



Generalized practical stability analysis of Filippov-type systems ☆

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ABSTRACT

In this paper, we study the generalized practical stability and the totally generalized practical stability of Filippov-type piecewise smooth systems. We also apply these properties to a brake model of bike to study its practical stability. Some numerical computations are presented to illustrate our theories.

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1. Introduction

It is well known that the theory of stability in the sense of Lyapunov [1–4] has been widely developed and has been vastly applied in many fields. However, in some cases, a system may be stable or even asymptotically stable in the sense of Lyapunov and it may still be completely useless in practice [5], since the domain of attraction may not be large enough to allow the desired deviation to conceal out. On the other hand, a system may be unstable in the sense of Lyapunov and it may oscillate sufficiently near a state whose performance is acceptable in practice [6]. For example, many aircrafts and missiles behave in this manner. Thus, from practical consideration, a notion which is neither weaker nor stronger than Lyapunov stability is proposed by Lasalle and Lefschetz [5]. Motivated by this, Weiss and Infante [7] introduced the concept of finite time stability. Based on this, Michel [8] studied the type of stability over the infinite time interval and Bernfeld and Lakshmikantham [9] studied similar theory in finite- as well as infinite-dimensional Banach spaces. A systematic study on the practical stability is collected in the book [10].

Meanwhile, Michel [11], Michel and Porter [12] extended the practical stability to piecewise smooth systems. Chen [13] improved and extended the corresponding results [11]. He dealt with a piecewise smooth system with external perturbation [14], and in various decomposition forms [15]. He also studied discontinuous large-scale systems [16,17] and the strong practical stability [18–20]. Ruan [21] studied practical stability for retarded piecewise smooth Filippov systems. In this paper, one purposes is to extend the general practical stability theories to Filippov-type systems.

The concept of practical stability is usually defined in terms of neighborhoods of a point [10] and is applied to the local stability property. In general, it is not enough to study the stability near a trivial solution. Consequently, it is useful to extend the practical stability notions relative to arbitrary sets or more generally, arbitrary tubes. In [10,22], the practical stability is generalized to deal with two arbitrary sets $\Omega_1 \subset \Omega_2$, which are specified for the initial state and the entire system state, respectively. Obviously, the two sets Ω_1 and Ω_2 do not have to include a trivial solution and they can be specified flexibly in real applications. Then, this generalized practical stability (GP-stability for short) requires that if the initial state is in Ω_1 ,

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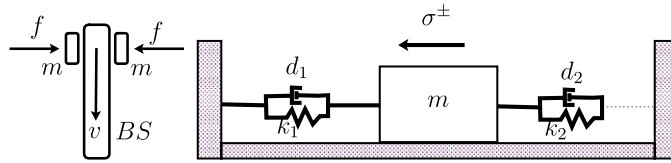


Fig. 1.1. A brake system for a bike and its simplified model.

then the system state should always stay in Ω_2 . The GP-stability is a significant extension of practical stability. We then apply it to analyze the generalized practical stability of a brake model of a bike, which is either asymptotically stable or asymptotically unstable.

Our research interest in this paper emanates from studying the stability of the brake model of a bike. The corresponding brake equipment is sketched in the left-hand side of Fig. 1.1, where BS stands for a rotating rim with constant velocity v , and m is a brake patch. This structure can be simplified to a model shown in the right-hand side of Fig. 1.1.

In this model, the mass m rests on a smooth surface and is connected to the walls by springs (k_1 and k_2) and dampers (d_1 and d_2). We assume that the steady state of the mass m ($x_1 = 0$) corresponds to the situation that both springs k_1, k_2 and dampers d_1, d_2 are unloaded. Furthermore, if the mass moves toward left-hand side from the steady state (i.e. $x_1 < 0$), both springs and dampers take effect; if the mass moves towards the right-hand side from the steady state (i.e. $x_1 > 0$), only the spring k_1 and the damper d_1 take effect.

By Newton's second law, we obtain a piecewise smooth system

$$\begin{aligned} m\ddot{x}_1 + d_1\dot{x}_1 + k_1v &= \sigma^+(x_1, \dot{x}_1, \lambda), \quad \text{if } x_1 > 0, \\ m\ddot{x}_1 + (d_1 + d_2)\dot{x}_1 + (k_1 + k_2)x_1 &= \sigma^-(x_1, \dot{x}_1, \lambda), \quad \text{if } x_1 < 0, \end{aligned} \quad (1.1)$$

where σ^\pm represents the total external force added to the mass and the parameter λ controls its magnitude. Denoting by $x_2 = \dot{x}_1$, Eq. (1.1) can be rewritten as the following first order system

$$\dot{x} = f(x, \lambda), \quad x = (x_1, x_2)^T, \quad (1.2)$$

where the function $f = (f_1, f_2)^T : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ is piecewise defined as

$$f(x, \lambda) = \begin{cases} f^+(x, \lambda), & x_1 > 0, \\ f^-(x, \lambda), & x_1 < 0. \end{cases}$$

Recent years, the Filippov type of discontinuous system have been widely studied in many ways, see [23,24] and people have paid much attentions to the bifurcation properties of this type system [25–27]. In [28] the existence of bifurcating periodic orbits of Eq. (1.2) is obtained via generalized Hopf bifurcation. In this paper we concentrate ourselves on investigating the GP-practical stability for these bifurcating piecewise smooth periodic orbits.

In applications when the brake system is taking action, perturbations are impossible to be avoid, which lead to the following more general system

$$\dot{x} = f(x, \lambda) + u(x, t, \lambda), \quad (1.3)$$

where u represents the external input. We also study the totally GP-stability property of Eq. (1.3) under small perturbation.

This paper is organized as follows. In Section 2 we introduce all the basic assumptions for this work and then we collect some definitions on practical stability concepts. In Section 3 we study the GP-stability and GP-unstability of Eq. (1.2) and in Section 4 we investigate the totally GP-stability of Eq. (1.3). In Section 5 we apply our theories to study the uniformly GP-stability and totally GP-stability of the bifurcating periodic solutions of Eq. (1.2) and Eq. (1.3), respectively. Finally, in Section 6 we present some numerical computations to illustrate our theories.

