}(a_k + x)}{\prod_{k=1}^{q}(b_k + x)}$$

with positive a_k, b_k. Let $e_m(c_1, \ldots, c_q)$ denote m-th elementary symmetric polynomial,

$$\begin{aligned} &e_1(c_1, \ldots, c_q) = c_1 + c_2 + \cdots + c_q, \\ &e_2(c_1, \ldots, c_q) = c_1c_2 + c_1c_3 + \cdots + c_1c_q + c_2c_3 + c_2c_4 + \cdots + c_2c_q + \cdots + c_{q-1}c_q, \\ &\quad\vdots \\ &e_q(c_1, \ldots, c_q) = c_1c_2\cdots c_q. \end{aligned}$$

Lemma 2. *The function $R(x)$ is monotone increasing on $(0,\infty)$ if*

$$\frac{e_q(b_1,\dots,b_q)}{e_q(a_1,\dots,a_q)} \geqslant \frac{e_{q-1}(b_1,\dots,b_q)}{e_{q-1}(a_1,\dots,a_q)} \geqslant \dots \geqslant \frac{e_1(b_1,\dots,b_q)}{e_1(a_1,\dots,a_q)} \geqslant 1 \tag{16}$$

and monotone decreasing if

$$\frac{e_q(b_1,\dots,b_q)}{e_q(a_1,\dots,a_q)} \leqslant \frac{e_{q-1}(b_1,\dots,b_q)}{e_{q-1}(a_1,\dots,a_q)} \leqslant \dots \leqslant \frac{e_1(b_1,\dots,b_q)}{e_1(a_1,\dots,a_q)} \leqslant 1. \tag{17}$$

Proof. We have

$$A(x) \equiv \prod_{k=1}^{q}(a_k+x) = \sum_{k=0}^{q} e_k(a_1,\dots,a_q)x^{q-k} = \sum_{k=0}^{q} e_{q-k}(a_1,\dots,a_q)x^k,$$
$$B(x) \equiv \prod_{k=1}^{q}(b_k+x) = \sum_{k=0}^{q} e_k(b_1,\dots,b_q)x^{q-k} = \sum_{k=0}^{q} e_{q-k}(b_1,\dots,b_q)x^k,$$
$$R'(x) = \frac{A'(x)B(x)-B'(x)A(x)}{[B(x)]^2}.$$

Hence by Lemma 1 we can assert that

$$1 = \frac{e_0(a_1,\dots,a_q)}{e_0(b_1,\dots,b_q)} \geqslant \frac{e_1(a_1,\dots,a_q)}{e_1(b_1,\dots,b_q)} \geqslant \frac{e_2(a_1,\dots,a_q)}{e_2(b_1,\dots,b_q)} \geqslant \dots \geqslant \frac{e_q(a_1,\dots,a_q)}{e_q(b_1,\dots,b_q)}$$

which is the same as (16), is sufficient for $R'(x) > 0$. Similarly for $R'(x) < 0$. $\square$

Remark 9. Conditions $b_i > a_i$, $i = 1,\dots,q$, used in [5], are clearly sufficient but not necessary for (16).

Theorem 9. *Put*

$$f(\alpha,x) = {}_{q+1}F_q\big(\alpha,(a_q);(b_q);x\big)$$

and suppose $a_i, b_i > 0$, $i = 1,\dots,q$, $\beta > \alpha > 0$. Then

a) *for any fixed $\delta > 0$ the function*

$$x \mapsto f(\alpha+\delta,x)f(\beta,x) - f(\beta+\delta,x)f(\alpha,x)$$

has negative power series coefficients if (16) *holds and positive power series coefficients if* (17) *holds*;

b) *under condition* (16) *and $x \in (0,1)$ the function $\alpha \mapsto f(\alpha+\delta,x)/f(\alpha,x)$ is monotone increasing on* $[0,\infty)$ *for any fixed $\delta > 0$ so that the function $\alpha \mapsto f(\alpha,x)$ is log-convex*;

c) *under condition* (17) *and $x \in (0,1)$ the function $\alpha \mapsto f(\alpha+\delta,x)/f(\alpha,x)$ is monotone decreasing on* $[0,\infty)$ *for any fixed $\delta > 0$ so that the function $\alpha \mapsto f(\alpha,x)$ is log-concave*;

d) *under condition* (17) *and $x \in (0,1)$*

$$\frac{\Gamma(\alpha+\delta)\Gamma(\beta)}{\Gamma(\beta+\delta)\Gamma(\alpha)} < \frac{f(\beta+\delta,x)f(\alpha,x)}{f(\alpha+\delta,x)f(\beta,x)} < 1.$$