2. Basic assumptions

In this section we describe and discuss the basic assumptions for this work.

For any given $t_0 \in \mathbb{R}$ and $T \in \mathbb{R}$, let $I = [t_0, t_0 + T)$ if $T > 0$ or $I = (t_0 + T, t_0]$ if $T < 0$. Here the number T may be finite or infinite. For convenience, in this paper we will state and prove our results in the case $T > 0$, and all results apply to the case $T < 0$.

A property will be said to hold almost everywhere (abbreviated as a.e.) if the set of points where it fails is a set of measure zero.

Let $\Omega \subset \mathbb{R}^n$ be an open disk centered around the origin with radius $\rho > 0$ split into two open disjoint subdomains Ω^+ and Ω^- by a piecewise continuously differentiable surface Σ , such that $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$, where the surface Σ is defined by a scalar indicator function $h(x)$. Then the domains Ω^+ , Ω^- and Σ can be described by

$$\Omega^\pm = \{x \in \Omega \mid \pm h(x) > 0\}, \quad \Sigma = \{x \in \Omega \mid h(x) = 0\}.$$

We assume that

(H1) The function $h(x)$ is piecewise C^r ($r \geq 2$) smooth in Ω and the gradient $\nabla h(x) \neq 0$ at its smooth points.

Consider the following dynamical system

$$\dot{x}(t) = f(x, t), \quad (x, t) \in \Omega \times I, \quad (2.1)$$

where the function f is piecewise defined as

$$f(x, t) = \begin{cases} f^+(x, t), & x \in \Omega^+, \\ f^-(x, t), & x \in \Omega^-. \end{cases}$$

(H2) The functions f^\pm are C^r ($r \geq 2$) smooth in Ω^\pm up to its boundary.

Define a set-valued function by

$$F(x, t) = \begin{cases} \{f^+(x, t)\}, & x \in \Omega^+, \\ \{\alpha f^+(x, t) + (1 - \alpha)f^-(x, t), \alpha \in [0, 1]\}, & x \in \Sigma, \\ \{f^-(x, t)\}, & x \in \Omega^-. \end{cases} \quad (2.2)$$

Then Eq. (2.1) is embedded in the following differential inclusion

$$\dot{x} \in F(x, t). \quad (2.3)$$

Maximal existence and uniqueness in forward time or in backward time of a solution of the differential inclusion (2.3) are guaranteed by the assumptions (H1) and (H2) (cf. [29,30]).

Consider the system (2.1) with permanently acting perturbation (external input) of the form

$$\dot{x}(t) = f(x, t) + u(x, t), \quad (2.4)$$

where $u : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$ represents the external inputs satisfying the following assumption.

(H3) $u(x, t)$ satisfies Carathéodory property. That is, $u(x, t)$ is continuous in x for almost all $t \in I$, and is measurable in t for each $x \in \Omega$, and for any closed bounded domain $D \subset \Omega$ there exists a summable function $n(t)$ such that almost everywhere in D there holds $\|u(x, t)\| \leq n(t)$.

For convenience we introduce some notations. For a set $\Omega \subset \mathbb{R}^n$, we use $\bar{\Omega}$ and Ω^c to denote its closure and its complement, respectively. For two sets $\Omega_1 \subset \Omega_2 \subset \Omega$, we use $[\Omega_2 - \Omega_1]$ to denote the set $\{x \in \mathbb{R}^n : x \in \Omega_2, x \notin \Omega_1\}$.

We will use Lyapunov-like functions to investigate the GP-stability of Eq. (2.1). Let a real-valued function $V : \Omega \times I \rightarrow \mathbb{R}^1$ possess continuous first order partial derivatives on Ω .

In the case that the systems (2.1) and (2.4) are smooth, by $\dot{V}(x, t)$ we denote the derivative of the function $V(x, t)$ with respect to t along a solution $x(t)$ of Eq. (2.1), which is defined by

$$\dot{V}(x, t) = \nabla_x V(x, t) \cdot f(x, t) + \frac{\partial V}{\partial t}, \quad (2.5)$$

where $\nabla_x V$ denotes the gradient vector of V , and \cdot denotes the inner product. Similarly, if $x(t)$ is a solution of the perturbed Eq. (2.4), the corresponding derivative along this solution is

$$\dot{V}(x, t) = \dot{V}(x, t)|_{u=0} + \nabla_x V(x, t) \cdot u(x, t). \quad (2.6)$$

In the case that the systems (2.1) and (2.4) are piecewise smooth, the counterpart of the similar Lyapunov-like function in Eq. (2.5) and Eq. (2.6) are valid almost everywhere.

The function $V(x, t)$ is absolutely continuous if it is locally Lipschitz continuous. For any solution $x(t)$ of Eq. (2.1) or Eq. (2.4), the function $V(x, t)$ has a derivative $\dot{V}(x(t), t)$ almost everywhere along the solution $x(t)$. Therefore, we have

$$V(x(t), t) = V(x(t_0), t_0) + \int_{t_0}^t \dot{V}(x(\tau), \tau) d\tau, \quad t \geq t_0. \quad (2.7)$$

Next, we collect some definitions [22] on GP-stability for systems (2.1) and (2.4). In the following definitions we always require that the two sets $\Omega_1 \subset \Omega_2$ are open.

Definition 2.1. System (2.1) is GP-stable with respect to $(\Omega_1, \Omega_2, t_0, T)$, if $x(t_0) \in \Omega_1$ implies $x(t) \in \Omega_2$ for all $t \in I$.

Definition 2.2. System (2.1) is uniformly GP-stable with respect to (Ω_1, Ω_2, T) , if for each $t_i \in I$, $x(t_i) \in \Omega_1$ implies $x(t) \in \Omega_2$ for all $t = [t_i, T)$.

Definition 2.3. System (2.1) is GP-unstable with respect to $(\Omega_1, \Omega_2, t_0, T)$, if there exists $x(t_0) \in \Omega_1$ and a time $t_c \in (t_0, t_0 + T)$ such that $x(t_c) \notin \Omega_2$.

Definition 2.4. System (2.4) is totally GP-stable with respect to $(\Omega_1, \Omega_2, \varepsilon, t_0, T)$, if $x(t_0) \in \Omega_1$ and $\|u(x, t)\| \leq \varepsilon$ for $x \in \Omega_2$ and $t \in I$ a.e., imply $x(t) \in \Omega_2$ for all $t \in I$.

Definition 2.5. System (2.4) is totally uniformly GP-stable with respect to $(\Omega_1, \Omega_2, \varepsilon, T)$, if for each $t_i \in I$, $x(t_i) \in \Omega_1$ and $\|u(x, t)\| \leq \varepsilon$ for $x \in [\Omega_2 - \bar{\Omega}_1]$, $t \in I$, a.e., imply $x(t) \in \Omega_2$ for all $t \in [t_i, t_0 + T)$.

Remark 2.6. (1) In applications, the set Ω_2 usually represents the possible trajectory region of a system under consideration and the set Ω_1 is the collection of initial values such that the corresponding solutions with initial values in Ω_1 remain in Ω_2 during the develop time period I .

(2) A system which is Lyapunov stable may be unstable in the above sense of definitions, and vice versa.

(3) If system (2.1) is autonomous, GP-stability and uniformly GP-stability are equivalent.

3. GP-stability analysis for Filippov-type systems

In this section we study sufficient conditions for determining the GP-stability or instability of the following Filippov-type system

$$\dot{x}(t) = f(x, t), \quad (x, t) \in \Omega \times I. \quad (3.1)$$

Theorem 3.1. Assume (H1) and (H2). System (3.1) is GP-stable with respect to $(\Omega_1, \Omega_2, t_0, T)$, if there exist a real-valued Lyapunov-like function $V(x, t)$ satisfying local Lipschitz continuity in $\Omega \times I$, and a real-valued function $\phi(t)$ being Lebesgue-integrable on I , such that

$$\begin{aligned} \text{(i)} \quad & \dot{V}(x(t), t) \leq \phi(t), \quad \text{a.e. } t \in I, \quad x(t) \in \Omega_2, \\ \text{(ii)} \quad & \int_{t_0}^t \phi(\tau) d\tau < \inf_{x \in \Omega_2^c} V(x, t) - \sup_{x \in \bar{\Omega}_1} V(x, t_0), \quad \forall t \in I. \end{aligned}$$

Proof. Let $x(t)$ be a solution of Eq. (3.1) with initial value $x(t_0) \in \Omega_1$. For contradiction, we assume that there exists a first exit time $t_1 \in I$ such that $x(t) \in \Omega_2$ for $t_0 \leq t < t_1$ and $x(t_1) \notin \Omega_2$. Since the set Ω_2 is open, the point $x(t_1)$ lies on the boundary of the set Ω_2 .

By virtue that $x(t)$ is absolutely continuous and $V(x, t)$ satisfies the Lipschitz continuity, we know the composed function $V(x(t), t)$ is absolutely continuous, therefore

$$V(x(t_1), t_1) = V(x(t_0), t_0) + \int_{t_0}^{t_1} \dot{V}(x(\tau), \tau) d\tau.$$

According to hypothesis (i), we have

$$\sup_x V(x(t_1), t_1) \leq \sup_{x \in \bar{\Omega}_1} V(x, t_0) + \int_{t_0}^{t_1} \phi(\tau) d\tau.$$

From hypothesis (ii) it follows that

$$V(x(t_1), t_1) < \sup_{x \in \bar{\Omega}_1} V(x, t_0) + \inf_{x \in \Omega_2^c} V(x, t_1) - \sup_{x \in \bar{\Omega}_1} V(x, t_0) = \inf_{x \in \Omega_2^c} V(x, t_1),$$

which implies $x(t_1) \in \Omega_2$. This is a contradiction. Therefore, $x(t) \in \Omega_2$ holds for all $t \in I$. This completes the proof of the theorem. \square

Theorem 3.2. Assume (H1) and (H2). The system (3.1) is uniformly GP-stable with respect to (Ω_1, Ω_2, T) , if there exist a real-valued Lyapunov-like function $V(x, t)$ satisfying local Lipschitz continuity in $[\Omega - \bar{\Omega}_1] \times I$, and a real-valued function $\phi(t)$ being Lebesgue-integrable on I , such that

- (i) $\dot{V}(x(t), t) \leq \phi(t), \quad \text{a.e. } t \in I, x \in [\Omega_2 - \bar{\Omega}_1],$
- (ii) $\int_{t_1}^{t_2} \phi(\tau) d\tau < \inf_{x \in \Omega_2^c} V(x, t_2) - \sup_{x \in \Omega_1} V(x, t_1), \quad \forall t_1, t_2 \in I, t_2 \geq t_1.$

Proof. For contradiction, we assume that there exists a point $x(t_i) \in \Omega_1$ and two times $t_i \in I$ and t_2 , such that the solution $x(t)$ passing through the point $(x(t_i), t_i)$ satisfies $x(t) \in \Omega_2$ for $t_i \leq t < t_2$ and $x(t_2) \notin \Omega_2$. Because the solution $x(t)$ is absolutely continuous and $\Omega_1 \subset \Omega_2$, there exists a time $t_1 \in (t_i, t_2)$ such that $x(t_1) \in \partial\Omega_1$. Therefore, $x(t_1) \in [\Omega_2 - \bar{\Omega}_1]$.