Remark 10. For ${}_3F_2(\alpha,a_1,a_2;b_1,b_2;x)$ conditions (16) read:

$$\frac{b_1b_2}{a_1a_2} \geqslant \frac{b_1+b_2}{a_1+a_2} \geqslant 1$$

and for condition (17) both inequalities are reversed. According to Remark 7 after Lemma 1 in this case these conditions are both necessary and sufficient for the increase or decrease of the function $x \mapsto (a_1+x)(a_2+x)/[(b_1+x)(b_2+x)]$.

We can extend Theorem 9b) to negative x using the generalized Stieltjes transform representation

$${}_{q+1}F_q\left(\begin{matrix}\alpha,(a_q)\\(b_q)\end{matrix}\,\middle|\,x\right) = \int_0^1 \frac{\rho((a_q);(b_q);s)\,ds}{(1-sx)^{\alpha}}, \tag{18}$$

valid for $b_i > a_i > 0$, $i = 1, \dots, q$, and $x < 1$, recently obtained by the authors in [15]. An explicit expression for the positive function $\rho(s)$ is given in [15]. Indeed, the inequality

$$f(\alpha+\delta, x) f(\beta, x) < f(\beta+\delta, x) f(\alpha, x)$$

is exactly the Chebyshev inequality [16, Chapter IX, formula (1.1)]. Hence, Theorem 9b) is true for all $x < 1$ if $b_i > a_i > 0$, $i = 1, \dots, q$.

The claim of Lemma 1 made under condition (15) clearly remains true when $a_n = a_{n-1} = a_{n-r} = 0$. Condition (17) then reads

$$\frac{e_q(b_1, \dots, b_q)}{e_{q-r-1}(a_1, \dots, a_{q-r-1})} \leqslant \frac{e_{q-1}(b_1, \dots, b_q)}{e_{q-r-2}(a_1, \dots, a_{q-r-1})} \leqslant \dots \leqslant \frac{e_{r+2}(b_1, \dots, b_q)}{e_1(a_1, \dots, a_{q-r-1})} \leqslant e_{r+1}(b_1, \dots, b_q).$$

This permits the application of Theorem 1 to ${}_pF_q$ with $p \leqslant q$. Of course in this case we will only have the decreasing sequence $\{F_n/F_{n-1}\}$ and log-concavity of ${}_pF_q$ in the upper parameters. For instance, an analogue of Theorem 4d) for ${}_qF_q$ is the inequality

$${}_{2q+2}F_{2q+1}\left(\begin{matrix} -m, \alpha, a_1, \dots, a_{q-1}, 1-b_1-m, \dots, 1-b_q-m, 1-\alpha m/(\alpha+\beta) \\ b_1, \dots, b_q, 1-a_1-m, \dots, 1-a_{q-1}-m, 1-\beta-m, -\alpha m/(\alpha+\beta) \end{matrix}\,\middle|\, -1\right) > 0$$

valid for $\alpha > \beta > 0$, integer $q \geqslant 1$, integer $m \geqslant 2$ and

$$\frac{e_q(b_1, \dots, b_q)}{e_{q-1}(a_1, \dots, a_{q-1})} \leqslant \frac{e_{q-1}(b_1, \dots, b_q)}{e_{q-2}(a_1, \dots, a_{q-1})} \leqslant \dots \leqslant \frac{e_2(b_1, \dots, b_q)}{e_1(a_1, \dots, a_{q-1})} \leqslant e_1(b_1, \dots, b_q),$$

where all $a_i, b_i > 0$. For ${}_2F_2(\alpha, a_1; b_1, b_2; x)$ these inequalities simplify to the single condition $a_1 \geqslant b_1 b_2/(b_1+b_2)$ which ensures log-concavity in α. Theorems 2 and 3 can be applied as well.