Since $V(x, t)$ is Lipschitz continuous, the composed function $V(x(t), t)$ is absolutely continuous as well. Therefore we obtain

$$V(x(t_2), t_2) = V(x(t_1), t_1) + \int_{t_1}^{t_2} \dot{V}(x(\tau), \tau) d\tau.$$

According to hypothesis (i), we have

$$V(x(t_2), t_2) \leq \sup_{x \in \Omega_1} V(x, t_1) + \int_{t_1}^{t_2} \phi(\tau) d\tau.$$

From hypothesis (ii) it follows that

$$V(x(t_2), t_2) < \sup_{x \in \Omega_1} V(x, t_1) + \inf_{x \in \Omega_2^c} V(x, t_2) - \sup_{x \in \Omega_1} V(x, t_1) = \inf_{x \in \Omega_2^c} V(x, t_2),$$

which implies $x(t_2) \in \Omega_2$. This is a contradiction and the proof is completed. \square

Theorem 3.3. Assume (H1) and (H2). The system (3.1) is GP-unstable with respect to $(\Omega_1, \Omega_2, t_0, T)$, if there exist a real-valued Lyapunov-like function $V(x, t)$ satisfying local Lipschitz continuity in $\Omega \times I$, a point $x_0 \in \Omega_1$, a time $t_1 \in (t_0, t_0 + T)$, a solution $x(t)$ passing through the point (t_0, x_0) , and a real-valued function $\phi(t)$ being Lebesgue-integrable on I , such that

- (i) $\dot{V}(x(t), t) \geq \phi(t), \quad \text{a.e. } t \in I, x \in \Omega_2,$
- (ii) $\int_{t_0}^{t_1} \phi(\tau) d\tau > \sup_{x \in \Omega_2^c} V(x, t_1) - V(x_0, t_0), \quad \forall t_1 \in I,$
- (iii) $V(x, t_1) \leq \sup_{x \in \Omega_2^c} V(x, t_1), \quad \forall t_1 \in I.$

Proof. Let $x_0 \in \Omega_1$ and the solution $x(t)$ pass through the point (x_0, t_0) . For contradiction, we assume that the solution $x(t)$ is always in Ω_2 for $t \in I$.

Since $x(t)$ is absolutely continuous and $V(x, t)$ is Lipschitz continuous, the composed function $V(x(t), t)$ is also absolutely continuous. Therefore for any $t_0 < t_1 < t_0 + T$

$$V(x(t_1), t_1) = V(x(t_0), t_0) + \int_{t_0}^{t_1} \dot{V}(x(\tau), \tau) d\tau.$$

According to hypothesis (i), we have

$$V(x(t_1), t_1) \geq V(x_0, t_0) + \int_{t_0}^{t_1} \phi(\tau) d\tau.$$

From hypothesis (ii) it follows that

$$V(x(t_1), t_1) > V(x_0, t_0) + \sup_{x \in \Omega_2^c} V(x, t_1) - V(x_0, t_0) = \sup_{x \in \Omega_2^c} V(x, t_1),$$

which contradicts with the hypothesis (iii), thus there exist a point x_0 and a time $t_1 \in (t_0, t_0 + T)$ such that $x(t_1) \notin \Omega_2$. This completes the proof. \square

4. Totally GP-stability analysis for perturbed Filippov-type systems

In the section we investigate the totally GP-stability of the following perturbed Filippov-type system

$$\dot{x}(t) = f(x, t) + u(x, t). \quad (4.1)$$

Theorem 4.1. Assume (H1)–(H3). System (4.1) is totally GP-stable with respect to $(\Omega_1, \Omega_2, \varepsilon, t_0, T)$, if there exist a real-valued Lyapunov-like function $V(x, t)$ satisfying local Lipschitz continuity in $\Omega \times I$, and two Lebesgue-integrable functions $\phi(t)$ and $\eta(t)$ on I , such that

- (i) $\dot{V}(x(t), t)|_{u=0} \leq \phi(t), \quad \text{a.e. } t \in I, x \in \Omega_2,$
- (ii) $\|\nabla V(x(t), t)\| \leq \eta(t), \quad \text{a.e. } t \in I, x \in \Omega_2,$
- (iii) $\int_{t_0}^t \phi(\tau) + \varepsilon \eta(\tau) d\tau < \inf_{x \in \Omega_2^c} V(x, t) - \sup_{x \in \bar{\Omega}_1} V(x, t_0), \quad \forall t \in I.$

Proof. Let $x(t)$ be a solution of Eq. (4.1) with initial condition $x(t_0) \in \Omega_1$. For contradiction, we assume that there exists a first exit time $t_1 \in I$ such that $x(t) \in \Omega_2$ for $t_0 \leq t < t_1$ and $x(t_1) \notin \Omega_2$. Therefore this point $x(t_1)$ is on the boundary of the open set Ω_2 . From the absolutely continuity of the functions $x(t)$ and $V(x, t)$, it follows that

$$V(x(t_1), t_1) = V(x(t_0), t_0) + \int_{t_0}^{t_1} \dot{V}(x(\tau), \tau) d\tau.$$

According to the estimates in hypotheses (i) and (ii), we obtain

$$V(x(t_1), t_1) \leq \sup_{x \in \bar{\Omega}_1} V(x, t) + \int_{t_0}^{t_1} (\phi(\tau) + \varepsilon \eta(\tau)) d\tau.$$

Finally, in view of the hypothesis (iii), we have

$$V(x(t_1), t_1) < \inf_{x \in \Omega_2^c} V(x, t_1),$$

which implies that $x(t_1) \in \Omega_2$. This is a contradiction. Therefore, $x(t) \in \Omega_2$ holds for all $t \in I$. \square

Theorem 4.2. Assume (H1)–(H3). System (2.4) is totally uniformly GP-stable with respect to $(\Omega_1, \Omega_2, \varepsilon, T)$, if there exist a real-valued Lyapunov-like function $V(x, t)$ satisfying local Lipschitz continuity in $\Omega \times I$, and two Lebesgue-integrable functions $\phi(t)$ and $\eta(t)$ on I , such that

- (i) $\dot{V}(x(t), t)|_{u=0} \leq \phi(t), \quad \text{a.e. } t \in I, x \in [\Omega_2 - \bar{\Omega}_1],$
- (ii) $\|\nabla V(x(t), t)\| \leq \eta(t), \quad \text{a.e. } t \in I, x \in [\Omega_2 - \bar{\Omega}_1],$
- (iii) $\int_{t_1}^{t_2} \phi(\tau) + \varepsilon \eta(\tau) d\tau < \inf_{x \in \Omega_2^c} V(x, t_2) - \sup_{x \in \bar{\Omega}_1} V(x, t_1), \quad \forall t_1, t_2 \in I, t_2 \geq t_1.$

Proof. Let u be a perturbation function satisfying $\|u(x, t)\| \leq \varepsilon$ for $x \in [\Omega_2 - \bar{\Omega}_1], t \in I$, a.e. For contradiction, we assume that there exist a point $(x_0, t_i) \in \Omega_1 \times I$, and a time t_2 such that the solution $x(t)$ of Eq. (4.1) passing through (x_0, t_i) satisfies $x(t) \in \Omega_2$ for $t_i < t < t_2$ and $x(t_2) \notin \Omega_2$. Because the solution $x(t)$ is absolutely continuous, there exists a first exit time $t_1 \in (t_0, t_2)$ such that $x(t_1) \in \partial\Omega_1$ which means $x(t_1) \in [\Omega_2 - \bar{\Omega}_1]$.

Since $V(x, t)$ is Lipschitz continuous, the composed function $V(x(t), t)$ is also absolutely continuous. Therefore we have

$$V(x(t_2), t_2) = V(x(t_1), t_1) + \int_{t_1}^{t_2} \dot{V}(x(\tau), \tau) d\tau.$$

According to hypotheses (i) and (ii), we have

$$V(x(t_2), t_2) \leq \sup_{x \in \bar{\Omega}_1} V(x, t_1) + \int_{t_1}^{t_2} (\phi(\tau) + \varepsilon \eta(\tau)) d\tau.$$

From hypothesis (iii) it follows that

$$V(x(t_2), t_2) < \sup_{x \in \bar{\Omega}_1} V(x, t_1) + \inf_{x \in \bar{\Omega}_2^c} V(x, t_2) - \sup_{x \in \bar{\Omega}_1} V(x, t_1) = \inf_{x \in \bar{\Omega}_2^c} V(x, t_2),$$

which implies $x(t_2) \in \bar{\Omega}_2$. This is a contradiction the proof is finished. \square

5. Applications to brake system

In this section, we apply our theory to investigate the uniformly GP-stability and the totally GP-stability of the brake model mentioned in Section 1.

Let J be an open interval containing origin in its interior. Define a semi-disk $\Omega^\pm = \Omega \cap \{(x_1, x_2)^T : \pm x_1 > 0\}$ and denote its closure by $\bar{\Omega}^\pm$. Rewrite Eq. (1.2) as

$$\dot{x} = f(x, \lambda), \quad x = (x_1, x_2)^T, \quad \lambda \in J. \quad (5.1)$$

In [28] the piecewise defined function $f = f^\pm$ for $\pm x_1 > 0$ is assumed to have a precise formulation as

$$f^\pm(x, \lambda) = A^\pm(\lambda)x + (g_1^\pm(x, \lambda), g_2^\pm(x, \lambda))^T, \quad (5.2)$$

where $A^\pm(\lambda)$ is a 2×2 matrix of the form

$$A^\pm(\lambda) = \begin{pmatrix} a_{11}^\pm(\lambda) & a_{12}^\pm(\lambda) \\ a_{21}^\pm(\lambda) & a_{22}^\pm(\lambda) \end{pmatrix}, \quad (5.3)$$

the higher order terms g^\pm are C^r smooth and satisfy $|g_{1,2}^\pm| = \mathcal{O}(x_1^2 + x_2^2)$ as $(x_1, x_2) \rightarrow 0$.

In [28] the existence of bifurcating periodic orbits of Eq. (5.1) is studied under the following assumptions.

(A1) $f^\pm(0, \lambda) \equiv 0$ for all $\lambda \in J$.

(A2) The matrix $A^\pm(\lambda)$ possesses a pair of complex eigenvalues $\alpha^\pm(\lambda) \pm i\omega^\pm(\lambda)$, and there exists a constant $\omega^* > 0$ such that $\omega^\pm(\lambda) \geq \omega^*$ for $\lambda \in J$.

(A3) $a_{12}^\pm(0) > 0$.

$$(A4) \quad \frac{\alpha^+(0)}{\omega^+(0)} + \frac{\alpha^-(0)}{\omega^-(0)} = 0, \quad \left(\frac{\alpha^+(\lambda)}{\omega^+(\lambda)} + \frac{\alpha^-(\lambda)}{\omega^-(\lambda)} \right)' \Big|_{\lambda=0} \neq 0.$$

In [28] the following theory is obtained.

Lemma 5.1. (See [28, Theorem 4.4].) Assume (H2) and (A1)–(A4). At $\lambda = 0$ there bifurcates a continuous branch of periodic orbits from the origin; i.e. there is a constant $\delta_0 > 0$ and a uniquely determined continuous function $\lambda^* : (-\delta_0, \delta_0) \rightarrow \mathbb{R}$ satisfying $\lambda^*(0) = 0$ such that for each $x_2 \in (-\delta_0, \delta_0)$ there is a periodic orbit of Eq. (5.1) passing through $(0, x_2)$ at the parameter $\lambda = \lambda^*(x_2)$ with period $\tilde{T}(x_2, \lambda^*(x_2))$. The function \tilde{T} is continuous and satisfies $\tilde{T}(0, 0) = \frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)}$. Moreover there is no other periodic orbit of system (5.1) locally near $x_1 = x_2 = 0$ and $\lambda = 0$.

In [28], it is pointed out that the bifurcating periodic orbits are asymptotically stable in forward time from numerical computation. Therefore, they are asymptotically unstable in backward time. In the following we investigate the uniformly GP-stability and totally GP-stability near the bifurcating periodic orbits in the interval $(t_0 + T, t_0]$ with $T < 0$.

Fix $0 < x_2^* < \delta_0$, there is a unique bifurcating periodic orbit Γ^* of Eq. (5.1) passing through the point $(0, x_2^*)$ at parameter value $\lambda = \lambda^*(x_2^*)$.

We choose the domains Ω_1 and Ω_2 as follows. Take some constants

$$0 < x_2^{2-} < x_2^{1-} < x_2^* < x_2^{1+} < x_2^{2+} < \delta_0.$$

From Lemma 5.1 it follows that there exist four periodic orbits $\Gamma^{1\pm}$ and $\Gamma^{2\pm}$ passing through the points $(0, x_2^{1\pm})$ and $(0, x_2^{2\pm})$, respectively, with corresponding parameters determined by the function $\lambda = \lambda^*(x_2)$. The phase portrait of these five periodic orbits is shown in Fig. 5.1 by virtue of numerical computations and the parameter correspondence is depicted in Fig. 5.2.