Remark 11. Note that some results related to log-convexity of the modified Struve functions expressible in terms of ${}_1F_2$ are given in [14].

Acknowledgments

We thank Fedor Nazarov whose idea led to the proof of Theorem 1, Árpád Baricz for providing us with the copies of [2–4,7,12] and many valuable comments, Matti Vuorinen for sharing the problem of log-concavity of ${}_1F_1$ with us and Michael Gordy for sending us the copies of [5,8] and other useful communication. The work of the first author has been supported by Far Eastern Branch of the Russian Academy of Sciences (grants 09-III-A-01-008 and 09-II-CO-01-003), Russian Basic Research Fund (grant 08-01-00028-a) and the Presidential Grant for Leading Scientific Schools (grant 2810.2008.1).

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, 1999.
[2] A. Baricz, Functional inequalities involving Bessel and modified Bessel functions of the first kind, Expo. Math. 26 (3) (2008) 279–293, doi:10.1016/j.exmath.2008.01.001.
[3] A. Baricz, Turán type inequalities for hypergeometric functions, Proc. Amer. Math. Soc. 136 (9) (2008) 3223–3229.
[4] A. Baricz, Turán type inequalities for generalized complete elliptic integrals, Math. Z. 256 (2007) 895–911.
[5] R.W. Barnard, M. Gordy, K.C. Richards, A note on Turán type and mean inequalities for the Kummer function, J. Math. Anal. Appl. 349 (1) (2009) 259–263, doi:10.1016/j.jmaa.2008.08.024.
[6] C. Berg, R. Szwarc, Bounds on Turán determinants, J. Approx. Theory 161 (1) (2009) 127–141, doi:10.1016/j.jat.2008.08.010.
[7] D.J. Bordelon, Inequalities for special functions, Problem 72-15 (proposed by D.K. Ross), SIAM Rev. 15 (3) (1973) 665–670.
[8] M. Carey, M.B. Gordy, The bank as grim reaper: Debt composition and recoveries on defaulted debt, preprint, 2007.
[9] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite–Hadamard Inequalities and Applications, RGMIA Monograph, Victoria University, 2000.
[10] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. 1, McGraw-Hill Book Company, Inc., New York, 1953.
[11] G. Gasper, On the extension of Turán's inequality to Jacobi polynomials, Duke Math. J. 38 (1971) 415–428.
[12] V. Heikkala, M.K. Vamanamurthy, M. Vuorinen, Generalized elliptic integrals, Comput. Methods Funct. Theory 9 (1) (2009) 75–109.
[13] M.E.H. Ismail, A. Laforgia, Monotonicity properties of determinants of special functions, Constr. Approx. 26 (1) (2007) 1–9.
[14] C.M. Joshi, S. Nalwaya, Inequalities for modified Struve functions, J. Indian Math. Soc. 65 (1998) 49–57.
[15] D. Karp, S.M. Sitnik, Inequalities and monotonicity of ratios for generalized hypergeometric function, J. Approx. Theory 161 (1) (2009) 337–352, doi:10.1016/j.jat.2008.10.002.
[16] D.S. Mitrinović, J.E. Pecarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.
[17] C.P. Niculescu, L.-E. Persson, Convex Functions and Their Applications. A Contemporary Approach, Springer Science+Business Media, Inc., 2006.
[18] G. Pólya, Location of zeros, in: R.P. Boas (Ed.), Collected Papers, vol. II, MIT Press, Cambridge, MA, 1974.
[19] R. Szwarc, Positivity of Turán determinants for orthogonal polynomials, in: K.A. Ross, et al. (Eds.), Harmonic Analysis and Hypergroups, Delhi, 1995, Birkhäuser, Boston–Basel–Berlin, 1998, pp. 165–182. Also available as http://arxiv.org/abs/0710.3389v1.
[20] G. Szegő, On an inequality of P. Turán concerning Legendre polynomials, Bull. Amer. Math. Soc. 54 (1948) 401–405.