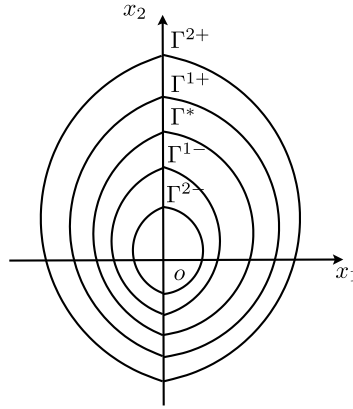


Fig. 5.1. The phase portrait of the five periodic orbits.

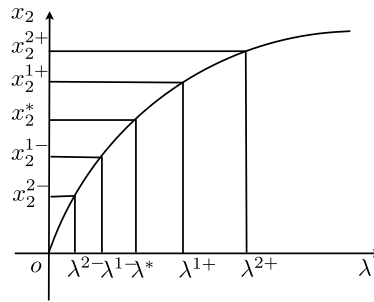


Fig. 5.2. Illustration of the parameter correspondence.

Let Ω_1 be the open region between the two periodic orbits Γ^{1+} and Γ^{1-} and Ω_2 be the open region between the two periodic orbits Γ^{2+} and Γ^{2-} .

Obviously, the symmetric matrix $A^+(\lambda)A^+(\lambda)^T$ possesses two real eigenvalues $\mu_{\pm}^+(\lambda)$ and direct computation gives

$$\mu_{\pm}^+(\lambda) = \pm(\alpha^+(\lambda)^2 + \beta^+(\lambda)^2).$$

Therefore $\|A^+(\lambda)\| = \sqrt{\alpha^+(\lambda)^2 + \beta^+(\lambda)^2}$. Here $\|\cdot\|$ represents the matrix norm associated with the standard Euclidian norm of a vector. Similarly we have $\|A^-(\lambda)\| = \sqrt{\alpha^-(\lambda)^2 + \beta^-(\lambda)^2}$. According to the smoothness assumption (H2) we know that there exists a constant $C^* > 0$ such that

$$\|A^{\pm}(\lambda)\| \leq C^*, \quad \text{for } \lambda \in J.$$

Noticing the asymptotically property of function g^{\pm} , we know there exists a constant $M > 0$ such that $\|g^{\pm}\| \leq M\|x\|^2$ for $x \in \Omega$, if necessary we may shrink the domain Ω .

For further analysis we introduce the polar coordinates (r, θ) and Eq. (5.1) is equivalent to

$$\dot{r} = \frac{(x_1, x_2)[A^{\pm}(\lambda)(x_1, x_2)^T + g^{\pm}(x_1, x_2, \lambda)]}{r}, \quad (5.4)$$

$$\dot{\theta} = \frac{(x_1, -x_2)[A^{\pm}(\lambda)(x_1, x_2)^T + g^{\pm}(x_1, x_2, \lambda)]}{r^2}. \quad (5.5)$$

By (r^*, θ) we denote the periodic orbit Γ^* in polar coordinates and we have the estimate

$$|dr^*/d\theta| \leq \tilde{C}^*, \quad (5.6)$$

where \tilde{C}^* is a constant depending on the orbit Γ^* . Define a Lyapunov-like function

$$V(r) = (r - r^*(\theta))^2, \quad (r, \theta) \in \Omega, \quad (5.7)$$

which is used to analysis the stabilities in the following. Now we are ready to state and prove our main results of this work.

Theorem 5.2. Assume (H1), (H2) and (A1)–(A4). For any bifurcating periodic orbit Γ^* of Eq. (5.1) there exist two domains Ω_1, Ω_2 ($\Gamma^* \subset \Omega_1 \subset \Omega_2 \subset \Omega$) and a time $T < 0$ such that the system (5.1) is uniformly GP-stable with respect to (Ω_1, Ω_2, T) .

Proof. Since system (5.1) is autonomous, uniformly GP-stability is the same as GP-stability. We will prove that system (5.1) is GP-stability with respect to $(\Omega_1, \Omega_2, t_0 = 0, T)$.

For any $x_0 \in \Omega_1$, let $x(t) = (r(t), \theta(t))$ in polar coordinates be the solution of Eq. (5.1) passing through x_0 at $t_0 = 0$. We calculate the derivative of the function V with respect to t along this orbit for $t < 0$,

$$\dot{V}(r) = 2(r - r^*(\theta)) \left(\dot{r} - \frac{dr^*}{d\theta} \dot{\theta} \right).$$

Noticing Eq. (5.4) and Eq. (5.5), directly calculation leads to the following estimates

$$|\dot{r}| \leq C^*r + Mr^2, \quad |\dot{\theta}| \leq C^* + Mr. \quad (5.8)$$

Let $d_1 = \max_{(r, \theta) \in [\Omega_2 - \Omega_1]} |r - r^*(\theta)|$, then we obtain

$$\dot{V}(x) \leq 2d_1 [C^*\beta + M\beta^2 + \tilde{C}^*(C^* + M\beta)],$$

where β is the maximal distance from the origin to the boundary of the domain Ω_2 .

Define

$$\phi(t) \equiv 2d_1 [C^*\beta + M\beta^2 + \tilde{C}^*(C^* + M\beta)].$$

Therefore the condition (i) in Theorem (3.1) is satisfied. Then the condition (ii) in theorem (3.1) is equivalent to

$$|t| \{2d_1 [C^*\beta + M\beta^2 + \tilde{C}^*(C^* + M\beta)]\} < d_2^2 - d_3^2,$$

where

$$d_2 = \min_{(r, \theta) \in [\Omega_2 - \Omega_1]} |r - r^*(\theta)|, \quad d_3 = \max_{(r, \theta) \in \Omega_1} |r - r^*(\theta)|.$$

Since the bifurcating periodic orbits continuously depend on the x_2 -variable, for any given $x_2^{2\pm}$ which determines the region Ω_2 , we may choose $x_2^{1\pm}$ close enough to x_2^* such that $d_3 \leq md_2$ ($m > 1$), which determines the region Ω_1 .

Choose $T < 0$ such that

$$|T| < \frac{d_2^2 - d_3^2}{2d_1 [C^*\beta + M\beta^2 + \tilde{C}^*(C^* + M\beta)]}. \quad (5.9)$$

Then all the conditions in Theorem 3.1 are satisfied and hence we complete the proof. \square

Next, we study the totally GP-stability of the following perturbed system

$$\dot{x} = f(x, \lambda) + u(x, t, \lambda), \quad x = (x_1, x_2)^T, \quad \lambda \in J. \quad (5.10)$$

Theorem 5.3. Assume (H1)–(H3) and (A1)–(A4). For any bifurcating periodic orbit Γ^* of Eq. (5.1) there exist two domains Ω_1 and Ω_2 such that $\Gamma^* \subset \Omega_1 \subset \Omega_2 \subset \Omega$, there is an $\varepsilon > 0$ such that $\|u(x, t, \lambda)\| \leq \varepsilon$, and for any given t_0 there is a time $T < 0$ such that the system (5.10) is totally GP-stable with respect to $(\Omega_1, \Omega_2, \varepsilon, t_0, T)$.

Proof. Fix $\varepsilon > 0$ and take any perturbation u satisfying $\|u(x, t, \lambda)\| \leq \varepsilon$. For any given $t_0 \in \mathbb{R}$ and any initial value $x_0 \in \Omega_1$, let $x(t) = (r(t), \theta(t))$ be the solution of Eq. (5.1) passing through x_0 at initial time t_0 . We calculate the derivative of the function V with respect to t along this orbit for $t < t_0$,

$$\dot{V}(x) = \dot{V}(x)|_{u=0} + (\nabla_x V(x))^T u(x, t, \lambda).$$

Let $\phi(t)$ be defined as in the proof of previous theorem, then the condition (i) in Theorem 4.1 is satisfied, i.e.

$$\dot{V}(x)|_{u=0} \leq \phi(t).$$

Clearly, $\nabla_x V(x) = (\partial V / \partial x_1, \partial V / \partial x_2)^T$, where for $i = 1, 2$,

$$\partial V / \partial x_i = 2(r - r^*(\theta)) (\dot{r} / \dot{x}_i - \dot{r}^* / \dot{x}_i). \quad (5.11)$$

According to the assumption (H2), (A1) and (A2), we know that $x = 0$ is the only trivial solution of Eq. (5.1) in the domain Ω . By virtue of the definition of domain Ω_2 we know that there exists a constant $1/N$ such that

$$|f_1^\pm(x, \lambda)| \geq 1/N, \quad |f_2^\pm(x, \lambda)| \geq 1/N, \quad \forall x \in \Omega_2, \quad (5.12)$$

where $f^\pm = (f_1^\pm, f_2^\pm)$ is the vector field of Eq. (5.1).

Noticing Eq. (5.11) and together with the estimates in Eq. (5.8) and Eq. (5.12), we obtain

$$\|\nabla_x V(x)\| \leq 2d_1[1 + N(C^*\|\Gamma^*\| + M\|\Gamma^*\|^2)],$$

where $\|\Gamma^*\|$ denotes the largest distance of the orbit Γ^* to the origin. Define a function

$$\eta(t) \equiv 2d_1[1 + N(C^*\|\Gamma^*\| + M\|\Gamma^*\|^2)],$$

which ensures the condition (ii) in Theorem 4.1.

The condition (iii) in Theorem 4.1 is equivalent to the following inequality for $t < t_0$,

$$(t_0 - t)\{2d_1[C^*\beta + M\beta^2 + \tilde{C}^*(C^* + M\beta) + \varepsilon + \varepsilon N(C^*\|\Gamma^*\| + M\|\Gamma^*\|^2)]\} < d_2^2 - d_3^2.$$

Similarly, we can choose the domain Ω_1 such that $d_3 \leq md_2 (m > 1)$. Take a $T < 0$ such that the following estimate holds

$$|T| < \frac{d_2^2 - d_3^2}{2d_1[C^*\beta + M\beta^2 + \tilde{C}^*(C^* + M\beta) + \varepsilon + \varepsilon N(C^*\|\Gamma^*\| + M\|\Gamma^*\|^2)]}.$$

Then all the conditions of Theorem 4.1 are satisfied. Therefore we complete the proof. \square

6. Numerical computation

In this section, we investigate two examples to illustrate the properties in previous sections.

Example 1. Consider the equation

$$\dot{x} = f^\pm(x, \lambda) = A^\pm(\lambda)x + g^\pm(x, \lambda), \quad \pm x_1 > 0 \quad (6.1)$$

with the following settings

$$A^\pm(\lambda) = \begin{pmatrix} 0 & 1 \\ -a^\pm & -b^\pm(\lambda) \end{pmatrix}, \quad g^\pm(x, \lambda) = \begin{pmatrix} 0 \\ -\kappa^\pm x_2^3 \end{pmatrix}.$$

Here, for simplicity, we take $a^+ = 0.1$, $a^- = 0.2$, $b^\pm(\lambda) = b_0^\pm - b_1^\pm \lambda$, where $b_0^+ = 0.05$, $b_0^- = -\frac{\sqrt{2}}{20}$, $b_1^\pm = 1$ and $\kappa^\pm = 1$.

In paper [28], by numerical computations it is observed that the bifurcating periodic orbits of Eq. (6.1) are asymptotically stable as time t increases. Therefore it is asymptotically unstable with exponential rate as time t decreases. In the previous section we proved that Eq. (6.1) is uniformly GP-stable in an interval $(T, 0]$ with some $T < 0$. In the following we will estimate the value T using numerical methods.

At $\lambda^* = 0.1827$ a bifurcating periodic orbit Γ^* with initial value $(0, x_2^* = 0.54444315280860)$ is obtained by numerical computation which is drawn in Fig. 6.1.

Take $\lambda^{2+} = 0.37$ and $\lambda^{2-} = 0.05$. By numerical computation we obtain two bifurcating periodic orbits Γ^{2+} and Γ^{2-} , which respectively intersect the positive x_2 -axis at

$$x_2^{2+} = 0.73236191493926, \quad x_2^{2-} = 0.29182489775345.$$

Let Ω_2 be the domain between these two periodic orbits. Similarly, we take $\lambda^{1+} = 0.192$ and $\lambda^{1-} = 0.175$. By numerical computation we obtain two bifurcating periodic orbits Γ^{1+} and Γ^{1-} , which respectively intersect the positive x_2 -axis at

$$x_2^{1+} = 0.55669715288258, \quad x_2^{1-} = 0.53395208960266.$$

Let Ω_1 be the domain between these two periodic orbits. See Fig. 6.2 for illustration of these domains.

We trace 35 orbits of Eq. (6.1) with initial values in the boundary of domain Ω_1 in backward time till they reach the boundary of domain Ω_2 , which are drawn in Fig. 6.3. Numerically we calculate the smallest time interval with $(T, 0] = (-10.03212965188638, 0]$.

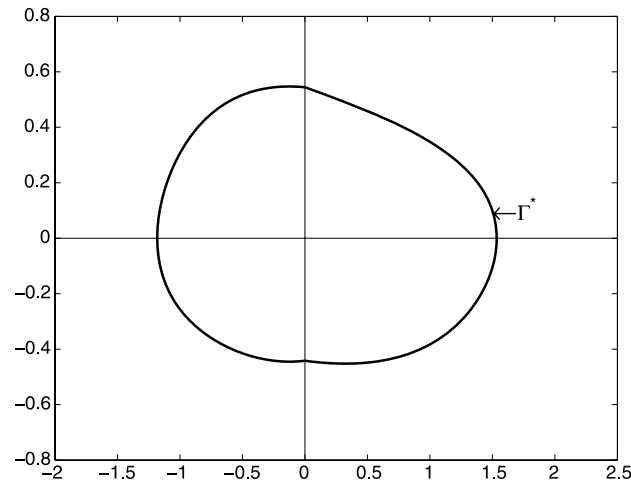


Fig. 6.1. The bifurcating periodic orbit Γ^* .

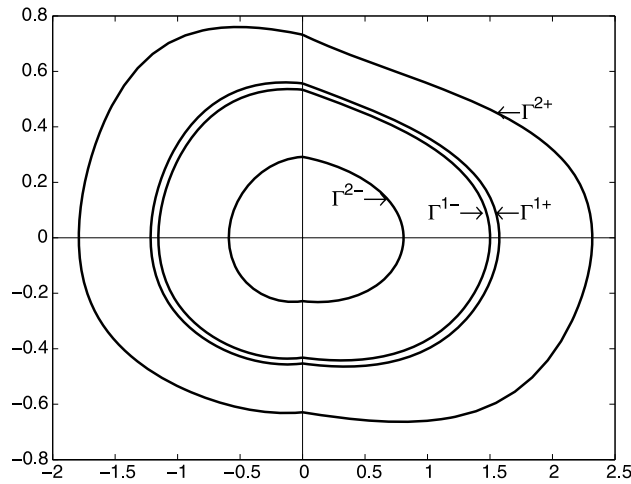


Fig. 6.2. Illustration of the domains Ω_1 and Ω_2 .

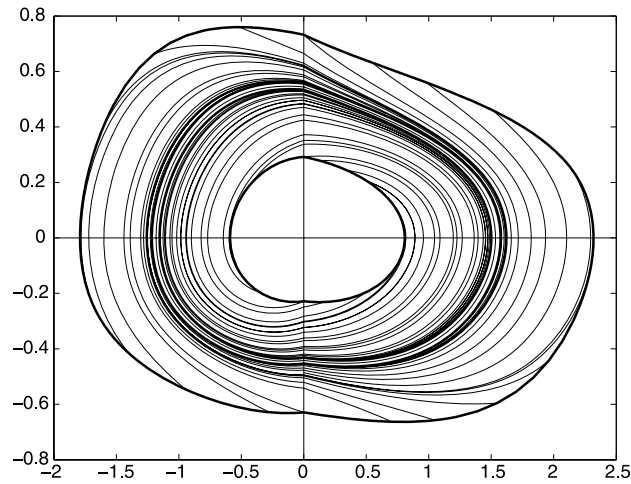


Fig. 6.3. Illustration of the uniformly GP-stability of Eq. (6.1).

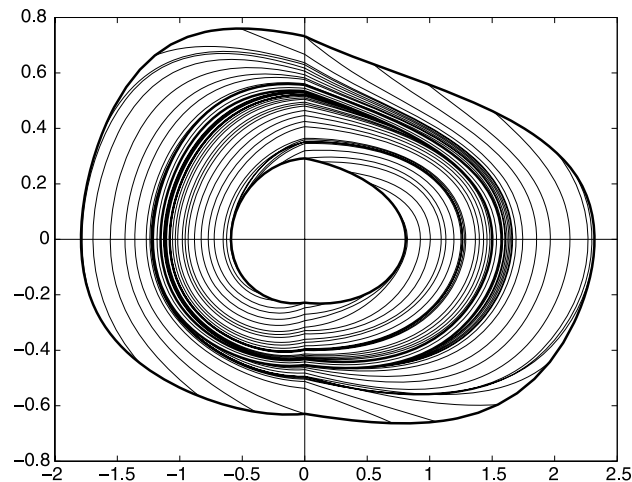


Fig. 6.4. Illustration of the totally GP-stability of Eq. (6.2).

Example 2. Consider the equation

$$\dot{x} = f^{\pm}(x, \lambda) + u(x, t, \lambda), \quad \pm x_1 > 0 \quad (6.2)$$

where the function f^{\pm} is defined as in Example 1 and $u = (0, \varepsilon \sin t)^T$. We also take the same domains Ω_1 and Ω_2 as in Example 1, cf. Fig. 6.2.

In this example we will illustrate the totally GP-stability of Eq. (6.2) by numerical simulations. Take the perturbation amplitude $\varepsilon = 0.01$ and initial time $t_0 = 0$. We trace 35 orbits of Eq. (6.2) with initial values in the boundary of domain Ω_1 in backward time till they reach the boundary of domain Ω_2 , which are drawn in Fig. 6.4. Numerically we calculate the smallest time interval with $(T, 0] = (-7.55591680870733, 0]$.

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